Light quark mass dependence of nucleon mass to two-loop order

Siwei Hu
Institute of High Energy Physics (IHEP)

In collaboration with Longbin Chen, Yu Jia, Zhewen Mo

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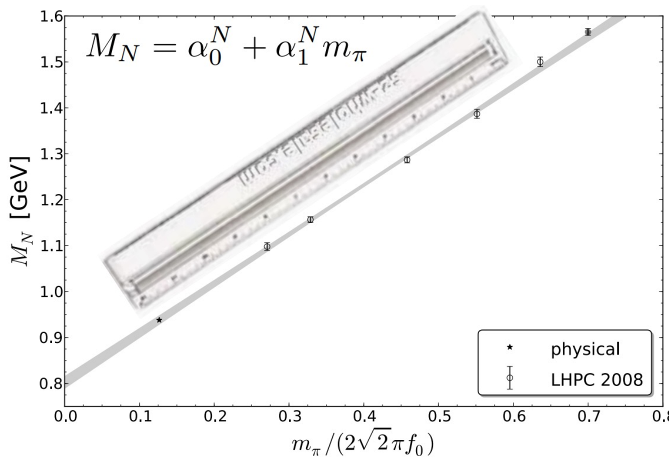
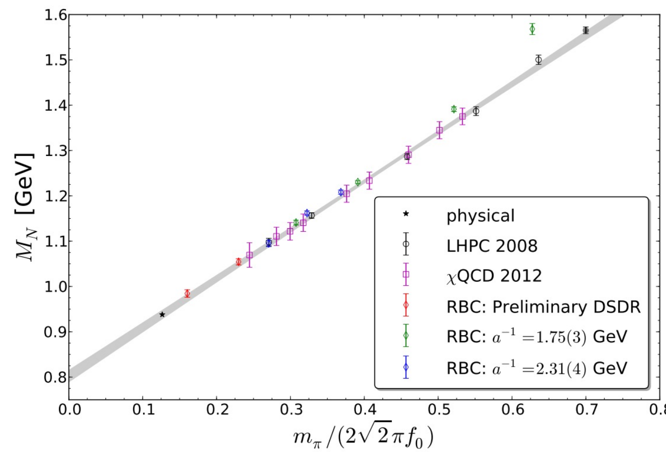
Contents

Nucleon mass

- Nucleon mass
  - $m_{p,n} = 938\ MeV$
  - Ruler approximation: $m_N = 800\ \text{MeV} + m_\pi$

$m_\pi \sim \sqrt{m_q}$





A. Walker-Load PoS LATTICE2013 (2014), 013

- Works well over a large range
- Wrong pion mass dependence — Wrong leading term, free of $\log M$
- Lattice cannot approach chiral limit

Chiral perturbative theory (ChPT)

- ChPT —— low energy EFT   Steven Weinberg Physica A 96 (1979)
  - Degree of freedom —— meson and baryon (pion and nucleon)
  - Mesonic sector   J. Gasser and H. Leutwyler *Annals Phys.* 158 (1984)

$$\mathcal{L}_{\pi\pi}^{(2)} = \frac{F^2}{4} Tr\left(\partial_\mu U \partial^\mu U\right) + \frac{F^2}{4} Tr\left(\chi U^\dagger + U^\dagger \chi\right), \quad U = \exp(i\frac{\vec{\pi}\cdot\vec{\sigma}}{F})$$

$$SU(2)_R \times SU(2)_L$$

  - Power counting -- consist with dimension

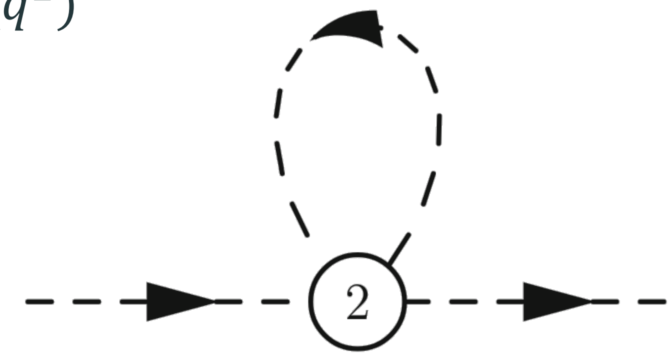
$$M_\pi = 136 \text{ MeV} \ll \Lambda_\chi = O(1\text{GeV}), \quad q \sim M_\pi/\Lambda_\chi$$

Well defined expansion

$$\partial_\mu \sim O(q), \quad M_\pi \sim O(q), \quad \chi = M_\pi^2 \sim O(q^2)$$

- $D = dim \times N_{Loop} - 2 \times N_I + \sum i \, N_{V_i}$

$$dim \times 1 - 2 \times 1 + 2 \rightarrow O(q^4)$$



Baryon Chiral perturbative theory (BChPT)

- baryon sector

J. Gasser et al. Nucl.Phys.B 307 (1988)

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\not{\partial} - m + \frac{g_A}{2F}\gamma_5 \not{\partial}\vec{\pi}\cdot\vec{\tau}\right)\Psi + \cdots$$

- Power counting breaking (PCB)
  - $M_\pi \ll \Lambda_\chi$, $m_N \sim \Lambda_\chi$
  - $\partial_\mu \pi \sim O(q)$, $M_\pi \sim O(q)$, $m_N \sim O(q^0)$, $\partial_\mu \Psi \sim O(q^0)$

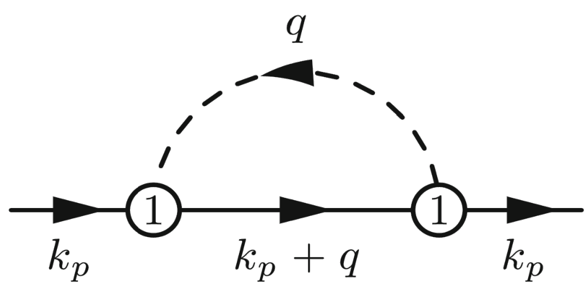
$$iS_F(p) = \frac{i}{\not{p} - m + i0^+} \quad \Rightarrow \quad D = dim\times N_L - 2\times N_{I,\pi} - 1\times N_{I,N} + \sum i\, N_{V_i}$$

$$dim\times 1 - 2\times 1 - 1\times 1 + 2 \rightarrow O(q^3)$$

PCB terms

$$\underbrace{2m^3\left[\log\left(\frac{\mu^2}{m^2}\right)\right]}_{O(q^0)} + \underbrace{2mM_\pi^2\left[1 + \log\left(\frac{\mu^2}{m^2}\right)\right]}_{O(q^2)}$$

$$\underbrace{-\frac{1}{m}\left\{\log\left(\frac{M_\pi^2}{m^2}\right)M_\pi^4 + 2M_\pi^3\sqrt{4m^2 - M_\pi^2}\arccos\left(\frac{M_\pi}{2m}\right)\right\}}_{O(q^3)+O(q^4)+\cdots}.$$



Power counting breaking

A new scale get involved in the Feynman Integral

Renormalization

- PCB – ill-defined cutoff of Feynman diagram
- Heavy Baryon ChPT (HBChPT)

  V. Bernard et al. Nucl.Phys.B 388 (1982)

  E.E. Jenkins and A.V. Manohar Phys.Lett.B 255 (1991)

  - $p = mv + k, \quad v = (1,0,0,0), \quad m \to \infty$
  - Non-relativistic

$$iS_F(p) = \frac{i}{\not{p} - m + i0^+} \Longrightarrow i\frac{1+\not{v}}{2v\cdot k + i\varepsilon}$$

- Infrared Regularization (IR)

  T. Becher and H. Leutwyler Eur.Phys.J. 9 (1999)

  - Discard the contribution near the pole $p^2 = m^2$ (regular part).
  - Introduce un-physical cut, poor convergence
- Extended-on-mass-shell scheme (EOMS)

  J. Gegelia and G. Japaridze Phys.Rev.D 60 (1999)

  T. Fuchs et al. Phys.Rev.D 68 (2003)

  - counterterms absorb only the divergence and PCB terms
  - Keeps the relativity and analyticity
  - Phenomenological success

  L. Durand and P. Ha Phys.Rev.D 58 (1998)

  L.S Geng et al. Phys.Rev.Lett. 101 (2008)

  L.S Geng Front.Phys.(Beijing) 8 (2013)

Nucleon self-energy

- Nucleon mass:

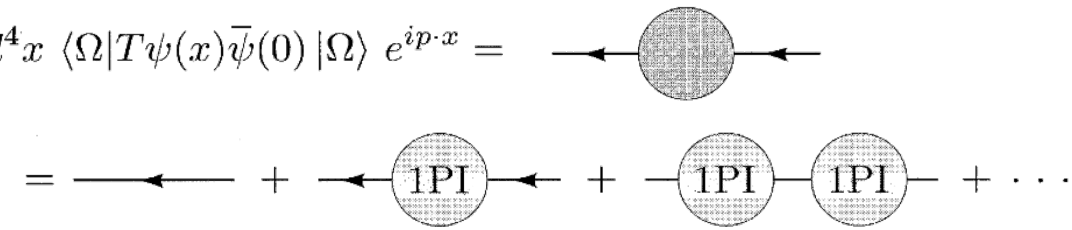
$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4$$ ⟸ Tree and one loop

$$+ \; k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 + \; k_7 M^6 \ln^2\frac{M}{\mu} + k_8 M^6 \ln\frac{M}{\mu} + k_9 M^6$$ ⟸ Two loop

- HBChPT:
  - Up to $O(q^5)$, the correction can be absorbed by g_A. J.A. McGovern and M.C. Birse Phys.Lett.B 446 (1999)
- IR:
  - Up to $O(q^6)$ M.R. Schindler et al. Phys.Lett.B 649 (2007)
- EOMS:
  - One Loop T. Fuches et al. Phys.Rev.D 68 (2003)
  - Two loop N.D. Conard et al. PoS CD2021 (2024)

- m_N
  - 1PI self-energy

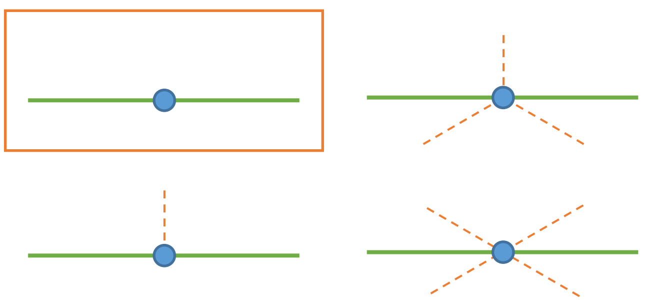
$$\int d^4x \, \langle \Omega | T \psi(x) \bar{\psi}(0) | \Omega \rangle \, e^{ip\cdot x} =$$



$$\{ \not{p} - m_{\mathrm{B}} - \Sigma(\not{p}, m_{\mathrm{B}}) \} \big|_{\not{p} = m_N} = 0$$

$$m_N = m_{\mathrm{B}} + \Sigma_{\mathrm{c}} + \hbar \Sigma^{(1)} + \hbar^2 \Sigma^{(2)} + \mathcal{O}(\hbar^3)$$

Contact term



$$\Sigma_c = -4c_1 M^2 + \hat{e}_1 M^4 + \hat{g}_1 M^6$$

Absorb the contact terms into propagator mass $\tilde{m} = m_{\mathrm{B}} + \Sigma_c$

$$\frac{i}{\not{p} - m - \Sigma_c} = \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m} i\Sigma_c \frac{i}{\not{p} - m} + \ldots$$

- $\Sigma^{(2)}$

  Three scales $(m_N, \tilde{m}, M)$ ➡ Two scales

$$\hbar^2\tilde{\Sigma}^{(2)}(m_N, \tilde{m}) = \hbar^2\tilde{\Sigma}^{(2)}(\tilde{m} + \mathcal{O}(\hbar), \tilde{m})$$
$$= \hbar^2\tilde{\Sigma}^{(2)}(\tilde{m}, \tilde{m}) + \mathcal{O}(\hbar^3)$$

- Method of region (MOR)

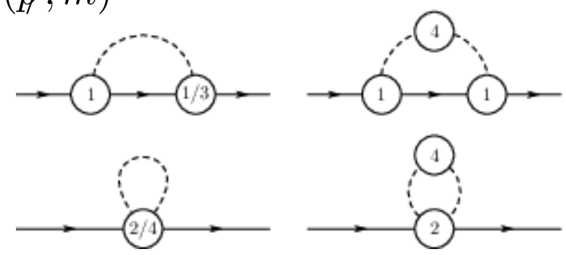
  two regions for each integration

  $l \sim M_\pi$ ➡ soft $\sim$ IR singular part

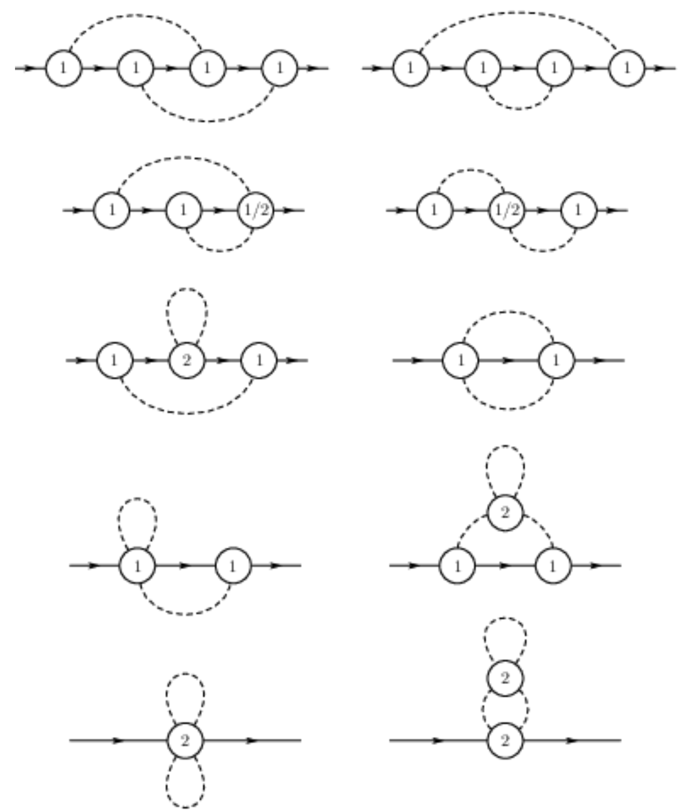
  $l \sim m_N$ ➡ hard $\sim$ IR regular part

  Contain PCB

$\tilde{\Sigma}^{(1)}(\not{p}, \tilde{m})$

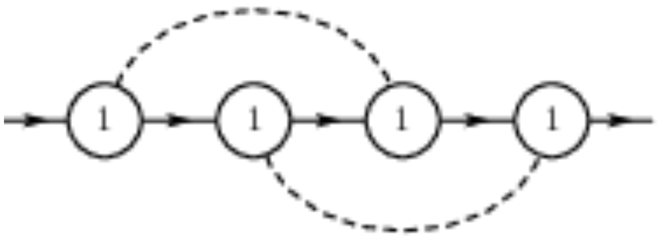


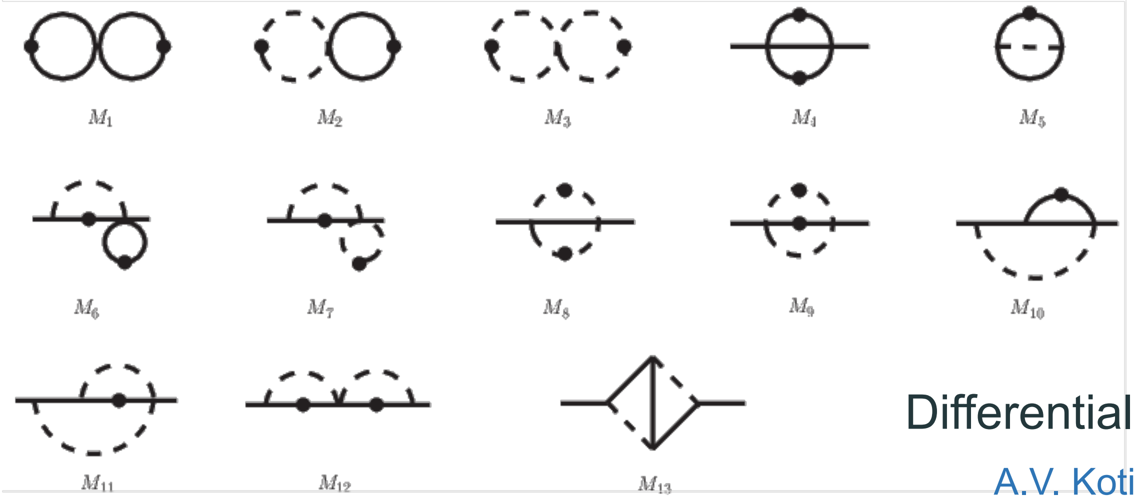
$\tilde{\Sigma}^{(2)}(\not{p}, \tilde{m})$



Analytic feynman integrals

- Modern technology
  - Using IBP reduce to MIs
  - DEs for MIs
  - AMFlow for numeric check





Differential equation method

A.V. Kotikov Phys.Lett.B 254 (1991)

A.V. Kotikov Phys.Lett.B 267 (1991)

J.M. Henn Phys.Rev.Lett. 110 (2013)

Numeric check: AMFlow

L. Xiao and Y.Q. Ma
Comput.Phys.Commun. 283 (2023)

Renormalization (EOMS)

- Bare perturbative theory
  - Substitute the bare quantity

$$l_{i,\mathrm{B}} = l_{i,\mathrm{R}} + \delta_{\mathrm{div}} l_i, \quad \} \text{ mesonic sector}$$

$$\left.\begin{aligned} x_{j,\mathrm{B}} &= x_{j,\mathrm{R}} + \delta_{\mathrm{div}} x_j + \delta_{\mathrm{eoms}} x_j, \qquad x_j \in \{c_j, d_j, e_j\} \\ m_{\mathrm{B}} &= m + \delta_{\mathrm{div}} m + \delta_{\mathrm{eoms}} m, \\ \mathrm{g}_{\mathrm{A},\mathrm{B}} &= \mathrm{g}_{\mathrm{A}} + \delta_{\mathrm{div}} \mathrm{g}_{\mathrm{A}} + \delta_{\mathrm{eoms}} \mathrm{g}_{\mathrm{A}}, \end{aligned}\right\} \text{ baryon sector}$$

  - Use δ_{div} to cancel the divergence
  - Find out PCB terms for each diagram, and use δ_{eoms} to cancel

- πN scattering v.s. Nucleon self-energy

D.L. Yao et al. Phys.Rev.D 87 (2013)

EOMS counterterms

- Bare perturbative theory

tree diagram

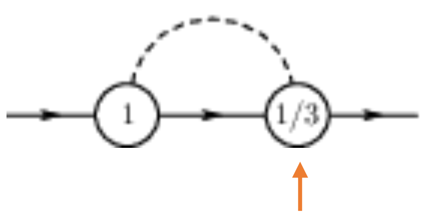
$$\delta_{div} = (\dots)\frac{\hbar}{\varepsilon} + (\dots)\frac{\hbar^2}{\varepsilon^2} + (\dots)\frac{\hbar^2}{\varepsilon}$$

$$\delta_{eoms} = (\dots)\hbar + (\dots)\hbar^2$$

one loop diagram

$$\delta_{div} = (\dots)\frac{\hbar}{\varepsilon}$$

$$\delta_{eoms} = (\dots)\hbar + (\dots)\hbar\varepsilon$$



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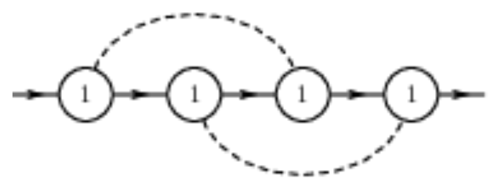


$$g_{A,B} = g_{A,R} + \delta_{div} g_A + \delta_{eoms} g_A$$

$$\frac{\hbar}{\varepsilon} \times (\dots)\hbar\varepsilon$$

loop × counterterms

+ Finite PCB term $(\dots)\hbar^2$

πN Coupling and $\pi N \rightarrow \pi N$ need to be calculated to $O(\varepsilon)$

- All PCB terms can be removed by counterterms which free of $\log M$

A non-trivial check

Result

- Tree level and one loop

$$
\begin{aligned}
m_N &= m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + \underline{k_4 M^4} \\
&+ k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 \\
&+ k_7 M^6 \ln^2\frac{M}{\mu} + k_8 M^6 \ln\frac{M}{\mu} + k_9 M^6,
\end{aligned}
$$

$\hat{e}_1 M^4$

$$
\begin{aligned}
k_1 &= -4c_1, \\
k_2 &= -\frac{3\mathtt{g_A}^2}{32F^2\pi}, \\
k_3 &= -\frac{3\mathtt{g_A}^2}{32\pi^2F^2m} + \frac{3(8c_1 - c_2 - 4c_3)}{32\pi^2F^2}, \\
k_4 &= \frac{3\mathtt{g_A}^2(1+4c_1m)}{32\pi^2F^2m} + \frac{3c_2}{128\pi^2F^2} \boxed{- \hat{e}_1,}
\end{aligned}
$$

T. Fuches et al. Phys.Rev.D 68 (2003)

Result

- Two loop

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + \underbrace{k_4 M^4}_{\hat{e}_1 M^4} + k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 + k_7 M^6 \ln^2\frac{M}{\mu} + k_8 M^6 \ln\frac{M}{\mu} + \underbrace{k_9 M^6}_{\hat{g}_1 M^6},$$

$$k_5 = \frac{3g_A{}^2}{1024\pi^3 F^4}\left(16g_A{}^2 - 3\right),$$

$$k_6 = \frac{17 g_A{}^4}{512\pi^3 F^4} - \frac{3\hat{d}_{16} g_A}{8\pi F^2} + \frac{3g_A{}^2\left[\pi^2 F^2 + 2m^2 + 8m^2\pi^2\left(2l_4 - 3l_3\right)\right]}{256\pi^3 F^4 m^2},$$

$$k_7 = -\frac{3}{256\pi^4 F^4 m}\left[g_A{}^2 - m(6c_1 - c_2 - 4c_3)\right],$$

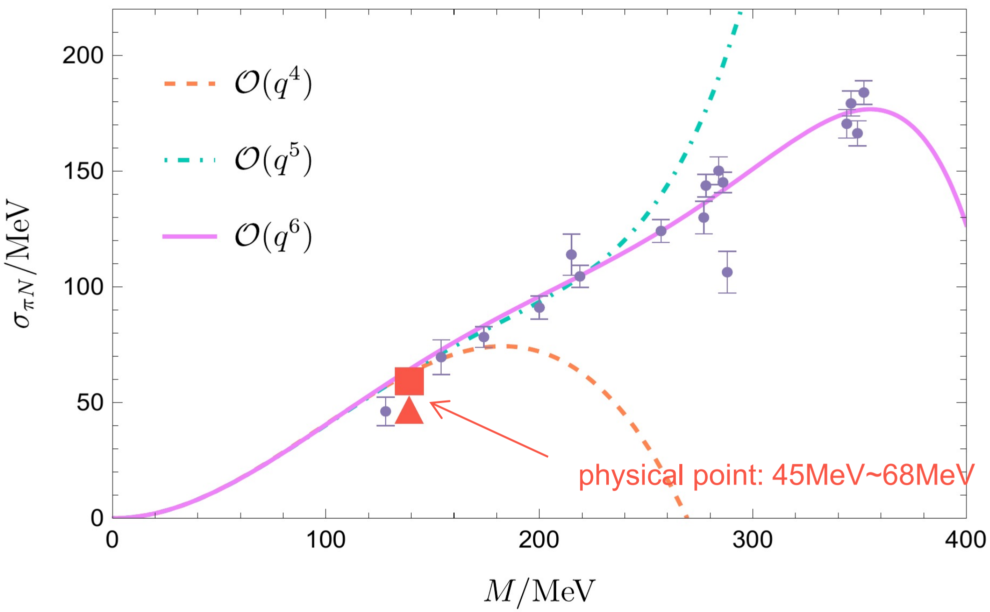
$$\begin{aligned} k_8 = &\frac{-3}{8\pi^2 F^2 m^2}\left[c_1 g_A{}^2 + m\left(\hat{d}_{16} g_A - 2c_1 c_2\right) + \left(8e_{14} + 2e_{15} + e_{16} + 2\hat{e}_{20} + 4\hat{e}_{36} + 4\hat{e}_{38}\right) m^2\right] \\ &+ \frac{1}{3072 F^4 m^2 \pi^4}\left[114 g_A{}^4 + \left(51 - 576\pi^2\left(2l_3 - l_4\right)\right) g_A{}^2\right. \\ &+ 36 c_1 m\left(-6 - 25 g_A{}^4 + 10 g_A{}^2 + 128\pi^2\left(l_3 - l_4\right)\right) - c_2 m\left(576\pi^2\left(2l_3 - l_4\right) + 23 g_A{}^2\right) \\ &\left.+ 4c_3 m\left(-9 + 41 g_A{}^2 - 576\pi^2\left(2l_3 - l_4\right)\right) - 4c_4 m\left(21 + 13 g_A{}^2\right) - 18\right], \end{aligned}$$

$$\begin{aligned} k_9 = &\hat{g}_1 + \frac{c_1\left(768 l_3 + 7\right) - 4\left(96 c_3 l_3 + 12 c_2 l_4 + c_4\right)}{1024\pi^2 F^4} + 5\frac{\left(225 c_1 + 4c_4\right) m + 6}{12288\pi^4 F^4 m} \\ &+ \frac{g_A{}^2\left(704 c_1^3 + 3\hat{e}_1\right)}{32\pi^2 F^2} + \frac{g_A{}^2\left(144 c_1\left(128 l_3 - 128 l_4 - 3\right) + 83 c_2 + 512 c_3 - 928 c_4\right)}{24576\pi^2 F^4} \\ &- \frac{\left(6210 c_1 - 1147 c_2 + 616 c_3 - 948 c_4\right) g_A{}^2}{12288\pi^4 F^4} - \frac{g_A{}^2}{128\pi^2 F^2 m^3} \\ &+ \frac{g_A{}^2\left(-1 + 18 l_3 - 12 l_4\right)}{64\pi^2 F^4 m} + \frac{c_1 g_A{}^4\left(96\pi^2 \ln 2 - 144\zeta_3 - 164\pi^2 + 1779\right)}{4096\pi^4 F^4} + \frac{3c_1 \hat{d}_{16} g_A}{2\pi^2 F^2} \\ &+ \frac{g_A{}^2\left(18 - \left(41\pi^2 + 145\right) g_A{}^2\right)}{2048\pi^4 F^4 m} + \frac{6e_{15} + 5e_{16} + 6\hat{e}_{20}}{32\pi^2 F^2} + \frac{-24 c_1^2 g_A{}^2 + 6\hat{d}_{16} g_A - 3c_1 c_2}{16\pi^2 F^2 m}, \end{aligned}$$

- m, $\hat{e}_1$ and $\hat{g}_1$ are undetermined
- Leading logarithms consist with IR and HBChPT

Sigma term

- $\sigma_{\pi N} = \langle N|H_{s.b.}|N\rangle$
  - Directly related to the chiral symmetry breaking
  - $\sigma_{\pi N} = \langle N|m_q(\bar{u}u + \bar{d}d)|N\rangle = M^2 \frac{\partial m_N}{\partial M^2}$ (Hellmann-Feynman theorem)



$m_N = m - 4c_1M^2 + \cdots$

- m, $\hat{e}_1$ and $\hat{g}_1$ are undetermined
- Use $\sigma_{\pi N}$ to fit $\hat{e}_1$ and $\hat{g}_1$

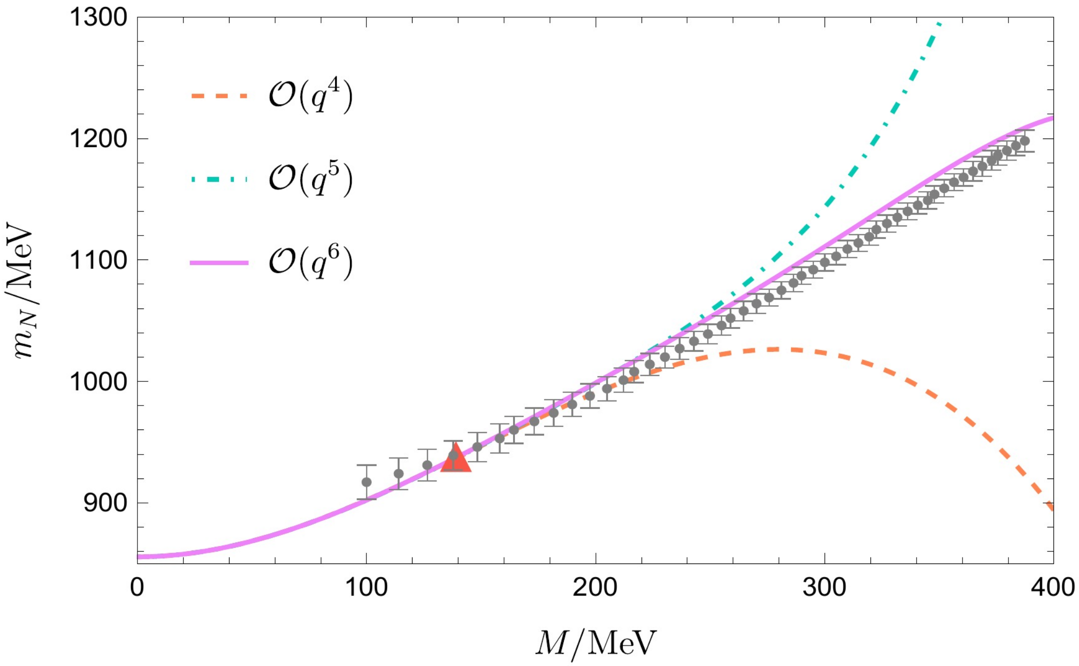
$c_{1\sim4}$, d_{18}, $e_{14\sim16}$, $\hat{e}_{20,36,38}$
from one loop πN scattering fitting
D.L. Yao et al. Phys.Rev.D 87 (2013)

Lattice data points and ▲ :
A. Agadjanov et al. Phys.Rev.Lett. 131 (2023)

■ Roy-Steiner equation:
M. Hoferichter et al. Phys.Rev.Lett. 115 (2015)

Nucleon mass

- BChPT vs lattice
  - $m = 856.6 \pm 1.7\ MeV$



$O(q^6)$ consistent with lattice result

$$
\begin{aligned}
m_N &= m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4 \\
&+ k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 \\
&+ k_7 M^6 \ln^2\frac{M}{\mu} + k_8 M^6 \ln\frac{M}{\mu} + k_9 M^6,
\end{aligned}
$$

$$
\begin{aligned}
& m_N/\mathrm{MeV} \\
&= \underbrace{856.6}_{m} + \underbrace{111}_{\mathcal{O}(q^2)} + \underbrace{(-14.4)}_{\mathcal{O}(q^3)} \\
&+ \underbrace{(-9.40) + (-4.48)}_{\mathcal{O}(q^4)} + \underbrace{(-4.02) + 4.42}_{\mathcal{O}(q^5)} \\
&+ \underbrace{0.775 + 1.97 + (-2.39)}_{\mathcal{O}(q^6)}.
\end{aligned}
$$

Good convergence

Lattice data points:

Y.B. Yang et al. Phys.Rev.Lett. 121 (2018)

Summary

- Analytic two-loop calculation of m_N
- Verify the validity of EOMS scheme
- The most precise prediction of nucleon mass in chiral limit
- Pion mass dependence consistent with lattice prediction

Thank you for your attention

Back up

$$\mathscr{L}_{\mathscr{M}} = -\bar{q}_R \mathscr{M} q_L - \bar{q}_L \mathscr{M}^\dagger q_R, \quad \mathscr{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$$\mathscr{M} \mapsto R \mathscr{M} L^\dagger$$

$$\mathscr{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \text{Tr}(\mathscr{M} U^\dagger + U \mathscr{M}^\dagger) \quad \Rightarrow \quad M_\pi^2 = 2 B_0 \hat{m}$$

$$\frac{\partial E(\lambda)}{\partial \lambda} = \left\langle \alpha(\lambda) \left| \frac{\partial H(\lambda)}{\partial \lambda} \right| \alpha(\lambda) \right\rangle$$

$$\begin{aligned} \lambda &\rightarrow \hat{m}, \\ |\alpha(\lambda)\rangle &\rightarrow |N(\hat{m})\rangle, \\ E(\lambda) &\rightarrow m_N(\hat{m}), \\ \frac{\partial H}{\partial \lambda} &\rightarrow \frac{\partial \mathscr{H}_{\text{QCD}}}{\partial \hat{m}} = \bar{u}u + \bar{d}d. \end{aligned}$$

$$A + B m_\pi^2 + C m_\pi^3 + D m_\pi^3 + E m_\pi^4 \log m_\pi$$

The expansion of nucleon mass

- Lagrangian

$$\mathcal{L}_{BchPT} = \mathcal{L}_{\pi\pi}^{(2)} + \mathcal{L}_{\pi\pi}^{(4)} + \sum_{i=1}^{6} \mathcal{L}_{\pi N}^{(i)}$$

  - Baryon sector
    - up to $O(q^4)$ N. Fettes et al. Annals Phys. 283 (2000)
    - up to $O(q^5)$ C.Q. Song et al. [2404.15047 [hep-ph]]
  - High order lagrangian contribute a tree level term $\hat{g}_1 M^6$

$$\mathcal{L}_{\pi N}^{(2)} = c_1\langle\chi_+\rangle\bar{N}N - \frac{c_2}{4m^2}\langle u^\mu u^\nu\rangle(\bar{N}D_\mu D_\nu N + h.c.) + \frac{c_3}{2}\langle u^\mu u_\mu\rangle\bar{N}N - \frac{c_4}{4}\bar{N}\gamma^\mu\gamma^\nu[u_\mu, u_\nu]N\,,$$

$$\begin{aligned}\mathcal{L}_{\pi N}^{(3)} = \bar{N}\Big\{ & -\frac{d_1+d_2}{4m}([u_\mu,[D_\nu,u^\mu]+[D^\mu,u_\nu]]D^\nu + h.c.) \\ & + \frac{d_3}{12m^3}([u_\mu,[D_\nu,u_\lambda]](D^\mu D^\nu D^\lambda + sym.) + h.c.) + i\frac{d_5}{2m}([\chi_-,u_\mu]D^\mu + h.c.) \\ & + i\frac{d_{14}-d_{15}}{8m}(\sigma^{\mu\nu}\langle[D_\lambda,u_\mu]u_\nu - u_\mu[D_\nu,u_\lambda]\rangle D^\lambda + h.c.) \\ & + \frac{d_{16}}{2}\gamma^\mu\gamma^5\langle\chi_+\rangle u_\mu + \frac{id_{18}}{2}\gamma^\mu\gamma^5[D_\mu,\chi_-]\Big\} N\,,\end{aligned}$$

$M^5 = m_q^{\frac{5}{2}}$, not analytic in quark mass ×

$$\begin{aligned}\mathcal{L}_{\pi N}^{(4)} = \bar{N}\Big\{ & e_{14}\langle h_{\mu\nu}h^{\mu\nu}\rangle - \frac{e_{15}}{4m^2}\left(\langle h_{\lambda\mu}h^\lambda{}_\nu\rangle D^{\mu\nu} + \text{h.c.}\right) + \frac{e_{16}}{48m^4}\left(\langle h_{\lambda\mu}h_{\nu\rho}\rangle D^{\lambda\mu\nu\rho} + \text{h.c.}\right) \\ & + \frac{ie_{17}}{2}[h_{\lambda\mu},h^\lambda{}_\nu]\sigma^{\mu\nu} - \frac{ie_{18}}{8m^2}\left([h_{\lambda\mu},h_{\nu\rho}]\sigma^{\mu\nu}D^{\lambda\rho} + \text{h.c.}\right) + e_{19}\langle\chi_+\rangle\langle u\cdot u\rangle \\ & - \frac{e_{20}}{4m^2}\left(\langle\chi_+\rangle\langle u_\mu u_\nu\rangle D^{\mu\nu} + \text{h.c.}\right) + \frac{ie_{21}}{2}\langle\chi_+\rangle[u_\mu,u_\nu]\sigma^{\mu\nu} + e_{22}[D_\mu,[D^\mu,\langle\chi_+\rangle]] \\ & - \frac{ie_{35}}{4m^2}\left(\langle\widetilde{\chi}_- h_{\mu\nu}\rangle D^{\mu\nu} + \text{h.c.}\right) + ie_{36}\langle u_\mu[D^\mu,\widetilde{\chi}_-]\rangle - \frac{e_{37}}{2}[u_\mu,[D_\nu,\widetilde{\chi}_-]]\sigma^{\mu\nu} + e_{38}\langle\chi_+\rangle\langle\chi_+\rangle \\ & + \frac{e_{115}}{4}\langle\chi_+^2 - \chi_-^2\rangle - \frac{e_{116}}{4}\left(\langle\chi_-^2\rangle - \langle\chi_-\rangle^2 + \langle\chi_+^2\rangle - \langle\chi_+\rangle^2\right)\Big\} N\,,\end{aligned}$$

Result

- Two loop

$$m_N = m + k_1 M^2 + k_2 M^3 + k_3 M^4 \ln\frac{M}{\mu} + k_4 M^4 + k_5 M^5 \ln\frac{M}{\mu} + k_6 M^5 + k_7 M^6 \ln^2\frac{M}{\mu} + k_8 M^6 \ln\frac{M}{\mu} + k_9 M^6,$$

$k_4 M^4$: $\hat{e}_1 M^4$; $k_9 M^6$: $\hat{g}_1 M^6$

$$k_5 = \frac{3 g_A{}^2}{1024\pi^3 F^4}\left(16 g_A{}^2 - 3\right),$$

$$k_6 = \frac{17 g_A{}^4}{512\pi^3 F^4} - \frac{3\hat{d}_{16} g_A}{8\pi F^2} + \frac{3 g_A{}^2\left[\pi^2 F^2 + 2m^2 + 8m^2\pi^2\left(2l_4 - 3l_3\right)\right]}{256\pi^3 F^4 m^2},$$

$$k_7 = -\frac{3}{256\pi^4 F^4 m}\left[g_A{}^2 - m(6c_1 - c_2 - 4c_3)\right],$$

$$k_8 = \frac{-3}{8\pi^2 \ldots}\left[\ldots 4\hat{e}_{38}\right) m^2\Big]$$

$$+\frac{\ldots}{3072\ldots} \ldots g_A{}^2)$$

$$+36c_1 \ldots$$

$$+4c_3 m \ldots$$

$$k_9 = \hat{g}_1 + \ldots$$

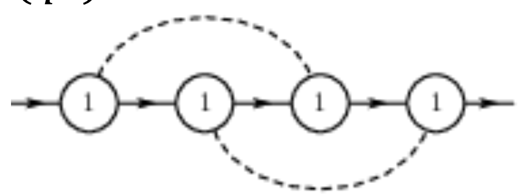
$$+\frac{g_A{}^2(\ldots}{\ldots}$$

$$-\frac{(6210c_1 - 1147c_2 + 616c_3 - 948c_4)\, g_A{}^2}{12288\pi^4 F^4} - \frac{g_A{}^2}{128\pi^2 F^2 m^3}$$

$$+\frac{g_A{}^2(-1 + 18l_3 - 12l_4)}{64\pi^2 F^4 m} + \frac{c_1 g_A{}^4 (96\pi^2\ln 2 - 144\zeta_3 - 164\pi^2 + 1779)}{4096\pi^4 F^4} + \frac{3c_1\hat{d}_{16} g_A}{2\pi^2 F^2}$$

$$+\frac{g_A{}^2(18 - (41\pi^2 + 145) g_A{}^2)}{2048\pi^4 F^4 m} + \frac{6e_{15} + 5e_{16} + 6\hat{e}_{20}}{32\pi^2 F^2} + \frac{-24c_1^2 g_A{}^2 + 6\hat{d}_{16} g_A - 3c_1 c_2}{16\pi^2 F^2 m},$$

$O(q^5)$:



$O(q^4)$?

$$\tilde{m} M^4 \zeta_3 = m M^4 \zeta_3 - 4c_1 M^6 \zeta_3 + \cdots$$

$$\tilde{m} = m - 4c_1 M^2 + \cdots$$

PCB PC preserved

- m, $\hat{e}_1$ and $\hat{g}_1$ are undetermined
- Leading logarithms consist with IR and HBChPT
- Expansion of $\tilde{m}$