量子色动力学初步

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QCD与重离子碰撞物理暑期学校，8月05-23，2024，复旦大学，上海

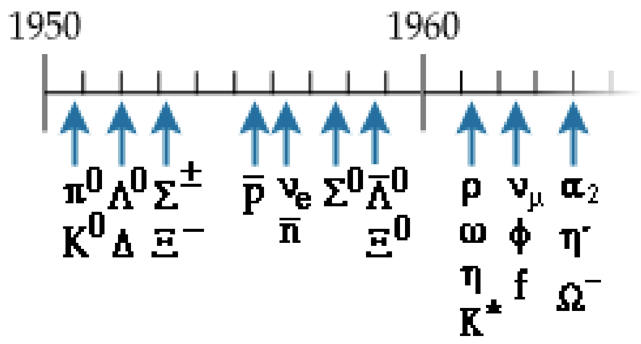
Outline:

- The advent of QCD
- QCD Lagrangian and Perturbative QCD
- The applications of pQCD(DGLAP)

鸣谢：邵鼎煜，曹杉杉，贾宇，刘晓辉，杨磊

The advent of QCD I

- **Quark model:** Gell-Mann, Nishijima, Ne'eman and Zweig, 1964, newly discovered many hadrons can be classified in a neat way



- **Color:** 1964/1965 Greenberg, Nambu and Han
necessary to introduce a new degree of freedom to understand the structure of a spin 3/2 hadron

$$|\Delta^{++}\rangle \sim |u\uparrow u\uparrow u\uparrow\rangle$$

- **Parton model:** 1968/1969, Feynman, Bjorken
Deep inelastic scattering at SLAC indicates that electron scatter off **point-like** constituents.

Quantum number: Spin ½ , e_u=2/3; e_d=e_s=-1/3

The advent of QCD II

- **Yang-Mills theory,** 1954

  Non-Abelian gauge theory, classical level

- **Ghost method,** Faddeev, Popov 1967, Quantization of non-Abelian field theory.
- **Renormalizable theory,** 't Hooft, Veltman, 1971
- **SU(3) gauge group(QCD Lagrangian),** Gell-Mann, Fritzsch, 1972
- **Asymptotic freedom,** Gross, Wilczek, Politzer,1973

QCD Lagrangian

$$L_{QCD} = -\frac{1}{2} Tr F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{n_f} \bar{q}^i (i\gamma^\mu D_\mu - m_i) q^i$$

Covariant derivative

$$D_\mu(x) = \partial_\mu - igt^a A_\mu^a$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \equiv F_{\mu\nu}^a t^a,$$

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c$$

SU(3) gauge symmetry:

- fundamental representation(3), quark sector
- adjoint representation(8), gluon sector

A representation is: $t^A = \frac{1}{2}\lambda^A$,

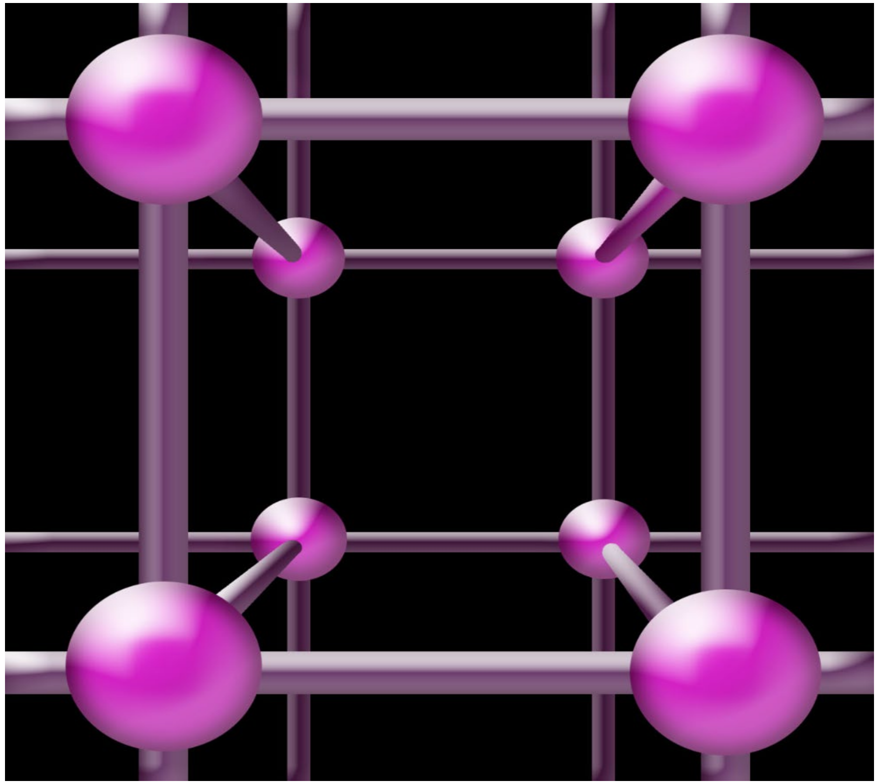
$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix}$$

Solving QCD

- **Perturbative QCD,** expand in terms of small coupling constant
- **Non-perturbative QCD,** Lattice QCD, K.G. Wilson 1974

  numerical solution with discretized space time



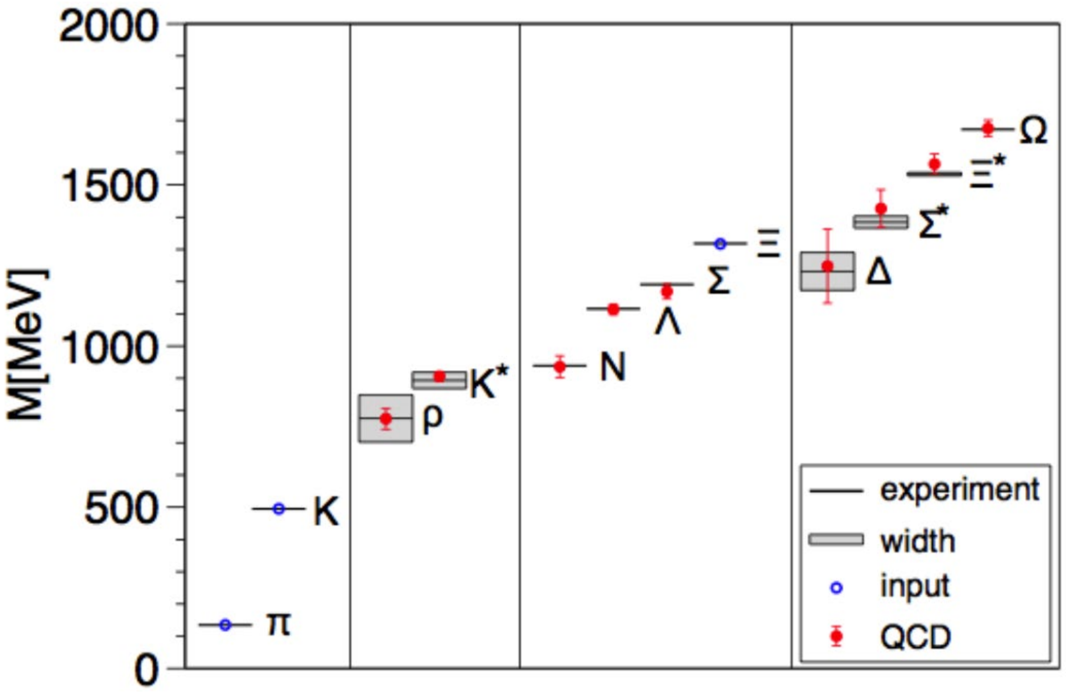
Lattice formulation of QCD

- **Analytical method**: Cornell potential $V(r) = A + \frac{B}{r} + \sigma r$

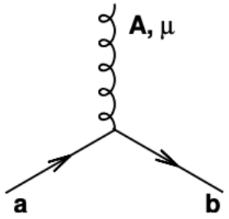
  Strong coupling expansion: $V(r) \sim -\frac{r}{a^2} \log \frac{\beta}{2N_c^2}(1 + O(\beta))$ $\beta = \frac{2N_c}{g^2}$

- **Perturbative quantity:** running coupling, EM form factor

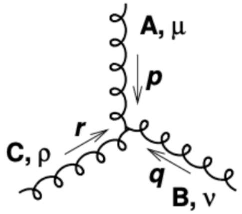
- **Numerical simulation, hadron spectrum:**



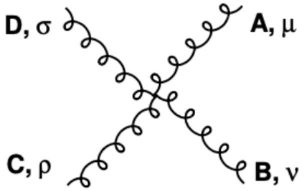
Perturbative QCD: Feynman rules



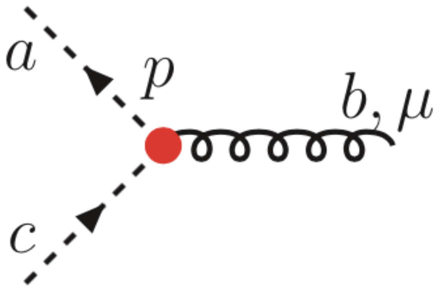
$-ig_s t^A_{ba}\gamma^\mu$



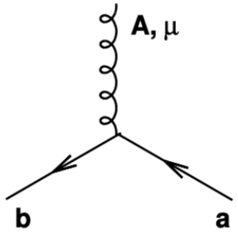
$-g_s f^{ABC}[(p-q)^\rho g^{\mu\nu} + (q-r)^\mu g^{\nu\rho} + (r-p)^\nu g^{\rho\mu}]$



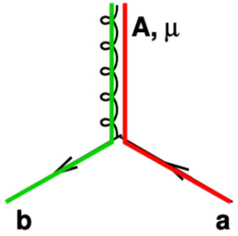
$-ig_s^2 f^{XAC} f^{XBD}[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\gamma}] + (C,\gamma) \leftrightarrow (D,\rho) + (B,\nu) \leftrightarrow (C,\gamma)$



$g_s f_{abc} p_\mu$

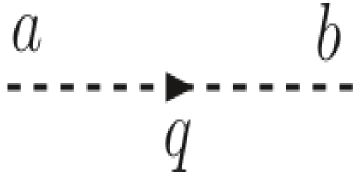


$\bar{\psi}_b(-ig_s t^A_{ba}\gamma^\mu)\psi_a$



A gluon emission repaints the quark colour.
A gluon itself carries colour and anti-colour.

ghost :



$-i\delta_{ab}/q^2$

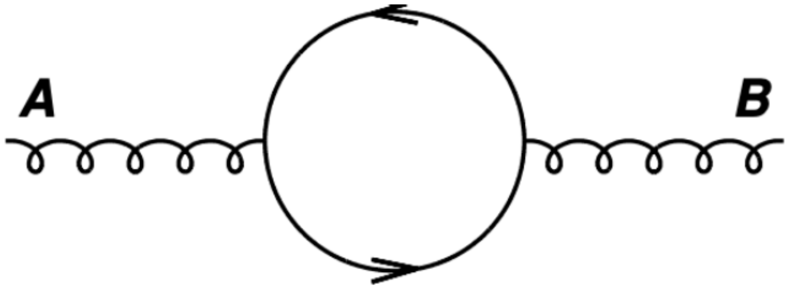
$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}}_{\bar{\psi}_b} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{t^1_{ab}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\psi_a}$$

Faddeev-Popov ghost field
(only appear in internal propagator)

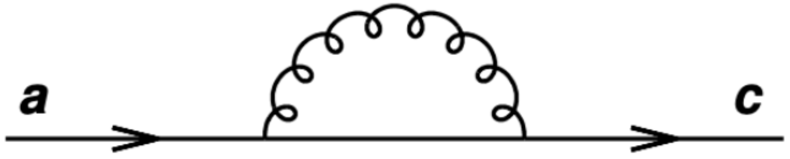
$$r\bar{b}, r\bar{g}, b\bar{r}, b\bar{g}, g\bar{r}, g\bar{b}, (r\bar{r} - b\bar{b})/\sqrt{2}, (r\bar{r} + b\bar{b} - 2g\bar{g})/\sqrt{6}$$

Color algebra

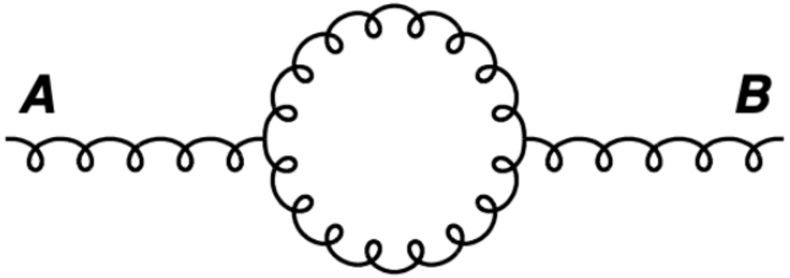
$$\mathrm{Tr}(t^A t^B) = T_R \delta^{AB}\,, \qquad T_R = \tfrac{1}{2}$$



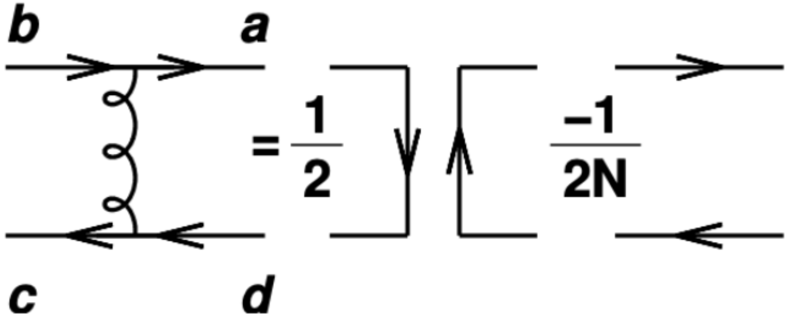
$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}\,, \qquad C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$$



$$\sum_{C,D} f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \qquad C_A = N_c = 3$$



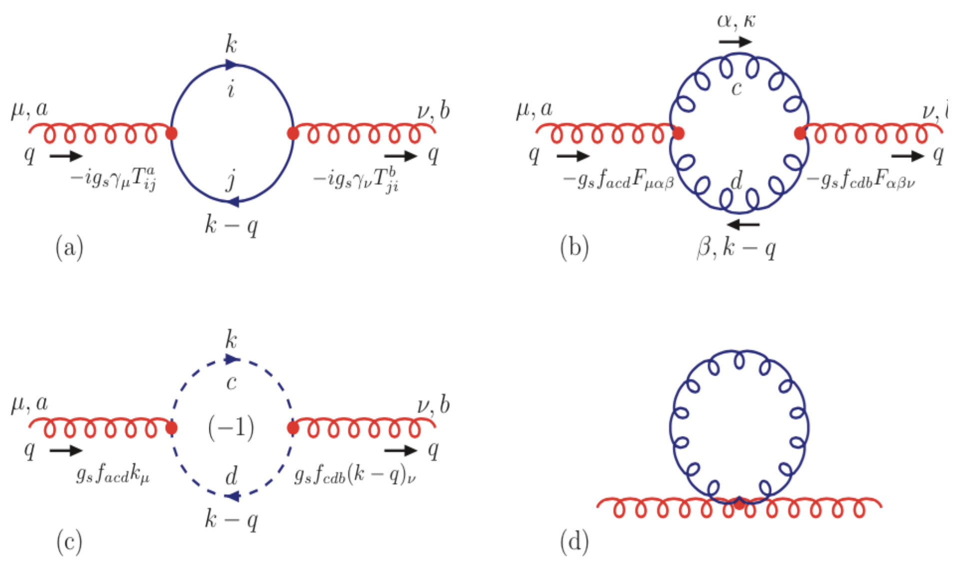
$$t^A_{ab} t^A_{cd} = \frac{1}{2}\delta_{bc}\delta_{ad} - \frac{1}{2N_c}\delta_{ab}\delta_{cd} \quad \text{(Fierz)}$$



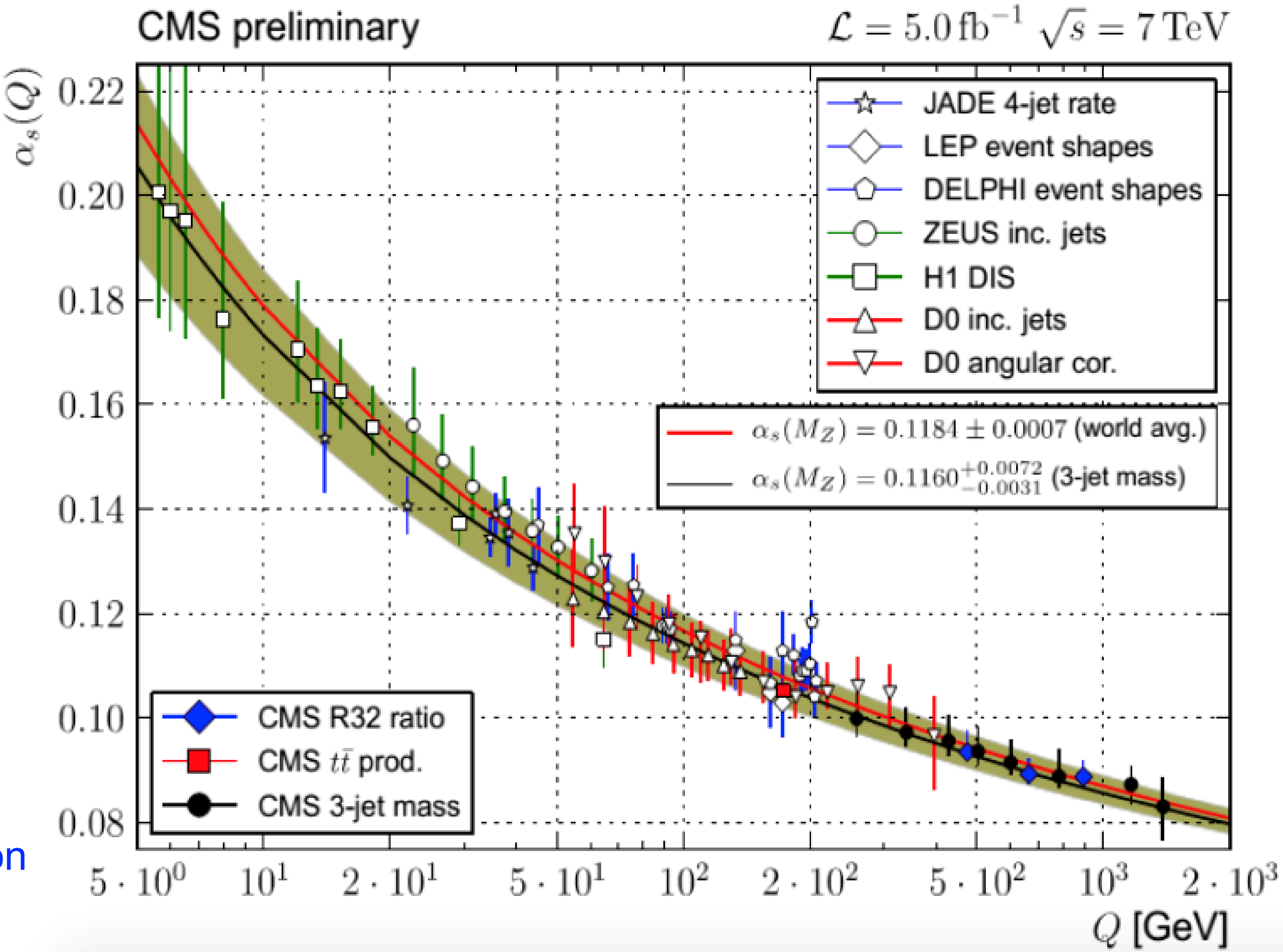
Running coupling

$$\alpha_s(\mu) = \frac{\alpha_s(M)}{1 + \frac{\alpha_s(M)}{2\pi}(11 - \frac{2}{3}n_f)\ln\frac{\mu}{M}}$$

Gluon vacuum polarization:



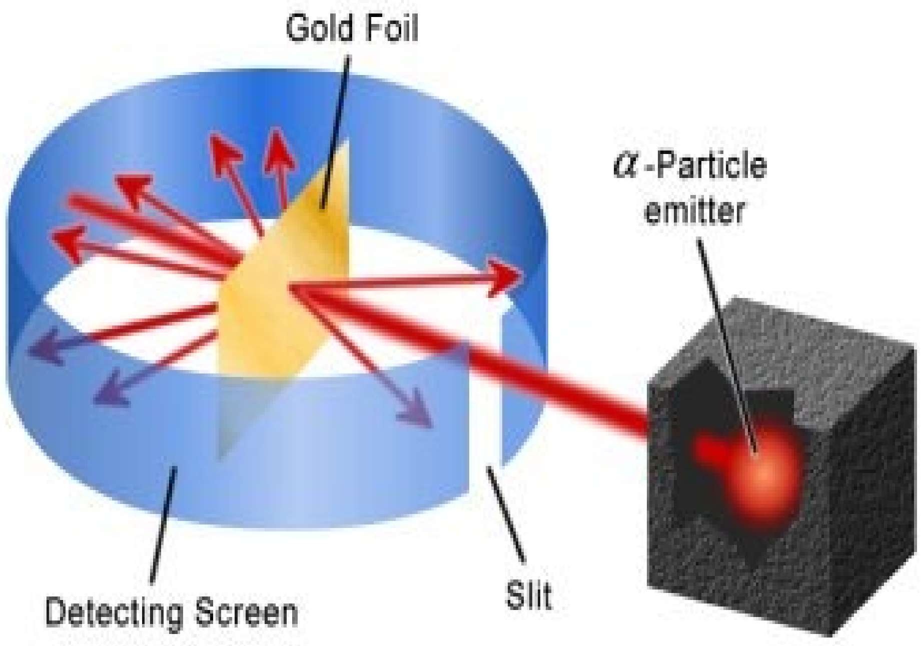
Asymptotic freedom due to gluon self-interaction



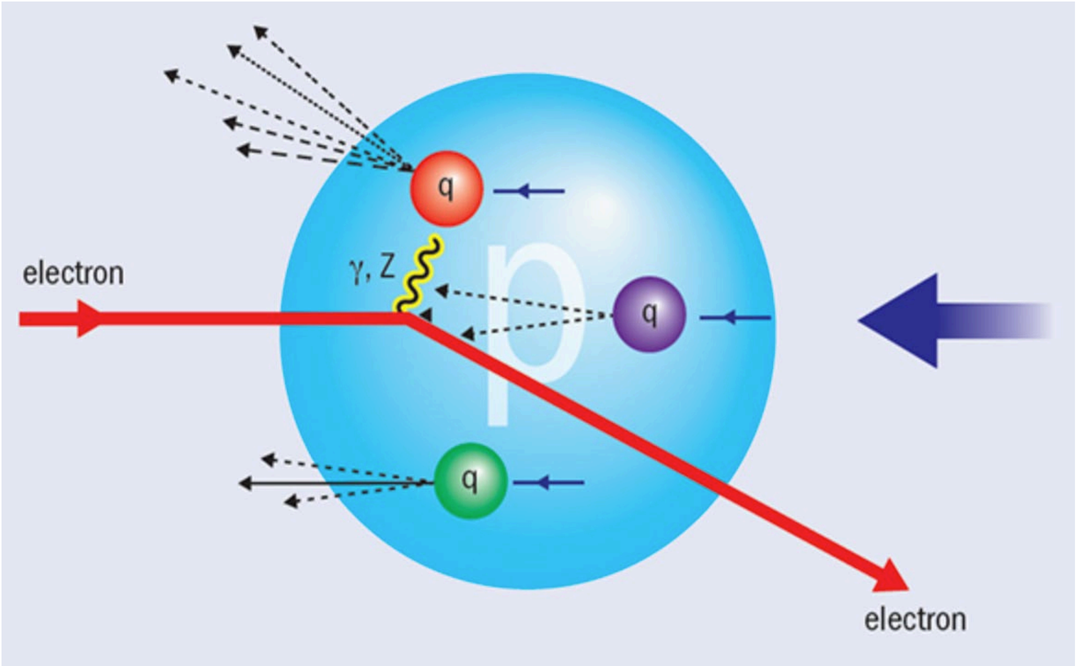
The application of pQCD

Deeply inelastic scattering(DIS) and parton distribution

Rutherford's experiment

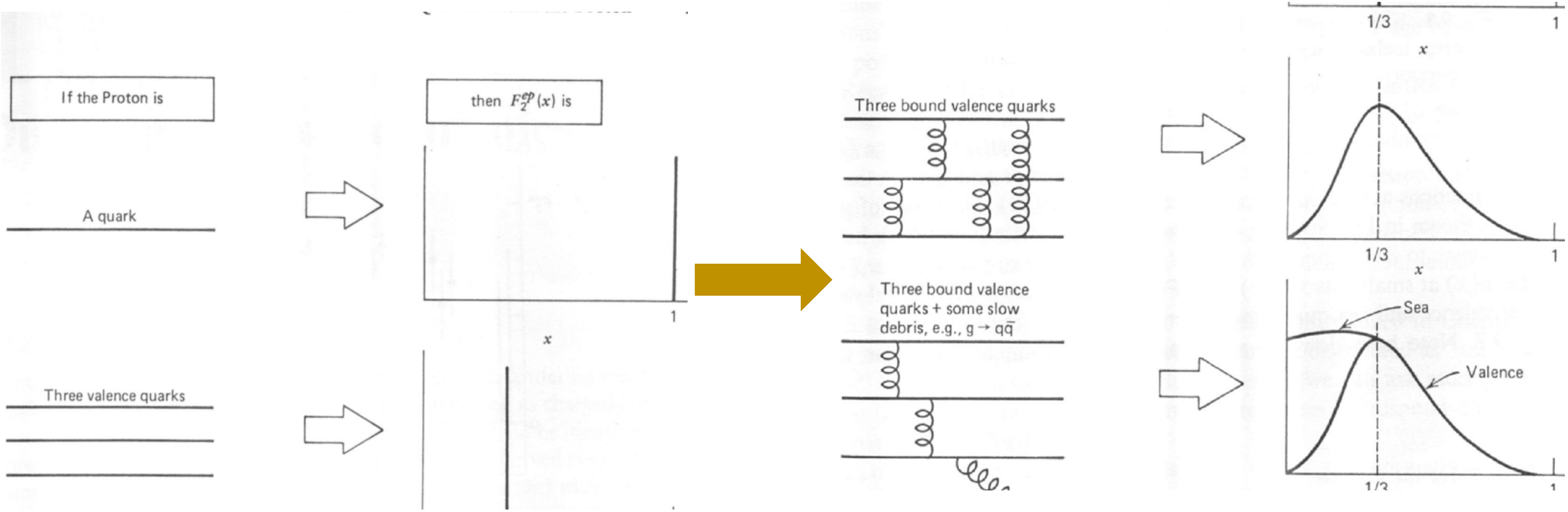


Modern Rutherford's experiment(DIS)



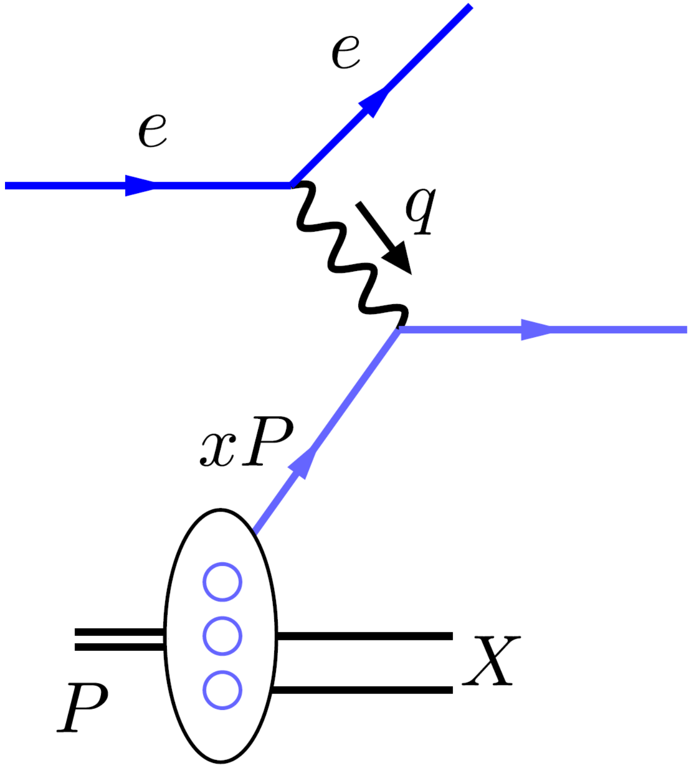
Parton distribution function(PDF) $f_i(x)$

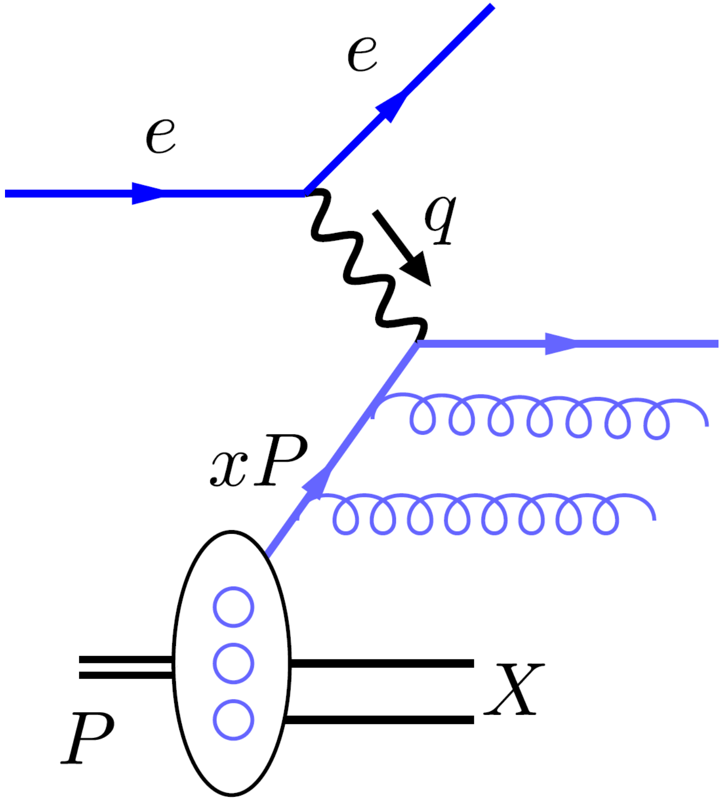
Parton distribution functions



When x→0, PDFs are dominated by gluons and sea quarks, gluons carry ~50% momentum fraction of proton.

QCD improved parton model: Scale dependence of PDFs

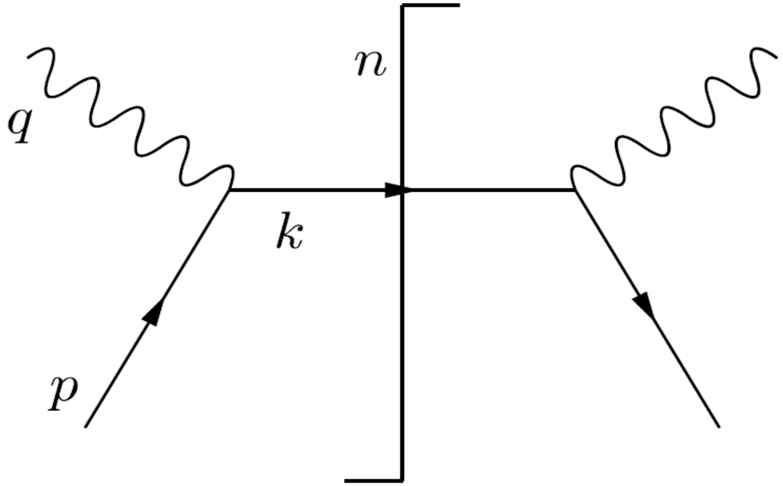




- **PDF at the initial scale**, non-perturbative object
- **The scale dependence of PDF**, perturbative calculable

Born cross section

Tree level: $\gamma^* + q \to q$



Virtual photon: $q^\mu = -x_B P^+ \bar{n}^\mu + n^\mu$, with $n^\mu = [0^+, q^-, 0_T]$

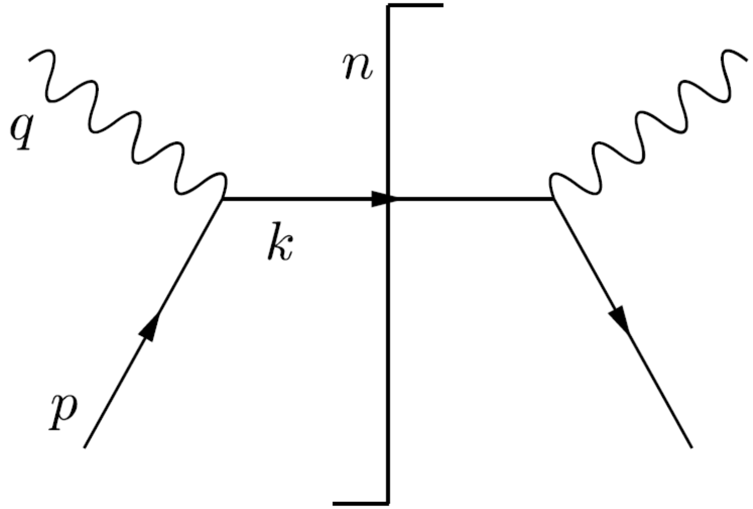
Initial quark: $p^\mu = [xP^+, 0^-, 0_T]$

Virtual photon polarization vector: $\varepsilon_T^\mu = [0,0,\varepsilon_T]$

Born cross section

The amplitude:

$$i\mathcal{M}_{fi} = ie_q e\bar{u}(k)\not{\epsilon}_T u(p,\lambda)$$



The squared amplitude:

$$\sum_q \frac{1}{2}\sum_{\lambda\lambda'} |\mathcal{M}|^2 = \frac{1}{2} e_q^2 e^2 \bar{u}(k)\not{\epsilon}_T u(p,\lambda)\bar{u}(p,\lambda')\not{\epsilon}_T^* u(k)$$

Simplified as,

$$\sum_q \frac{1}{2}|\mathcal{M}|^2 = \sum_q \frac{1}{2} e_q^2 e^2 \mathrm{Tr}[\not{n}\not{\epsilon}_T \not{p}\not{\epsilon}_T^*]$$

Born cross section

$$q^\mu = -x_B P^+ \bar{n}^\mu + n^\mu$$
$$P^\mu = [P^+, 0^-, 0_T]$$
$$p^\mu = [xP^+, 0^-, 0_T]$$
$$\varepsilon_T^\mu = [0, 0, \varepsilon_T]$$
$$n = [0^+, q^-, 0_T]$$

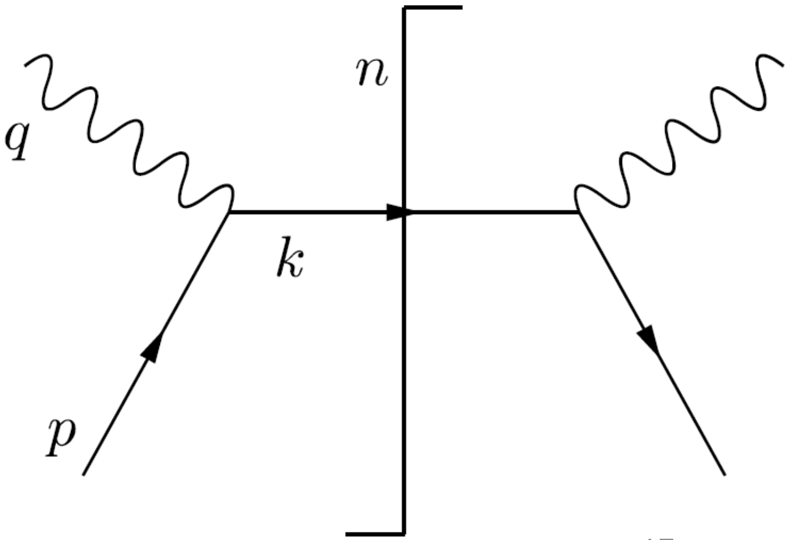
The cross section reads

$$\sigma_{\text{Born}} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k^+} \frac{1}{F} \sum_q \frac{1}{2} \sum_{\lambda\lambda'} |\mathcal{M}|^2 \, (2\pi)^4 \delta^{(4)}(p + q - k) f_q(x)$$

$$= \frac{1}{2}\frac{1}{2s} \sum_q e_q^2 e^2 \mathrm{Tr}[\not{p} \not{\varepsilon}_T^* \not{n} \not{\varepsilon}_T] \, (2\pi)\delta\left[(p+q)^2 - k^2\right] f_q(x)$$

$$= \frac{1}{2}\frac{1}{2s} \sum_q e_q^2 e^2 \, 4p\cdot n (2\pi) \frac{1}{2P\cdot q} \delta(x - x_B) \, f_q(x)$$

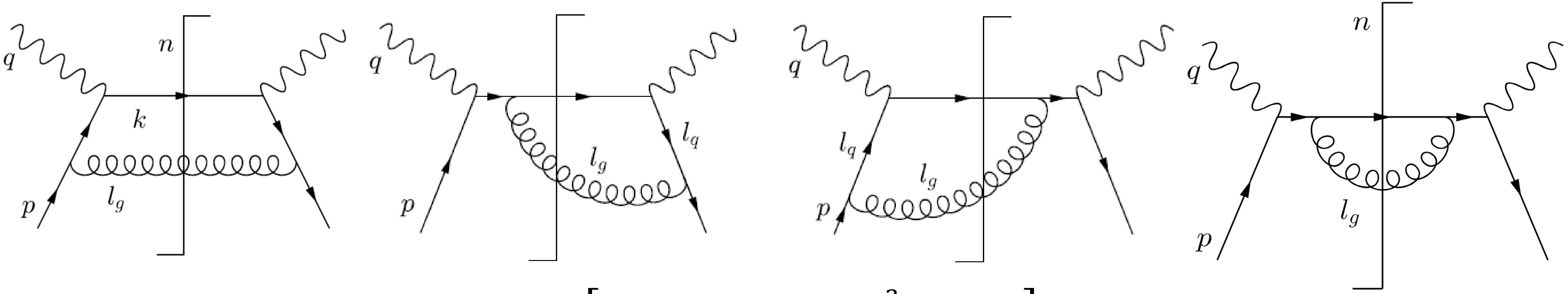
$$= \frac{4\pi^2 \alpha_e}{s} \sum_q e_q^2 x \, \delta(x - x_B) \, f_q(x)$$



The coupling constant: $\alpha_e = \frac{e^2}{4\pi}$

Real correction

Real correction: $\gamma + q \to g + q$.



Radiated gluon four momentum: $l_g = \left[(1-z)xP^+, \frac{l_\perp^2}{2(1-z)xP^+}, l_\perp\right]$

The squared amplitude from diagram (a),

$$\sum_q \frac{1}{2} \sum_{\lambda\lambda'} |\mathcal{M}|^2 = -\sum_q \frac{1}{2} e_q^2 e^2 g_s^2 \frac{Tr[t^a t^a]}{N_c} \frac{1}{\left(p - l_g\right)^4} \mathrm{Tr}[\gamma^\mu (\not{p} - \not{l}_g) \not{\epsilon}_T^* \not{n} \not{\epsilon}_T (\not{p} - \not{l}_g) \gamma^\nu] g_{\mu\nu}$$

Real correction

The numerator is

$$-Tr[\not{p}\gamma^{\mu}(\not{p}-\not{l}_g)\not{\varepsilon}_T^{*}\not{n}\not{\varepsilon}_T(\not{p}-\not{l}_g)\gamma^{\nu}]g_{\mu\nu}$$
$$= -16(\varepsilon_T\cdot\varepsilon_T^{*})(l_g\cdot n)(l_g\cdot p)$$
$$= 8l_{\perp}^{2}(p\cdot n)$$

Using the trace formula

$$Tr\{\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\} = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

The simplification of the denominator

$$\frac{1}{(p-l_g)^4} = \frac{1}{(-2p\cdot l_g)^2} = \frac{1}{\left(-2xP^{+}\frac{l_{\perp}^{2}}{2(1-z)xP^{+}}\right)^2} = \frac{(1-z)^2}{l_{\perp}^{4}}$$

Real correction

The phase space integration

$$\int \mathrm{d}\mathcal{P}.\mathcal{S}\,(2\pi)^4\delta^{(4)}\left(p+q-k-l_g\right)$$

$$=\int \frac{\mathrm{d}^3 l_g}{(2\pi)^3 2 l_g^0}\frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0}(2\pi)^4\delta^{(4)}\left(p+q-k-l_g\right)$$

$$=\int \frac{\mathrm{d}^2 l_g d l_g^+}{(2\pi)^3 2 l_g^+}(2\pi)\delta\left[\left(p-l_\mathrm{g}+q\right)^2\right]$$

$$\simeq \int \mathrm{dzd}l_\perp^2 \frac{xP^+\pi}{(2\pi)^2 2(1-z)xP^+}\frac{1}{2(P\cdot q)}\delta(zx-x_B)$$

Ignoring the power suppressed term $\mathcal{O}\left(l_\perp^2/Q^2\right)$

Real correction

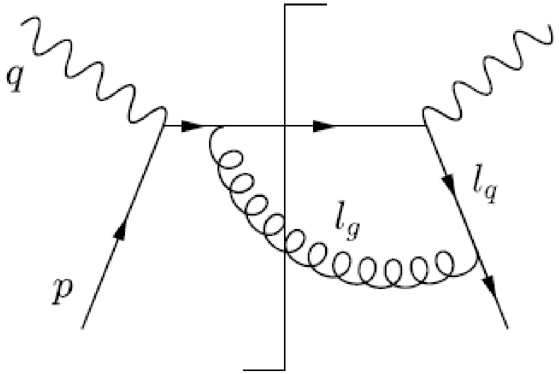
The real correction from the diagram (a) is cast into the form,

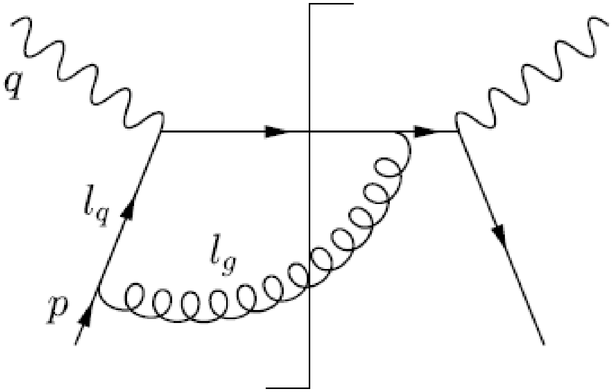
$$= \int \mathrm{dzd}l_\perp^2 \sum_q f_q(x) \frac{1}{2s} \frac{1}{2} e_q^2 e^2 g_s^2 C_F \frac{8 l_\perp^2 (p \cdot n)(1-z)^2}{l_\perp^4} \frac{1}{4\pi 2(1-z)} \frac{\delta(zx - x_B)}{2(P \cdot q)}$$

$$= \int \mathrm{dzd}l_\perp^2 \sum_q f_q(x) \frac{1}{2s} C_F e_q^2 g_s^2 \alpha_e \frac{x(1-z)}{l_\perp^2} \delta(zx - x_B)$$

$$= \sigma_{\mathrm{Born}} \left(\frac{\alpha_s}{2\pi}\right) \int \frac{dz}{z} \frac{\mathrm{d}l_\perp^2}{l_\perp^2} f_q\left(\frac{x_B}{z}\right)(1-z)$$

Real correction





$$\int \frac{\mathrm{d}^3 l_g}{(2\pi)^3 2 l_g^0} \frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0} \frac{1}{F} \sum_q f_q(x) \frac{1}{2} \sum_{\lambda\lambda'} |\mathcal{M}|^2 \, (2\pi)^4 \delta^{(4)}\left(p+q-k-l_g\right)$$

where

$$\frac{1}{2}\sum_{\lambda\lambda'}|\mathcal{M}|^2 = -\frac{1}{2}\sum_q e_q^2 e^2 g_s^2 \frac{\mathrm{Tr}[t^a t^a]}{N_C} Tr\left\{\not{p}\gamma^\mu \frac{\not{p}-\not{l}_g}{\left(p-l_g\right)^2} \not{\varepsilon}_T^* \not{n} \gamma^\nu \frac{\not{n}+\not{l}_g}{\left(n+l_g\right)^2} \not{\varepsilon}_T\right\} g_{\mu\nu}$$

$$= -\frac{1}{2}\sum_q e_q^2 e^2 g_s^2 C_A \frac{16(p\cdot n)\left(l_g\cdot n - l_g\cdot p - p\cdot n\right)(\varepsilon_T\cdot\varepsilon_T^*)}{\left(-2p\cdot l_g\right)\left(2n\cdot l_g\right)}$$

$$\simeq \sum_q e_q^2 e^2 \frac{8z(p\cdot n)^2}{2x(P\cdot q)l_\perp^2}$$

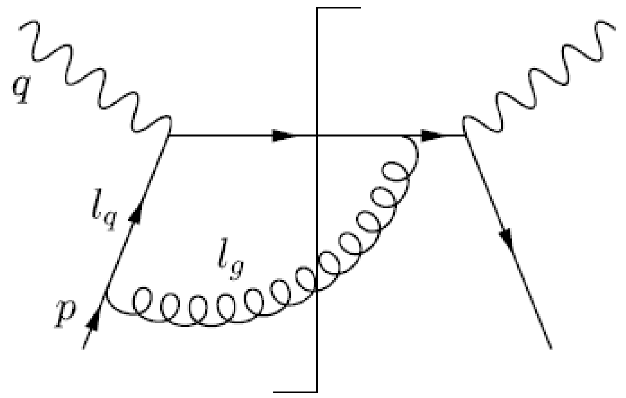
Real correction

$$\int \frac{\mathrm{d}^3 l_g}{(2\pi)^3 2 l_g^0} \frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0} \frac{1}{F} \sum_q f_q(x) e_q^2 e^2 \frac{8z(p\cdot n)^2}{2x(P\cdot q) l_\perp^2} (2\pi)^4 \delta^{(4)}(p+q-k-l_g)$$

$$= \int \frac{\mathrm{d}^3 l_g}{(2\pi)^2 2(1-z) x P^+} \frac{1}{2s} \sum_q f_q(x) e_q^2 e^2 \frac{8z(p\cdot n)^2}{2x(P\cdot q) l_\perp^2} \frac{1}{2(P\cdot q)} \delta(zx - x_B)$$

$$= \int \mathrm{d}z \mathrm{d}l_\perp^2 \frac{1}{2s} \sum_q f_q(x) e_q^2 \alpha_e \frac{x}{l_\perp^2} \frac{z}{(1-z)} \delta(zx - x_B)$$

$$= \sigma_{\mathrm{Born}} \left(\frac{\alpha_s}{2\pi}\right) \int \frac{dz}{z} \frac{\mathrm{d}l_\perp^2}{l_\perp^2} \frac{z}{1-z} f_q\left(\frac{x_B}{z}\right)$$



The splitting kernel from the real correction: $\mathrm{P}_{qq} = 1 - z + \frac{2z}{1-z} = \frac{1+z^2}{1-z}$

Virtual correction

- Assuming target is a single quark, one has:

$$\int_0^1 dx \left[\delta(1-x) + \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{dz}{z} \int_{\mu^2}^{Q^2} \frac{dl_\perp^2}{l_\perp^2} \frac{1+z^2}{1-z} \delta\left(1 - \frac{x}{z}\right) + \delta(1-x) a\right] = 1$$

$$\frac{\alpha_s}{2\pi} \int_{\mu^2}^{Q^2} \frac{dl_\perp^2}{l_\perp^2} \int_0^1 dz \frac{1+z^2}{1-z} + a = 0 \quad \Rightarrow \quad a = -\frac{\alpha_s}{2\pi} C_F \int_0^1 dz \frac{1+z^2}{1-z}$$

- real+virtual correction

$$\int_{\mu^2}^{Q^2} \frac{dl_\perp^2}{l_\perp^2} \left[\int \frac{dz}{z} \left(\frac{1+z^2}{1-z}\right) f\left(\frac{x}{z}\right) - \int dy \frac{1+y^2}{1-y} f(x)\right]$$

Introduce "+" notation:

$$\int_0^1 dz \frac{1}{(1-z)_+} f(z) = \int_0^1 dz \frac{1}{1-z} [f(z) - f(1)]$$

DGLAP evolution equation

◆ Real+virtual correction is re-organized as:

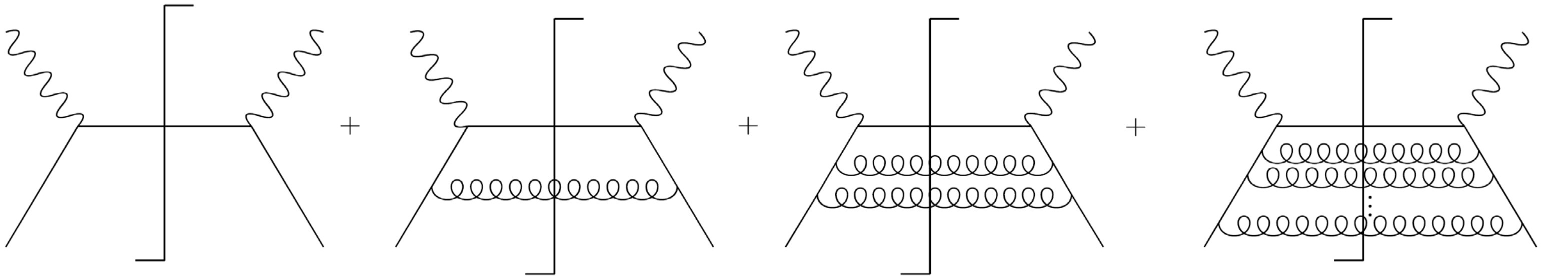
$$\frac{\alpha_s}{2\pi} C_F \int_{\mu^2}^{Q^2} \frac{dl_\perp^2}{l_\perp^2} \int_0^1 \frac{dz}{z} P_{qq}(z)$$

With, $P_{qq}(z) = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)$

➢ NLO cross section

$$\sigma_{\mathrm{Born}} f(x_B) + \sigma_{\mathrm{Born}} \frac{\alpha_s}{2\pi} C_F \int_{\mu^2}^{Q^2} \frac{dl_\perp^2}{l_\perp^2} \int_0^1 dz P_{qq}(z) f\left(\frac{x_B}{z}\right)$$

$$= \sigma_{\mathrm{Born}} \left[f(x_B) + \frac{\alpha_s}{2\pi} C_F \int_{\mu^2}^{Q^2} \frac{dl_\perp^2}{l_\perp^2} \int_0^1 dz P_{qq}(z) f\left(\frac{x_B}{z}\right) \right] = \boldsymbol{\sigma_{\mathrm{Born}} f(x_B, Q^2)}$$



➢ DGLAP equation:

$$\frac{\partial f(x, Q^2)}{\partial \ln Q^2} = \frac{\alpha_s}{2\pi} C_F \int_{x_B}^1 dz\, P_{qq}(z) f\left(\frac{x_B}{z}\right)$$

Automatic calculation with Feynarts and FeynCalc

```mathematica
(*Loading FeynCalc and FeynArts...*)
$LoadAddOns = {"FeynArts"};
<< FeynCalc`
$FAVerbose = 0;
Clear ["Global`*"]
清除
diags = InsertFields[CreateTopologies[0, 2 → 2], {V[1], F[3, {1}]} ->
      {F[3, {1}], V[5]}, InsertionLevel -> {Classes}, Model -> "SMQCD"];
Paint[diags, ColumnsXRows -> {1, 1}, Numbering -> Simple,
    ImageSize -> {200, 150}];
    图像尺寸
Diagram[i_] := DiagramExtract[diags, i];
(*Generating Amplitude*)
amp[j_] := FCFAConvert[CreateFeynAmp[Diagram[j]], IncomingMomenta -> {q, p},
    OutgoingMomenta -> {k, lg}, LorentzIndexNames -> {μ, ν}, UndoChiralSplittings -> True, ChangeDimension -> 4,
                                                                                     真
    List -> False, SMP -> True, Contract -> True, DropSumOver -> True, Prefactor -> 3 / 2 * e_q] /. {SMP["m_u"] → 0};
    列表     假            真               真                  真
```

```
(*Inputing momenta*)
REPLACE = {Momentum[p] → x P Momentum[nb],

    Momentum[lg] → (1 - z) x P Momentum[nb] + lperp^2/(2 (1 - z) x P) Momentum[n]/qminus + Momentum[lt]};

(*Defining scalar product*)
Pair[Momentum[n], Momentum[nb]] = qminus;
Pair[Momentum[n], Momentum[n]] = Pair[Momentum[nb], Momentum[nb]] = 0;
Pair[Momentum[n], Momentum[lt]] = Pair[Momentum[nb], Momentum[lt]] = 0;
Pair[Momentum[lt], Momentum[lt]] = -lperp^2;
Pair[Momentum[n], Momentum[Polarization[q, -ⅈ]]] = Pair[Momentum[nb], Momentum[Polarization[q, -ⅈ]]] = 0;
Pair[Momentum[n], Momentum[Polarization[q, ⅈ]]] = Pair[Momentum[nb], Momentum[Polarization[q, ⅈ]]] = 0;
Pair[Momentum[Polarization[q, -ⅈ]], Momentum[Polarization[q, ⅈ]]] = -1;
Pair[Momentum[lg], Momentum[lg]] = 0;
```

```
(*Second diagram contribution*)
SquarAmp1 =
 (((1/SUNN .amp[2] ComplexConjugate[amp[2]]) // FeynAmpDenominatorExplicit // FermionSpinSum[#, ExtraFactor → 1/2] &) //
      DoPolarizationSums[#, lg, 0] & // DiracSimplify // SUNSimplify // ReplaceAll[#, REPLACE] & //
                                                                          全部替换
    ScalarProductExpand // Contract // Simplify) /. {Momentum[k] → Momentum[n]};
                                      化简

(*Interference diagram contribution*)
SquarAmp2 =
 (((1/SUNN . (amp[1] × ComplexConjugate[amp[2]] + amp[2] × ComplexConjugate[amp[1]])) // FeynAmpDenominatorExplicit //
       FermionSpinSum[#, ExtraFactor → 1/2] &) // DoPolarizationSums[#, lg, 0] & // DiracSimplify // SUNSimplify //
    ReplaceAll[#, REPLACE] & // ScalarProductExpand // Contract // Simplify) /. {Momentum[k] → Momentum[n]};
    全部替换                                                  化简

(*First diagram contribution*)
SquarAmp3 =
 (((1/SUNN . (amp[1] × ComplexConjugate[amp[1]])) // FeynAmpDenominatorExplicit //
       FermionSpinSum[#, ExtraFactor → 1/2] &) // DoPolarizationSums[#, lg, 0] & // DiracSimplify // SUNSimplify //
    ReplaceAll[#, REPLACE] & // ScalarProductExpand // Contract // Simplify) /. {Momentum[k] → Momentum[n]};
    全部替换                                                  化简
```

```
(*Inputing phase space*)
Phase = x P π / ((2 π)^2 2 (1 - z) x P) * 1/(2 P qminus) * 1/z;
(*separating constant and 1/z*)
Splitting =
  ((Series[SquarAmp2, {lperp, 0, -2}] // Normal) * Phase + (Series[SquarAmp1, {lperp, 0, -2}] // Normal) * Phase) /
    级数                               转换为普通表达式        级数                               转换为普通表达式
   (1/z (CF x SMP["e"]^2 SMP["g_s"]^2 e_q^2)/(4 lperp^2 π)) // Simplify;
                                                                 化简
```

```
(*Printing amplitude*)
Print["M1", "=", amp[1]]; Print["M2", "=", amp[2]];
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(*Printing squired amplitudes*)
Print["|M2|^2", "=", SquarAmp1 /. {SMP["e"] → Sqrt[4 π αe], SMP["g_s"] → Sqrt[4 π αs], lperp → "l⊥", qminus → "q⁻"} // Simplify];
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Print["M1M2*+M1*M2", "=", SquarAmp2 /. {SMP["e"] → Sqrt[4 π αe], SMP["g_s"] → Sqrt[4 π αs], lperp → "l⊥", qminus → "q⁻"} // Simplify];
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Print["|M1|^2", "=", SquarAmp3 /. {SMP["e"] → Sqrt[4 π αe], SMP["g_s"] → Sqrt[4 π αs], lperp → "l⊥", qminus → "q⁻"} // Simplify];
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(*Printing cross section*)
Print["σ2", "=", (SquarAmp1 * Phase * 1/(2 s)) /. {SMP["e"] → Sqrt[4 π αe], lperp → "l⊥", qminus → "q⁻"} // Simplify]
打印 平方根 化简

Print["σInterference", "=", (Series[SquarAmp2 * Phase * 1/(2 s), {lperp, 0, -2}] // Normal) /. {SMP["e"] → Sqrt[4 π αe], lperp → "l⊥", qminus → "q⁻"} //
打印 级数 转换为普通表达式 平方根

  Simplify]
  化简

(*Printing Splitting function*)
Print["Pgq(z)", "=", Splitting];
打印
```
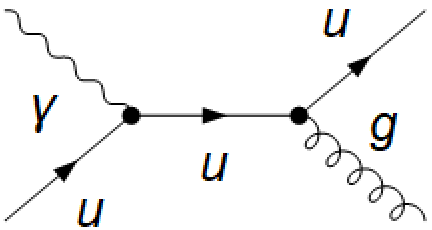
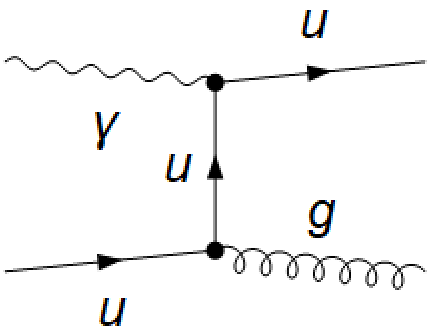

$\gamma \quad u \rightarrow u \quad g$



1

$\gamma \quad u \rightarrow u \quad g$



2

$$M_1 = -\frac{(\varphi(\bar{k})).(\bar{\gamma}\cdot\bar{\varepsilon}^*(\mathrm{lg})).(\bar{\gamma}\cdot(\bar{k}+\overline{\mathrm{lg}})).(\bar{\gamma}\cdot\bar{\varepsilon}(q)).(\varphi(\bar{p}))\ \mathrm{e}\ g_s\ e_q\ T^{\mathrm{Glu4}}_{\mathrm{Col3\ Col2}}}{(-\bar{k}-\overline{\mathrm{lg}})^2}$$

$$M_2 = -\frac{(\varphi(\bar{k})).(\bar{\gamma}\cdot\bar{\varepsilon}(q)).(\bar{\gamma}\cdot(\bar{p}-\overline{\mathrm{lg}})).(\bar{\gamma}\cdot\bar{\varepsilon}^*(\mathrm{lg})).(\varphi(\bar{p}))\ \mathrm{e}\ g_s\ e_q\ T^{\mathrm{Glu4}}_{\mathrm{Col3\ Col2}}}{(\overline{\mathrm{lg}}-\bar{p})^2}$$

$$|M_2|^2 = \frac{64\, q^-\, C_F\, P\, \pi^2\, x\,(z-1)^2\, e_q^2\, \alpha_e\, \alpha_s}{l_\perp{}^2}$$

$$M_1 M_2{}^* + M_1{}^* M_2 = -\frac{64\, C_F\, \pi^2\left(l_\perp{}^2 - 2\, q^-\, P\, x\,(z-1)\, z\right) e_q^2\, \alpha_e\, \alpha_s}{l_\perp{}^2\,(z-1)}$$

$$|M_1|^2 = \frac{16\, l_\perp{}^2\, C_F\, \pi^2\, e_q^2\, \alpha_e\, \alpha_s}{q^-\, P\, x\,(z-1)^2}$$

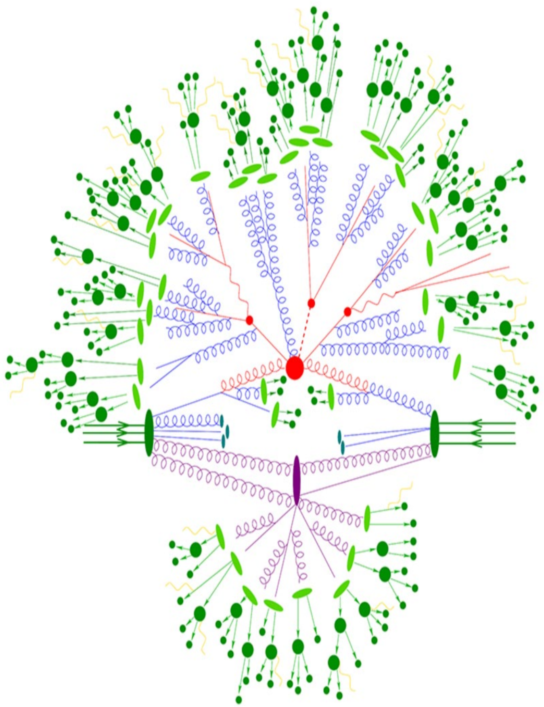
$$\sigma_2 = -\frac{C_F\, x\,(z-1)\, g_s^2\, e_q^2\, \alpha_e}{2\, l_\perp{}^2\, s\, z}$$

$$\sigma_{\mathrm{Interference}} = -\frac{C_F\, x\, g_s^2\, e_q^2\, \alpha_e}{l_\perp{}^2\, s\,(z-1)}$$

$$P_{\mathrm{gq}}(\mathrm{z}) = \frac{z^2+1}{1-z}$$

Monte Carlo implementation

Monte Carlo implementation



- Start with a highly virtual parton ($Q^2 = p^2 - m^2$), consider $a \to bc$ splitting

DGLAP equation for FF:
$$\frac{\partial}{\partial Q^2} D(z, Q^2) = \frac{\alpha_s}{2\pi}\frac{1}{Q^2}\int_z^1 \frac{dy}{y} P(y) D\left(\frac{z}{y}, Q^2\right)$$

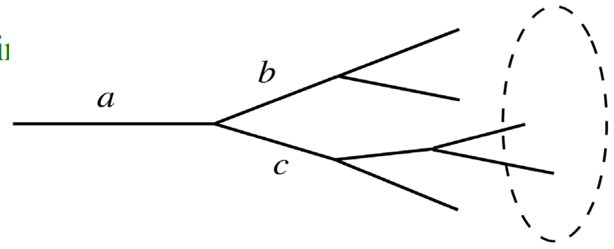
Sudakov form factor:
$$\Delta_a(Q^2_{\max}, Q^2_a) = \prod_i \Delta_{ai}(Q^2_{\max}, Q^2_a) = \prod_i \exp\left[-\int_{Q_a^2}^{Q^2_{\max}} \frac{dQ^2}{Q^2}\frac{\alpha_s(Q^2)}{2\pi}\int_{z_{\min}}^{z_{\max}} dz P_{ai}(z, Q^2)\right]$$

Probability of no splitting between $Q_{\max}$ and Q_a

i: splitting channel
P: splitting function
z: fractional energy or momentum

Monte Carlo implementation

- Random number $r \in (0,1)$
- If $r \leq \Delta(Q^2_{\max}, Q^2_{\min})$, particle is stable, no splitting ($Q_{\min}$: minimum allowed virtuality)
- Otherwise, splitting happens
- If $r \leq \Delta(Q^2_{\max}, Q^2_a) = \dfrac{\Delta(Q^2_{\max}, Q^2_{\min})}{\Delta(Q^2_a, Q^2_{\min})}$, no splitting above Q_a, or splitting happens at or below Q_a
- Solve $r = \Delta(Q^2_{\max}, Q^2_a)$ to obtain Q_a, virtuality at which a splits
- Determine the splitting channel, use branching ratio from $\mathrm{BR}_{ai}(Q^2_a) = \int_{z_{\min}}^{z_{\max}} dz P_{ai}(z, Q^2_a)$

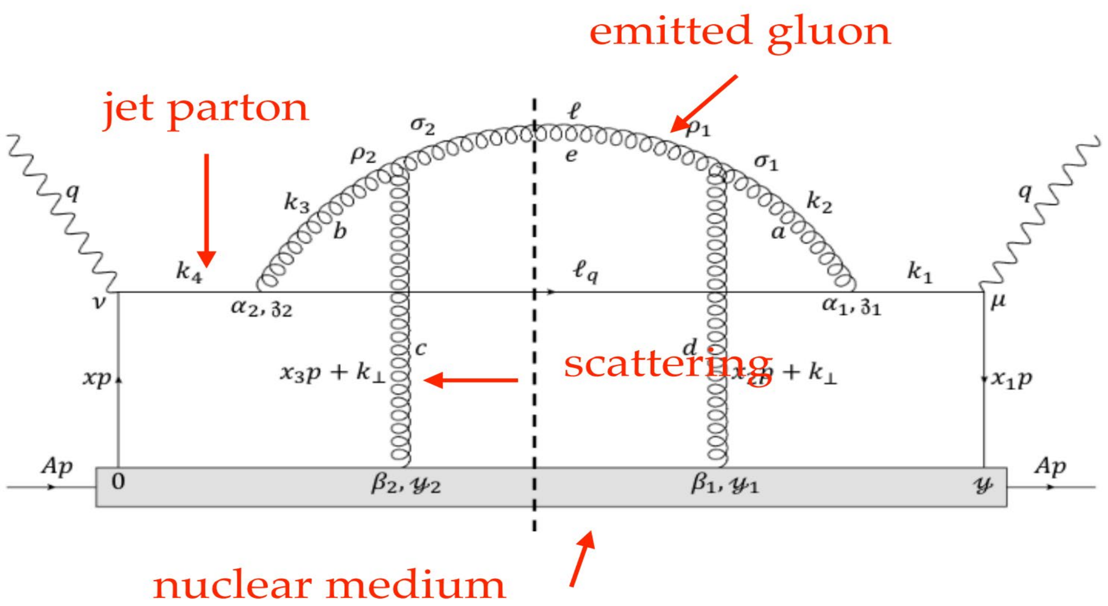


- For a given channel, sample z using $P_{ai}(z, Q^2_a)$
- zQ_a and $(1-z)Q_a$ are new $Q_{\max}$'s for determining Q_b and Q_c in $a \to bc$ splitting

Medium effect

Recall: Sudakov $$\Delta_a(Q^2_{\rm max}, Q^2_a) = \prod_i \Delta_{ai}(Q^2_{\rm max}, Q^2_a) = \prod_i \exp\left[-\int_{Q^2_a}^{Q^2_{\rm max}} \frac{dQ^2}{Q^2}\frac{\alpha_s(Q^2)}{2\pi}\int_{z_{\rm min}}^{z_{\rm max}} dz P_{ai}(z, Q^2)\right]$$

Splitting function: $$P_{ai}(z, Q^2) = P^{\rm vac}_{ai}(z) + P^{\rm med}_{ai}(z, Q^2)$$ (vacuum part + medium-induced part)



$$P^{\rm med}_{ai}(z, Q^2) = \frac{C_A}{C_2(a)}\frac{P^{\rm vac}_{ai}(z)}{z(1-z)Q^2}$$

$$\times \int_0^{\tau_f^+} d\zeta^+ \hat{q}_a\left(\vec{r} + \hat{n}\frac{\zeta^+}{\sqrt{2}}\right)\left[2 - 2\cos\left(\frac{\zeta^+}{\tau_f^+}\right)\right]$$

interference effect

eHING, Xin-nian Wang's lecture