QCD 相变与输运

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Outline

1. Quantum statistics, spectral density, KMS relation
2. QCD phase transitions
3. Relativistic hydrodynamics, linear response
4. Kinetic theory, transport theory

1. Quantum statistics, spectral density, KMS relation

Ref:
Le Bellac, Thermal Field Theory

Imaginary time formalism

transition probability

$$F(q', t'; q, t) = \left\langle q' \left| e^{-i\hat{H}(t'-t)} \right| q \right\rangle$$

thermal patition function

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_n e^{-\beta E_n} = \int dq \left\langle q \left| e^{-\beta\hat{H}} \right| q \right\rangle$$

analytic continuation $\quad t \to -i\tau,\ t' \to -i\tau'$

$$Z(\beta) = \int dq\, F(q, -i\beta; q, 0)$$

Path integral representation

$$F(q', t'; q, t) = \int \mathcal{D} q(t)\, e^{i S[q(t)]}$$

For point particle action

$$S[q(t)] = \int dt \left(\frac{1}{2} m \dot{q}(t)^2 - V(q(t)) \right) = \int dt\, (T - V)$$

$$t \to -i\tau,\; t' \to -i\tau'$$

$$S[q(\tau)] = -i \int d\tau \left(-\frac{1}{2} m \dot{q}(\tau)^2 - V(q(\tau)) \right) = i \int d\tau\, (T + V)$$

$$F(q', i\tau'; q, i\tau) = \int \mathcal{D} q(\tau)\, e^{i S[q(\tau)]} = \int \mathcal{D} q(\tau)\, e^{-S_E[q(\tau)]}$$

probability for Monte Carlo simulation

Partition function again

$$F(q', i\tau'; q, i\tau) = \int \mathcal{D}q(\tau)\, e^{iS[q(\tau)]} = \int \mathcal{D}q(\tau)\, e^{-S_E[q(\tau)]}$$

$$Z(\beta) = \int dq\, F(q, -i\beta; q, 0)$$

$$Z(\beta) = \int \mathcal{D}q(\tau) \exp\left[-\int_0^\beta d\tau \left(\frac{1}{2} m \dot{q}(\tau)^2 + V(q(\tau))\right)\right]$$

probability for Monte Carlo simulation

periodic boundary condition $q(\beta) = q(0)$

QM → QFT

path → field configuration

Correlation functions

$$D^{>}(t,t') = \left\langle \hat{O}(t)\,\hat{O}(t') \right\rangle_\beta$$

$$D^{<}(t,t') = \left\langle \hat{O}(t')\,\hat{O}(t) \right\rangle_\beta = D^{>}(t',t)$$

$$\langle \ldots \rangle_\beta \equiv \mathrm{Tr}\left(e^{-\beta \hat{H}} \ldots\right) / \mathrm{Tr}\left(e^{-\beta \hat{H}}\right)$$

insert complete eigenstates of H

O hermitian

$$D^{>}(t,t') = \frac{1}{Z(\beta)} \sum_{n,m} e^{-\beta E_n} e^{i E_n (t-t')} e^{-i E_m (t-t')} \left| \left\langle n \right| \hat{O}(0) \left| m \right\rangle \right|^2$$

$$D^{>}(k_0) = \frac{2\pi}{Z(\beta)} \sum_{n,m} e^{-\beta E_n} \delta(k_0 + E_n - E_m) \left| \left\langle n \right| \hat{O}(0) \left| m \right\rangle \right|^2$$

manifestly real & positive

KMS relation

$$e^{-\beta \hat{H}} \hat{O}(t) e^{\beta \hat{H}} = \hat{O}(t + i\beta)$$

$$D^{>}(t, t') = D^{<}(t + i\beta, t')$$

Kubo-Martin-Schwinger (KMS) relation

KMS Fourier transformed

$$D^{<}(k_0) = e^{-\beta k_0} D^{>}(k_0)$$

requirements:

- hermitian operator
- translational invariance in time

KMS in a broader sense: fluctuation-dissipation theorem

Spectral function

$$\rho(k_0) = D^>(k_0) - D^<(k_0)$$

$$D^<(k_0) = e^{-\beta k_0} D^>(k_0)$$

$\longrightarrow$ $D^>(k_0) = (1 + f(k_0))\,\rho(k_0),\ D^<(k_0) = f(k_0)\,\rho(k_0)$

Bose-Einstein distribution $f(k_0) = \dfrac{1}{e^{\beta k_0} - 1}$

$$\rho(k_0) = \frac{2\pi}{Z(\beta)} \sum_{n,m} e^{-\beta E_n} \left(\delta(k_0 + E_n - E_m) - \delta(k_0 + E_m - E_n)\right) \left| \langle n | \hat{O}(0) | m \rangle \right|^2$$

properties:

- odd function, "particles"+"anti-particles"
- $\epsilon(k_0)\,\rho(k_0) > 0,$
- temperature dependent in general

Free theory spectral function

Position operator for harmonic oscillator

$$\hat{O}(t) = \frac{1}{\sqrt{2\omega}}\left(a\, e^{-i\omega t} + a^{\dagger} e^{i\omega t}\right)$$

$$\rho(k_0) = \frac{2\pi}{Z(\beta)} \sum_{n,m} e^{-\beta E_n} \left(\delta(k_0 + E_n - E_m) - \delta(k_0 + E_m - E_n)\right) \left| \langle n | \hat{O}(0) | m \rangle \right|^2$$

$$\rightarrow \quad \rho_F(k_0) = 2\pi\,\epsilon(k_0)\,\delta(k_0{}^2 - \omega^2)$$

spectrum of harmonic oscillator

temperature independence: special feature for free theory

Imaginary time correlation function

$$\Delta(\tau) = D^{>}(t = -i\tau, 0) \qquad 0 < \tau < \beta$$ evolution along imaginray time

$$\Delta(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n \tau} \Delta(\tau)$$

$$\omega_n = 2\pi T n,\ n = 0, \pm 1, \pm 2 \ldots$$ Matsubara frequency

F.T. defined on discrete purely imaginary frequencies only

$$\Delta(\tau) = T \sum_n e^{-i\omega_n \tau} \Delta(i\omega_n)$$

frequence sum for inverse F.T.

Spectral representation: imaginray time

$$\Delta(i\,\omega_n) = \int_0^\beta d\tau\, e^{i\,\omega_n\,\tau}\, D^>(-i\,\tau)$$

$$= \int_0^\beta d\tau\, e^{i\,\omega_n\,\tau} \int \frac{dk_0}{2\,\pi}\, e^{-k_0\,\tau}\, D^>(k_0) \;=\; -\int \frac{dk_0}{2\,\pi}\, \frac{\rho(k_0)}{i\,\omega_n - k_0}$$

free theory $\Delta_F = \dfrac{1}{\omega_n{}^2 + \omega^2}$

analytic continuation $i\,\omega_n \rightarrow z$ z away from real axis

$$\Delta(z) = -\int \frac{dk_0}{2\,\pi}\, \frac{\rho(k_0)}{z - k_0}$$

Spectral representation: retarded/time-ordered

$$D_R(t) = i\,\theta(t)\left\langle\left[\hat{O}(t), \hat{O}(0)\right]\right\rangle_\beta = i\,\theta(t)\left(D^>(t) - D^<(t)\right)$$ retarded

$$\theta(t) = i\int\frac{dk_0'}{2\pi}\frac{e^{-i k_0' t}}{k_0' + i\eta}$$

→ $$D_R(k_0) = -\int\frac{dk_0'}{2\pi}\frac{\rho(k_0')}{k_0 - k_0' + i\eta}$$

$$D(k_0) = \int dt\, e^{i k_0 t}\left(\theta(t)\,D^>(t) + \theta(-t)\,D^<(t)\right)$$ time-ordered

$$= i\int\frac{dk_0'}{2\pi}\frac{\rho(k_0')}{k_0 - k_0' + i\eta} + f(k_0)\,\rho(k_0)$$

Fluctuation-dissipation theorem (FDT)

$$D^{sym}(t) = \frac{1}{2}\left\langle \hat{O}(t)\,\hat{O}(0) + \hat{O}(0)\,\hat{O}(t)\right\rangle_\beta = \frac{1}{2}\left(D^>(t) + D^<(t)\right)$$

$$D^{sym}(k_0) = \left(\frac{1}{2} + f(k_0)\right)\rho(k_0)$$

spectrum of fluctuation

$$O(t) = \int dt'\, D_R(t-t')\, j(t')$$

imaginary part of retarded function: dissipation

$$\mathrm{Im}\, D_R(k_0) = \int \frac{dk_0'}{2\pi}\rho(k_0')\,\pi\,\delta(k_0 - k_0') = \frac{1}{2}\rho(k_0)$$

$$D^{sym}(k_0) = (1 + 2f(k_0))\,\mathrm{Im}\, D_R(k_0)$$

FDT

Brownian motion

$$\frac{dp}{dt} = -\eta\, p + \xi(t)$$

$$\langle \xi(t)\, \xi(t') \rangle = \kappa\, \delta(t - t')$$

$$\langle p(t)^2 \rangle = \int_{-\infty}^{t} dt_1\, dt_2\, e^{\eta(t-t_1)}\, e^{\eta(t-t_2)}\, \langle \xi(t_1)\, \xi(t_2) \rangle \overset{t\to\infty}{=} \frac{\kappa}{2\eta}$$

dissipation

particle reaches equilibrium $\left\langle \frac{p(t)^2}{2m} \right\rangle = \frac{1}{2} T$

fluctuation

$$\frac{\kappa}{2\eta} = m\, T$$

Einstein relation

Brownian motion from FDT

$$\langle p(t)\, p(0)\rangle^{sym} = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{0} dt_2\, e^{\eta(t-t_1)}\, e^{\eta(-t_2)} \langle \xi(t_1)\, \xi(t_2)\rangle$$

$$= \frac{\kappa}{2\eta}\left(e^{-\eta t}\,\theta(t) + e^{\eta t}\,\theta(-t)\right)$$

$$\rightarrow \quad D^{sym}(k_0) = \frac{\kappa}{{k_0}^2 + \eta^2}$$

spectrum of fluctuation

FDT for $\beta k_0 << 1$

$$\rho(k_0) = \frac{\beta\kappa k_0}{{k_0}^2 + \eta^2} = 2\,\mathrm{Im} D^R(k_0)$$

An educated guess $D^R = \dfrac{\beta\kappa}{2}\dfrac{1}{-i k_0 + \eta}$

dissipation

From QM to QFT

Derivations applicable to bosonic operator

Change due to Fermi statistics for fermionic operator

$$S^{>}(t, t') = \left\langle \hat{O}_f(t)\, \hat{O}_f(t') \right\rangle_\beta$$

$$S^{<}(t, t') = -\left\langle \hat{O}_f(t')\, \hat{O}_f(t) \right\rangle_\beta$$

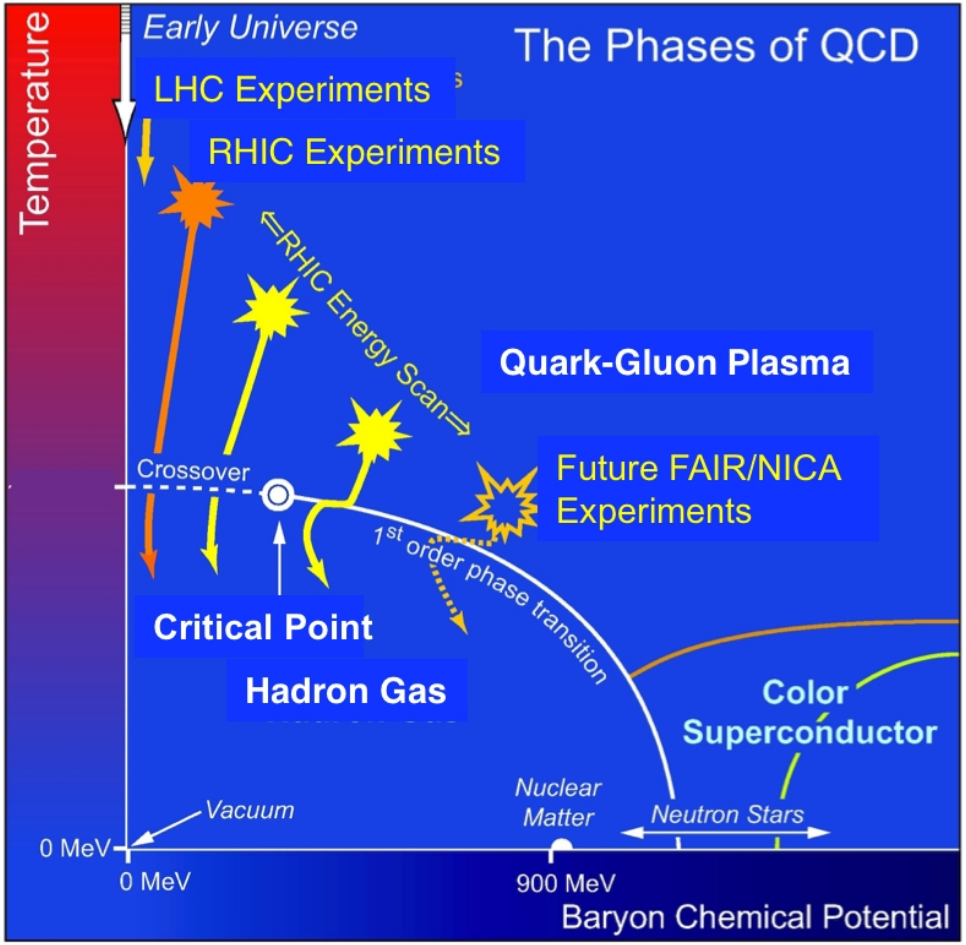
KMS relation $S^{<}(k_0) = e^{-\beta k_0} S^{>}(k_0)$

$$\longrightarrow \quad S^{>}(k_0) = \left(1 - \tilde{f}(k_0)\right) \rho(k_0), \; S^{<}(k_0) = -\tilde{f}(k_0)\, \rho(k_0)$$

Fermi-Dirac distribution $\tilde{f}(k_0) = \dfrac{1}{e^{\beta k_0} + 1}$

2. QCD phase transitions

Ref:
Kapusta, Gale, Finite-Temperature Field Theory Principles and Applications
Schwartz, Quantum Field Theory and the Standard Model

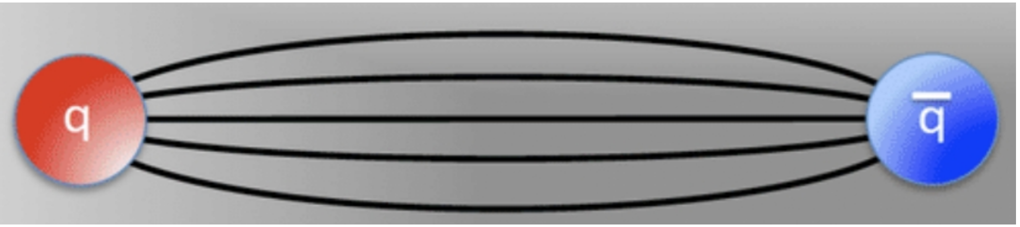


accessible by HIC: deconfinement & chiral phase transitions

A measure of confinement

quark-anti-quark pair interaction

vacuum



$T \gg \Lambda$ asymptotic freedom : flux tube disappears

Heavy quark limit

- Potential meaningful
- Depends on gluon only

Polyakov loop

A gauge invariant order parameter

$$L\left(\vec{x}\right) = \frac{1}{N} \mathrm{tr}\, \mathcal{P} \left\{ \exp\left[i\, g \int_0^{\beta} d\tau\, A_4\left(\tau, \vec{x}\right)\right]\right\} \equiv \frac{1}{N} \mathrm{tr}\, W\left(\beta, \vec{x}\right)$$

Polyakov loop: Wilson line along imaginary path

quark in gauge field $\partial_\mu \to \partial_\mu - i g A_\mu$

Virtue of heavy quark limit

1. quark moves along imaginary time $g A_4$: effective chemical potential
2. spin irrelevant

Heavy quark free energy

$$\psi(\tau, \vec{x}) = W(\tau, \vec{x})\,\psi(0, \vec{x})$$ evolution along imaginary time

$$\exp(-\beta F_q) = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_{nq} \langle nq \,|\, e^{-\beta \hat{H}} \,|\, nq \rangle$$

$$= \frac{1}{N} \sum_{a} \sum_{n} \langle n \,|\, \psi_a(0, \vec{x})\, e^{-\beta \hat{H}}\, \psi_a{}^{\dagger}(0, \vec{x}) \,|\, n \rangle$$

$$= \frac{1}{N} \sum_{a} \sum_{n} \langle n \,|\, e^{-\beta \hat{H}}\, \psi_a(\beta, \vec{x})\, \psi_a{}^{\dagger}(0, \vec{x}) \,|\, n \rangle$$

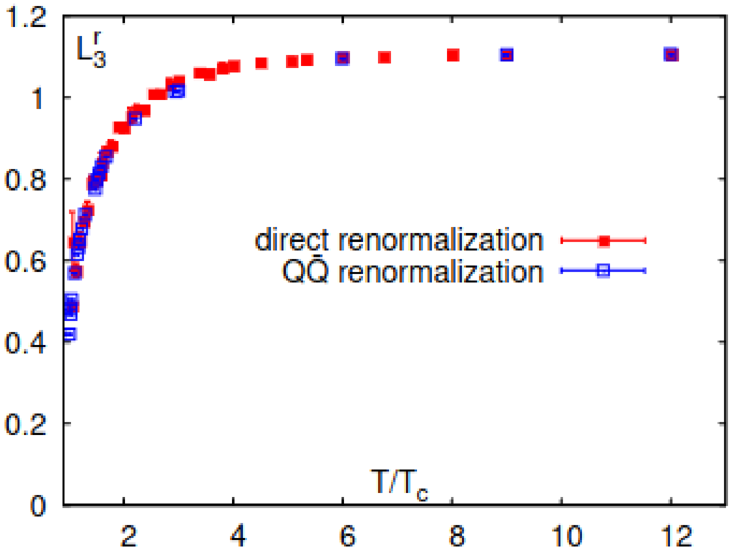
$$= \frac{1}{N} \sum_{a,b} \sum_{n} \langle n \,|\, e^{-\beta \hat{H}}\, W_{ab}(\beta, \vec{x})\, \psi_b(0, \vec{x})\, \psi_a{}^{\dagger}(0, \vec{x}) \,|\, n \rangle = \langle L(\vec{x}) \rangle$$

n: gauge field states   a, b: color labels

Heavy quark pair free energy

$$\exp\left(-\beta F_{q\bar{q}}\right) = \left\langle L\left(\vec{x}\right) L^{\dagger}\left(\vec{y}\right)\right\rangle$$

$$\left|\vec{x}-\vec{y}\right| \rightarrow \infty, \quad \exp\left(-\beta F_{q\bar{q}}\right) \rightarrow \left\langle \left| L\left(\vec{x}\right) \right|^2 \right\rangle$$



first order PT

$$T < T_c, \quad \left| L\left(\vec{x}\right) \right| \rightarrow 0, \; F_q \rightarrow \infty$$

$$T > T_c, \quad \left| L\left(\vec{x}\right) \right| \text{ finite}, \; F_q \text{ finite}$$

0711.2251

Center symmetry

SU(N) gauge theory

$$A_\mu(x) \to V(x)\left(A_\mu + \frac{i}{g}\partial_\mu\right)V(x)^\dagger$$

gauge parameter with

$$V\left(\tau=\beta, \vec{x}\right) = V\left(\tau=0, \vec{x}\right)z$$ constant $z = e^{i\theta}$

preserve periodic boundary condition

$\theta = 2\pi n/N$ belongs to SU(N) $\det z = \left(e^{i\theta}\right)^N = 1$

$z = e^{i2\pi n/N}$ Z_N center of SU(N)

Center gange transformation

$$V\left(\tau=\beta,\vec{x}\right)=V\left(\tau=0,\vec{x}\right)z \qquad z=e^{i2\pi n/N}$$

Polyakov loop (order parameter) transforms as

$$W\left(\vec{x}\right)\rightarrow V(0,x)\,W\left(\vec{x}\right)V(\beta,x)^{\dagger}$$

$$L\left(\vec{x}\right)\rightarrow L\left(\vec{x}\right)z$$

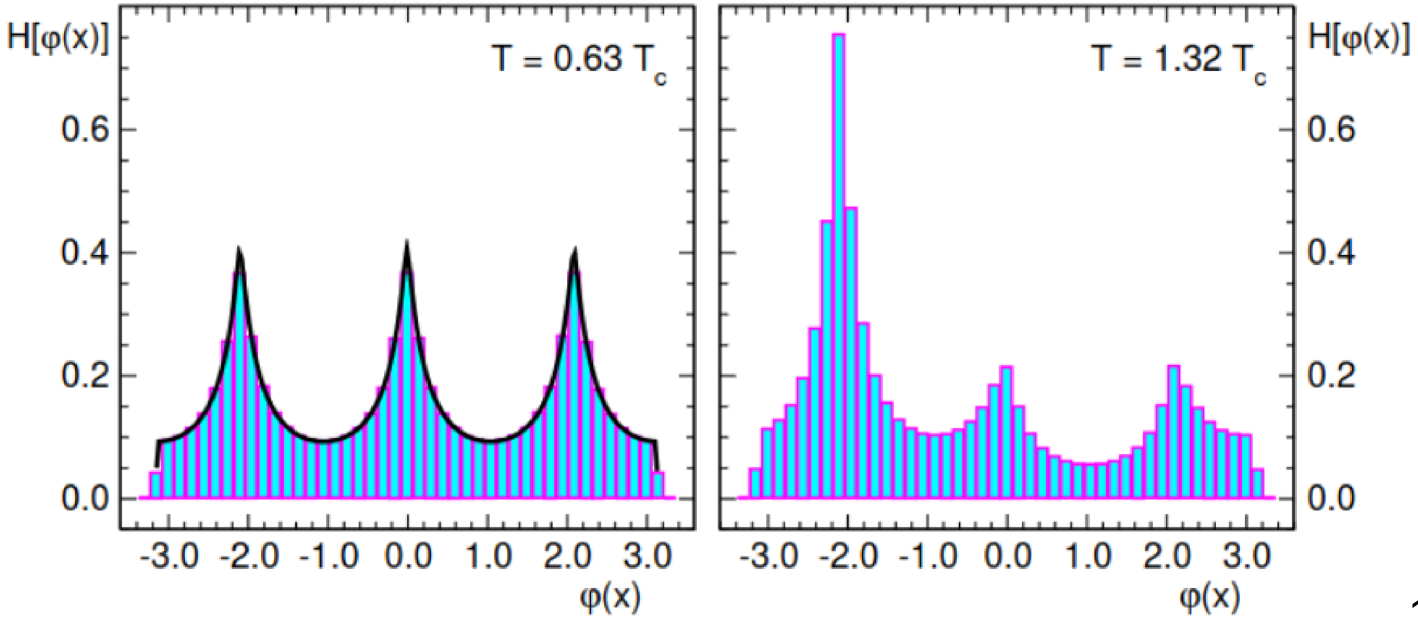
$L\left(\vec{x}\right)=0$ center symmetry unbroken $T<T_c$

$L\left(\vec{x}\right)\neq 0$ center symmetry broken $T>T_c$

Phase of Polyakov loop

$$L(\vec{x}) \rightarrow L(\vec{x})\, z$$

$$L(\vec{x}) = |L(\vec{x})|\, e^{i\varphi(\vec{x})} \qquad \varphi(\vec{x}) \rightarrow \varphi(\vec{x}) + \frac{2\pi n}{N}$$



N=3

$T < T_c$

three phases equal

$T > T_c$

one phase dominates (for a gauge configuration)

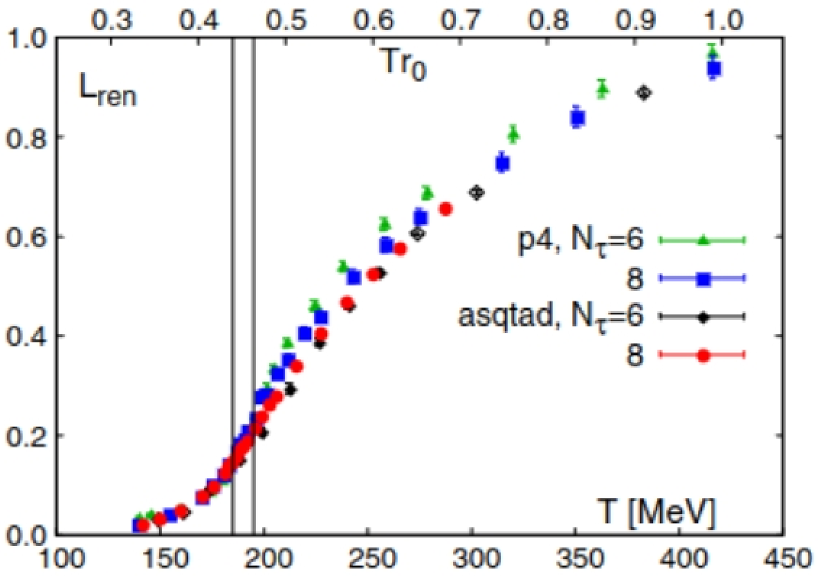
1011.2329

Center symmetry in QCD

$$V\left(\tau=\beta,\vec{x}\right)=V\left(\tau=0,\vec{x}\right)z \qquad \psi(\tau,x)\rightarrow V\left(\tau,\vec{x}\right)\psi(\tau,x)$$

preserve anti-periodic boundary condition requires $z=1$

center symmetry lost in QCD



crossover

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Phenomenology (mean field)

$$A_4^{ab} = \frac{2\pi T}{g} q_a \delta^{ab} \equiv \frac{Q_a}{g} \delta^{ab}$$

"diagonal gauge" 1011.3820

$g A_4$: effective chemical potential

quark with color "a" has
chemical potential $2\pi T q_a$

$$\sum_a q_a = 0.$$

Minkowski space

$A_4 = i A_0,$

$$n_a = \frac{1}{e^{\beta(E - i Q_a)} + 1}, \quad n_{\bar{a}} = \frac{1}{e^{\beta(E + i Q_a)} + 1}$$

purely imaginary
chemical potential

Symmetries of 2-flavor massless QCD

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu}{}^a)^2 + i\,\overline{u}^L \not{D} u^L + i\,\overline{u}^R \not{D} u^R + i\,\overline{d}^L \not{D} d^L + i\,\overline{d}^R \not{D} d^R$$

$$\psi^{R/L} = \frac{1}{2}(1 \pm \gamma^5) \qquad \psi = u,\, d$$

$$q^L \to g_L\, q^L,\; q^R \to g_R\, q^R \qquad q^{L/R} = \begin{pmatrix} u^{L/R} \\ d^{L/R} \end{pmatrix}$$

$$g_L \;\&\; g_R \quad U(2) \to SU(2) \times U(1)$$

$$J_\mu{}^{R,a} = \overline{q}\,\tau^a\,\gamma^\mu\,\frac{1}{2}(1+\gamma^5)\,q \qquad J_\mu{}^{R} = \overline{q}\,\gamma^\mu\,\frac{1}{2}(1+\gamma^5)\,q$$

$$J_\mu{}^{L,a} = \overline{q}\,\tau^a\,\gamma^\mu\,\frac{1}{2}(1-\gamma^5)\,q \qquad J_\mu{}^{L} = \overline{q}\,\gamma^\mu\,\frac{1}{2}(1-\gamma^5)\,q$$

Chiral symmetry

$$SU(2)_R \times SU(2)_L \times U(1)_R \times U(1)_L \to SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$$

isospin chiral vector axial

isospin: unbroken — almost degenerate pseudoscalars (140 MeV)

chiral: spontaneously broken — no light scalars — Nambu-Goldstone Boson

vector: quark number conserved

axial: broken by anomaly

Order parameter: quark condensate

$$\langle \bar{q} q \rangle = \langle q_L{}^{\dagger} q_R + q_R{}^{\dagger} q_L \rangle \to \langle q_L{}^{\dagger} g_L{}^{\dagger} g_R q_R + c.c. \rangle = \langle tr(g_L{}^{\dagger} g_R q_R q_L{}^{\dagger} + c.c.) \rangle$$

invariant under $SU(2)_V$ $g_L = g_R$

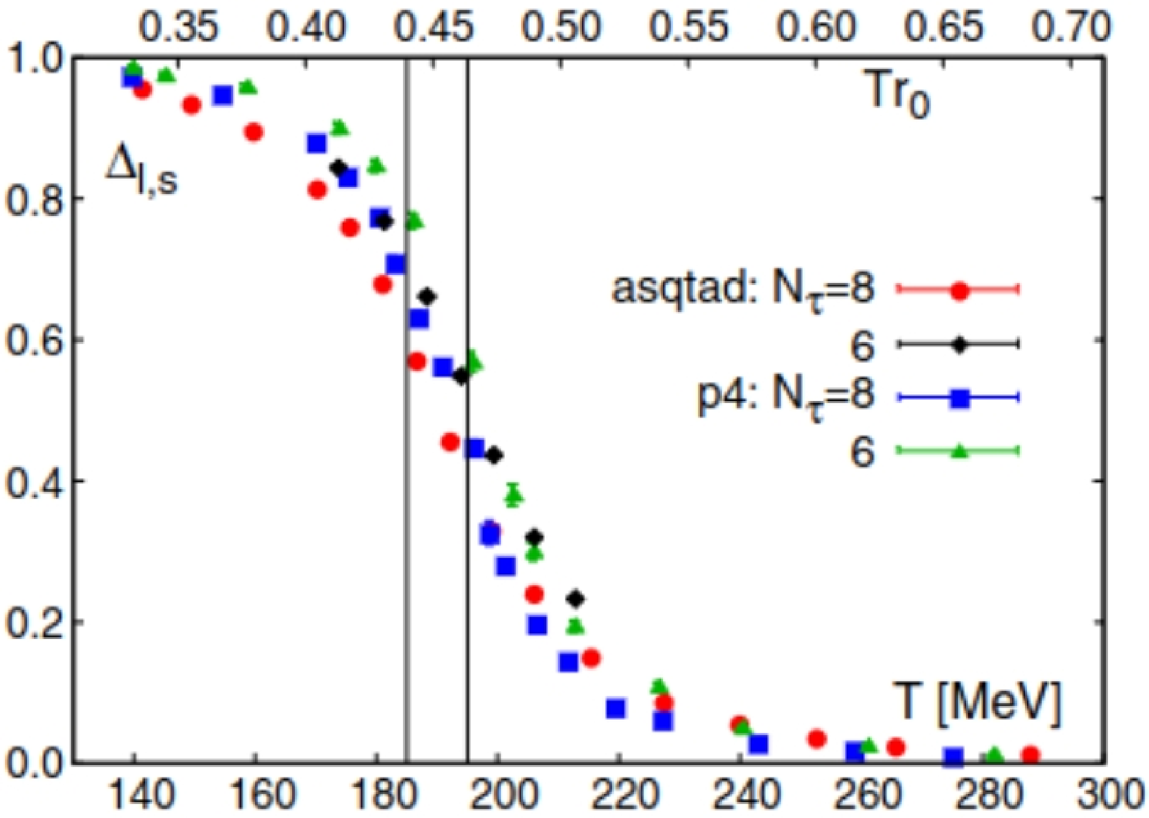
transform under $SU(2)_A$ $g_L = g_R{}^{\dagger}$

Bose Einstein condensation

$\langle \bar{q} q \rangle$ pairing of quark&anti-quark

superfluid ^{3}He pairing of electron&^{3}He atom

Chiral symmetry restoration



$$\Delta_{l,s}(T) = \frac{\langle\bar{\psi}\psi\rangle_{l,T} - \frac{m_l}{m_s}\langle\bar{\psi}\psi\rangle_{s,T}}{\langle\bar{\psi}\psi\rangle_{l,0} - \frac{m_l}{m_s}\langle\bar{\psi}\psi\rangle_{s,0}} .$$

Low energy effective dof

Effective dof $SU(2)_A$ superfluid phases

$$U = \exp\left(i\,\tau^a \pi^a / F\right)$$

$$\tau^a \pi^a = \begin{pmatrix} \pi^0 & \pi^1 - i\,\pi^2 \\ \pi^1 + i\,\pi^2 & -\pi^0 \end{pmatrix} = \begin{pmatrix} \pi^0 & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 \end{pmatrix}$$

$$U \to g_R\, U\, g_L{}^\dagger$$

$$g_{R/L} = \exp\left(i\,\theta_{R/L}{}^a\, \tau^a\right)$$

$$\pi^a \to \pi^a + F\left(\theta_R{}^a - \theta_L{}^a\right) - \frac{1}{2}\epsilon^{abc}\left(\theta_R{}^a + \theta_L{}^a\right)\pi^c$$

$SU(2)_A$ $\qquad$ $SU(2)_V$

Low energy effective action

Effective action invariant under $SU(2)_V \times SU(2)_A$

$$\mathcal{L} = \frac{1}{4} F^2 \, \mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger + 2m \frac{B}{F^2}(U^\dagger + U)\right]$$

kinetic quark mass

kinetic: invariant under $SU(2)_V \times SU(2)_A$

mass: invariant under $SU(2)_V$ only

expanding to quadratic in pions → $m_\pi^2 = 2mB$

matching $\mathcal{L}_m = -m\bar{q}q$ → $\frac{B}{F^2} = -\langle \bar{u}u \rangle = -\langle \bar{d}d \rangle$

3. Relativistic hydrodynamics, linear response

Ref:
Kovtun, 1205.5040

Hydrodynamic dof

Conserved quantities decays slowest, survive in the limit $k \to 0, \omega \to 0$

Conserved quantities $\epsilon, p^i\ n$ coarse-graining

EOM $\partial_\mu T^{\mu\nu} = 0, \partial_\mu J^\mu = 0$

HIC reality
- Finite system size/evolution time
- Phase transition

Hydrodynamics applicable if
- System close to equilibrium
- Mean free path/time << system size/time
- Sufficient away from phase transition

Constitutive relations

Hydrodynamic dof $\epsilon, p^i\ n$ use instead T, μ, v^i 5 fields

EOM $\partial_\mu T^{\mu\nu} = 0, \partial_\mu J^\mu = 0$ $T^{\mu\nu}$ & J^μ 14 fields

Express 14 fields with 5 fields using gradient expansion

zeroth order (ideal hydrodynamics)

$$T^{\mu\nu} = (\epsilon + p)\, u^\mu u^\nu - p\, \eta^{\mu\nu} = \epsilon\, u^\mu u^\nu + \Delta^{\mu\nu} p,\ J^\mu = n\, u^\mu$$

$$\Delta^{\mu\nu} = u^\mu u^\nu - \eta^{\mu\nu} \qquad u^\mu = \gamma(1, v^i)$$

Equation of state $p(T, \mu)$, provided by microscopic theory

$$s = \frac{\partial p}{\partial T},\ n = \frac{\partial p}{\partial \mu},\ \epsilon = T s + \mu n - p$$

Entropy conservation in ideal hydro

$$\partial_\mu T^{\mu\nu} = 0, \; \partial_\mu J^\mu = 0$$

$$\partial_\mu\left((\epsilon + p)\, u^\mu\right) = u^\mu\, \partial_\mu p$$
$$\partial_\mu\left(n\, u^\mu\right) = 0$$

$$\partial_\mu\left((\epsilon + p)\, u^\mu - \mu\, n\, u^\mu\right) = u^\mu\, \partial_\mu p - n\, u^\mu\, \partial_\mu \mu$$

$$T\, \partial_\mu\left(s\, u^\mu\right) + s\, u^\mu\, \partial_\mu T = u^\mu\, \partial_\mu p - n\, u^\mu\, \partial_\mu \mu = u^\mu\, s\, \partial_\mu T$$

$$\partial_\mu\left(s\, u^\mu\right) = 0$$

ideal entropy current

Constitutive equation at next order

Definitions of T, μ, v^i ambiguous out of equilibrium

Fixing the ambiguity: hydrodynamic frame

Landau frame: absorb $\delta T^{00}, \delta T^{0i}, \delta J^0$ into redefinitions of T, v^i, μ.

→ $\delta T^{\mu\nu} u_\mu = \delta J^\mu u_\mu = 0$

$$\delta T^{\mu\nu} = \eta \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} \eta_{\alpha\beta} \partial_\lambda u^\lambda \right) - \zeta \Delta^{\mu\nu} \partial_\lambda u^\lambda$$

$$\delta J^\mu = -\sigma T \Delta^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) + \chi_T \Delta^{\mu\nu} \partial_\nu T$$

$\chi_T = 0$, from non-negative entropy production

η, ζ, σ provided by microscopic theory

Linear response

$$\delta O(x) = -i\int d^4x'\,\theta(t,t')\left\langle\left[\hat{O}(x),\hat{O}(x')\right]\right\rangle_\beta J(x')$$

$$= -\int d^4x'\,D_R(x,x')\,J(x')$$

$$H_{int} = \int d^3x\,O(x)\,J(x)$$

$$\delta O(q) = -D_R(q)\,J(q)$$

Applications in hydrodynamics

$$O \to J^\mu,\; J \to A_\mu$$

$$O \to \frac{1}{2}T^{\mu\nu},\; J \to h_{\mu\nu}$$

Example: conductivity

$$\delta J^{\mu} = -\sigma T \Delta^{\mu\nu} \partial_{\nu}\left(\frac{\mu}{T}\right).$$

$$-\sigma T \Delta^{\mu\nu} \partial_{\nu}\left(\frac{\mu}{T}\right) \sim -\sigma \partial_i \mu \sim -\sigma \partial_i A_0 \sim -\sigma(\partial_i A_0 - \partial_0 A_i) = \sigma E_i$$

fix conductivity from current response to EM fields

- direct response: Ohmic current
- indirect response: induced $\mu, T \propto \partial_i$

homogeneous, non-static E field

$$\delta J^{i} = -\sigma(\partial_i A_0 - \partial_0 A_i) = \sigma \partial_0 A_i$$

$$D_R^{ii}(q_0 \to 0, q = 0) = i \sigma q_0$$

Example: shear viscosity

$$\delta T^{\mu\nu} = \eta\,\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\eta_{\alpha\beta}\partial_\lambda u^\lambda\right) - \zeta\,\Delta^{\mu\nu}\partial_\lambda u^\lambda$$

off-diagonal $\delta T^{ij} = \eta\,(\partial_i u_j + \partial_j u_i)$

$$\delta T^{ij} = \eta\,(\nabla_i u_j + \nabla_j u_i) = \eta\left(\partial_i u_j + \partial_j u_i - 2\,\Gamma_{ij}{}^{\lambda} u_\lambda\right)$$

fix shear viscosity from EMT response to metric

- direct response: Christoffel
- indirect response: induced $u_i \propto \partial_i$

homogeneous, non-static metric field

$$\Gamma_{ij}{}^{0} = -\frac{1}{2} h_{ij,0} \qquad u_\lambda = \left(1, \vec{0}\right)$$

$$\delta T^{ij} = -i\,\eta\,q_0\,h_{ij} \longrightarrow D_R{}^{ij,ij}(q_0, q = 0) = i\,\eta\,q_0$$

Spectral functions in hydro limit

$$D_R^{ii}(q_0 \to 0, q = 0) = i\,\sigma\,q_0$$

$$D_R^{ij,ij}(q_0 \to 0, q = 0) = i\,\eta\,q_0$$

KMS relation $\mathrm{Im}\,D_R(q_0) = \dfrac{1}{2}\rho(q_0)$

→ $\rho^{i}(q_0) = 2\,\sigma\,q_0,\ \rho^{ij}(q_0) = 2\,\eta\,q_0$

consistent with general property of spectral function

Difficult to calculate for QCD!

Calculation using lattice

$$\Delta(i\,\omega_n) = -\int \frac{dk_0}{2\,\pi} \frac{\rho(k_0)}{i\,\omega_n - k_0}$$

In principle possible by inverting the spectral representation

Difficulties in practice

- Lattice output at Matusbara frequencies
- Insensitivity to spectral function in the hydro regime. Accuracy?

Naive calculation using diagrams

Example: conductivity

$$D_R^{ii}\,(q_0 \to 0,\, q = 0) = i\,\sigma\, q_0$$

One-loop photon self-energy in HTL approximation

$$D_R^{ii} = m^2\left(\frac{q_0}{q}\right)\left[\left(1-\left(\frac{q_0}{q}\right)^2\right)Q_0\left(\frac{q_0}{q}\right)+\frac{q_0}{q}\right]$$

$$m^2 = \frac{1}{6}\sum_f Q_f^2\, e^2\, T^2$$

$$Q_0\left(\frac{q_0}{q}\right) = \frac{1}{2}\ln\left|\frac{q_0+q}{q_0-q}\right| - i\,\frac{\pi}{2}\,\theta\left(q^2 - q_0^2\right)$$

Vanishing spectral function in time-like momentum!

Parametric estimate in weak coupling

$$j \sim e\,n\,v \sim e\,n\,a\,\tau \sim e^2\,n\,\frac{E\,\tau}{m} \longrightarrow \sigma \sim n\,e^2\,\frac{\tau}{m}$$

$$T^{ij} \sim n\,m\,v^i\,v^j \sim n\,m\,(a^i\,v^j + a^j\,v^i)\,\tau$$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}{}^{\mu}\,\frac{dx^\alpha}{d\tau}\,\frac{dx^\beta}{d\tau} = 0 \longrightarrow a^i \sim h_{ij,0}\,v_j \sim \sigma_{ij}\,v_j$$

$$T^{ij} \sim n\,m\,\sigma_{ik}\,v_k\,v_j\,\tau \sim n\,m\,\sigma_{ij}\,\tau \longrightarrow \eta \sim n\,m\,\tau$$

$$n \sim T^3,\ m \sim T \qquad \tau \sim \frac{1}{n\,\sigma_{cross}\,v} \sim \frac{1}{g^4\,T}$$

$$\longrightarrow \sigma \sim \frac{e^2\,T}{g^4},\ \eta \sim \frac{T^3}{g^4}$$

Very difficult diagrammatically
Simplified with kinetic theory

Hydro: from deterministic to stochastic

Deterministic hydro: dissipation but no fluctuation

consider hydro fluctuations only

Fluctuations in Brownian motion

$$\frac{dp}{dt} = -\eta p + \xi(t)$$

$$\langle \xi(t)\, \xi(t') \rangle = \kappa\, \delta(t - t')$$

Fluctuations in hydrodynamics

$$\partial_\mu T^{\mu\nu} = 0,\ \partial_\mu J^\mu = 0$$

EOM can't change

$$\delta T^{\mu\nu} = \eta\, \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} \eta_{\alpha\beta} \partial_\lambda u^\lambda \right) - \zeta\, \Delta^{\mu\nu} \partial_\lambda u^\lambda$$ +noise

$$\delta J^\mu = -\sigma T \Delta^{\mu\nu} \partial_\nu \left(\frac{\mu}{T} \right) + \chi_T \Delta^{\mu\nu} \partial_\nu T$$ +noise

Fix noise

- origin of noise: averaging over fast dof → noise localized in hydro cell
- form of noise: chosen to be compatible with FDT

Example: diffusion in neutral system

$$D_R^{00} = \frac{i\,\sigma\, q^2}{q_0 + i\, D\, q^2} \quad \longrightarrow \quad D_{sym}^{00} = \frac{2\, T\, \sigma\, q^2}{q_0^2 + (D\, q^2)^2}$$

$$\delta J^i = -\sigma\, \partial_i \mu + \xi_i = -D\, \partial_i n + \xi_i \quad \longrightarrow \quad \left(-i\, q_0 + D\, q^2\right) n + i\, q_i\, \xi_i = 0$$

to reproduce $\langle n(q_0, q)\, n(q_0', q')\rangle_{sym} = \dfrac{2\, T\, \sigma\, q^2}{q_0^2 + (D\, q^2)^2}\, \delta(q_0 + q_0')\, \delta^3(q + q')$

$$\langle \xi_i(q_0, q)\, \xi_j(q_0', q')\rangle = 2\, T\, \sigma\, \delta_{ij}\, q_i\, q_j\, \delta(q_0 + q_0')\, \delta^3(q + q')$$

$$\langle \xi_i(t, x)\, \xi_j(t', x')\rangle = 2\, T\, \sigma\, \nabla^2\, \delta(t - t')\, \delta^3(x - x')$$

Solving stochastic hydro

$$\partial_t n_B(\mathbf{x},t) = \frac{D}{n_c}\left(m^2\nabla^2 n_B - K\nabla^4 n_B\right)$$
$$+ D\nabla^2\left(\frac{\lambda_3}{n_c^2}\left(\Delta n_B\right)^2 + \frac{\lambda_4}{n_c^3}\left(\Delta n_B\right)^3 + \frac{\lambda_6}{n_c^5}\left(\Delta n_B\right)^5\right)$$
$$+ \sqrt{2Dn_c}\nabla\cdot\boldsymbol{\zeta}(\mathbf{x},t) \qquad D = \Gamma T/n_c.$$
$$\langle\zeta_i(\mathbf{x},t)\zeta_j(\mathbf{x}',t')\rangle = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\delta_{ij}.$$

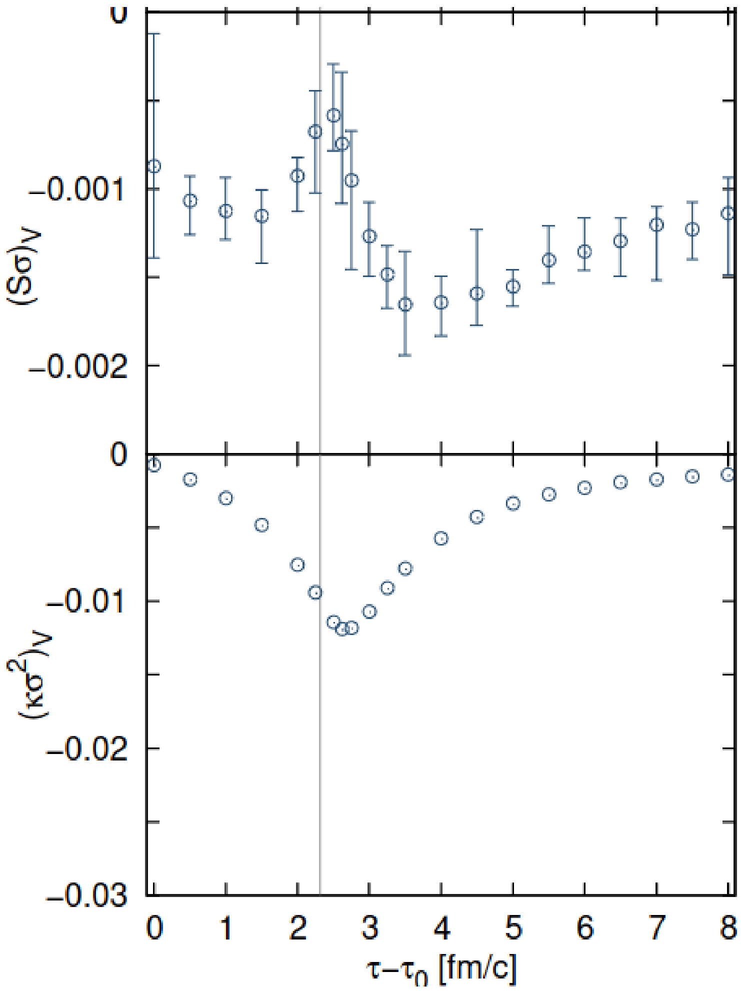
- Solving like Langevin equation
- Nonlinearity leads to possible discretization dependence

Relation to HIC measurements

$$\sigma^2 = \langle \delta N_B{}^2 \rangle = V T^3 \chi_2$$

$$S = \frac{\langle \delta N_B{}^3 \rangle}{\sigma^3} = \frac{V T^3 \chi_3}{(V T^3 \chi_2)^{3/2}}$$

$$\kappa = \frac{\langle \delta N_B{}^4 \rangle}{\sigma^4} - 3 = \frac{V T^3 \chi_4}{(V T^3 \chi_2)^2}$$



Evolution of baryon density in fixed temperature profile

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau} \right)$$

1804.05728

4. Kinetic theory, transport theory

Ref:
Arnold-Moore-Yaffe, hep-ph/0209353
Lin, 2109.00184

Boltzmann equation for general system

$$\left(\partial_t + \hat{p}\cdot\partial_x + \dot{\vec{p}}\cdot\partial_p\right) f\left(t, \vec{x}, \vec{p}\right) = C[f] \qquad \dot{\vec{p}} = Q\, e\left(\vec{E} + \hat{p}\times\vec{B}\right)$$

free-streaming acceleration collision force from EM fields

Assumptions & simplifications:

- Quasi-particles exist, lifetime >> time scale of interest
- Spin averaged
- Collision term system dependent

example:
conductivity in metal,
electron-phonon scattering

Boltzmann equation for QCD

$$\left(\partial_t + \hat{p}\cdot\partial_x + \dot{\vec{p}}\cdot\partial_p\right) f\left(t, \vec{x}, \vec{p}\right) = C[f] \qquad \dot{\vec{p}} = Q\,e\left(\vec{E} + \hat{p}\times\vec{B}\right)$$

- DOF: quarks & gluons
- Spin/color averaged
- No long range chromo-E&B fields
- Collision term: quark-gluon scattering

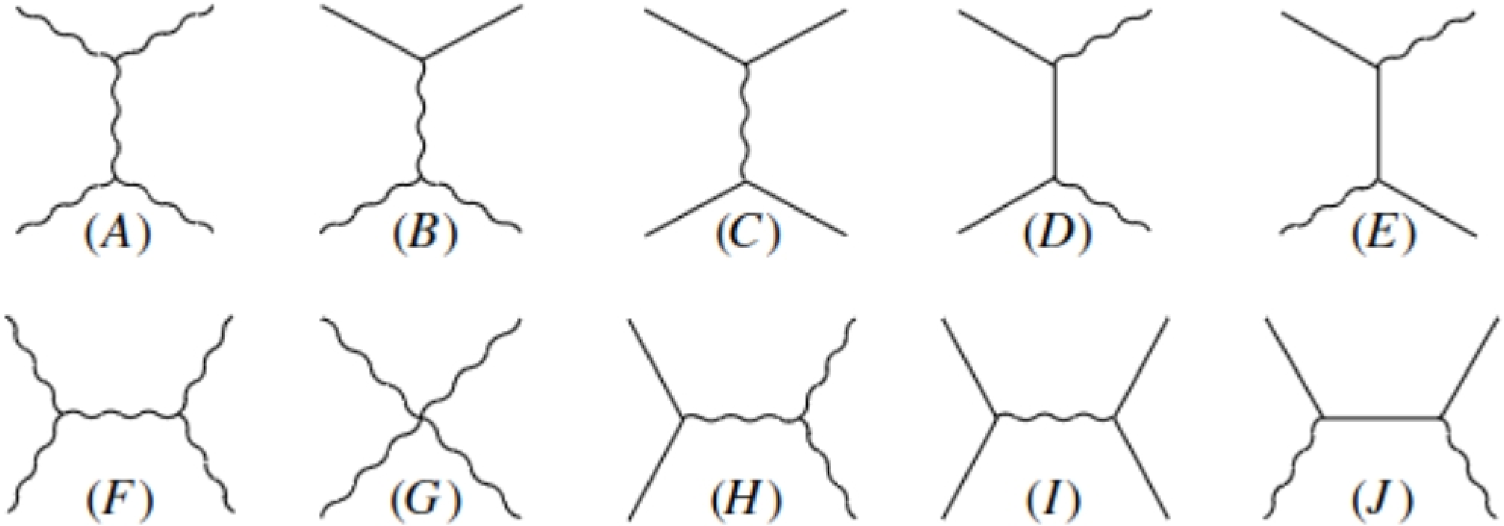
QCD: 2 to 2 scattering

$$C_a^{2\leftrightarrow 2}[f] = \frac{1}{4\,|\,p\,|\;\nu_a} \sum_{bcd} \int_{k,p',k'} \left|\, \mathcal{M}_{cd}{}^{ab}\,(p,\,k,\,p',\,k')\,\right|^2 (2\,\pi)^4\,\delta^4$$

$$(P + K - P' - K')[f_a\,(p)\,f_b\,(k)\,(1 \pm f_c\,(p'))\,(1 \pm f_d\,(k')) - f_c\,(p')\,f_d\,(k')\,(1 \pm f_a\,(p))\,(1 \pm f_b\,(k))]$$

$$\int_p = \int \frac{d^3\,p}{2\,|\,p\,|\,(2\,\pi)^3}$$

ν_a number of spin/color states



Example: electric conductivity of QGP

$$\left(\partial_t + \hat{p}\cdot\partial_x + \dot{\vec{p}}\cdot\partial_p\right) f\left(t, \vec{x}, \vec{p}\right) = C[f] \qquad \dot{\vec{p}} = Q\,e\left(\vec{E} + \hat{p}\times\vec{B}\right)$$

$$D_R^{ii}\,(q_0 \to 0,\ q = 0) = i\,\sigma\,q_0$$

homogeneous electric field $E \sim e^{-i\,q_0\,t}$ hep-ph/0607172

$$f_a\left(t, \vec{x}, \vec{p}\right) = f_{a0}\,(p) + \delta f_a\left(t, \vec{x}, \vec{p}\right) \qquad \delta f_a\left(t, \vec{x}, \vec{p}\right) = \delta f_a\left(\vec{p}\right) e^{-i\,q_0\,t}$$

equilibrium response to E

$$\left(-i\,q_0\,\delta f_a\left(t, \vec{p}\right) + Q_a\,e\,\vec{E}\cdot\partial_p f_{a0}\right) = -\delta C[f] \qquad \delta C[f] \propto \delta f$$

solve the linearized Boltzmann equation

Simplifying collision term

rotational invariance

$$\delta f_a\left(\vec{p}\right) = f_{a0}\,(1 \mp f_{a0})\,f_{a1}\,(p) = \frac{Q\,e\,\vec{E}\cdot\hat{p}}{T^2}\,f_{a0}\,(1 \mp f_{a0})\,\chi_a\,(p)$$

$$f_a\,(p)\,f_b[k]\,(1 \pm f_c\,(p'))\,(1 \pm f_d\,(k')) - f_c\,(p')\,f_d\,(k')\,(1 \pm f_a\,(p))\,(1 \pm f_b\,(k)) =$$

$$f_{a0}\,(p)\,f_{b0}[k]\,(1 \pm f_{c0}\,(p'))\,(1 \pm f_{d0}\,(k'))\,(f_{a1}\,(p) + f_{b1}\,(k) - f_{c1}\,(p') - f_{d1}\,(k'))$$

$$Q_a\,T = -i\,q_0\,Q_a\,\chi_a\,(p) + \frac{1}{f_{a0}\,(1 - f_{a0})}\,\frac{1}{4\,p\,v_a}\sum_{bcd}\int_{k,p',k'} \left|\,\mathcal{M}_{cd}{}^{ab}\,(p, k, p', k')\,\right|^2 (2\,\pi)^4\,\delta^4$$

$$(P + K - P' - K')[f_{a0}\,(p)\,f_{b0}\,(k)\,(1 \pm f_c\,(p'))\,(1 \pm f_d\,(k'))\,(f_{a1}\,(p) + f_{b1}\,(k) - f_{c1}\,(p') - f_{d1}\,(k'))]$$

symbolically

$$S\,(p) = \left[-i\,q_0 + \hat{C}\right]\chi\,(p)$$ integral equation

Conductivity from kinetic theory

$$J^i = \sum_a \nu_a \int_p Q_a \, e \, \hat{p}_i \, \delta f_a \left(t, \vec{x}, \vec{p}\right)$$

$$= \sum_a \nu_a \int_p Q_a{}^2 e^2 \frac{E_i}{3T^2} f_{a0} (1 \mp f_{a0}) \chi_a (p)$$

$$\sigma = \sum_a \nu_a \int_p Q_a{}^2 e^2 \frac{1}{3T^2} f_{a0} (1 \mp f_{a0}) \chi_a (p)$$

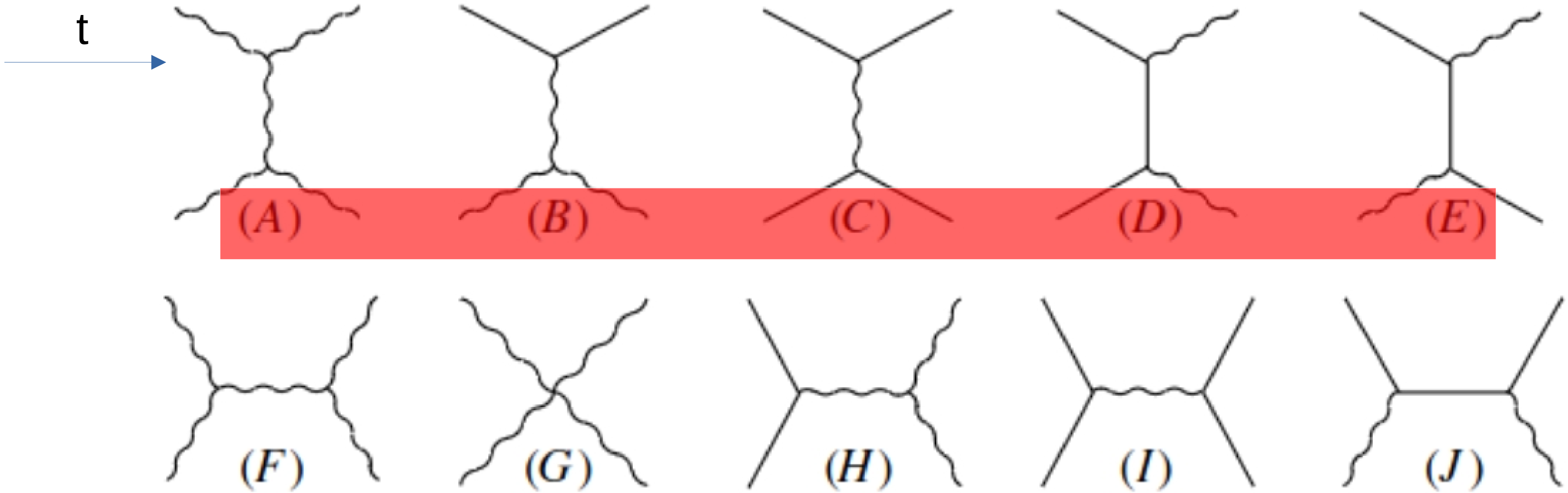
parametric dependence

$$q_0 \to 0 \qquad S(p) = [-i q_0 + \hat{C}] \chi(p) \to \hat{C} \chi(p)$$

$$C \sim g^4 \to \chi(p) \sim \frac{1}{g^4} \quad \Longrightarrow \quad \sigma \sim \frac{e^2}{g^4}$$

steady state far from equilibrium
in weak coupling limit

IR divergence



exchanged soft virtual particle $\int \frac{dq}{q}$

origin of divergence: long range interaction mediated by gluon & quark

cut off by screening effect through self-energies

Leading logarithmic result

$$\int \frac{dq}{q} \sim \int_{gT}^{T} \frac{dq}{q} \sim \ln \frac{1}{g}$$

gT: thermal mass of gluon&quark

$$\sigma \sim \frac{e^2}{g^4 \ln \frac{1}{g}}$$

enhancement of
cross section by log

Comparison with conductivity in metal

- metal: large angle electron-phonon scattering
- QGP: large to small angle quark-gluon scattering

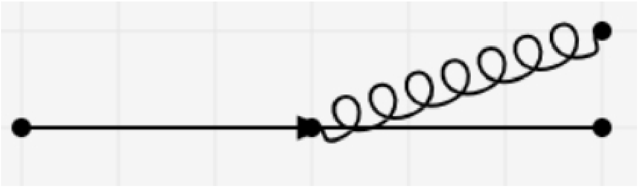
1 to 2 collinear splitting

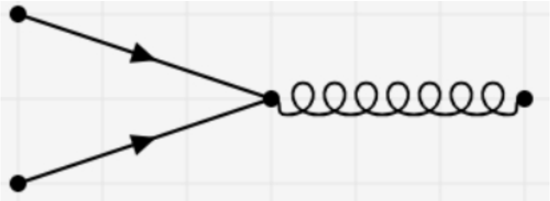
Recap of assumptions

- DOF are real particles without self-energy correction
- Exchanged virtual particles are seflf-energy corrected to regularize IR divergence

Physical difference between real particle & almost real particle?

A new type of process possible with self-energy correction: collinear splitting





$\theta \sim g$.

power counting: vertex $\sim g$, phase space $\sim g^2$

cross section g^4 same as 2 to 2

1 to 2 collinear splitting

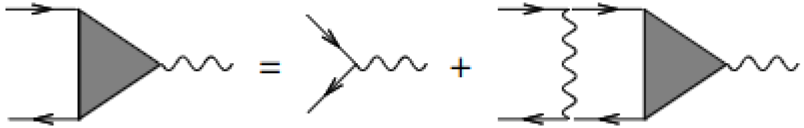
$$C_a^{1\leftrightarrow 2}[f] = \frac{(2\pi)^3}{2\,|p|^2\,\nu_a}\sum_{bc}\int dp'\,dk'\,\delta(|p|-p'-k')\,\gamma_{bc}^{a}(p,p',k')[f_a(p)(1\pm f_b(p'))(1\pm f_c(k')) - f_b(p')f_c(k')(1\pm f_a(p))]$$

$$+\frac{(2\pi)^3}{|p|^2\,\nu_a}\sum_{bc}\int dp'\,dk'\,\delta(|p|+k-p')\,\gamma_{ab}^{c}(p',p,k)[f_a(p)f_b(k)(1\pm f_c(p')) - f_c(p')(1\pm f_a(p))(1\pm f_b(k))]$$

splitting vertex γ_{bc}^{a}

daughter particles remain coherent for long,
multiple scatterings needs be resummed

resummation with
an integral equation



Kadanoff-Baym equation (QED example)

$$S_{\alpha\beta}{}^{<}(x, p) = \int d^4 s\, e^{i P\cdot s} \left(-\left\langle \overline{\psi}_\beta \left(x - \frac{s}{2} \right) \psi_\alpha \left(x + \frac{s}{2} \right) \right\rangle \right)$$

$$\Pi_{\mu\nu}{}^{<}(x, P) = \int d^4 s\, e^{i P\cdot s} \left(\left\langle A_\nu \left(x - \frac{s}{2} \right) A_\mu \left(x + \frac{s}{2} \right) \right\rangle \right)$$

x: coarse-grained coordinate

Kadanoff-Baym equation (effective expansion in ∂_x)

$$\not{\partial} S^{<}(x, P) = \Sigma^{>}(x, p)\, S^{<}(x, p) - \Sigma^{<}(x, p)\, S^{>}(x, p)$$

$$\left[-2\, P\cdot\partial\, \eta^{\mu\nu} + \left(\partial^\mu P^\nu + \partial^\nu P^\mu \right) - \frac{1}{\xi} \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha P_\beta + \partial_\beta P_\alpha \right) \right] D_{\nu\rho}{}^{<} = \Pi^{\mu\nu >} D_{\nu\rho}{}^{<} - \Pi^{\mu\nu <} D_{\nu\rho}{}^{>}$$

$$= \Pi^{\mu\nu >} D_{\nu\rho}{}^{<} - \Pi^{\mu\nu <} D_{\nu\rho}{}^{>}$$

From K-B to Boltzmann

zeroth order Wigner function

$$S^{<}(x,P) = -2\pi\epsilon(P\cdot u)\,\delta(P^2-m^2)\,(\not{P}+m)\,f_e(x,P)$$

$$D_{\mu\nu}{}^{<}(x,P) = 2\pi\epsilon(P\cdot u)\,\delta(P^2)\,P_{\mu\nu}{}^{T}\,f_\gamma(x,P)$$

spin-averaged distributions

$$\not{\partial}S^{<}(x,P) = \Sigma^{>}(x,p)\,S^{<}(x,p) - \Sigma^{<}(x,p)\,S^{>}(x,p)$$

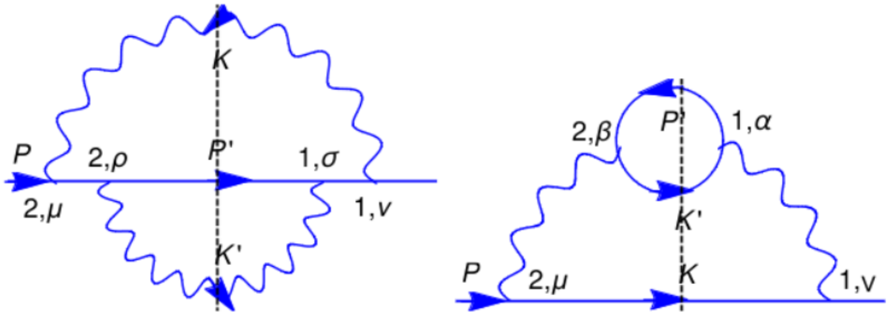
Tr → Boltzmann equation for electron

$$\left[-2P\cdot\partial\,\eta^{\mu\nu} + (\partial^\mu P^\nu + \partial^\nu P^\mu) - \frac{1}{\xi}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\alpha P_\beta + \partial_\beta P_\alpha)\right]D_{\nu\rho}{}^{<} = \Pi^{\mu\nu>}D_{\nu\rho}{}^{<} - \Pi^{\mu\nu<}D_{\nu\rho}{}^{>}$$

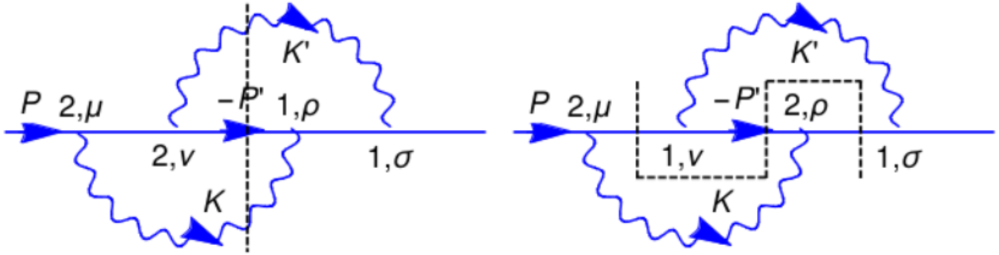
$$= \Pi^{\mu\nu>}D_{\nu\rho}{}^{<} - \Pi^{\mu\nu<}D_{\nu\rho}{}^{>}$$

$P_{\mu\nu}{}'^{T}$ → Boltzmann equation for photon

Collision term from self-energy: 2 to 2

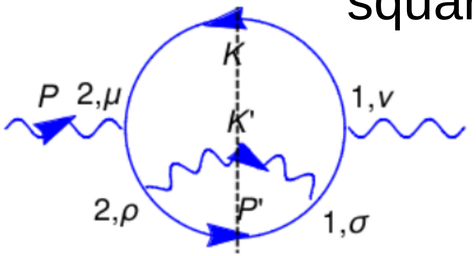


square

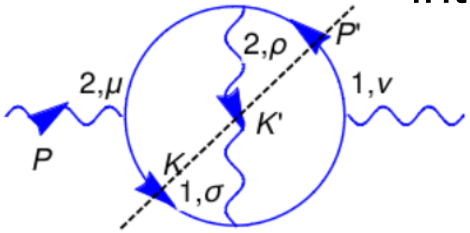


interference

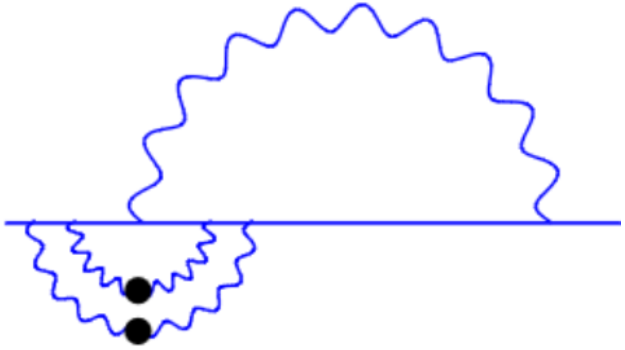
square



interference



Collision term from self-energy: 1 to 2



interference

interference

Spin polarization from K-B

$$\frac{i}{2}\partial\!\!\!/ S^{<(0)} + \frac{P\!\!\!/ - m}{\hbar} S^{<(1)} = \frac{i}{2}\left(\Sigma^{>(0)} S^{<(0)} - \Sigma^{<(0)} S^{>(0)}\right)$$

off-diagonal components determine spin polarization

$$S^{<(1)}(P) = \gamma^5 \gamma_\mu \mathcal{A}^\mu + \frac{i[\gamma_\mu, \gamma_\nu]}{4} \mathcal{S}^{\mu\nu}$$

$$\mathcal{A}^\mu = -2\pi\hbar\epsilon(P\cdot u)\frac{\epsilon^{\mu\nu\rho\sigma} P_\rho u_\sigma \mathcal{D}_\nu f_e}{2(P\cdot u + m)}\delta(P^2 - m^2), \qquad \mathcal{D}_\nu = \partial_\nu - \Sigma_\nu^> - \Sigma_\nu^< \frac{1-f_e}{f_e}$$

$$D_{\lambda\rho}^{<(1)} = -2\pi\epsilon(P\cdot u)\delta(P^2)\frac{iP_{\lambda\alpha}P^{\nu\beta}P^\alpha \partial_\beta P_{\nu\rho}^T f_\gamma(P)}{2(-P^2 + (P\cdot u)^2)} + 2\pi\epsilon(P\cdot u)\delta(P^2)P_{\nu\rho}^T \times \frac{i\hbar u_\lambda u_\mu \left(\Pi^{\mu\nu>(0)} f_\gamma(P) - \Pi^{\mu\nu<(0)}(1 + f_\gamma(P))\right)}{2(-P^2 + (P\cdot u)^2)} - (\lambda \leftrightarrow \rho).$$

kinematical
collisional

Open questions

Theory

- QCD generalization
- Gauge dependence of Wigner function
- Collision term in external EM fields

Phenomenology

- collinear splitting contribution