2024复旦大学“QCD与重离子碰撞物理”暑期学校

QCD自旋物理基础
Basics for QCD Spin Physics

梁作堂 (Liang Zuo-tang)
山东大学(Shandong University)
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2024复旦大学“QCD与重离子碰撞物理”暑期学校

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Contents of Part I

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Introduction: The concept of electron spin

原子光谱学与量子物理理论发展

- 分立谱(Balmer's formulae) ⟶ 量子力学
- 精细结构(fine structure) ⟶ 电子自旋
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Introduction: The concept of the electron spin

电子自旋的发现

Die Naturwissenschaften 13, 953–954 (1925)

Heft 47. 20. II. 1925] Zuschriften und vorläufige Mitteilungen. 953

Ersetzung der Hypothese vom unmechanischen Zwang durch eine Forderung bezüglich des inneren Verhaltens jedes einzelnen Elektrons.

in Übereinstimmung zu kommen, muß man also diesem Modell die folgenden Forderungen stellen:

a) Das Verhältnis des magnetischen Momentes des Elektrons zum mechanischen muß für die Eigenrotation doppelt so groß sein als für die Umlaufsbewegung[2]).

b) Die verschiedenen Orientierungen vom R zur Bahnebene (oder K) des Elektrons muß, vielleicht in Zusammenhang mit einer Heisenberg-Wentzelschen Mittelungsvorschrift[3]), die Erklärung des Relativitätsdoubletts liefern können.

G. E. Uhlenbeck und S. Goudsmit.

Leiden, den 17. Oktober 1925.
Instituut voor Theoretische Natuurkunde.

Es ist mir ein Bedürfnis, festzustellen, daß Prof. W. J. de Haas mir schon vor einigen Monaten die Apparatur für ein sehr interessantes Experiment zeigte, das sich ebenfalls mit dem Problem der inneren Rotation des Elektrons beschäftigt. Obwohl mir die betreffenden Ideen von Prof. de Haas seit längerer Zeit bekannt waren, hatten die Herren Uhlenbeck und Goudsmit, als sie mir kürzlich die obigen Überlegungen mitteilten, davon keinerlei Kenntnis.

P. Ehrenfest.

George Uhlenbeck (left) with Hendrik Kramers (middle) and Samuel Goudsmit in Ann Arbor, Michigan. This photograph was taken around 1928, three years after Uhlenbeck and Goudsmit proposed the idea that each electron rotates with an angular momentum $\hbar/2$ and carries a magnetic moment of one Bohr magneton, $e\hbar/2mc$.

264 *NATURE* [February 20, 1926

Letters to the Editor.

[*The Editor does not hold himself responsible for opinions expressed by his correspondents. Neither can he undertake to return, nor to correspond with the writers of, rejected manuscripts intended for this or any other part of* Nature. *No notice is taken of anonymous communications.*]

Spinning Electrons and the Structure of Spectra.

this moment of momentum is given by $Kh/2\pi$, where $K=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. The total angular momentum of the atom is $Jh/2\pi$, where $J=1, 2, 3$. The symbols K and J correspond to those used by Landé in his classification of the Zeeman effects of the optical multiplets. The letters S, P, D also relate to the analogy with the structure of optical spectra which we consider below. The dotted lines represent the position of the energy levels to be expected in the absence of the spin of the electron. As the arrows in-

electron.

In conclusion, we wish to acknowledge our indebtedness to Prof. Niels Bohr for an enlightening discussion, and for criticisms which helped us distinguish between the essential points and the more technical details of the new interpretation.

G. E. Uhlenbeck.
S. Goudsmit.

Instituut voor Theoretische Natuurkunde,
Leyden, December 1925.

Having had the opportunity of reading this inter-

the correspondence between classical mechanics and the quantum theory.

N. Bohr.

Copenhagen, January 1926.

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Introduction: The concept of the electron spin

GEORGE UHLENBECK AND THE DISCOVERY OF ELECTRON SPIN

How two young Dutchmen, one with only a master's degree, the other a graduate student, made a most important finding in theoretical atomic physics.

Abraham Pais

The owl depicted on the signet ring George Uhlenbeck used to wear—"Uhlenbeck" in German means "owl's brook"—derives from his family's coat of arms. The shield reads, in the language of heraldry: Azure, on a tree trunk proper rising from water argent, an owl contourné, head affronty. In plain language, it depicts an owl with its head turned toward you, sitting on a tree trunk in natural color, which rises up out of a silvery brook. (I owe the transcription of the Dutch blazon into English heraldry to Michael Maclagan, the Richmond Herald in the College of Arms, in London.)

Abraham Pais is Detlev Bronk Professor Emeritus at Rockefeller University, in New York. He based this article on his presentation at APS's Uhlenbeck Memorial Symposium, held in Baltimore on 3 May 1989.

34 PHYSICS TODAY DECEMBER 1989

假设	问题	解答
电子具有（绕自身转动的）额外的自由度 $s=1/2$	表面速度远大于光速？	量子效应 (Bohr)
自旋磁矩 郎德因子$g_s=2$	电子磁矩与自身运动产生的磁场相互作用？ (Bohr)	相对运动，外场 (Einstein)
自旋轨道耦合 $V_{ls}(r)=-\frac{1}{2m^2}\frac{dV}{rdr}\hat{\vec{l}}\cdot\hat{\vec{s}}$	因子2的差别？	相对论运动学效应 托马斯进动 Thomas precession

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Introduction: The concept of the electron spin

在相对论量子力学中，这些问题得到完美的解决

Dirac equation: $i\partial_t\psi=\hat{H}\psi$ $\quad \hat{H}=\vec{\alpha}\cdot\hat{\vec{p}}+\beta m$ $\quad \psi=\begin{pmatrix}\varphi\\ \eta\end{pmatrix}$

（1）Dirac粒子是自旋1/2的费米子

Even for a free Dirac particle:

$$\left[\hat{H},\hat{\vec{L}}\right]=-i\vec{\alpha}\times\hat{\vec{p}}\neq 0$$

$$\left[\hat{H},\vec{\Sigma}\right]=2i\vec{\alpha}\times\hat{\vec{p}}\neq 0 \qquad \vec{\Sigma}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

$$\left[\hat{H},\hat{\vec{J}}\right]=0 \qquad \hat{\vec{J}}=\hat{\vec{L}}+\frac{\vec{\Sigma}}{2}$$

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Introduction: The concept of the electron spin

（2）Dirac粒子的磁矩$g_s = 2$

The magnetic moment:

$$\hat{\vec{v}} = \frac{d\hat{\vec{r}}}{dt} = \frac{1}{i}[\vec{r}, \hat{H}] = \vec{\alpha}$$

$$\hat{\vec{M}} = \frac{1}{2}q\vec{r}\times\hat{\vec{v}} = \frac{1}{2}q\vec{r}\times\vec{\alpha} = \frac{q}{2}\begin{pmatrix} 0 & \vec{r}\times\vec{\sigma} \\ \vec{r}\times\vec{\sigma} & 0 \end{pmatrix}$$

考查自由的Dirac粒子

$$\psi = \begin{pmatrix}\varphi \\ \eta\end{pmatrix} \quad \hat{H}\psi = E\psi \quad \begin{cases}(E-m)\varphi = \vec{\sigma}\cdot\hat{\vec{p}}\eta \\ (E+m)\eta = \vec{\sigma}\cdot\hat{\vec{p}}\varphi\end{cases} \quad \eta = \frac{\vec{\sigma}\cdot\hat{\vec{p}}}{E+m}\xi$$

$$\langle\psi|\hat{\vec{M}}|\psi\rangle = \frac{q}{2}\int d^3r\,(\varphi^\dagger\vec{r}\times\vec{\sigma}\eta + \eta^\dagger\vec{r}\times\vec{\sigma}\varphi) = \frac{q}{E+m}\int d^3r\,\varphi^\dagger(\hat{\vec{L}}+\vec{\sigma})\varphi$$

Non-relativistic limit: $E\sim m \gg |\vec{p}|\sim V(r)$

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Introduction: The concept of the electron spin

（3）Dirac粒子的spin-orbit coupling

考查中心力场中运动的Dirac粒子 $\hat{H} = \vec{\alpha}\cdot\hat{\vec{p}} + \beta m + V(r) \quad \hat{H}_{eff}\varphi = E\varphi$

$$\hat{H}_{eff} = m + \frac{\hat{\vec{p}}^2}{E+m-V} + V + \frac{dV}{rdr}\frac{\vec{\sigma}\cdot\hat{\vec{L}}}{(E+m-V)^2} - i\frac{dV}{rdr}\frac{\vec{r}\cdot\hat{\vec{p}}}{(E+m-V)^2}$$

$$\approx m + \frac{\hat{\vec{p}}^2}{2m} + V + \frac{1}{4m^2}\frac{dV}{rdr}\vec{\sigma}\cdot\hat{\vec{L}} - i\frac{dV}{rdr}\frac{\vec{r}\cdot\hat{\vec{p}}}{4m^2}$$

but this is NOT the non-relativistic equation! ξ is not normalized, $\hat{H}_{eff}$ is not Hermitian
The correct form is obtained by using the Foldy-Wouthuysen transformation.

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Introduction: The concept of the electron spin

Orbit angular momentum is non-zero even if the quark is in the ground state.

$$\hat{H}\psi = E\psi \qquad \hat{H} = \vec{\alpha}\cdot\hat{\vec{p}} + \beta m + V(r)$$

The stationary state is taken as the eigenstate of $(\hat{H}, \hat{\vec{J}}^2, \hat{J}_z, \hat{\pi})$, where $\hat{\pi}$ is parity:

$$\psi_{E_n jm\pi}(r,\theta,\phi,S) = \begin{pmatrix} f_{nl}(r)\Omega^l_{jm}(\theta,\phi) \\ (-1)^{(l-l'+1)/2}g_{nl'}(r)\Omega^{l'}_{jm}(\theta,\phi)\end{pmatrix}$$

$j = l \pm \frac{1}{2}, l - l' = \mp 1$, and $\pi = (-1)^l$

$\Omega^l_{jm}(\theta,\phi)$ is the eigenstate of $(\hat{\vec{J}}^2, \hat{\vec{L}}^2, \hat{J}_z)$:

$$\Omega^l_{jm}(\theta,\varphi) = \sqrt{\frac{j+m}{2j}}Y_{l\,m-\frac{1}{2}}(\theta,\varphi)\xi\left(\frac{1}{2}\right) + \sqrt{\frac{j-m}{2j}}Y_{l\,m+\frac{1}{2}}(\theta,\varphi)\xi\left(-\frac{1}{2}\right) \qquad j = l + \frac{1}{2}$$

$$\Omega^l_{jm}(\theta,\varphi) = -\sqrt{\frac{j-m+1}{2j+2}}Y_{l\,m-\frac{1}{2}}(\theta,\varphi)\xi\left(\frac{1}{2}\right) + \sqrt{\frac{j+m+1}{2j+2}}Y_{l\,m+\frac{1}{2}}(\theta,\varphi)\xi\left(-\frac{1}{2}\right) \qquad j = l - \frac{1}{2}$$

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Introduction: The concept of the electron spin

Ground state： $E = E_0, j = \frac{1}{2}, \pi = +1\ (l = 0), m = \pm\frac{1}{2}$

$$\psi_0 \equiv \psi_{E_0\frac{1}{2}m+}(r,\theta,\varphi,S) = \begin{pmatrix} f_{00}(r)\Omega^0_{\frac{1}{2}m}(\theta,\varphi) \\ -g_{01}(r)\Omega^1_{\frac{1}{2}m}(\theta,\varphi)\end{pmatrix}$$

$$\Omega^0_{\frac{1}{2}m}(\theta,\varphi) = Y_{00}(\theta,\varphi)\xi(m) = \frac{1}{\sqrt{4\pi}}\xi(m)$$

$$\Omega^1_{\frac{1}{2}m}(\theta,\varphi) = \sqrt{\frac{3+2m}{6}}Y_{1,m+\frac{1}{2}}(\theta,\varphi)\xi\left(-\frac{1}{2}\right) - \sqrt{\frac{3-2m}{6}}Y_{1,m-\frac{1}{2}}(\theta,\varphi)\xi\left(\frac{1}{2}\right)$$

The magnetic moment $\langle\psi_0|\hat{\vec{M}}|\psi_0\rangle = \mu\xi^+(m)\vec{\sigma}\xi(m) \qquad \mu = -\frac{2}{3}e\int dr\,r^3 f_{00}(r)g_{01}(r)$

The average value of the orbital angular momentum

$$\langle\psi_0|\hat{\vec{L}}^2|\psi_0\rangle = 2\int dr\,r^2 g^2_{01}(r) \qquad \langle\psi_0|\hat{L}_z|\psi_0\rangle = \frac{5m}{3}\int dr\,r^2 g^2_{01}(r)$$

ZTL and Meng Ta-chung（孟大中）, Z. Phys. A344, 177 (1992).

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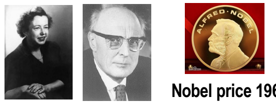
Spin-orbit coupling in systems under strong interaction

At the hadron level

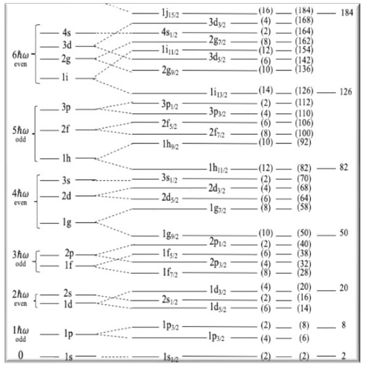
Nuclear shell model



Nobel price 1963

M.G. Mayer, J.H.D. Jensen（1948）

LS-coupling ⟹ “magic numbers”

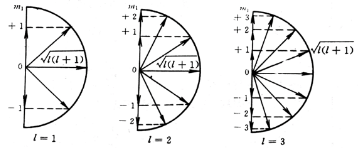


M.G. Mayer and J.H.D. Jensen, “Elementary Theory of Nuclear Shell Structure”, Wiley, New York and Chapman Hall, London, 1955.

Introduction: The concept of electron spin

Three characteristics of spin

- 量子
- 相对论
- 自旋轨道耦合



空间量子化示意图

By the way

$g = 2$, point-like; g-2 experiments, test of QED, new physics.

Anomalous magnetic moment:

$g \neq 2$ significantly different from 2, composite nature of particles;

e.g. $\mu_p = 2.97\mu_N$, $\mu_n = -1.91\mu_N$ the first signature of structure of nucleon.

Contents of Part I

Description of spin states: spin-1/2 particles

单粒子状态, 非相对论情形

$$\hat{\vec{s}} = \frac{1}{2}\vec{\sigma} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_z \xi_z(m) = m\xi_z(m) \qquad \xi_z(+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi_z(-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For any $\vec{n} = \sin\theta\cos\varphi\,\vec{e}_x + \sin\theta\sin\varphi\,\vec{e}_y + \cos\theta\,\vec{e}_z$, we have

$$\sigma_n = \vec{\sigma}\cdot\vec{n} \qquad \sigma_n \xi_n(m) = m\xi_n(m) \qquad \xi_n(+) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}$$

For any $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$, we have $\sigma_n \xi = \xi$ $\quad \tan\frac{\theta}{2} = \frac{|b|}{|a|} \quad e^{i\varphi} = \frac{|a|b}{|b|a}$

For any $\hat{O}$, we have $\hat{O} = \hat{O}_s I + \hat{\vec{O}}_V \cdot \vec{\sigma} \qquad \hat{O}_s = \frac{1}{2}\mathrm{Tr}\hat{O} \qquad \hat{\vec{O}}_V = \frac{1}{2}\mathrm{Tr}(\vec{\sigma}\hat{O})$

Description of spin states: spin-1/2 particles

单粒子状态，相对论情形

$$\psi_{ps}(x)=u(p,s)e^{ipx} \qquad u(p,s)=N\begin{pmatrix}\xi_z(m)\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi_z(m)\end{pmatrix}$$

$$\sigma_z\xi_z(m)=m\xi_z(m) \qquad \Sigma_z u(p,S)\neq m u(p,S)$$

Helicity（螺旋度） $\hat{h}\equiv\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$ $\qquad \hat{h}u(p,\lambda)=\lambda u(p,\lambda)$

$$u(p,\lambda)=N\begin{pmatrix}\xi_h(\lambda)\\ \lambda\sqrt{\frac{E-m}{E+m}}\xi_h(\lambda)\end{pmatrix} \qquad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi_h(\lambda)=\lambda\xi_h(\lambda)$$

$$\xi_h(+)=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{-i\varphi}\end{pmatrix} \qquad \xi_h(-)=\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}e^{-i\varphi}\end{pmatrix}$$

$$\xrightarrow{m\,=\,0} \quad u(p,\lambda)=\begin{pmatrix}\xi_h(\lambda)\\ \lambda\xi_h(\lambda)\end{pmatrix}$$

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Description of spin states: spin-1/2 particles

Helicity（螺旋度） $\hat{h}\equiv\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$

ANNALS OF PHYSICS: 7, 404–428 (1959)

On the General Theory of Collisions for Particles with Spin*

M. JACOB† AND G. C. WICK

周光召

This has been done by Stapp (6) for collisions between spin-½ particles and by Chao and Shirokov (7)[1] for particles of arbitrary spin. In either case, the authors

7. CHOU KUANG-CHAO AND M. I. SHIROKOV, *J. Exptl. Theoret. Phys. (U.S.S.R.)* **34,** 1230 (1958); translation: *Soviet Phys. JETP* **7,** 851 (1958).

* *Note added in proof*. We have recently received a copy of a paper by Chou Kuang-Chao [*J. Exptl. Theoret. Phys.* (U.S.S.R.) **36,** 909 (1959)] in which a treatment is given which applies when one of the incident particles has zero mass.

SOVIET PHYSICS JETP VOLUME 36 (9), NUMBER 3 SEPTEMBER, 1959

REACTIONS INVOLVING POLARIZED PARTICLES OF ZERO REST MASS

CHOU KUANG-CHAO
Joint Institute of Nuclear Studies

Submitted to JETP editor May 31, 1958
J. Exptl. Theoret. Phys. (U.S.S.R.) 36, 909-918 (March, 1959)

In this paper the group-theoretical point of view is used for the description of the spin states of particles of zero rest mass. Complete sets of operators and of their eigenfunctions for a system of two particles are found in the representation of the momenta and spins. The statistical tensors for the particles produced in a reaction of the type a + b → c + d or a → c + d are obtained in the case in which one of these particles has no rest mass. The most general selection rules are derived for the reaction a + b → c + d in the form of relations between the statistical tensors, under the condition that the space and time parities are conserved. The wave functions are calculated for a system of two identical particles of zero rest mass.

SOVIET PHYSICS JETP VOLUME 34 (7), NUMBER 5 NOVEMBER, 1958

THE RELATIVISTIC THEORY OF REACTIONS INVOLVING POLARIZED PARTICLES

CHOU KUANG-CHAO and M. I. SHIROKOV
Joint Institute for Nuclear Research

Submitted to JETP editor December 6, 1957
J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 1230-1239 (May, 1958)

It is shown that, in the rest system, the relativistic formulas for the angular distribution and the polarization vectors and tensors for a reaction of the type a + b → c + d are essentially the same as the nonrelativistic formulas, if the spin of a particle is defined as its internal angular momentum around its center of mass. The square of this internal angular momentum is Lorentz invariant. The spins of the particles are arbitrary, and their rest masses are nonvanishing.

The main difference from the nonrelativistic case is that the description of the spin state is not the same in different Lorentz reference systems. Therefore for cascades of reactions (for example, for experiments on double scattering) corrections must be applied to the nonrelativistic formal theory. The relativistic changes in the angular correlations are indicated for successive reactions of the type π + p → Y + K, Y → N + π.

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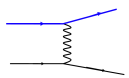
Description of spin states: spin-1/2 particles

Helicity（螺旋度） $\hat{h}\equiv\frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$

① Only for particles with given $\vec{p}$

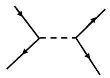
② Neither additive nor multiplicative

③ Lorentz invariant for $m=0$, helicity=chirality



④ Helicity conservation：

- scattering: $h_{\text{in}}=h_{\text{final}}$
- pair creation/annihilation: $h_{\text{particle}}=-h_{\text{antiparticle}}$



$$\psi=\psi_L+\psi_R \qquad \psi_{L/R}=\frac{1}{2}(1\pm\gamma_5)\psi$$

$$\bar{\psi}\psi=\bar{\psi}_L\psi_R+\bar{\psi}_R\psi_L \qquad \bar{\psi}\gamma_\mu\psi=\bar{\psi}_L\gamma_\mu\psi_L+\bar{\psi}_R\gamma_\mu\psi_R$$

The most well-known example: $\sigma(\pi^-\to\mu^-\bar{\nu}_\mu)\gg\sigma(\pi^-\to e^-\bar{\nu}_e)$

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Description of spin states: spin-1/2 particles

Chirality and helicity

（1）Chirality（手征性）定义与性质

$$\gamma_5=i\gamma_0\gamma_1\gamma_2\gamma_3 \qquad \gamma_5^\dagger=\gamma_5 \qquad \{\gamma_5,\gamma_\mu\}=0 \qquad \gamma_5^2=1 \qquad \gamma_5=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$$

$$\gamma_5\psi=\lambda\psi \qquad \lambda=\pm1 \Longleftrightarrow \psi_{L/R}=\frac{1}{2}(1\pm\gamma_5)\psi \qquad \psi=\psi_L+\psi_R$$

$$\psi^\dagger\psi=\psi_L^\dagger\psi_L+\psi_R^\dagger\psi_R \qquad \psi_L^\dagger\psi_R=\psi_R^\dagger\psi_L=0$$

$$\bar{\psi}\psi=\bar{\psi}_L\psi_R+\bar{\psi}_R\psi_L \qquad \bar{\psi}_L\psi_L=\bar{\psi}_R\psi_R=0$$

$$\bar{\psi}\gamma^\mu\psi=\bar{\psi}_L\gamma^\mu\psi_L+\bar{\psi}_R\gamma^\mu\psi_R \qquad \bar{\psi}_L\gamma^\mu\psi_R=\bar{\psi}_R\gamma^\mu\psi_L=0$$

（2）当$m=0$时， chirality=helicity

$$u(p,\lambda)=\begin{pmatrix}\xi(\lambda)\\ \lambda\xi(\lambda)\end{pmatrix} \qquad \gamma_5 u(p,\lambda)=\begin{pmatrix}\lambda\xi(\lambda)\\ \xi(\lambda)\end{pmatrix}$$

$$u(p,R)=\begin{pmatrix}\xi(+)\\ \xi(+)\end{pmatrix}=u(p,+) \qquad u(p,L)=\begin{pmatrix}\xi(-)\\ -\xi(-)\end{pmatrix}=u(p,-)$$

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Description of spin states: spin-1/2 particles

Dirac spinor的bilinear covariants（双线性协变量）

（1）The independent Γ-matrices

In the 2x2 case: $(I,\sigma_x,\sigma_y,\sigma_z)$ $\{\sigma_i,\sigma_j\}=2\delta_{ij}$ $\mathrm{Tr}\sigma_i=0$ $\mathrm{Tr}(\sigma_i\sigma_j)=2\delta_{ij}$

For a given $\hat{O}$: $\hat{O}=\hat{O}_s I+\hat{\vec{O}}_V\cdot\vec{\sigma}$ $\hat{O}_s=\frac{1}{2}\mathrm{Tr}(\hat{O})$ $\hat{\vec{O}}_V=\frac{1}{2}\mathrm{Tr}(\hat{O}\vec{\sigma})$

In the 4x4 case: $\Gamma_n=\{I,\gamma_5,\gamma_\mu,\gamma_5\gamma_\mu,\sigma_{\mu\nu}\}$ 16 independent Γ-matrices

$\mathrm{Tr}\Gamma_n=0$ besides $\Gamma_1=I$ $\Gamma_n^2=\pm I$ $\mathrm{Tr}\{\Gamma_a\Gamma_b\}=\pm 4\delta_{ab}$

$\gamma_5^\dagger=\gamma_5,\ \gamma_\mu^\dagger=\gamma_0\gamma_\mu\gamma_0,\ (\gamma_5\gamma_\mu)^\dagger=\gamma_0(\gamma_5\gamma_\mu)\gamma_0,\ \sigma_{\mu\nu}^\dagger=\gamma_0\sigma_{\mu\nu}\gamma_0$

For a given $\hat{O}$: $\hat{O}=\hat{O}_s I+\hat{O}_P\gamma_5+\hat{O}_{V\mu}\gamma^\mu+\hat{O}_{A\mu}\gamma_5\gamma^\mu+\hat{O}_{T\mu\nu}\sigma^{\mu\nu}$

$\hat{O}_n=\pm\frac{1}{4}\mathrm{Tr}(\hat{O}\Gamma_n)$

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Description of spin states: spin-1/2 particles

Dirac spinor的bilinear covariants（双线性协变量）

（2）The bilinear covariants $\psi\Gamma_n\psi$

$\psi\psi$	scalar	$\hat{P}\psi\psi=\psi\psi$
$\psi\gamma_5\psi$	pseudoscalar	$\hat{P}\psi\gamma_5\psi=-\psi\gamma_5\psi$
$\psi\gamma_\mu\psi$	vector	$\hat{P}\psi\gamma_\mu\psi=\psi\gamma^\mu\psi$
$\psi\gamma_5\gamma_\mu\psi$	axial vector	$\hat{P}\psi\gamma_5\gamma_\mu\psi=-\psi\gamma_5\gamma^\mu\psi$
$\psi\sigma_{\mu\nu}\psi$	(anti-symmetric) tensor	$\hat{P}\psi\sigma_{\mu\nu}\psi=\psi\sigma^{\mu\nu}\psi$

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Description of spin states: spin-1/2 particles

Polarization vector of a spin-1/2 particle system

The spin density matrix $\hat{\rho}=\sum_\alpha g_\alpha|\alpha\rangle\langle\alpha|$ normalization $\mathrm{Tr}\hat{\rho}=\sum_\alpha g_\alpha=1$

Average value of $\hat{O}$: $\langle\hat{O}\rangle=\mathrm{Tr}\,\hat{\rho}\hat{O}$

Probability in the state $|\psi\rangle$: $P_\psi=\langle\psi|\hat{\rho}|\psi\rangle$

We choose a basis, e.g., the helicity basis $|\lambda\rangle$, where $\lambda=\pm1$,

$\rho_{\lambda\lambda'}=\langle\lambda|\hat{\rho}|\lambda'\rangle=\sum_\alpha g_\alpha\langle\lambda|\alpha\rangle\langle\alpha|\lambda'\rangle$

$\hat{\rho}=\begin{pmatrix}\rho_{++} & \rho_{+-}\\ \rho_{-+} & \rho_{--}\end{pmatrix}$ is a 2x2 Hermitian matrix.

We decompose it as $\hat{\rho}=\frac{1}{2}(1+\vec{P}\cdot\vec{\sigma})$

$\vec{P}=\mathrm{Tr}(\hat{\rho}\vec{\sigma})=\langle\vec{\sigma}\rangle$ is the polarization vector of the system.

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Description of spin states: spin-1/2 particles

Polarization vector of a spin-1/2 particle system in a pure state $|p,n\rangle$

Non-relativistic, the spin state is given by the Pauli spinor $\xi(n)$

$\vec{\sigma}\cdot\vec{n}\xi(n)=\xi(n)$ $\xi(n)=\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{pmatrix}$

The helicity state is given by $\xi(\lambda)$ where $\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi_h(\lambda)=\lambda\xi_h(\lambda)$

$\hat{\rho}=|n\rangle\langle n|$ $\rho_{\lambda\lambda'}=\langle\lambda|\hat{\rho}|\lambda'\rangle=\langle\lambda|n\rangle\langle n|\lambda'\rangle$ $\langle\lambda|n\rangle=\xi_h^\dagger(\lambda)\xi(n)$

take $\vec{p}=|\vec{p}|\vec{e}_z$ as an example where we have $\xi_h(+)=\begin{pmatrix}1\\0\end{pmatrix}$ $\xi_h(-)=\begin{pmatrix}0\\1\end{pmatrix}$

$\xi_h^\dagger(+)\xi(n)=\cos\frac{\theta}{2}$

$\xi_h^\dagger(-)\xi(n)=\sin\frac{\theta}{2}e^{i\varphi}$

$\hat{\rho}=\begin{pmatrix}\cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta e^{i\varphi}\\ \frac{1}{2}\sin\theta e^{i\varphi} & \sin^2\frac{\theta}{2}\end{pmatrix}$ $\vec{P}=\mathrm{Tr}(\hat{\rho}\vec{\sigma})=\vec{n}$

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Description of spin states: spin-1/2 particles

Polarization vector of a spin-1/2 particle system in a pure state $|p,n\rangle$

Relativistic, the spin state is given by the Dirac spinor $|p,n\rangle$

$$|p,n\rangle = N\begin{pmatrix} \xi(n) \\ \dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\xi(n) \end{pmatrix} \qquad \text{where } \vec{\sigma}\cdot\vec{n}\xi(n)=\xi(n) \qquad \xi(n)=\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}$$

The helicity state $|p,\lambda\rangle$ $\quad |p,\lambda\rangle = N\begin{pmatrix} \xi_h(\lambda) \\ \dfrac{\lambda|\vec{p}|}{E+m}\xi_h(\lambda) \end{pmatrix}$ where $\dfrac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi_h(\lambda)=\lambda\xi_h(\lambda)$

$$\langle\lambda|n\rangle = \langle p,\lambda|p,n\rangle = N^2\left[\xi_h^\dagger(\lambda)\xi(n) + \xi_h^\dagger(\lambda)\frac{\lambda|\vec{p}|}{E+m}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi(n)\right] = \xi_h^\dagger(\lambda)\xi(n)$$

$$\Longrightarrow \quad \hat{\rho} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta e^{i\varphi} \\ \frac{1}{2}\sin\theta e^{i\varphi} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$\vec{P} = \mathrm{Tr}(\hat{\rho}\vec{\sigma}) = \vec{n}$

average of $\vec{\sigma}$ in the rest frame of the particle

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Description of spin states: spin-1/2 particles

Relativistic, the spin state is given by the Dirac spinor $|p,n\rangle$

Suppose the particle is in $|p_1,n_1\rangle$ and $|p_2,n_2\rangle$ with probabilities g_1 and g_2 respectively

$$\hat{\rho} = g_1|p_1,n_1\rangle\langle p_1,n_1| + g_2|p_2,n_2\rangle\langle p_2,n_2|$$

$$\rho_{\lambda\lambda'} = g_1\langle p,\lambda|p_1,n_1\rangle\langle p_1,n_1|p,\lambda'\rangle + g_2\langle p,\lambda|p_2,n_2\rangle\langle p_2,n_2|p,\lambda'\rangle$$

$$\rho_{1\lambda\lambda'}(p) = \langle p,\lambda|p_1,n_1\rangle\langle p_1,n_1|p,\lambda'\rangle$$
$$= \langle p_1,\lambda|p_1,n_1\rangle\langle p_1,n_1|p_1,\lambda'\rangle\delta^4(p-p_1) = \rho_{1\lambda\lambda'}(p_1,n_1)\delta^4(p-p_1)$$

$$\hat{\rho}(p) = g_1\hat{\rho}_1(p_1,n_1)\delta^4(p-p_1) + g_2\hat{\rho}_2(p_2,n_2)\delta^4(p-p_2)$$

$$= g_1\frac{1}{2}(1+\vec{\sigma}\cdot\vec{n}_1)\delta^4(p-p_1) + g_2\frac{1}{2}(1+\vec{\sigma}\cdot\vec{n}_2)\delta^4(p-p_2)$$

$$\vec{P} = \mathrm{Tr}(\hat{\rho}\vec{\sigma}) = g_1\mathrm{Tr}(\hat{\rho}_1\vec{\sigma}) + g_2\mathrm{Tr}(\hat{\rho}_2\vec{\sigma}) = g_1\vec{n}_1 + g_2\vec{n}_2$$

average of $\vec{\sigma}$ in the rest frame of the particle

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Description of quark spin correlations —— decomposition

For single particle, we decompose

$$\hat{\rho}^{(1)} = \frac{1}{2}(\mathbb{I} + P_{1i}\hat{\sigma}_{1i}) \qquad P_{1i} = \langle\hat{\sigma}_{1i}\rangle = \mathrm{Tr}[\hat{\rho}^{(1)}\hat{\sigma}_{1i}]$$

For two particle system (12),

we are used to $\quad \hat{\rho}^{(12)} = \frac{1}{2^2}\left(\mathbb{I}_1\otimes\mathbb{I}_2 + P_{1i}\hat{\sigma}_{1i}\otimes\mathbb{I}_2 + P_{2i}\mathbb{I}_1\otimes\hat{\sigma}_{2i} + t_{ij}^{(12)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\right)$

shortage: $t_{ij}^{(12)} = P_{1i}P_{2j} \neq 0$ if $\hat{\rho}^{(12)} = \hat{\rho}^{(1)}\otimes\hat{\rho}^{(2)}$

we propose $\quad \hat{\rho}^{(12)} = \hat{\rho}^{(1)}\otimes\hat{\rho}^{(2)} + \frac{1}{2^2}c_{ij}^{(12)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}$

$$c_{ij}^{(12)} = \langle\hat{\sigma}_{1i}\hat{\sigma}_{2j}\rangle - \langle\hat{\sigma}_{1i}\rangle\langle\hat{\sigma}_{2j}\rangle \qquad c_{ij}^{(12)} = 0 \text{ if } \hat{\rho}^{(12)} = \hat{\rho}^{(1)}\otimes\hat{\rho}^{(2)}$$

For three particle system (123)

$$\hat{\rho}^{(123)} = \hat{\rho}^{(1)}\otimes\hat{\rho}^{(2)}\otimes\hat{\rho}^{(3)} + \frac{1}{2^2}\left[c_{ij}^{(12)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\rho}^{(3)} + (1\to2\to3)\right]$$
$$+ \frac{1}{2^3}c_{ijk}^{(123)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k}$$

Ji-peng Lv, Zi-han Yu, ZTL, Qun Wang and Xin-Nian Wang, PRD109, 114003 (2024)

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Description of quark spin correlations —— α-dependence

Single particle: $\hat{\rho}^{(1)}(\alpha) = \frac{1}{2}[1 + P_{1i}(\alpha)\hat{\sigma}_{1i}]$

Two particle system A=(12): $\hat{\rho}^{(12)}(\alpha_1,\alpha_2) = \hat{\rho}^{(1)}(\alpha_1)\otimes\hat{\rho}^{(2)}(\alpha_2) + \frac{1}{2^2}c_{ij}^{(12)}(\alpha_1,\alpha_2)\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}$

Suppose A=(12) is in the state $|\alpha_{12}\rangle$, the α_{12}-dependent spin density matrix for A is

$$\hat{\rho}^{(12)}(\alpha_{12}) = \langle\alpha_{12}|\hat{\rho}^{(12)}(\alpha_1,\alpha_2)|\alpha_{12}\rangle = \sum_{\alpha_1\alpha_2}|\langle\alpha_1,\alpha_2|\alpha_{12}\rangle|^2\hat{\rho}^{(12)}(\alpha_1,\alpha_2)$$
$$= \hat{\rho}^{(1)}(\alpha_{12})\otimes\hat{\rho}^{(2)}(\alpha_{12}) + \frac{1}{2^2}c_{ij}^{(12)}(\alpha_{12})\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}$$

average inside A

$$\hat{\rho}^{(1)}(\alpha_{12}) = \sum_{\alpha_1\alpha_2}|\langle\alpha_1,\alpha_2|\alpha_{12}\rangle|^2\hat{\rho}^{(1)}(\alpha_1) = \frac{1}{2}[1 + P_{1i}(\alpha_{12})\hat{\sigma}_{1i}]$$

The polarization $\quad P_{1i}(\alpha_{12}) = \sum_{\alpha_1\alpha_2}|\langle\alpha_1,\alpha_2|\alpha_{12}\rangle|^2 P_{1i}(\alpha_1) \equiv \langle P_{1i}(\alpha_1)\rangle \quad$ equals to P_{1i} averaged inside A

However, the correlation $\quad c_{ij}^{(12)}(\alpha_{12}) \neq \left\langle c_{ij}^{(12)}(\alpha_1,\alpha_2)\right\rangle \quad$ does not equal to $c_{ij}^{(12)}$ averaged inside A

instead

$$c_{ij}^{(12)}(\alpha_{12}) = \langle c_{ij}^{(12)}(\alpha_1,\alpha_2)\rangle + c_{ij}^{(12;0)}(\alpha_{12})$$

"effective correlation" = "genuine correlation" + "induced correlation"

the observed — generated from the dynamical process — caused by the average over α_i

$$c_{ij}^{(12;0)}(\alpha_{12}) \equiv \langle P_{1i}(\alpha_1)P_{2j}(\alpha_2)\rangle - \langle P_{1i}(\alpha_1)\rangle\langle P_{1i}(\alpha_1)\rangle$$

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Spin density matrix for V from quark combination

For $q_1 + q_2 \to V$, in general $\hat{\rho}^V = \hat{\mathcal{M}}\hat{\rho}^{(q_1 q_2)}\hat{\mathcal{M}}^\dagger$ $\hat{\mathcal{M}}$: transition matrix

$$|i\rangle \to |f\rangle = \hat{\mathcal{M}}|i\rangle, \quad \hat{\rho}^f = \sum_\alpha g_\alpha |f_\alpha\rangle\langle f_\alpha| = \sum_\alpha g_\alpha \hat{\mathcal{M}}|i_\alpha\rangle\langle i_\alpha|\hat{\mathcal{M}}^\dagger = \hat{\mathcal{M}}\hat{\rho}^i\hat{\mathcal{M}}^\dagger$$

If only spin degree of freedom is considered

$$\rho^V_{mm'} = \langle jm|\hat{\mathcal{M}}\hat{\rho}^{(q_1q_2)}\hat{\mathcal{M}}^\dagger|jm'\rangle$$

$$= \sum_{m_1m_2,m'_1m'_2} \langle jm|\hat{\mathcal{M}}|m_1m_2\rangle\langle m_1m_2|\hat{\rho}^{(q_1q_2)}|m'_1m'_2\rangle\langle m'_1m'_2|\hat{\mathcal{M}}^\dagger|jm'\rangle$$

$$= N \sum_{m_1m_2,m'_1m'_2} \langle jm|m_1m_2\rangle\langle m_1m_2|\hat{\rho}^{(q_1q_2)}|m'_1m'_2\rangle\langle m'_1m'_2|jm'\rangle$$

independent of $\hat{\mathcal{M}}$

since $\langle jm|\hat{\mathcal{M}}|m_1m_2\rangle = \sum_{j'm'}\langle jm|\hat{\mathcal{M}}|j'm'\rangle\langle j'm'|m_1m_2\rangle$

$= \langle jm|\hat{\mathcal{M}}|jm\rangle\langle jm|m_1m_2\rangle$

$= N_j\langle jm|m_1m_2\rangle$

space rotation invariance demands
①angular momentum conservation $j = j', m = m'$
② $\langle jm|\hat{\mathcal{M}}|jm\rangle$ is independent of m

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Description of spin states: spin-1/2 particles

Four dimensional polarization vector and spin projection operator of a spin-1/2 particle

In the rest frame $s = (0, \vec{s})$ $p \cdot s = 0$

In the moving frame $s = \left(\frac{\vec{p}\cdot\vec{s}}{m}, \vec{s} + \frac{(\vec{p}\cdot\vec{s})\vec{p}}{m(E+m)}\right)$

Longitudinal polarization $\vec{s} \parallel \vec{p}$: $s_{||} = \lambda\frac{1}{m}\left(|\vec{p}|, E\frac{\vec{p}}{|\vec{p}|}\right) = \lambda v\frac{p}{m} + \lambda\frac{m}{E}\left(0, \frac{\vec{p}}{|\vec{p}|}\right) \to \lambda\frac{p}{m}$

Transverse polarization $\vec{s} \perp \vec{p}$: $s_\perp = (0, \vec{s}_\perp, 0)$

$$s = s_{||} + s_\perp = \lambda v\frac{p}{m} + \lambda\frac{m}{E}\left(0, \frac{\vec{p}}{|\vec{p}|}\right) + s_\perp$$

Space reflection: $p^\mu = (p_0, \vec{p}) \to \tilde{p}^\mu = p_\mu = (p_0, -\vec{p})$

$s^\mu = (s_0, \vec{s}) \to -\tilde{s}^\mu = -s_\mu = (-s_0, \vec{s})$

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Description of spin states: spin-1/2 particles

The spin projection operator $u(p,s)u(p,s) = (\not{p} + m)\frac{1}{2}(1 + \gamma_5\not{s})$

$u(p,s) = N\begin{pmatrix}\xi(s) \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\xi(s)\end{pmatrix}$ where $\vec{\sigma}\cdot\vec{s}\xi(s) = \xi(s)$ $\xi(s)\xi^\dagger(s) = \frac{1}{2}(1 + \vec{\sigma}\cdot\vec{s})$

$$u(p,s)u(p,s) = \begin{pmatrix}\xi(s)\xi^\dagger(s) & -\xi(s)\xi^\dagger(s)\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\xi(s)\xi^\dagger(s) & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\xi(s)\xi^\dagger(s)\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2}(1+\vec{\sigma}\cdot\vec{s}) & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\frac{1}{2}[1-\vec{\sigma}\cdot\vec{s}+\frac{2}{\vec{p}^2}(\vec{p}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma})] \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\frac{1}{2}(1+\vec{\sigma}\cdot\vec{s}) & -\frac{E-m}{E+m}\frac{1}{2}[1-\vec{\sigma}\cdot\vec{s}+\frac{2}{\vec{p}^2}(\vec{p}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma})]\end{pmatrix}$$

$$= \begin{pmatrix}1 & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\frac{E-m}{E+m}\end{pmatrix}\begin{pmatrix}\frac{1}{2}(1+\vec{\sigma}\cdot\vec{s}) & 0 \\ 0 & \frac{1}{2}[1-\vec{\sigma}\cdot\vec{s}+\frac{2}{\vec{p}^2}(\vec{p}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma})]\end{pmatrix}$$

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Description of spin states: spin-1/2 particles

The spin projection operator $u(p,s)u(p,s) = (\not{p} + m)\frac{1}{2}(1 + \gamma_5\not{s})$

$$N^2\begin{pmatrix}1 & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\frac{E-m}{E+m}\end{pmatrix} = \frac{N^2}{E+m}\begin{pmatrix}E+m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E+m\end{pmatrix} = \frac{N^2}{E+m}(\gamma\cdot p + m)$$

$\gamma_5 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$ $\gamma_5\gamma_0 = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$ $\gamma_5\vec{\gamma} = \begin{pmatrix}-\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}$ $s = (\frac{\vec{p}\cdot\vec{s}}{m}, \vec{s} + \frac{(\vec{p}\cdot\vec{s})\vec{p}}{m(E+m)})$

$$\gamma_5\gamma\cdot s = \begin{pmatrix}\vec{\sigma}\cdot\vec{s} + \frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E+m)} & -\frac{\vec{p}\cdot\vec{s}}{m} \\ \frac{\vec{p}\cdot\vec{s}}{m} & -\vec{\sigma}\cdot\vec{s} - \frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E+m)}\end{pmatrix}$$

$$\begin{pmatrix}1+\vec{\sigma}\cdot\vec{s} & 0 \\ 0 & 1-\vec{\sigma}\cdot\vec{s}+\frac{2}{\vec{p}^2}(\vec{p}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma})\end{pmatrix} = 1 + \gamma_5\gamma\cdot s + \begin{pmatrix}-\frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E+m)} & \frac{\vec{p}\cdot\vec{s}}{m} \\ -\frac{\vec{p}\cdot\vec{s}}{m} & \frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E-m)}\end{pmatrix}$$

$$(\gamma\cdot p + m)\begin{pmatrix}-\frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E+m)} & \frac{\vec{p}\cdot\vec{s}}{m} \\ -\frac{\vec{p}\cdot\vec{s}}{m} & \frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E-m)}\end{pmatrix} = 0$$

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Description of spin states: spin-1 particles

Spin operator

$$\hat{\vec{S}} = \vec{\Sigma} \qquad \Sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \qquad \Sigma_y = \frac{i}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix} \qquad \Sigma_z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

The helicity basis $\quad |\lambda = 1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad |\lambda = -1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad |\lambda = 0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$

The polarization vector $\vec{\varepsilon}$

The basis: $|\varepsilon_{(x)}\rangle \equiv \frac{1}{\sqrt{2}}(|\lambda = -1\rangle - |\lambda = 1\rangle) \qquad |\varepsilon_{(y)}\rangle \equiv \frac{i}{\sqrt{2}}(|\lambda = -1\rangle + |\lambda = 1\rangle)$

$|\varepsilon_{(z)}\rangle \equiv |\lambda = 0\rangle$

The general form of a pure state: $|\varepsilon\rangle = \varepsilon_x|\varepsilon_{(x)}\rangle + \varepsilon_y|\varepsilon_{(y)}\rangle + \varepsilon_z|\varepsilon_{(z)}\rangle$

The polarization vector is defined as: $\vec{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_z)$

The spin polarization vector $\vec{S}$ $\qquad \vec{S} \equiv \langle\vec{S}\rangle = (\vec{S}_T, \lambda)$ in the rest frame of V

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Description of spin states: spin-1 particles

The spin density matrix

$$\hat{\rho} = \begin{pmatrix}\rho_{11} & \rho_{10} & \rho_{1-1}\\ \rho_{01} & \rho_{00} & \rho_{0-1}\\ \rho_{-11} & \rho_{-10} & \rho_{-1-1}\end{pmatrix}$$

Decomposition $\quad \hat{\rho} = \frac{1}{3}\left[1 + \frac{3}{2}S^i\Sigma^i + 3T^{ij}\Sigma^{ij}\right] \qquad \Sigma^{ij} = \frac{1}{2}(\Sigma^i\Sigma^j + \Sigma^j\Sigma^i) - \frac{2}{3}I\delta_{ij}$

$$T = \frac{1}{2}\begin{pmatrix}-\frac{2}{3}S_{LL} + S_{TT}^{xx} & S_{TT}^{xx} & S_{LT}^{x}\\ S_{TT}^{xy} & -\frac{2}{3}S_{LL} - S_{TT}^{xx} & S_{LT}^{y}\\ S_{LT}^{x} & S_{LT}^{y} & \frac{4}{3}S_{LL}\end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix}S_{LL} & 0 & 0\\ 0 & S_{LL} & 0\\ 0 & 0 & -2S_{LL}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 0 & S_{LT}^{x}\\ 0 & 0 & S_{LT}^{y}\\ S_{LT}^{x} & S_{LT}^{y} & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}S_{TT}^{xx} & S_{TT}^{xx} & 0\\ S_{TT}^{xy} & S_{TT}^{xx} & 0\\ 0 & 0 & 0\end{pmatrix}$$

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Description of spin states: spin-1 particles

Spin density matrix in terms of Lorentz covariants

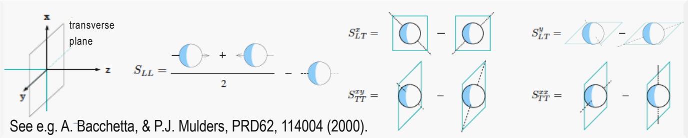
Spin polarization vector: $S = (0, S_x, S_y, S_z) = (0, S_T^x, S_T^y, S_L)$

Tensor polarization: Scalar S_{LL}

Vector $S_{LT} = (0, S_{LT}^x, S_{LT}^y, 0)$

Tensor $S_{TT} = \begin{pmatrix}0 & 0 & 0 & 0\\ 0 & S_{TT}^{xx} & S_{TT}^{xy} & 0\\ 0 & S_{TT}^{xy} & S_{TT}^{yy} & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$

$$\hat{\rho} = \begin{pmatrix}\frac{1+S_{LL}}{3} + \frac{S_L}{2} & \frac{(S_{LT}^x - iS_{LT}^y) + (S_T^x - iS_T^y)}{2\sqrt{2}} & \frac{S_{TT}^{xx} - iS_{TT}^{xy}}{2}\\ \frac{(S_{LT}^x + iS_{LT}^y) + (S_T^x + iS_T^y)}{2\sqrt{2}} & \frac{1-2S_{LL}}{3} & \frac{(-S_{LT}^x + iS_{LT}^y) + (S_T^x - iS_T^y)}{2\sqrt{2}}\\ \frac{S_{TT}^{xx} + iS_{TT}^{xy}}{2} & \frac{(-S_{LT}^x - iS_{LT}^y) + (S_T^x + iS_T^y)}{2\sqrt{2}} & \frac{1+S_{LL}}{3} - \frac{S_L}{2}\end{pmatrix}$$



See e.g. A. Bacchetta, & P.J. Mulders, PRD62, 114004 (2000).

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Description of spin states: spin-1 particles

Relationship between the spin polarization vector $\vec{S}$ and the polarization vector $\vec{\varepsilon}$

$\vec{S} = \mathrm{Im}(\vec{\varepsilon}^* \times \vec{\varepsilon})$

$\vec{S} = 0$ for any pure state with a real $\vec{\varepsilon}$

$\vec{\varepsilon}^{(\pm)} = \frac{1}{\sqrt{2}}(\mp 1, -i, 0) \Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0, 0, \pm 1)$

$\vec{\varepsilon}^{(0)} = (0, 0, 1) \Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0, 0, 0)$

pure state with $\rho_{00} = 1$ $\Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0, 0, 0) \Longleftrightarrow \vec{\varepsilon}^{(0)} = (0, 0, 1)$ in OZ direction

pure state with $\rho_{00} = 0$ $\Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0, 0, \sin 2\theta \sin\varphi) \Longleftrightarrow \vec{\varepsilon} = (\cos\theta, \sin\theta e^{i\varphi}, 0)$ perpendicular to OZ direction

ρ_{00}: vector meson spin alignment

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Description of spin states: spin-1 particles

The pure state:

$$|\lambda=1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad |\lambda=-1\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}\quad |\lambda=0\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$|\varepsilon_{(x)}\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}\quad |\varepsilon_{(y)}\rangle=\frac{i}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\quad |\varepsilon_{(z)}\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$|\varepsilon\rangle=\varepsilon_x|\varepsilon_{(x)}\rangle+\varepsilon_y|\varepsilon_{(y)}\rangle+\varepsilon_z|\varepsilon_{(z)}\rangle=\begin{pmatrix}\frac{1}{\sqrt{2}}(-\varepsilon_x+i\varepsilon_y)\\ \varepsilon_z\\ \frac{1}{\sqrt{2}}(\varepsilon_x+i\varepsilon_y)\end{pmatrix}$$

$$\langle\varepsilon|=\left(\frac{1}{\sqrt{2}}(-\varepsilon_x^*-i\varepsilon_y^*)\quad \varepsilon_z^*\quad \frac{1}{\sqrt{2}}(\varepsilon_x^*-i\varepsilon_y^*)\right)$$

$$\Sigma_x=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\quad \Sigma_y=\frac{i}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}\quad \Sigma_z=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$\vec{S}=\langle\vec{S}\rangle=(\langle\Sigma_x\rangle,\langle\Sigma_y\rangle,\langle\Sigma_z\rangle)=\frac{1}{2}\left(i(\varepsilon_z^*\varepsilon_y-\varepsilon_y^*\varepsilon_z),\ i(\varepsilon_z^*\varepsilon_x-\varepsilon_x^*\varepsilon_z),i(\varepsilon_x^*\varepsilon_y-\varepsilon_y^*\varepsilon_x)\right)=\mathrm{Im}(\vec{\varepsilon}^*\times\vec{\varepsilon})$$

Description of spin states: spin-1 particles

The spin density matrix:

$$\hat{\rho}(\varepsilon)=|\varepsilon\rangle\langle\varepsilon|=\begin{pmatrix}\frac{1}{2}|\varepsilon_x-i\varepsilon_y|^2 & \frac{1}{\sqrt{2}}(-\varepsilon_x+i\varepsilon_y)\varepsilon_z^* & \frac{1}{2}(-\varepsilon_x+i\varepsilon_y)(\varepsilon_x^*-i\varepsilon_y^*)\\ \frac{1}{\sqrt{2}}\varepsilon_z(-\varepsilon_x^*-i\varepsilon_y^*) & |\varepsilon_z|^2 & \frac{1}{\sqrt{2}}\varepsilon_z(\varepsilon_x^*-i\varepsilon_y^*)\\ \frac{1}{2}(\varepsilon_x+i\varepsilon_y)(-\varepsilon_x^*-i\varepsilon_y^*) & \frac{1}{\sqrt{2}}(\varepsilon_x+i\varepsilon_y)\varepsilon_z^* & \frac{1}{2}|\varepsilon_x+i\varepsilon_y|^2\end{pmatrix}$$

$$\rho_{00}=1\Longleftrightarrow\hat{\rho}(\varepsilon)=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\Longleftrightarrow\begin{matrix}\varepsilon_z=1\\ \varepsilon_x=\varepsilon_y=0\end{matrix}\Longleftrightarrow\vec{\varepsilon}=(0,0,1)\Longleftrightarrow\vec{S}=\langle\vec{S}\rangle=(0,0,0)$$

$$\rho_{00}=0\Longleftrightarrow\varepsilon_z=0\Longleftrightarrow\hat{\rho}(\varepsilon)=\begin{pmatrix}\frac{1}{2}|\varepsilon_x-i\varepsilon_y|^2 & 0 & \frac{1}{2}(-\varepsilon_x+i\varepsilon_y)(\varepsilon_x^*-i\varepsilon_y^*)\\ 0&0&0\\ \frac{1}{2}(\varepsilon_x+i\varepsilon_y)(-\varepsilon_x^*-i\varepsilon_y^*) & 0 & \frac{1}{2}|\varepsilon_x+i\varepsilon_y|^2\end{pmatrix}$$

Normalization $|\varepsilon_x|^2+|\varepsilon_y|^2=1$ $\quad \varepsilon_x=\cos\theta \quad \varepsilon_y=\sin\theta e^{i\varphi} \quad \vec{\varepsilon}=(\cos\theta,\sin\theta e^{i\varphi},0)$

$$\hat{\rho}(\varepsilon)=\begin{pmatrix}\frac{1}{2}(1+\sin2\theta\sin\varphi) & 0 & \frac{1}{2}(-\cos2\theta+i\sin2\theta\cos\varphi)\\ 0&0&0\\ \frac{1}{2}(-\cos2\theta-i\sin2\theta\cos\varphi) & 0 & \frac{1}{2}(1-\sin2\theta\sin\varphi)\end{pmatrix}\qquad \vec{S}=\langle\vec{S}\rangle=(0,0,\sin2\theta\sin\varphi)$$

Description of polarization of the photon

Circular polarization of the photon:

in the helicity state $\quad \rho_\gamma^{circ}=\frac{1}{2}\begin{pmatrix}1+P_{circ}&0&0\\0&0&0\\0&0&1-P_{circ}\end{pmatrix}$

Linear polarization of the photon:

linearly polarized along OX or OY $\Longleftrightarrow$ in state $|\varepsilon_{(x)}\rangle$ or $|\varepsilon_{(y)}\rangle$

$$\rho_\gamma^{lin(x)}=\frac{1}{2}\begin{pmatrix}1&0&-P_{lin}\\0&0&0\\-P_{lin}&0&1\end{pmatrix}\qquad \rho_\gamma^{lin(y)}=\frac{1}{2}\begin{pmatrix}1&0&P_{lin}\\0&0&0\\P_{lin}&0&1\end{pmatrix}$$

in OXY plane at an angle γ to OX $\quad \rho_\gamma^{lin}=\frac{1}{2}\begin{pmatrix}1&0&-P_{lin}e^{-2i\gamma}\\0&0&0\\-P_{lin}e^{2i\gamma}&0&1\end{pmatrix}$

$$\rho_\gamma^{lin}=\frac{1}{2}\begin{pmatrix}1&0&-P_{lin}e^{-2i\gamma}\\0&0&0\\-P_{lin}e^{2i\gamma}&0&1\end{pmatrix}\to\frac{1}{2}\begin{pmatrix}1&-P_{lin}e^{-2i\gamma}\\-P_{lin}e^{2i\gamma}&1\end{pmatrix}=\frac{1}{2}[1-P_{lin}(\sigma_x\cos2\gamma+\sigma_y\sin2\gamma)]$$

Description of polarization of particles with different spins

Spin 3/2 hadrons: $\quad \hat{\rho}=\frac{1}{4}\left(1+\frac{4}{5}S^i\Sigma^i+\frac{2}{3}T^{ij}\Sigma^{ij}+\frac{8}{9}R^{ijk}\Sigma^{ijk}\right)$

$$\Sigma^i=(\Sigma^x,\Sigma^y,\Sigma^z)\quad \Sigma^x=\frac{1}{2}\begin{pmatrix}0&\sqrt{3}&0&0\\\sqrt{3}&0&0&0\\0&0&0&\sqrt{3}\\0&0&\sqrt{3}&0\end{pmatrix}\quad \Sigma^y=\frac{1}{2}\begin{pmatrix}0&-i\sqrt{3}&0&0\\i\sqrt{3}&0&-i&0\\0&i&0&-i\sqrt{3}\\0&0&i\sqrt{3}&0\end{pmatrix}\quad \Sigma^z=\frac{1}{2}\begin{pmatrix}3&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-3\end{pmatrix}\quad S^i=(S_T^x,S_T^y,S_L)$$

$$\Sigma^{ij}=\frac{1}{2}(\Sigma^i\Sigma^j+\Sigma^j\Sigma^i)-\frac{5}{4}\delta^{ij}1\qquad T^{ij}=\begin{pmatrix}-S_{LL}+S_{TT}^{xx} & S_{TT}^{xy} & S_{LT}^x\\ S_{TT}^{xy} & -S_{LL}-S_{TT}^{xx} & S_{LT}^y\\ S_{LT}^x & S_{LT}^y & 2S_{LL}\end{pmatrix}$$

$$\Sigma^{ijk}=\frac{1}{3}(\Sigma^{ij}\Sigma^k+\Sigma^{jk}\Sigma^i+\Sigma^{ki}\Sigma^j)-\frac{4}{15}(\delta^{ij}\Sigma^k+\delta^{jk}\Sigma^i+\delta^{ki}\Sigma^j)$$

$$R^{ijk}=\frac{1}{4}\begin{pmatrix}\begin{pmatrix}-3S_{LLT}^x+S_{TTT}^{xxx} & -S_{LLT}^y+S_{TTT}^{yxx} & -2S_{LLL}+S_{LTT}^{xx}\\ -S_{LLT}^y+S_{TTT}^{yxx} & -S_{LLT}^x-S_{TTT}^{xxx} & S_{LTT}^{xy}\\ -2S_{LLL}+S_{LTT}^{xx} & S_{LTT}^{xy} & 2S_{LL}\end{pmatrix}\\ \begin{pmatrix}-S_{LLT}^y+S_{TTT}^{yxx} & -S_{LLT}^x-S_{TTT}^{xxx} & S_{LTT}^{xy}\\ -S_{LLT}^x-S_{TTT}^{xxx} & -3S_{LLT}^y-S_{TTT}^{yxx} & -2S_{LLL}-S_{LTT}^{xx}\\ S_{LTT}^{xy} & -2S_{LLL}-S_{LTT}^{xx} & 4S_{LLT}^y\end{pmatrix}\\ \begin{pmatrix}-2S_{LLL}+S_{LTT}^{xx} & S_{LTT}^{xy} & 4S_{LLT}^x\\ S_{LTT}^{xy} & -2S_{LLL}-S_{LTT}^{xx} & 4S_{LLT}^y\\ 4S_{LLT}^x & 4S_{LLT}^y & 4S_{LLL}\end{pmatrix}\end{pmatrix}$$

See e.g. Jing Zhao, Zhe Zhang, ZTL, Tianbo Liu, Ya-jin Zhou, PRD106, 094006 (2022)

Description of polarization of particles with different spins

Spin 1/2 hadrons:

The spin density matrix is 2x2: $\hat{\rho} = \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix} = \frac{1}{2}(1 + \vec{S} \cdot \vec{\sigma})$

Vector polarization: $S^\mu = (0, \vec{S}_T, \lambda)$

Spin 1 hadrons:

See e.g. A. Bacchetta, & P.J. Mulders, PRD62, 114004 (2000)

The spin density matrix is 3x3: $\rho = \begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix} = \frac{1}{3}(1 + \frac{3}{2}\vec{S} \cdot \vec{\Sigma} + 3T^{ij}\Sigma^{ij})$

Vector polarization: $S^\mu = (0, \vec{S}_T, \lambda)$ — 3

Tensor polarization: S_{LL}, $S_{LT}^i = (S_{LT}^x, S_{LT}^y)$, $S_{TT}^{ij} = \begin{pmatrix} S_{TT}^{xx} & S_{TT}^{xy} \\ S_{TT}^{xy} & -S_{TT}^{xx} \end{pmatrix}$ — 5

8 independent components

Spin 3/2 hadrons:

See e.g. Jing Zhao, Zhe Zhang, ZTL, Tianbo Liu, Ya-jin Zhou, PRD106, 094006 (2022)

The spin density matrix is 4x4: $\hat{\rho} = \frac{1}{4}\left(1 + \frac{4}{5}S^i\Sigma^i + \frac{2}{3}T^{ij}\Sigma^{ij} + \frac{8}{9}R^{ijk}\Sigma^{ijk}\right)$

Vector polarization: $S^\mu = (0, \vec{S}_T, \lambda)$ — 3

Tensor polarization (rank 2): S_{LL}, $S_{LT}^i = (S_{LT}^x, S_{LT}^y)$, $S_{TT}^{ij} = \begin{pmatrix} S_{TT}^{xx} & S_{TT}^{xy} \\ S_{TT}^{xy} & -S_{TT}^{xx} \end{pmatrix}$ — 5

Tensor polarization (rank 3): S_{LLL}, $S_{LLT}^i = (S_{LLT}^x, S_{LLT}^y)$, $S_{LTT}^{ij} = \begin{pmatrix} S_{LTT}^{xx} & S_{LTT}^{xy} \\ S_{LTT}^{xy} & -S_{LTT}^{xx} \end{pmatrix}$ $S_{TTT}^{ijx} = \begin{pmatrix} S_{TTT}^{xxx} & S_{TTT}^{yxx} \\ S_{TTT}^{yxx} & -S_{TTT}^{xxx} \end{pmatrix}$ — 7

15 independent components

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Contents of Part I

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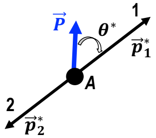
Polarization measurements: hyperon polarization

Two body decay $A \to 1 + 2$ In the rest frame of A

$p_A = (M_A, 0, 0, 0)$ $p_1 = (E_1^*, \vec{p}_1^*)$ $p_2 = (E_2^*, \vec{p}_2^*)$ $\vec{p}_1^* = -\vec{p}_2^* = \vec{p}^*$

$p_A = p_1 + p_2$ $E_1^* = (M_A^2 + m_1^2 - m_2^2)/2M_A$



For unpolarized (or spinless) A, the decay product is isotropic.

$$\frac{d^3N}{d^3p_1} = \frac{1}{4\pi\, p^{*2}}\delta(|\vec{p}_1| - |\vec{p}^*|) \qquad \frac{dN}{d\Omega} = \frac{1}{4\pi}$$

For parity conserved decays of A, the decay product is isotropic.

For parity violating decay of the hyperon,

$$\frac{dN}{d\Omega} = \frac{1}{4\pi}\left(1 + \alpha\vec{P} \cdot \frac{\vec{p}_1^*}{|\vec{p}_1^*|}\right) = \frac{1}{4\pi}(1 + \alpha P \cos\theta^*)$$

Spin self analyzing parity violating weak decay of the hyperon A.

α: the decay polarization parameter, listed in PDG.

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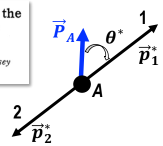
Polarization measurements: hyperon polarization

More completely

$A \to 1 + 2$, e.g., $\Lambda \to p\pi^-$

Phys. Rev. 106, 1645 (1957)

General Partial Wave Analysis of the Decay of a Hyperon of Spin $\frac{1}{2}$
T. D. Lee* and C. N. Yang
Institute for Advanced Study, Princeton, New Jersey
(Received October 22, 1957)



$$\frac{dN}{d\Omega} = \frac{1}{4\pi}(1 + \alpha P_A \cos\theta^*)$$

$$\vec{P}_1 = \frac{(\alpha + \vec{P}_A \cdot \vec{e}_1)\vec{e}_1 + \beta\vec{e}_1 \times \vec{P}_A + \gamma(\vec{e}_1 \times \vec{P}_A) \times \vec{e}_1}{1 + \alpha\vec{P}_A \cdot \vec{e}_1}$$

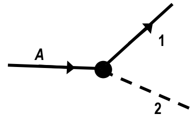
$$\alpha = \frac{2\mathrm{Re}(S^*P)}{|S|^2 + |P|^2} \qquad \beta = \frac{2\mathrm{Im}(S^*P)}{|S|^2 + |P|^2} \qquad \gamma = \frac{|S|^2 - |P|^2}{|S|^2 + |P|^2} \qquad \alpha^2 + \beta^2 + \gamma^2 = 1$$

Parity conserved: $S = 0$, so that $\alpha = 0, \beta = 0, \gamma = -1$

or using (α, ϕ): $\phi = \tan^{-1}\frac{\gamma}{\beta}$, $\beta = (1 - \alpha^2)^{\frac{1}{2}}\cos\phi$, $\gamma = (1 - \alpha^2)^{\frac{1}{2}}\sin\phi$

Parity conserved: $\alpha = 0, \phi = -\pi/2$

$\alpha, \beta, \gamma, \phi$: decay parameters listed in PDG.



Can also be derived from $M \sim u(A)(a - \gamma_5 b)u(1)$

$S \sim a$, $\quad P \sim b|\vec{p}_1|/(E_1 + m_1)$

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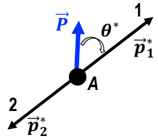
Polarization measurements: $A \to 1+2$

Consider $A \to 1+2$ in the rest frame of A

Suppose the spin density matrix of A is $\hat{\rho}_A = \sum_{M_A} g_{M_A} |S_A, M_A\rangle\langle S_A, M_A|$

The spin density matrix of the system (12) is $\hat{\rho}_{12} = \hat{U}\hat{\rho}_A \hat{U}^\dagger$



The angular distribution — helicity state of (12)

$$W(\theta,\varphi) = N \sum_{\lambda_1\lambda_2} \langle \vec{p};\lambda_1\lambda_2 | \hat{\rho}_{12} | \vec{p};\lambda_1\lambda_2\rangle = N \sum_{\lambda_1\lambda_2} \langle \vec{p};\lambda_1\lambda_2 | \hat{U}\hat{\rho}_A \hat{U}^\dagger | \vec{p};\lambda_1\lambda_2\rangle$$

$$= N \sum_{\lambda_1\lambda_2;M_A M_A'} \langle \vec{p};\lambda_1\lambda_2 | \hat{U} | S_A, M_A\rangle\langle S_A, M_A | \hat{\rho}_A | S_A, M_A'\rangle\langle S_A, M_A' | \hat{U}^\dagger | \vec{p};\lambda_1\lambda_2\rangle$$

$$= N \sum_{\lambda_1\lambda_2;M_A M_A'} A_{M_A}(\vec{p};\lambda_1\lambda_2) A^*_{M_A'}(\vec{p};\lambda_1\lambda_2) \langle S_A, M_A | \hat{\rho}_A | S_A, M_A'\rangle$$

$A_{M_A}(\vec{p};\lambda_1\lambda_2) = \langle \vec{p};\lambda_1,\lambda_2 | \hat{U} | S_A, M_A\rangle$ the decay amplitude of A in the state $|S_A, M_A\rangle$ to (12) in the state $|\vec{p},\lambda_1,\lambda_2\rangle$ with helicities λ_1 and λ_2

Polarization measurements: $A \to 1+2$

The decay amplitude is $A_m(\vec{p};\lambda_1,\lambda_2) = \langle \vec{p};\lambda_1,\lambda_2 | \hat{U} | S_A, M_A\rangle$

Space rotation invariance demands

- total angular momentum conservation

$$A_m(\vec{p};\lambda_1,\lambda_2) = \langle \vec{p};\lambda_1,\lambda_2 | E, S_A, M_A;\lambda_1,\lambda_2\rangle\langle E, J = S_A, M = M_A;\lambda_1,\lambda_2 | \hat{U} | S_A, M_A\rangle$$

- Wigner-Eckhard theorem

$$\langle S_A, M_A;\lambda_1,\lambda_2 | \hat{U} | S_A, M_A\rangle = \langle S_A;\lambda_1,\lambda_2 | \hat{U} | S_A\rangle = H_{S_A}(\lambda_1,\lambda_2)$$

Helicity amplitude, independent of M_A, independent of angles (θ,φ).

Hence $A_m(\vec{p};\lambda_1,\lambda_2) = \langle \vec{p};\lambda_1,\lambda_2 | E, S_A, M_A;\lambda_1,\lambda_2\rangle H_{S_A}(\lambda_1,\lambda_2)$

The angular dependence is determined by $\langle \vec{p};\lambda_1,\lambda_2 | E, S_A, M_A;\lambda_1,\lambda_2\rangle$

Polarization measurements: $A \to 1+2$

Calculation of $\langle p,\theta,\varphi;\lambda_1,\lambda_2 | p,J,M;\lambda_1,\lambda_2\rangle$

It can be shown that $\langle p,0,0;\lambda_1,\lambda_2 | p,J,M;\lambda_1,\lambda_2\rangle = \left(\frac{2J+1}{4\pi}\right)^{1/2}$

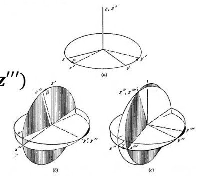
$$|p,\theta,\varphi;\lambda_1,\lambda_2\rangle = \hat{R}(\varphi,\theta,-\varphi)|p,0,0;\lambda_1,\lambda_2\rangle$$

Any rotation can be described by three Euler angles (α,β,γ)

(1) a rotation of angle α around z-axis $(x,y,z) \to (x',y',z')$

(2) a rotation of angle β around y'-axis $(x',y',z') \to (x'',y'',z'')$

(3) a rotation of angle γ around z''-axis $(x'',y'',z'') \to (x''',y''',z''')$



The rotation operator $\hat{R}_n(\alpha) = e^{-i\alpha\hat{J}_n}$

$$\hat{R}(\alpha,\beta,\gamma) = \hat{R}_{z''}(\gamma)\hat{R}_{y'}(\beta)\hat{R}_z(\alpha)$$

$$\hat{R}_{y'}(\beta) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z^{-1}(\alpha)$$

$$\hat{R}(\alpha,\beta,\gamma) = \hat{R}_z(\alpha)\hat{R}_y(\beta)\hat{R}_z(\gamma) = e^{-i\alpha\hat{J}_z}e^{-i\beta\hat{J}_y}e^{-i\gamma\hat{J}_z}$$

Polarization measurements: $A \to 1+2$

The Wigner rotation matrix $\langle jm' | \hat{R}(\alpha,\beta,\gamma) | jm\rangle$

$$\hat{R}(\alpha,\beta,\gamma)|jm\rangle = \sum_{m'} |jm'\rangle\langle jm' | \hat{R}(\alpha,\beta,\gamma)|jm\rangle = \sum_{m'} D^j_{mm'}(\alpha,\beta,\gamma)\,|jm'\rangle$$

$$D^j_{mm'}(\alpha,\beta,\gamma) = \langle jm' | \hat{R}(\alpha,\beta,\gamma)|jm\rangle = e^{-im'\alpha}e^{-im\gamma}\left\langle jm' \middle| e^{-i\beta\hat{J}_y} \middle| jm\right\rangle = e^{-im'\alpha-im\gamma} d^j_{mm'}(\beta)$$

$$d^j_{mm'}(\beta) = \langle jm' | e^{-i\beta\hat{J}_y} | jm\rangle$$

$$= [(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!]^{1/2}$$

$$\times \sum_{k=max\{m-m',0\}}^{min\{j+m,j-m'\}} \frac{\left(\cos\frac{\beta}{2}\right)^{2j}\left(\tan\frac{\beta}{2}\right)^{2k-m+m'}}{(j+m-k)!\,(j-m'-k)!\,k!\,(k-m'+m)!}$$

$$d^{1/2}(\beta) = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \qquad d^1(\beta) = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}$$

Polarization measurements: $A \to 1 + 2$

The inner product

$$\langle \vec{p}; \lambda_1, \lambda_2 | S_A, M_A; \lambda_1, \lambda_2 \rangle = \langle p, 0, 0; \lambda_1, \lambda_2 | R^\dagger(\varphi, \theta, -\varphi) | S_A, M_A; \lambda_1, \lambda_2 \rangle$$

$$= \sum_{M'_A} \langle p, 0, 0; \lambda_1, \lambda_2 | S_A, M'_A; \lambda_1, \lambda_2 \rangle \langle S_A, M'_A; \lambda_1, \lambda_2 | R^\dagger(\varphi, \theta, -\varphi) | S_A, M_A; \lambda_1, \lambda_2 \rangle$$

$$M'_A = \lambda_1 - \lambda_2 \equiv \lambda$$

$$= \langle p, 0, 0; \lambda_1, \lambda_2 | S_A, \lambda; \lambda_1, \lambda_2 \rangle \langle S_A, \lambda; \lambda_1, \lambda_2 | R^\dagger(\varphi, \theta, -\varphi) | S_A, M_A; \lambda_1, \lambda_2 \rangle$$

$$= \left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}} D^{S_A *}_{M_A \lambda}(\varphi, \theta, -\varphi)$$

The decay amplitude

$$A_m(\vec{p}; \lambda_1, \lambda_2) = \langle \vec{p}; \lambda_1, \lambda_2 | \hat{U} | S_A, M_A \rangle = \langle \vec{p}; \lambda_1, \lambda_2 | S_A, M_A; \lambda_1, \lambda_2 \rangle H_{S_A}(\lambda_1, \lambda_2)$$

$$= \left(\frac{2J+1}{4\pi}\right)^{\frac{1}{2}} D^{S_A *}_{M_A \lambda}(\varphi, \theta, -\varphi) H_A(\lambda_1, \lambda_2)$$

The angular distribution of 1 in $A \to 1 + 2$

$$W(\theta, \varphi) = N' \sum_{\lambda_1, \lambda_2; M_A, M'_A} |H_A(\lambda_1, \lambda_2)|^2 D^{S_A *}_{M_A \lambda}(\varphi, \theta, -\varphi) D^{S_A}_{M'_A \lambda}(\varphi, \theta, -\varphi) \langle M_A | \hat{\rho}_A | M'_A \rangle$$

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Polarization measurements: vector meson

For $V \to 1 + 2$, where 1 and 2 are two pseudoscalar mesons, we have $S_A = 1, \lambda_1 = \lambda_2 = 0$

e.g., $\rho \to \pi\pi$

$$W(\theta, \varphi) = N \sum_{M_A, M'_A} |H_A|^2 D^{1*}_{M_A 0}(\varphi, \theta, -\varphi) D^{1}_{M'_A 0}(\varphi, \theta, -\varphi) \langle M_A | \hat{\rho}_A | M'_A \rangle$$

$$= \frac{3}{8\pi}\{(1 - \rho_{00}) + (3\rho_{00} - 1)\cos^2\theta$$

$$-\sqrt{2}\sin 2\theta\,[\cos\varphi\,(\mathrm{Re}\rho_{10} - \mathrm{Re}\rho_{-10}) - \sin\varphi\,(\mathrm{Im}\rho_{10} + \mathrm{Im}\rho_{-10})]$$

$$-2\sin^2\theta\,(\cos 2\varphi\,\mathrm{Re}\rho_{1-1} + \sin 2\varphi\,\mathrm{Im}\rho_{1-1})\}$$

$$\int_0^{2\pi} d\varphi\, W(\theta, \varphi) = \frac{3}{4}[(1 - \rho_{00}) + (3\rho_{00} - 1)\cos^2\theta]$$

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Polarization measurements: vector meson

For $V \to 1 + 2$, where 1 and 2 are two spin-1/2 Fermions, i.e., $S_A = 1, \lambda_1 = \pm\frac{1}{2}, \lambda_2 = \pm\frac{1}{2}$

consider the case: (1) Helicity conservation: $\lambda_1 = -\lambda_2$, $\lambda = \pm 1$

(2) Space reflection invariance: $H_A(\lambda_1, \lambda_2) = H_A(-\lambda_1, -\lambda_2)$

only one independent helicity amplitude

e.g., $J/\psi \to e^+ e^-$

$$W(\theta, \varphi) = \frac{3}{4\pi(3 + \lambda_\theta)}[1 + \lambda_\theta \cos^2\theta + \lambda_\varphi \sin^2\theta\cos 2\varphi + \lambda_{\theta\varphi}\sin 2\theta\cos\varphi$$

$$+\lambda_\varphi^\perp \sin^2\theta \sin 2\varphi + \lambda_{\theta\varphi}^\perp \sin 2\theta \sin\varphi]$$

$$\lambda_\theta = \frac{1 - 3\rho_{00}}{1 + \rho_{00}} \qquad \lambda_\varphi = \frac{2\mathrm{Re}\rho_{1-1}}{1 + \rho_{00}} \qquad \lambda_{\theta\varphi} = \frac{2\mathrm{Im}\rho_{-11}}{1 + \rho_{00}}$$

$$\lambda_\varphi^\perp = \frac{\sqrt{2}\mathrm{Re}(\rho_{01} - \rho_{-10})}{1 + \rho_{00}} \qquad \lambda_{\theta\varphi}^\perp = \frac{\sqrt{2}\mathrm{Im}(\rho_{01} - \rho_{-10})}{1 + \rho_{00}}$$

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Polarization measurements: Spin-1/2 Hyperon

For $H \to N\pi$, $S_A = \frac{1}{2}, \lambda_1 = \pm\frac{1}{2}, \lambda_2 = 0$

$$W(\theta, \varphi) = \frac{1}{4\pi}(1 + \alpha P\cos\theta + \alpha\sin\theta\cos\varphi\,\mathrm{Re}\rho_{+-} + \alpha\sin\theta\sin\varphi\,\mathrm{Im}\rho_{+-})$$

$$\alpha = \frac{\left|H_A\left(\frac{1}{2}\right)\right|^2 - \left|H_A\left(-\frac{1}{2}\right)\right|^2}{\left|H_A\left(\frac{1}{2}\right)\right|^2 + \left|H_A\left(-\frac{1}{2}\right)\right|^2}$$

$$P = \rho_{++} - \rho_{--}$$

If space reflection invariance $H_A\left(\frac{1}{2}\right) = H_A\left(-\frac{1}{2}\right)$ $\alpha = 0$

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Polarization measurements: Spin-3/2 baryon

For $A \to 1+2$, $S_A = \frac{3}{2}$, $S_1 = \frac{1}{2}$, $S_2 = 0$, e.g. $\Delta \to N\pi$ or $\Omega \to \Lambda K$

$$W(\theta,\varphi) = \frac{1}{8\pi}\Big\{2 + (1-3\cos^2\theta)S_{LL} - (S^x_{LT}\cos\phi + S^y_{LT}\sin\phi)\sin 2\theta - \sin^2\theta(S^{xx}_{TT}\cos 2\phi + S^{xy}_{TT}\sin 2\phi)$$
$$+\alpha_A\Big[\frac{4}{5}(S_L\cos\theta + S^x_T\sin\theta\cos\phi + S^y_T\sin\theta\sin\phi)$$
$$-\frac{1}{2}(3\cos\theta + 5\cos 3\theta)S_{LLL} - \frac{3}{4}(\sin\theta + 5\cos 3\theta)(S^x_{LLT}\cos\phi + S^y_{LLT}\sin\phi)$$
$$-3\sin^2\theta\cos\theta(S^{xx}_{LTT}\cos 2\phi + S^{xy}_{LTT}\sin 2\phi) - \sin^3\theta(S^{xxx}_{TTT}\cos 3\phi + S^{xxy}_{TTT}\sin 3\phi)\Big]\Big\}$$

If integrate over ϕ

$$W(\theta) = \frac{1}{4}\Big\{2 + (1-3\cos^2\theta)S_{LL} + \alpha_A\Big[\frac{4}{5}S_L\cos\theta - \frac{1}{2}(3\cos\theta + 5\cos 3\theta)S_{LLL}\Big]\Big\}.$$

only longitudinal components can be measured

If parity conserved, $\alpha_A = 0$ $\quad W(\theta) = \frac{1}{4}\Big\{2 + (1-3\cos^2\theta)S_{LL}\Big\}.$

See, e.g., the appendix in Zhe Zhang, Ji-peng Lv, Zi-han Yu, ZTL, e-Print: 2407.06480 [hep-ph]

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Polarization measurements: Spin-3/2 baryon

Successive decays

For $A \to 1+2,\ 1 \to 3+4$, $S_A = \frac{3}{2}$, $S_1 = S_3 = \frac{1}{2}$, $S_2 = S_4 = 0$, e.g. $\Omega \to \Lambda K$, $\Lambda \to p\pi^-$,

$$W(\theta_1,\theta_3) = \frac{1}{4}\Big\{(1+\alpha_1\alpha_A\cos\theta_3)\Big[1 - \frac{1}{4}S_{LL}(1+3\cos 2\theta_1)\Big] + \frac{2}{5}S_L\cos\theta_1(\alpha_A + \alpha_1\cos\theta_3)$$
$$-\frac{1}{4}S_{LLL}(3\cos\theta_1 + 5\cos 3\theta_1)(\alpha_A + \alpha_1\cos\theta_3)\Big\}.$$

For $A \to 1+2,\ 1 \to 3+4,\ 3 \to 5+6$, $S_A = \frac{3}{2}$, $S_1 = S_3 = S_5 = \frac{1}{2}$, $S_2 = S_4 = S_6 = 0$,
e.g. $\Xi^* \to \Xi\pi$, $\Xi \to \Lambda\pi$, $\Lambda \to p\pi^-$,

$$W(\theta_\Xi,\theta_\Lambda,\theta_p) = \frac{1}{8}\Big\{1 + \alpha_\Lambda\alpha_\Xi\cos\theta_p + \frac{2}{5}S_L(\alpha_\Xi + \alpha_\Lambda\cos\theta_p)\cos\theta_\Xi\cos\theta_\Lambda$$
$$-\frac{1}{4}S_{LL}(1+3\cos 2\theta_\Xi)(1+\alpha_\Lambda\alpha_\Xi\cos\theta_p)$$
$$-\frac{1}{4}S_{LLL}(\alpha_\Xi + \alpha_\Lambda\cos\theta_p)\cos\theta_\Lambda(3\cos\theta_\Xi + 5\cos 3\theta_\Xi)\Big\}.$$

See, e.g., the appendix in Zhe Zhang, Ji-peng Lv, Zi-han Yu, ZTL, e-Print: 2407.06480 [hep-ph]

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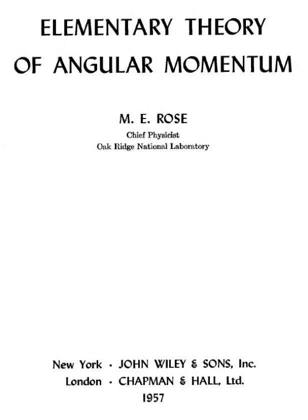
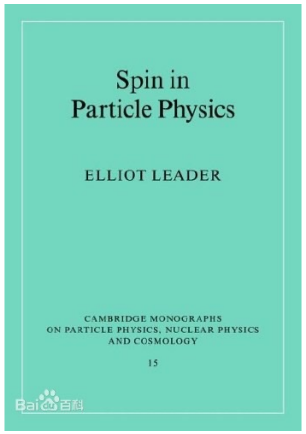
Two books





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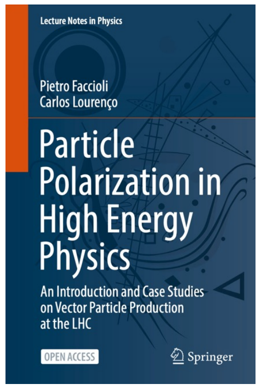
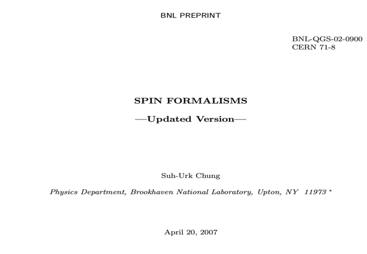
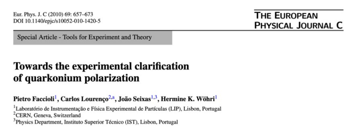
Other references







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