REVISITING $O(N)$ SIGMA MODEL AT UNPHYSICAL m_π AND FINITE TEMPERATURE

Zhiguang Xiao
Collaborated with: Lyv Yuan-Lin, Li Qu-Zhi, Zheng Han-Qing
Based on: PRD,109,094026; arxiv:2405.11313

Talk given at The 9TH WORKSHOP ON CHIRAL EFFECTIVE FIELD THEORY, Chang Sha, 2024-10-21

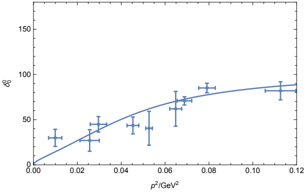
2024 年 10 月 21 日

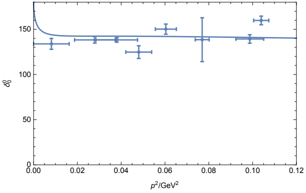
Analysing the lattice QCD data $m_\pi \neq 139$MeV: $m_\pi = 236$MeV, [Dudek, et al., PRD86,034031]; $m_\pi = 391$MeV[Briceno,PRL118,022002]

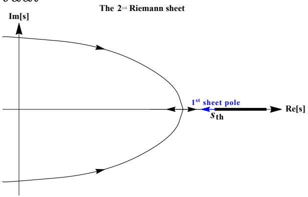
- PKU + crossing using BNR relation: [X. L. Gao,Z.H.Guo,ZX,ZZ, PRD 105 (2022)9,094002]

at $m_\pi = 236$MeV:$m_\sigma = 610 \pm 11$MeV,$\Gamma_\sigma = 327 \pm 8$MeV;

at $m_\pi = 391$MeV: $m_{bound} = 774 \pm 6$MeV,$m_{virtual} = 716 \pm 28$MeV

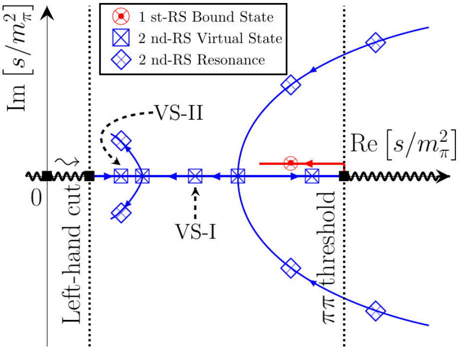




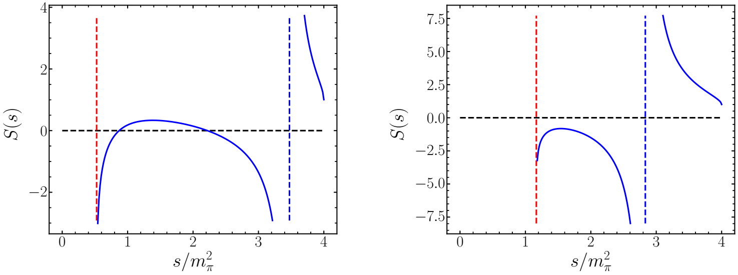


- Roy equation:[X.H. Cao, et al.,PRD108(2023)3,034009]
  at $m_\pi = 236$MeV: $\quad c\sqrt{s_\sigma} = 543 - i250$ MeV
  at $m_\pi = 391$MeV: $\quad m_{bound} = 759$MeV, $\sqrt{s_{sub}} = (269 - i211)$ MeV

  Propose the pole trajectory:



The proposed explanation:



The reason for the appearence of the subthreshold resonance poles:

- When the VS move through the threshold and becomes a bound state, the branch point of the l.h.c. move right.
- From the positivity of the residue of the bound state pole, it can be proved that near the branch point $S(s)$ tends to negative infinty.
- The picture is like above. For small m_π, there are two zero point of $S(s)$ below threshold. For larger m_π, $S(s)$ become smaller, no zero points $\to$ two resonance pole.

Introduction: $O(N)$ sigma model

- Lagrangian: $a = 1, \ldots, N$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}\mu_0^2 \phi_a \phi_a - \frac{\lambda_0}{8N}(\phi_a \phi_a)^2 + \alpha \phi_N \, ,$$

- Classical level: when $\alpha = 0$, No explicit breaking
  $\mu^2 > 0$, no SSB;
  $\mu^2 < 0$, SSB. $\phi^2 = -2\mu^2 N/\lambda \equiv \langle \phi \rangle^2, \langle \phi \rangle > 0$
- Classical level: when $\alpha \neq 0$, vacuum solution

$$(-\mu_0^2 - \frac{\lambda_0}{4N}|\phi|^2)\phi_a = 0 \Rightarrow \phi_a = 0, \quad \text{for } a = 1, \ldots, N-1$$

$$(-\mu_0^2 - \frac{\lambda_0}{4N}|\phi|^2)\phi_N + \alpha = 0$$

$\langle \phi_N \neq 0 \rangle \sim \mathcal{O}(N^{1/2})$, $\alpha \sim \mathcal{O}(N^{1/2})$.

- To count the N order: Introduce an auxiliary field χ,

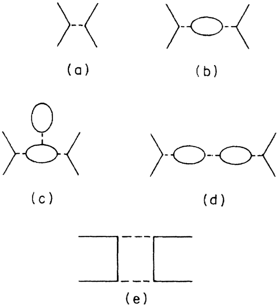
$$\begin{aligned}\mathcal{L} \to & \mathcal{L} + \frac{N}{2\lambda_0}\left(\chi - \frac{\lambda_0}{2N}\phi_a\phi_a - \mu_0^2\right)^2 \\ = & \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \alpha\phi_N + \frac{N}{2\lambda_0}\chi^2 - \frac{1}{2}\chi\phi_a\phi_a - \frac{N\mu_0^2}{\lambda_0}\chi\,,\end{aligned}$$

Integrate out χ, come back to the previous path integral.

- Count the order: χ propagator: $i\lambda_0/N \sim \mathcal{O}(1/N)$; closed ϕ loop: $\mathcal{O}(N)$; $\phi \sim \mathcal{O}(1)$

(a)(b)(c)(d) $\mathcal{O}(1/N)$,
(e) $\mathcal{O}(1/N^2)$ not the same topological structure as (b)

Leading order, only ϕ propagator in loop, 1PI one loop, No χ loop.



Effective action:

$$\Gamma(\phi,\chi) = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \alpha\phi_N + \frac{N}{2\lambda_0}\chi^2 - \frac{1}{2}\chi\phi_a\phi_a - \frac{N\mu_0^2}{\lambda_0}\chi \right) + \frac{i}{2}N\mathrm{Tr}\log(\partial^2 + \chi - i\epsilon)\,, \quad (1)$$

Effective potential: (ϕ, χ constants)

$$V(\phi,\chi) = -\alpha\phi_N - \frac{N}{2\lambda_0}\chi^2 + \frac{1}{2}\chi\phi_a\phi_a + \frac{N\mu_0^2}{\lambda_0}\chi - \frac{i}{2}N\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\log(-\ell^2 + \chi - i\epsilon \quad (2)$$

Renormalization condition:

$$\frac{\mu(M)^2}{\lambda(M)} = \frac{\mu_0^2}{\lambda_0} + \frac{i}{2}\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{\ell^2 + i\epsilon}\,, \quad (3)$$

$$\frac{1}{\lambda(M)} = \frac{1}{\lambda_0} - \frac{i}{2}\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{(\ell^2 + i\epsilon)(\ell^2 - M^2 + i\epsilon)}\,. \quad (4)$$

After renormalization:

$$V(\phi,\chi) = -\alpha\phi_N + \frac{1}{2}\chi\phi^2 + \frac{N\mu^2(M)}{\lambda(M)}\chi - \frac{N}{64\pi^2}\chi^2\left(\log\frac{M^2}{\chi} + \frac{1}{2}\right)\,, \quad (5)$$

$\frac{\partial V}{\partial \chi} = 0$ and $\frac{\partial V}{\partial \phi_a} = 0$

$$\frac{\partial V}{\partial \chi} = 0 \Rightarrow \quad \phi_a \phi_a = \frac{2N}{\lambda}\chi - \frac{2N\mu^2}{\lambda} - \frac{N}{16\pi^2}\chi \log \frac{\chi}{M^2}\,, \qquad (6)$$

$$\frac{\partial V}{\partial \phi_a} = 0 \Rightarrow \quad \chi \phi_a = 0 \ (a < N), \quad \chi \phi_N - \alpha = 0\,. \qquad (7)$$

- For $\alpha \neq 0$, With explicit broken, one solution: $\chi \neq 0$, $\phi_a = 0$, for $a = 1, \ldots N-1$, $\chi = \frac{\alpha}{\phi_N} \neq 0$.

$$\phi_N^3 = \frac{2N\alpha}{\lambda} - \frac{2N\mu^2}{\lambda}\phi_N - \frac{N\alpha}{16\pi^2} \log \frac{\alpha}{M^2 \phi_N}$$

$\langle \phi_N \rangle \sim \mathcal{O}(N^{1/2})$, $\langle \chi \rangle \sim \mathcal{O}(1)$, $\alpha \sim \mathcal{O}(N^{1/2})$.

- minimum condition is $\chi > 0$ and $\frac{32\pi^2\phi_N/N}{\log(M^2/\chi)-1} + \chi > 0$
- π mass $m_\pi^2 = \chi$.
- For $N = 4$, compared with PCAC: $\partial^\mu A_\mu^a = m_\pi^2 f_\pi \pi^a$, and with explicit SSB, $\partial^\mu A_\mu^a = \alpha\pi^a$. From $\chi = \frac{\alpha}{\phi_N} = m_\pi^2$, we see $\langle\phi_N\rangle = f_\pi \sim \mathcal{O}(N^{1/2})$.

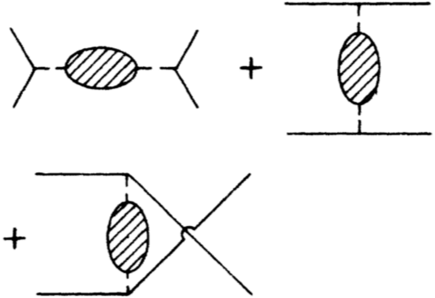


图: $O(N)$ model leading order $\pi\pi$ amplitude [Coleman,et.al.,PRD10,2491]

$\pi\pi$ amplitude to the leading $1/N$ order: (σ, τ have mixing)

$$\mathcal{T}_{\pi_a\pi_b\to\pi_c\pi_d} = iD_{\tau\tau}(s)\delta_{ab}\delta_{cd} + iD_{\tau\tau}(t)\delta_{ac}\delta_{bd} + iD_{\tau\tau}(u)\delta_{ad}\delta_{bc}\,, \tag{8}$$

$$D^{-1}(p^2) = -\,i\begin{pmatrix} p^2 - m_\pi^2 & -f_\pi \\ -f_\pi & N/\lambda_0 + NB_0(p^2, m_\pi) \end{pmatrix}\,, \tag{9}$$

$$\tau \equiv \chi - \langle\chi\rangle, \quad B_0(p^2, m_\pi) = \frac{-i}{2}\int \frac{\mathrm{d}^4\ell}{(2\pi)^4}\frac{1}{(\ell^2 - m_\pi^2 + i\epsilon)((\ell+p)^2 - m_\pi^2 + i\epsilon)}\,.$$

After renormalization:

$$\frac{1}{\lambda_0} + B_0(p^2, m_\pi) = \frac{1}{\lambda(M)} + B(p^2, m_\pi, M)\,, \tag{10}$$

$$B(s, m_\pi, M) = \frac{1}{32\pi^2}\left(1 + \rho(s)\log\frac{\rho(s)-1}{\rho(s)+1} - \log\frac{m_\pi^2}{M^2}\right)\,, \tag{11}$$

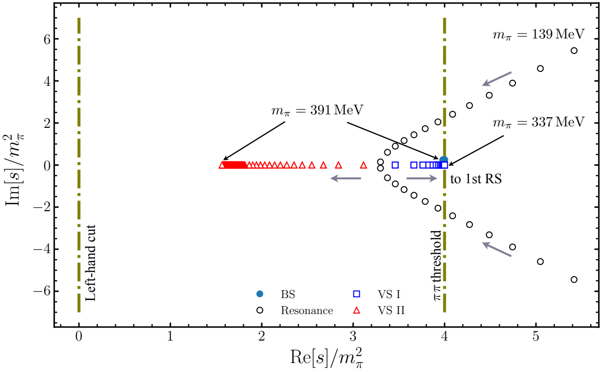
After projection to $IJ = 00$ channel, $\mathcal{O}(N)$ amplitude $\mathcal{T}_{00}^{LO}$ (define M such that $1/\lambda(M) = 0$),

$$\mathcal{T}_{00}^{LO}(s) = \frac{iND_{\tau\tau}(s)}{32\pi} = \frac{N}{32\pi}\mathcal{A}(s),$$
$$\mathcal{A}(s) = \frac{m_\pi^2 - s}{(s - m_\pi^2)B(s, m_\pi, M) - f_\pi^2/N},$$

- Adler zero: $s_A = m_\pi^2$.
- σ pole : solve

$$(s - m_\pi^2)B^{II}(s, m_\pi, M) - f_\pi^2/N = 0\,.$$

m_π DEPENDENCE OF σ AT LEADING $1/N$



- Adjust the $M \sim 550MeV$, at $m_\pi = 139MeV$, $\sqrt{s_\sigma} = 356 - i148MeV$.
- As m_π increases, $\sqrt{s_\sigma}$ moves $\rightarrow$ real axis $\rightarrow$ two virtual state poles $\rightarrow$ one virtual move left, the other move right $\rightarrow$ ($m_c \simeq 337\,\mathrm{MeV}$) one virtual, one bound
- No crossing symmetry: $\sigma \rightarrow$ bound state, l.h.c brach point $\rightarrow s = 4m_\pi^2 - m_\sigma^2$.
- Direct adding t and u channel contribution: violate unitarity.
- Unitarization method: IAM or K matrix, No control of the spurious poles, we resort to N/D.

N/D: UNITARITY WITH PARTIAL RECOVERY OF CROSSING

N/D Method: Basic ideas

- T matrix: $T(s) = \frac{N(s)}{D(s)}$: $N(s)$ only has left-hand cut, D only has right-hand cut.
- Since $\mathrm{Im}_R\,\mathcal{T}^{-1} = -\rho$, we have

$$\mathrm{Im}_R\, D(s) = -\rho(s) N(s)\,, \tag{12}$$

$$\mathrm{Im}_L\, N(s) = D(s)\,\mathrm{Im}_L\,\mathcal{T}(s)\,. \tag{13}$$

- Write down the dispersion relation of $N(s)$ and $D(s)$ (twice subtracted)

$$D(s) = \frac{s-s_A}{s_0-s_A} + g_D\frac{s-s_0}{s_A-s_0} - \frac{(s-s_0)(s-s_A)}{\pi}\int_R \frac{\rho(s')N(s')}{(s'-s)(s'-s_0)(s'-s_A)}\mathrm{d}s'\,, \tag{14}$$

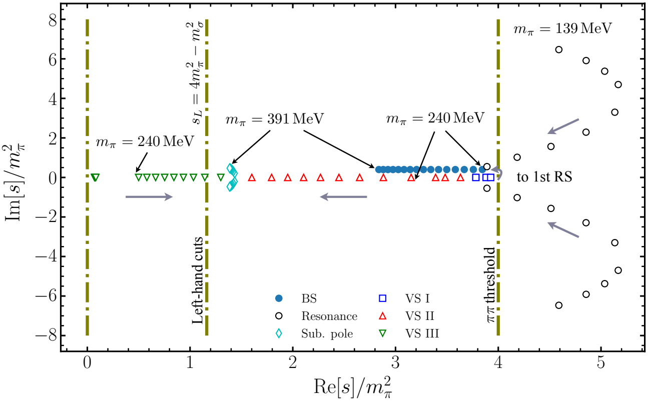
$$N(s) = b_0\frac{s-s_A}{s_0-s_A} + g_N\frac{s-s_0}{s_A-s_0} + \frac{(s-s_0)(s-s_A)}{\pi}\int_L \frac{D(s')\mathrm{Im}_L\mathcal{T}(s')}{(s'-s)(s'-s_0)(s'-s_A)}\mathrm{d}s'\,, \tag{15}$$

- Using the $O(N)$ result $\mathrm{Im}_L \mathcal{T}(s)$.
- Subtraction, $s_0 = 4m_\pi^2$, $s_A = m_\pi^2$, $D(s_0) = 1$: require $\mathcal{T}$ to recover $O(N)$ result at the leading $1/N$ order

$$b_0 = \mathcal{T}_{00}(s_0) = \frac{N-1}{32\pi}\mathcal{A}^{LO}(s_0) + I_{tu}(s_0)\,, \tag{16}$$

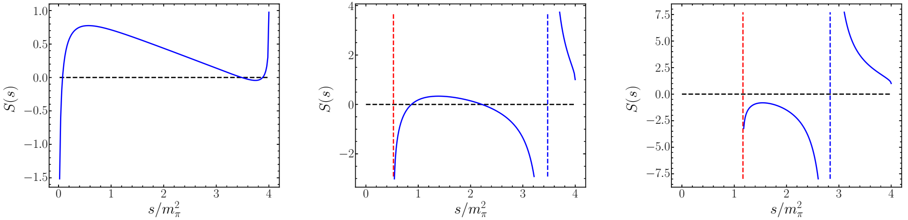
$$g_D = \frac{32\pi f_\pi^2 b_0}{N(s_0 - s_A)}\,, \quad \frac{g_N}{g_D} = \mathrm{Re}\, I_{tu}(s_A)\,. \tag{17}$$

- When $m_\pi > m_c$, we require $D(m_\sigma^2) = 0$, for the sigma pole to be consistent with the branch point of the left hand cut.
- Solve the integral equation numerically.



- As m_π increases:
  σ approaches the real axis$\rightarrow$ two virtual states, $\rightarrow$
  $\left\{ \begin{array}{l} \text{one moving right} \rightarrow \text{first sheet bound state} \\ \left. \begin{array}{l} \text{one moving left} \\ \text{l.h.c.} \quad \rightarrow \text{another virtual state} \end{array} \right\} \text{hit} \rightarrow \text{resonant poles} \end{array} \right.$
- The VSIII is generated when the Adler zero hit the left-hand cut

σ POLE TRAJECTORY WITH VARYING m_π

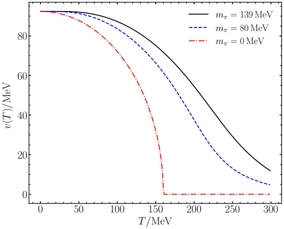


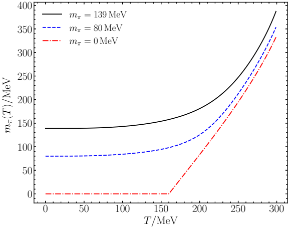
Physical-sheet S-matrix below threshold

- The left graph ($m_\pi = 207$ MeV) : two virtual states in the near threshold region and one additional virtual state pole generated close to the left-hand cut.
- The middle graph ($m_\pi = 283$ MeV) : one bound state with two virtual states.
- The right graph ($m_\pi = 391$ MeV): two virtual state poles have become a pair of resonance poles.

Temperature dependence at leading $1/N$

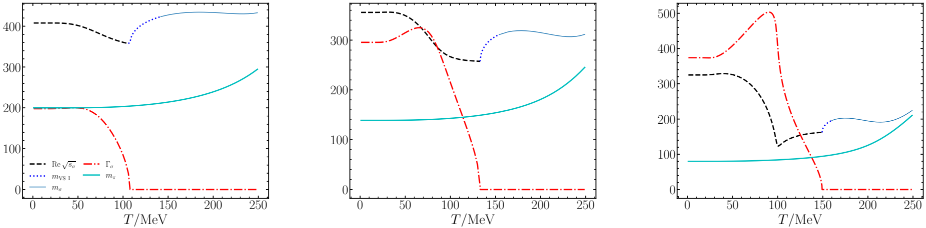
- It is well known: under high temperature, chiral symmetry recovers.[R. D. Pisarski and F. Wilczek, PRD 29,338(1984);A. Bazavov et al., PRD85,054503(2012)] $\lim_{T\to\infty} v(T) \to 0$
- It is expected that $m_\sigma \to m_\pi$ at high temperature.
- ChPT: intrincically in broken phase, break down at high temperature, $T \sim f_\pi$.
- In $O(N)$ model: v.e.v. $v(T)$ evolves with T,[J.O.Anderson, et. al. PRD70,116007]





- No explicit breaking $\alpha = 0$:at $T < T_c \sim 160$MeV, $m_\pi(T) = 0$, $v(T) \neq 0$ SSB; at $T > T_c$, $v(T) = 0$, $m_\pi(T) \neq 0$.
- With $\alpha \neq 0$, $v(T) \to 0$.

At the leading $1/N$ order, $N = 4$:



From left to right, $m_\pi(0) = 200$, 139 and 80 MeV respectively.

- Scattering amplitude at T in the center of mass frame.

$$\mathcal{T}_{00}^{T}(s) = -\frac{1}{32\pi} \frac{s - m_\pi^2(T)}{\left(s - m_\pi^2(T)\right) \; B^T\left(s, m_\pi(T), M\right) - v^2(T)/N} ,$$

$$B^T\left(s, m_\pi(T), M\right) = B\left(s, m_\pi(T), M\right) + B^{T\neq 0}\left(s, m_\pi(T)\right) ,$$

$$B^{T\neq 0}\left(s, m_\pi(T)\right) = \int_0^\infty \frac{\mathrm{d}k\, k^2}{8\pi^2 \omega_k^2} n_B(\omega_k) \left(\frac{1}{E + 2\omega_k} - \frac{1}{E - 2\omega_k}\right) ,$$

- σ resonance on the second sheet, $\rightarrow$ virtual state, $\rightarrow$ bound state $\rightarrow$ tends to m_π.

N/D WITH TEMPERATURE

- Unitarity with two particle intermediate states for $IJ = 00$ channel,

$$\mathrm{Im}\,\mathcal{T}^T(s) = \rho^T(s)\left|\mathcal{T}^T\right|^2 , \tag{18}$$

- Lorentz symmetry is broken by the temperature: Center of mass system in s channel is different from t channel — crossing is also broken.
- $IJ = 00$ thermal amplitude:

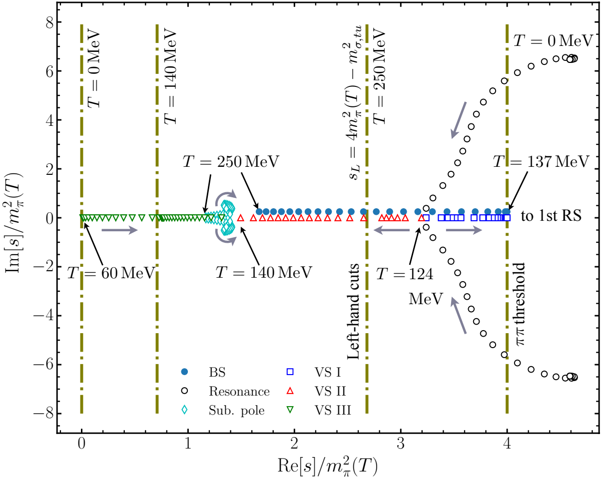
$$\mathcal{T}_{00}^T(s) = -\frac{1}{32\pi}\frac{s - m_\pi^2(T)}{\left(s - m_\pi^2(T)\right)\, B^T\left(s, m_\pi(T), M\right) - v^2(T)/N},$$

$$B^T\left(s, m_\pi(T), M\right) = B\left(s, m_\pi(T), M\right) + B^{T\neq 0}\left(s, m_\pi(T)\right) ,$$

$$B^{T\neq 0}\left(s, m_\pi(T)\right) = \int_0^\infty \frac{\mathrm{d}k\, k^2}{8\pi^2\omega_k^2}\, n_B(\omega_k)\left(\frac{1}{E + 2\omega_k} - \frac{1}{E - 2\omega_k}\right) ,$$

- N/D can be done: substitute the corresponding temperature dependent amplitudes.

N/D with temperature: σ trajectory



- $T = 0,\ m_\pi = 139$MeV.
- $T = 60$MeV, VSIII generated from the l.h.c.
- σ resonance $\to$ virtual states (I, II), (T=124MeV)
- VS1$\to$ BS (T=137MeV),
- $T = 140$MeV, VSII meets VSIII$\to$ subthreshold resonance.
- $m_\sigma^2 \to m_\pi^2$.

Vacuum structure: m_π dependence

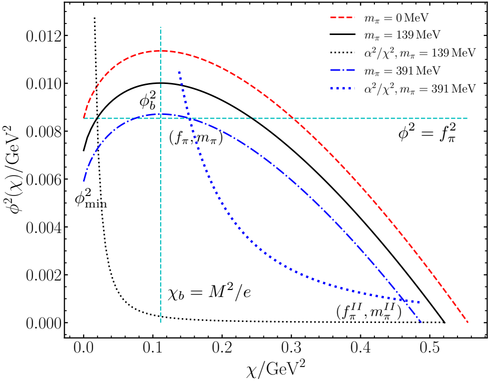
Solve $\chi(\phi)$, insert into $V(\phi,\chi)$, to obtain the effective potential $V_{eff}(\phi)$.

$$V(\phi,\chi) = -\alpha\phi_N + \frac{1}{2}\chi\phi^2 + \frac{N\mu^2(M)}{\lambda(M)}\chi - \frac{N}{64\pi^2}\chi^2\left(\log\frac{M^2}{\chi}+\frac{1}{2}\right),$$

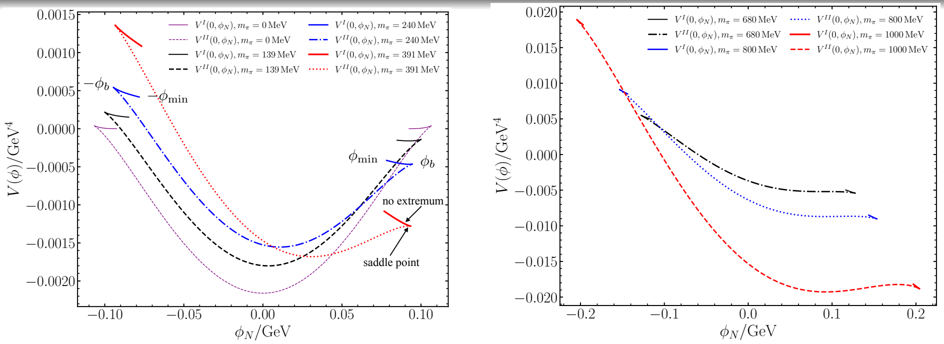
$$\frac{\partial V}{\partial \chi} = 0 \Rightarrow \quad \phi^2 = f_\pi^2 + \frac{N}{16\pi^2}\left(m_\pi^2 \log\frac{m_\pi^2}{M^2} - \chi\log\frac{\chi}{M^2}\right),$$

$$\frac{\partial V}{\partial \phi_a} = 0 \Rightarrow \quad \chi\phi_a = 0\ (a<N), \quad \chi\phi_N - \alpha = 0\,.$$

- Two solutions branches: separated at χ_b
- Left one: With Chiral SSB in the Chiral limit, determine f_π fixed, m_π.
- Right one : No chiral SSB in the Chiral limit.
- V become complex for $|\phi|^2 > \phi_b^2$: the system not stable. $\phi^2 < \phi^2_{min}$.

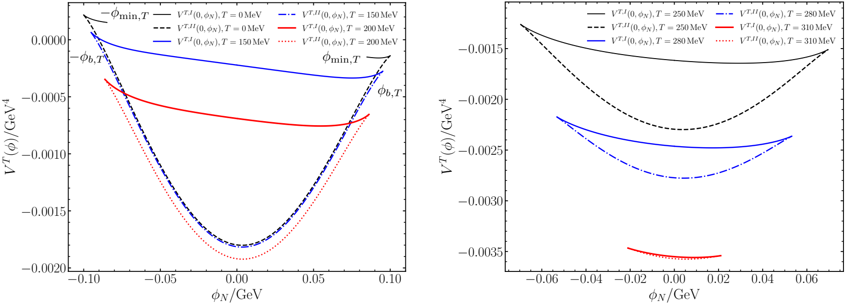


Two branches of the vacuum

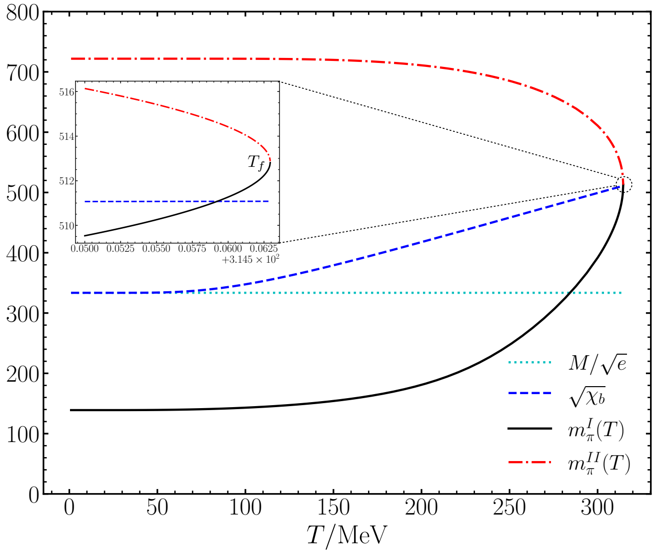


- $m_\pi^2 < \chi_b \sim 333$MeV, solution I: the local minimum on the first branch, false vacuum. There is a tachyon.
- Global minimum: Solution II on the second branch.
- As m_π increases, solution I moves towards the second branch.
- $m_\pi^2 > \chi_b$, no local minimum on the first branch. Solution I moves on the second branch $\to$ saddle point.
- $m_\pi > (32\pi^2 f_\pi^2/(N y_0))^{1/2} \sim 680$MeV: solution I $\leftrightarrow$ Solution II.

Vacuum structure: Finite temperature



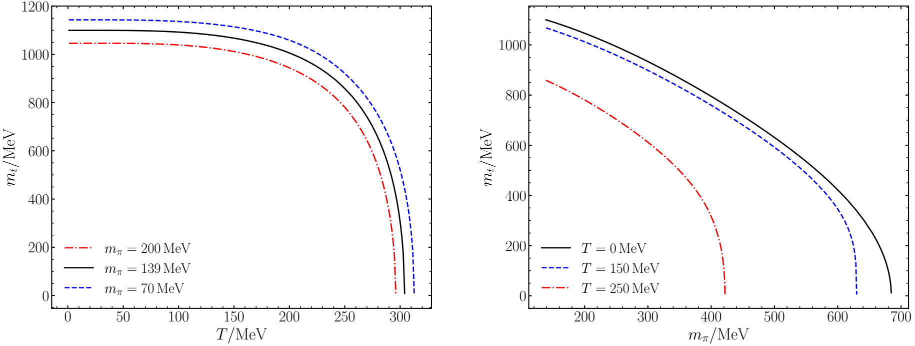
- T increases: $|\phi_b|$ and $|\phi_{min}| \to$ smaller.
- T_f: there is no solution for the gap equations. $T_f \sim 314\mathrm{MeV}$
- T_b: $\phi_b = 0$.
- The two branches: Effective potential V get closer.
- $T_c < T_f < T_b$. $T > T_f$, no vacuum, the system is already unstable. Difference $T_b - T_f \sim$ keV.



At high temperature: the Solution I will move to the second branch, becoming a saddle point.

Tachyon: m_π and T dependence

There could be a tachyon for solution I.



- Plays a role of another cutoff of the theory:$m_t^2 \ll s \ll s + m_t^2$. ($m_t \sim 1.1$GeV for physical mass, $T = 0$) [R. S. Chivukula and M. Golden, PLB 267, 233]
- $s = -m_t^2 < 0$: m_t decreases with temperature and m_π.
- Tachyon has positive residue in the $\sigma - \sigma$ propogator, similar to bound state.
- Tachyon $\rightarrow$ bound state transition $\leftrightarrow$ the point of exchanging the two solution.

Summary

- σ pole trojectory in leading $O(N)$ and N/D modified $O(N)$: with varying m_π and temperature.
- Subthreshold resonance pole generation: After crossing symmtry partially recovered.
- Vacuum structure: with varying m_π and temperture. Phenominological favored one is the first branch.