Hydrodynamics for heavy-ion collisions

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Fudan University, Shanghai, Aug. 20, 2024

Outline

1. Ideal hydrodynamics applied to heavy-ion collisions
2. History: How hydrodynamics became standard
3. Scale invariance of hydro seen in experimental data
4. Viscous corrections, qualitative effects, scaling laws

Goal of this lecture

Hydrodynamics is successful, therefore, it has been studied a lot, and hydrodynamic modeling has become rather complex.

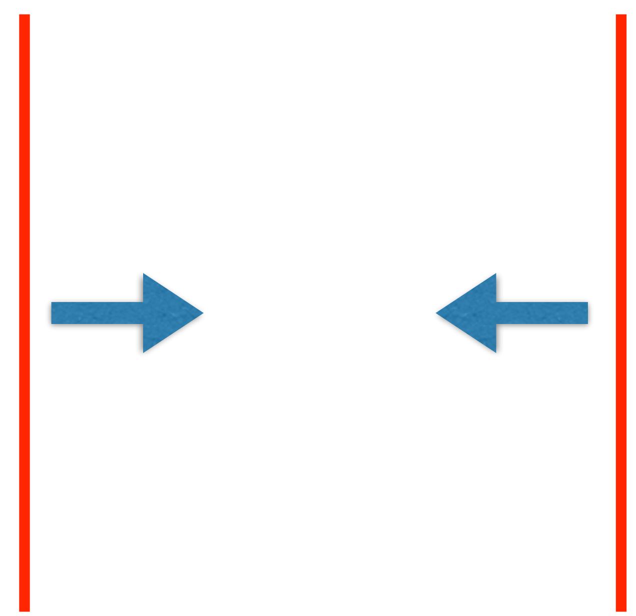
I want to dispel the idea that a hydrodynamic model is so complex that you can always match experimental data just by adjusting free parameters.

On the contrary, I want to show that the simplest description, namely, boost-invariant ideal hydrodynamics, is extremely robust, and that the only freedom lies in details, which never alter the main picture.

Scope of this lecture

- Large systems, where hydrodynamics provides a well-defined, quantitative description (e.g. fairly central Pb+Pb or Xe+Xe collision).
- Ultrarelativistic energies (LHC, top RHIC energy), where due to the strong Lorentz contraction, collision time << expansion time. In addition, matter/antimatter asymmetry is negligible.
- Hydrodynamics is « thermodynamic equilibrium in motion ». Addresses bulk observables: Focus on 99% of the particles and exclude the 1% rarest (electromagnetic probes, heavy flavours, high-p_t particles) which are not in thermal equilibrium.

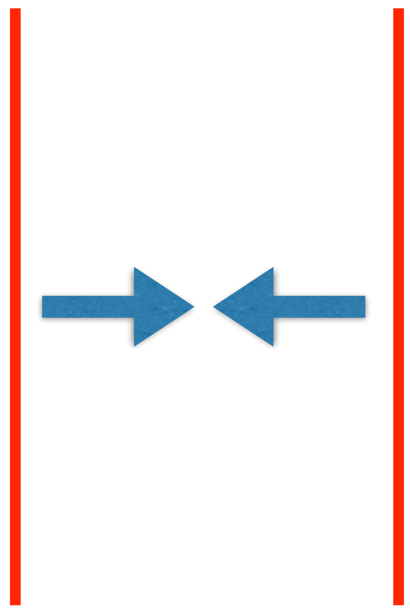
Head-on Pb+Pb collision at the LHC



Relativistic length contraction in the direction of motion, by a factor ~2700 at LHC

➔ Colliding spherical nuclei appears as disks

Head-on Pb+Pb collision at the LHC



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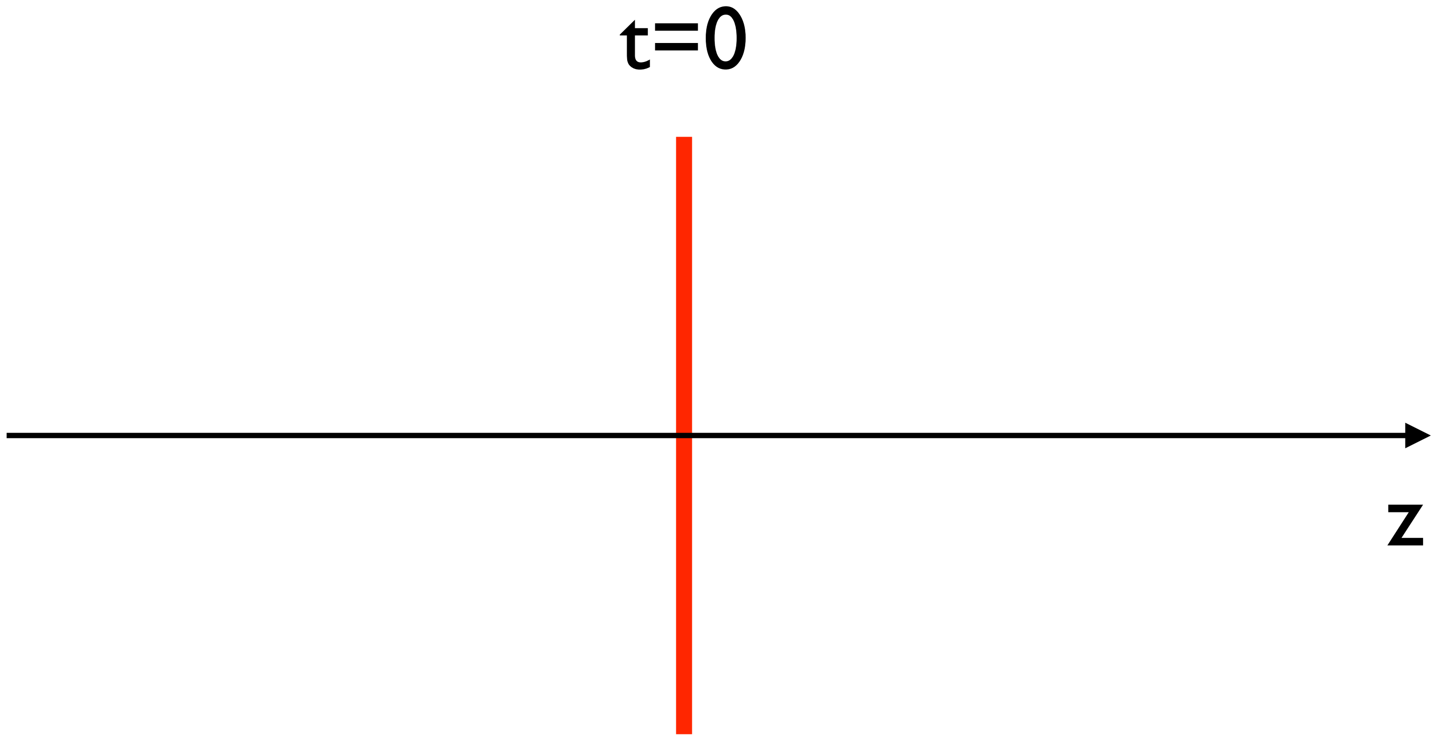
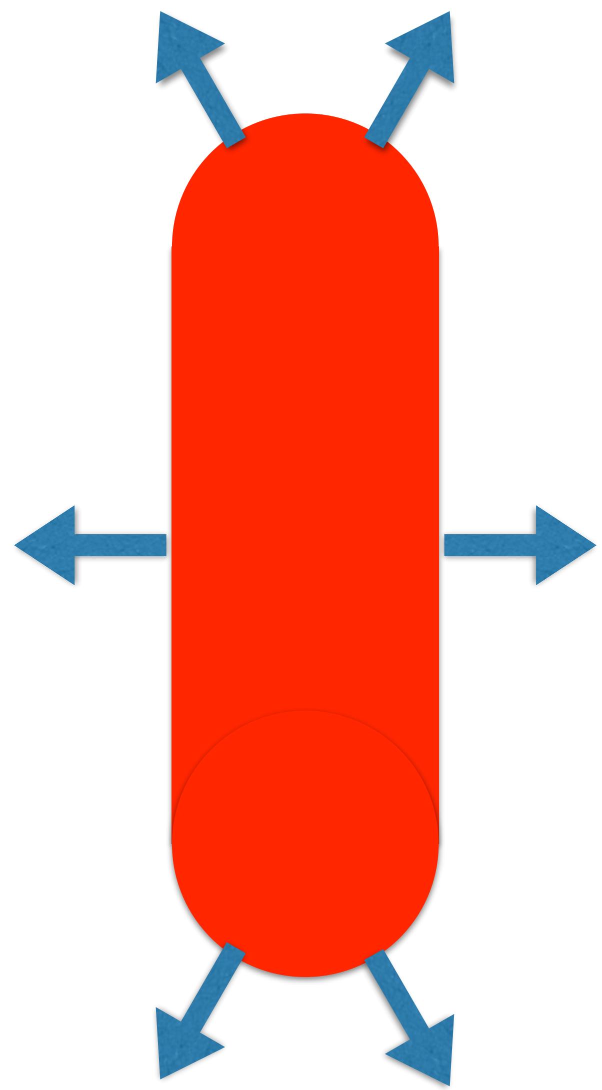


Collision = instantaneous process at z=t=0

Head-on Pb+Pb collision at the LHC



- Strongly-coupled quark-gluon matter is created.
- Expands into the vacuum at ~ velocity of light.
  This is what hydrodynamics is about.

Part I:

Ideal hydrodynamics applied to heavy-ion collisions

Thermodynamics reminder

- Fundamental identity $dE=-PdV+TdS+\mu dN$
- Extensivity: $E(\lambda V,\lambda S,\lambda N)=\lambda E(V,S,N)$
- *Exercise 1*:
  Take derivative with respect to λ, set $\lambda=1$, and show that $E=-PV+TS+\mu N$
- *Exercise 2*:
  Show that $-VdP+SdT+Nd\mu=0$ (Gibbs-Duhem eq.)

Matter/antimatter symmetry: N=0

- $dE=-PdV+TdS$
- $E=-PV+TS$ $e+P=Ts$ where $e=E/V, s=S/V$
- $-VdP+SdT=0$ $dP=sdT$
- This implies $de=Tds$
- side note: speed of sound c_s is defined by $c_s^2=dP/de=d\ln T/d\ln s$.

All thermodynamic quantities (e, s, P) are functions of a single variable, the temperature T. This is generally referred to as *equation of state*.

Definition of ideal hydrodynamics

Consider an (infinitesimally) small part of the system, which I call a *fluid cell*.

There is a Lorentz frame in which this *fluid cell* has no net momentum, called the local rest frame.

Fluid velocity $\mathbf{v}(\mathbf{x},t)$ = velocity of local rest frame relative to laboratory frame.

Ideal hydrodynamics = the *fluid cell* is in thermal equilibrium in the local rest frame.

Fluid temperature $T(\mathbf{x},t)$ defined in local rest frame

Evolution equations

The equations of ideal hydrodynamics specify how these two fields, $\mathbf{v}(\mathbf{x},t)$ and $T(\mathbf{x},t)$, evolve as a function of time.

Remember that all thermodynamic quantities, $e(\mathbf{x},t)$ $s(\mathbf{x},t)$, $P(\mathbf{x},t)$, are related to $T(\mathbf{x},t)$ through the equation of state.

There are two equations: energy, and momentum.

Energy equation

- The energy of a fluid cell evolves according to $dE=-PdV$ (work of pressure forces).
- Using the thermodynamic identity $dE=-PdV+TdS$, this implies $dS=0$ (only work, no heat transfer): entropy of the fluid cell is conserved.
- Relation to fluid velocity: during a time dt,
  $dV = dt \oiint \mathbf{v}(\mathbf{x},t).\, d\mathbf{S} = dt \iiint \nabla.\, \mathbf{v}(\mathbf{x},t)\, d^3x.$
  $= V\, dt\, \nabla.\, \mathbf{v}(\mathbf{x},t)$ for a small fluid cell.
- *Exercise 3*:
  Write $E=e(\mathbf{x},t)V$ and derive $de/dt=-(e+P)\, \nabla.\, \mathbf{v}$

Momentum equation

- Newton's second law: $d\mathbf{p}/dt=\mathbf{F}$.
- For a fluid cell of mass m,
  nonrelativistic : $d\mathbf{p}/dt= m d\mathbf{v}/dt= V\ \rho(\mathbf{x},t)\ d\mathbf{v}(\mathbf{x},t)/dt$
  relativistic :
  naive expectation, use $E=mc^2$, replace ρ with e.
  (in natural units where $c=1$)
  In fact, one must replace ρ with $e+P$, not just e.
- $\mathbf{F} = -\oiint P(\mathbf{x},t)\ d\mathbf{S} = \iiint \nabla P(\mathbf{x},t)\ d^3x$
  $= V\ \nabla P(\mathbf{x},t)$ for a small fluid cell.
- One finally obtains: $(e+P)d\mathbf{v}/dt=-\nabla P$.

Initial conditions

- In order to solve these equations, one must specify the initial conditions, and the equation of state.
- The collision happens at t=0.
- The matter does not thermalize immediately, and what happens before (called *pre-equilibrium dynamics*) is now carefully modeled

Kurkela Mazeliauskas Paquet Schlichting Teaney 1805.01604

- But it's actually OK to start hydrodynamics immediately, at an arbitrarily small time t_0: makes no difference at all on observables.

Vredevoogd Pratt 0810.4325

- Initial conditions = fluid velocity, entropy density

Initial conditions: fluid velocity

- Collision happens at z=t=0.
- Assume that the fluid comes from z=t=0 with constant velocity: $v_z=z/t_0$. (with $|z|<t_0$)

Bjorken Phys.Rev.D 27 (1983) 140

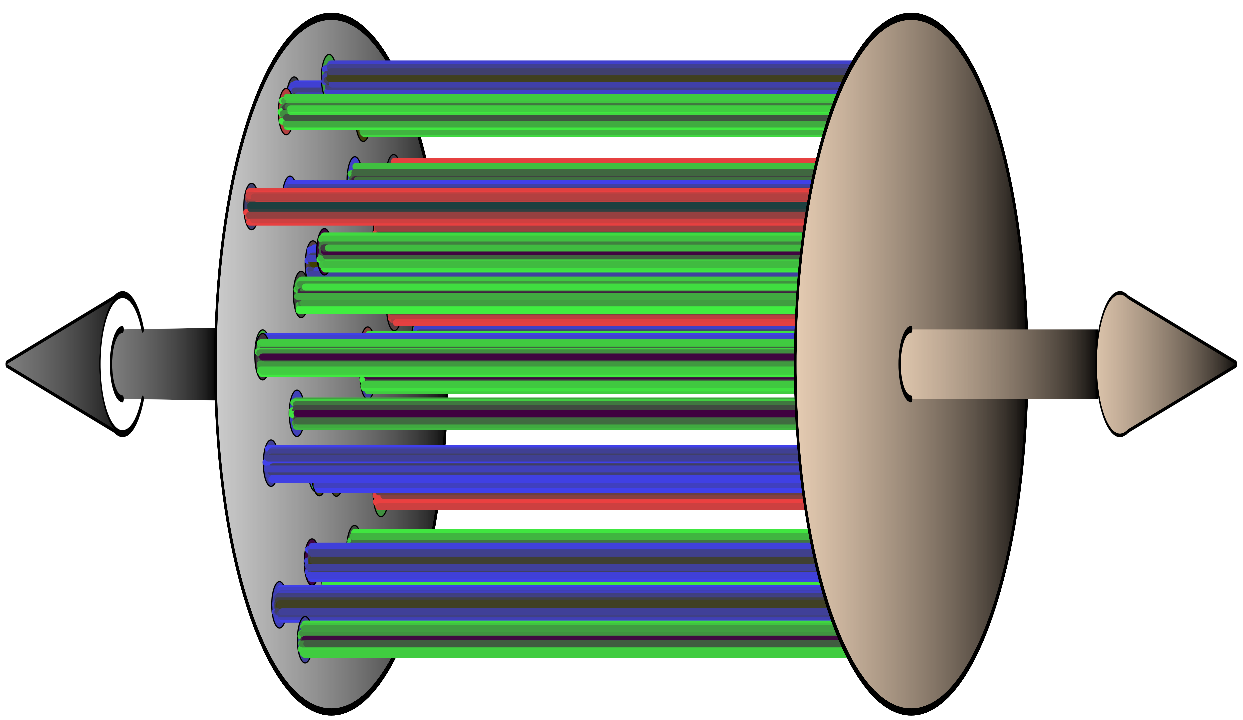
- Initial transverse fluid velocity $v_x=v_y=0$.
  Produced particles can have transverse momenta, but they average to 0. No collective transverse motion.
- *Exercise 4*: show that, at the initial time,
  $\nabla . \mathbf{v}(\mathbf{x},t_0) = 1/t_0$.

Initial conditions: entropy density

- In practice, most hydro codes specify the energy density $e(x,t_0)$, rather than the entropy density $s(x,t_0)$.
- It is a matter of convention, since e is related to s through the equation of state.
- I find entropy density more convenient because entropy is conserved.

Initial conditions: entropy density

$s(x,y,z,t_0)$ is independent of z.
(more precisely, of space-time rapidity)
Motivated by the dynamics of strong interactions (strings in the pre-QCD era, color flux tubes in the modern CGC approach)



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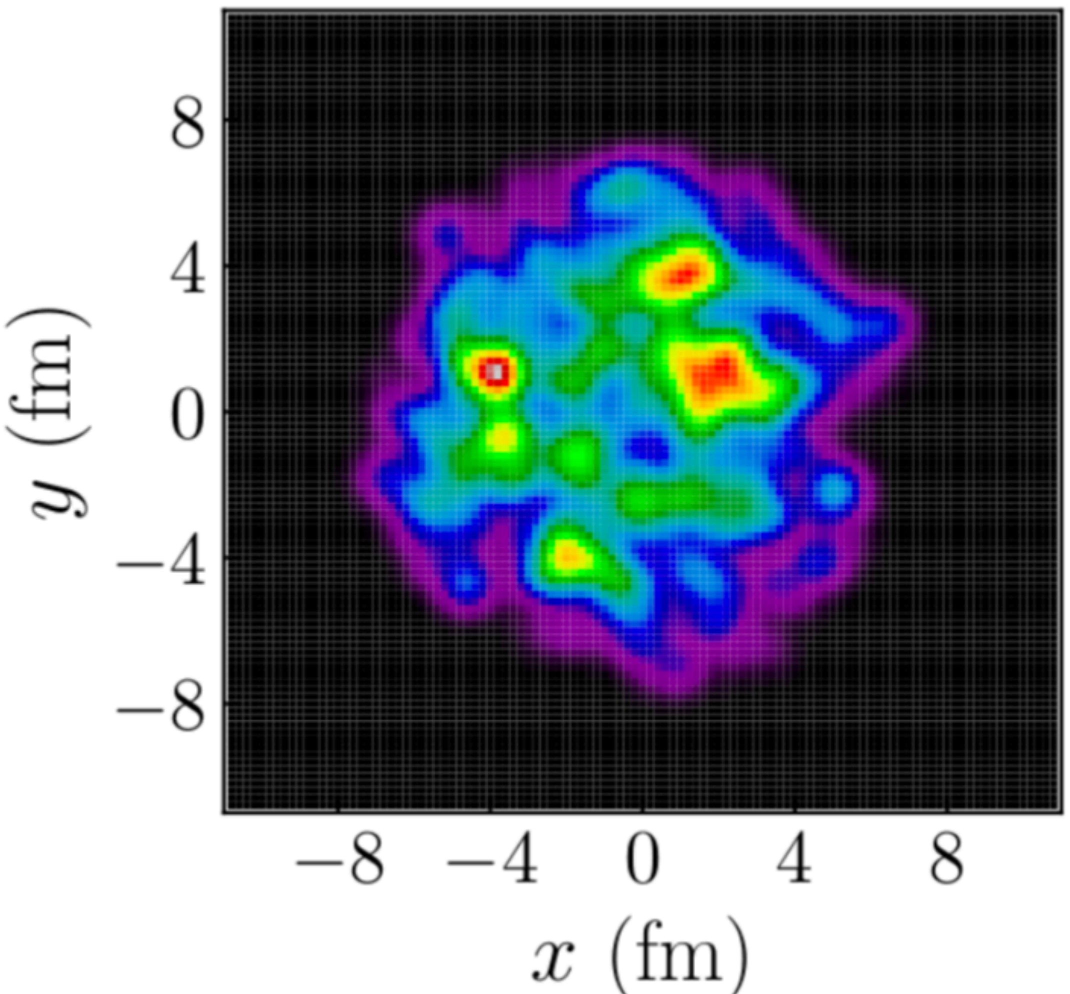
Then, the initial condition $v_z=z/t$ is preserved by the evolution.
This is called boost-invariant hydrodynamics

Bjorken Phys.Rev.D 27 (1983) 140

Initial conditions: entropy density

The only real model dependence is in the dependence on transverse coordinates, s(x,y).



courtesy Giuliano Giacalone

One adjusts the normalization by hand so as to reproduce the measured particle multiplicity.

The size is known: overlap between colliding nuclei.

Only freedom: how entropy is distributed across this area. Density profile is irregular: fluctuates event to event

Hadronization

- As the temperature decreases, the quark-gluon plasma turns into a hadron gas.
- At low enough temperatures, the equation of state of QCD calculated on the lattice is well approximated by an ideal gas of all hadron species (see lectures by Eulogio Oset).
- Therefore, each fluid cell is an ideal hadron gas, boosted by the fluid velocity.
- The simplest is to stop the hydro at a fixed temperature, the freeze-out temperature T_F. Typical value T_F =135 MeV.

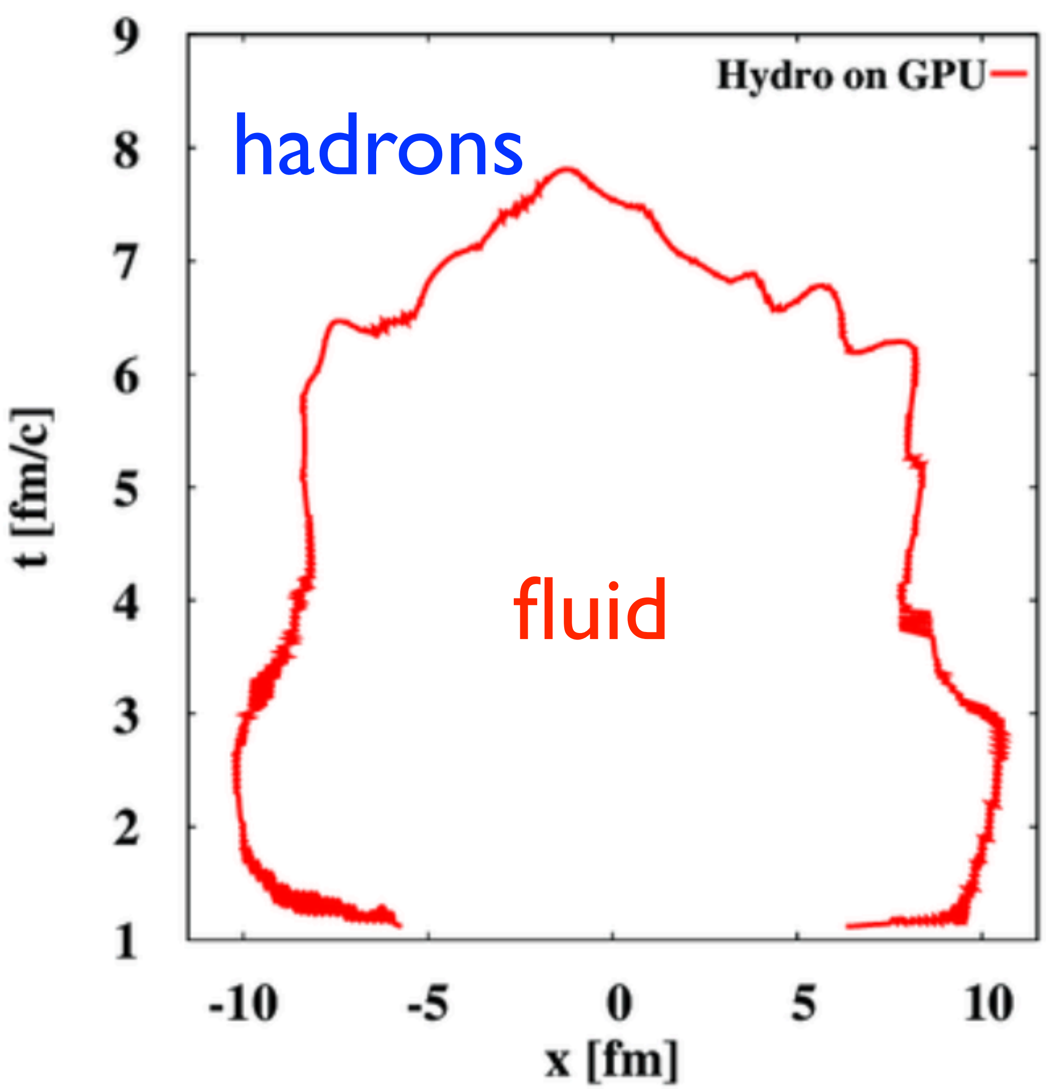
Guillen Ollitrault 2012.07898

- Freeze-out also produces unstable hadrons which then decay.

Mazeliauskas Flörchinger Grossi Teaney 1809.11049

Cooper-Frye freeze-out

Cooper Frye Phys. Rev/ D 10, 186 (1974)



Słodkowski et al.
Symmetry 2021, 13(3), 50

- The condition $T=T_F$ defines a *space-time hypersurface.*
- For a horizontal part (constant t) one just counts hadrons at that time.
- For a vertical part (constant x) one counts hadrons crossing a surface (flux proportional to v_x).
- How to do this right has been known for exactly 50 years

Output of the hydrodynamic calculation

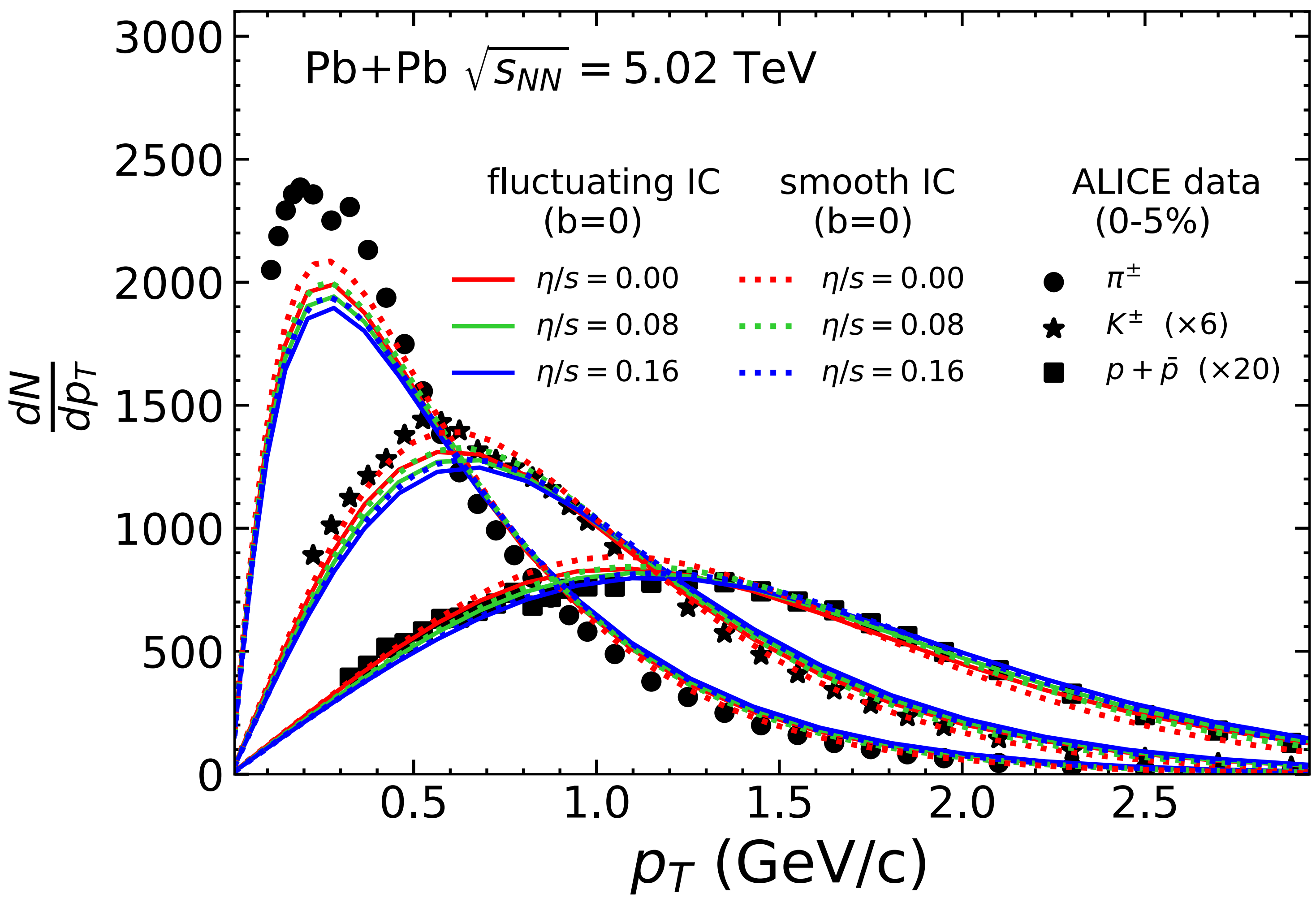
- Single-particle momentum distribution dN/d**p** for all stable hadrons.
- Momentum **p** depends on 3 variables
  1. rapidity y
  2. transverse momentum p_t
  3. azimuthal angle φ
- Boost invariance ⇒ $dN/dp_t d\varphi dy$ independent of y.

- Dependence on φ can be decomposed as a Fourier series $dN/dp_t d\varphi dy = (1/2\pi)\, dN/dp_t dy \sum_n V_n(p_t)\exp(-in\varphi)$ where $V_{-n}=V_n^*$ because the distribution is real, and $V_0=1$. V_n is the anisotropic flow (lecture by Pengfei Zhuang) in complex notation.

Hydro event *versus* collision event

- Hydrodynamics is a continuous description.
- central Pb+Pb collision at the LHC : discrete particles *(question to audience: how many?)*
- Hadronization is a random process: induces statistical fluctuations (and correlations due to resonance decays)
- ***A hydrodynamic event* is an *ensemble of collision events*** with the same initial conditions.
- It makes perfect sense to *compute* V_n for a single hydrodynamic event.
- But *measuring* V_n in a single collision event is not interesting because of large statistical fluctuations. Measurements are always averaged over many events.

Hydro versus experiment: p_t spectra of identified hadrons



Parida Samanta Ollitrault 2407.17313

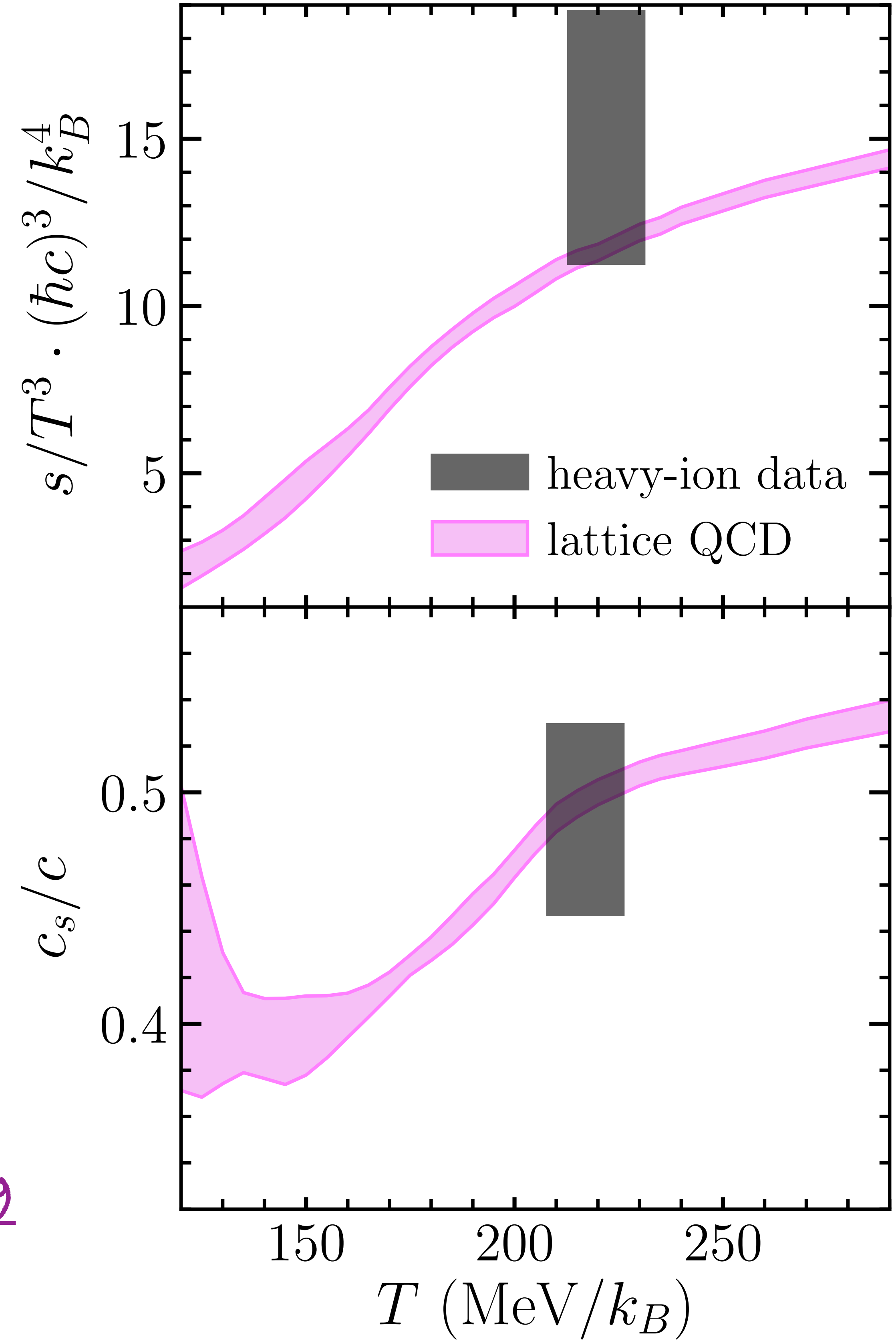
Only inputs of hydro calculation:

- Total hadron multiplicity (normalization of initial entropy adjusted by hand)
- Equation of state (taken from lattice QCD)
- Freeze-out temperature (here T_F=130 MeV)

Boost-invariant ideal hydro is not perfect, but simple and robust.

Equation of state from LHC data

- The equation of state is the most important ingredient of ideal hydrodynamics.
- Therefore, comparison between LHC data and hydro calculations constrains the equation of state.
- LHC data are compatible with lattice QCD



Gardim Giacalone Luzum Ollitrault
Nature Physics 16 (2020) 6, 615-619

Part 2:

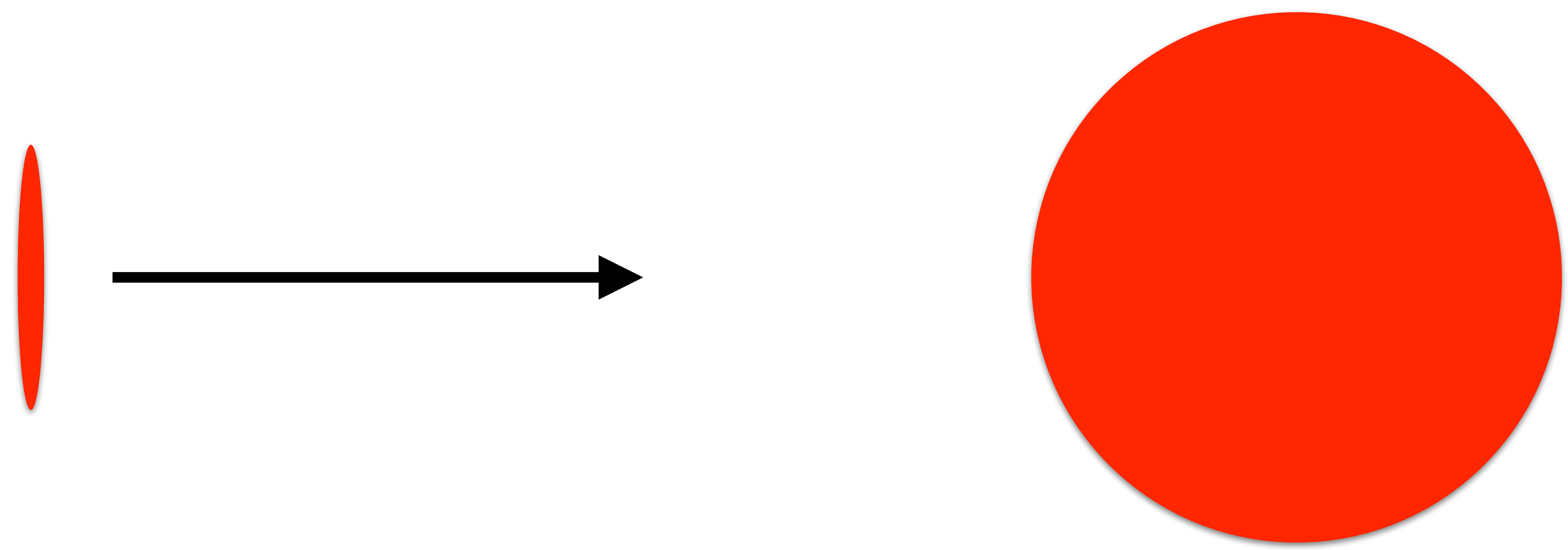
History ;

How hydrodynamics became standard

1. Elliptic flow (2000)
2. The ridge (2010)

History of ultrarelativistic nucleus-nucleus collisions

20th century: nuclei accelerated on fixed (heavy) target nuclei



Late 1980s: light nuclei (^{16}O, ^{32}S)

- Brookhaven National Laboratory (New York): AGS accelerator
- CERN: SPS accelerator

History of ultrarelativistic nucleus-nucleus collisions

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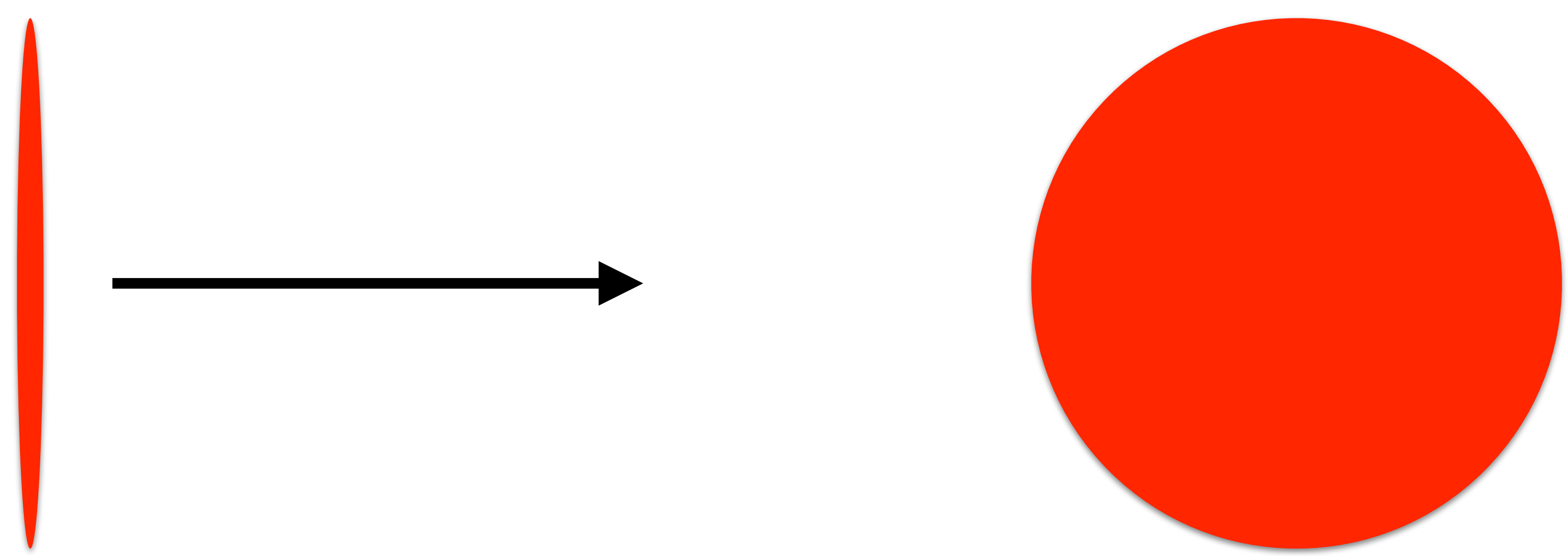
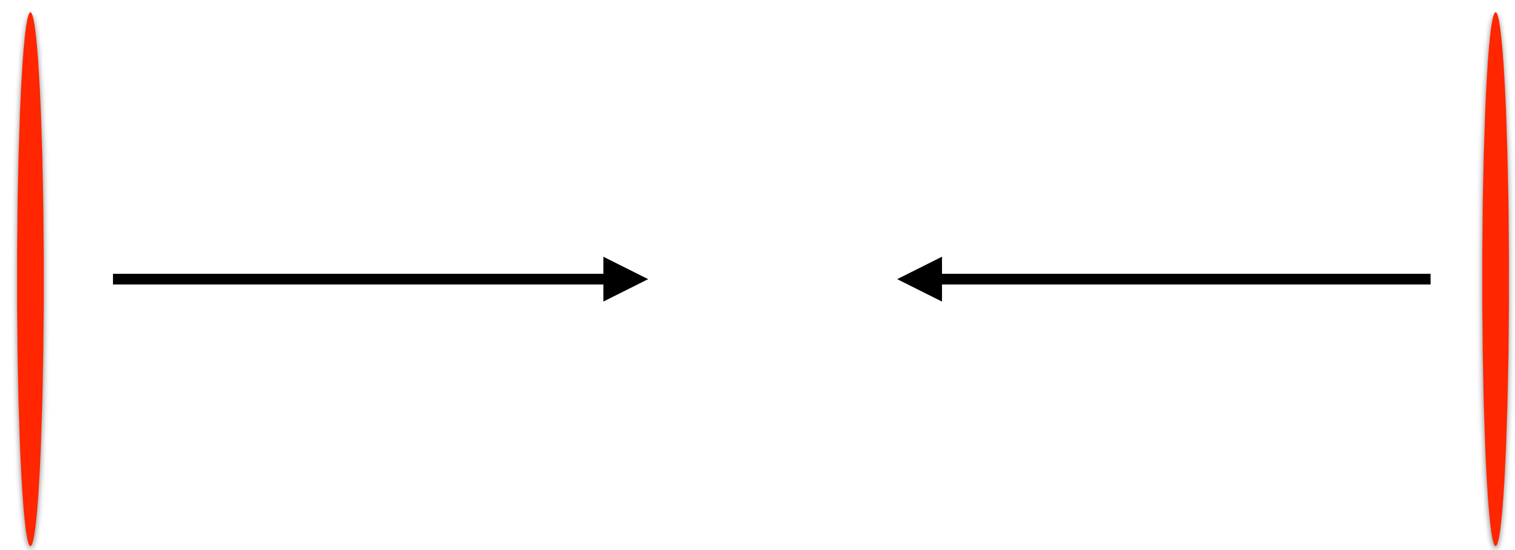


1990s: heavy nuclei (^{197}Au, ^{208}Pb)

- Brookhaven National Laboratory (New York): AGS accelerator
- CERN: SPS accelerator

History of ultrarelativistic nucleus-nucleus collisions

21th century: Nuclei accelerated in both directions

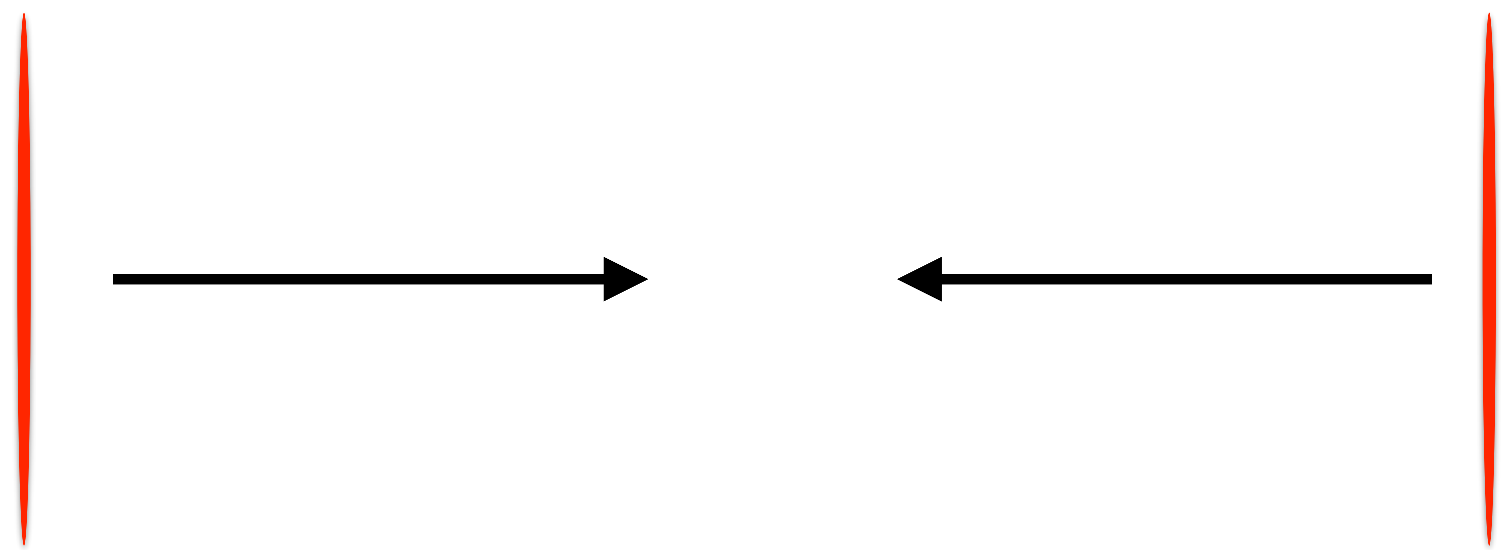


2000-present

- Brookhaven National Laboratory
  RHIC = Relativistic Heavy Ion Collider

History of ultrarelativistic nucleus-nucleus collisions

21th century: Nuclei accelerated in both directions



2010-present

- CERN: LHC = Large Hadron Collider (energy = 25xRHIC)

What was the initial motivation?

1. Back in the 1980s it was thought that the equation of state of strong interactions (QCD) had a first-order phase transition to a quark-gluon plasma. This later proved wrong.
2. Léon van Hove argued in 1982 that one could measure this equation of state in nucleus-nucleus collisions if the produced matter thermalizes, and that the transverse momentum per particle $\langle p_T \rangle$ would be proportional to the temperature T (proved right ~35 years later).
3. My first assignment as a PhD student in 1985 was to write a hydrodynamic code and check the correspondence postulated by Van Hove.

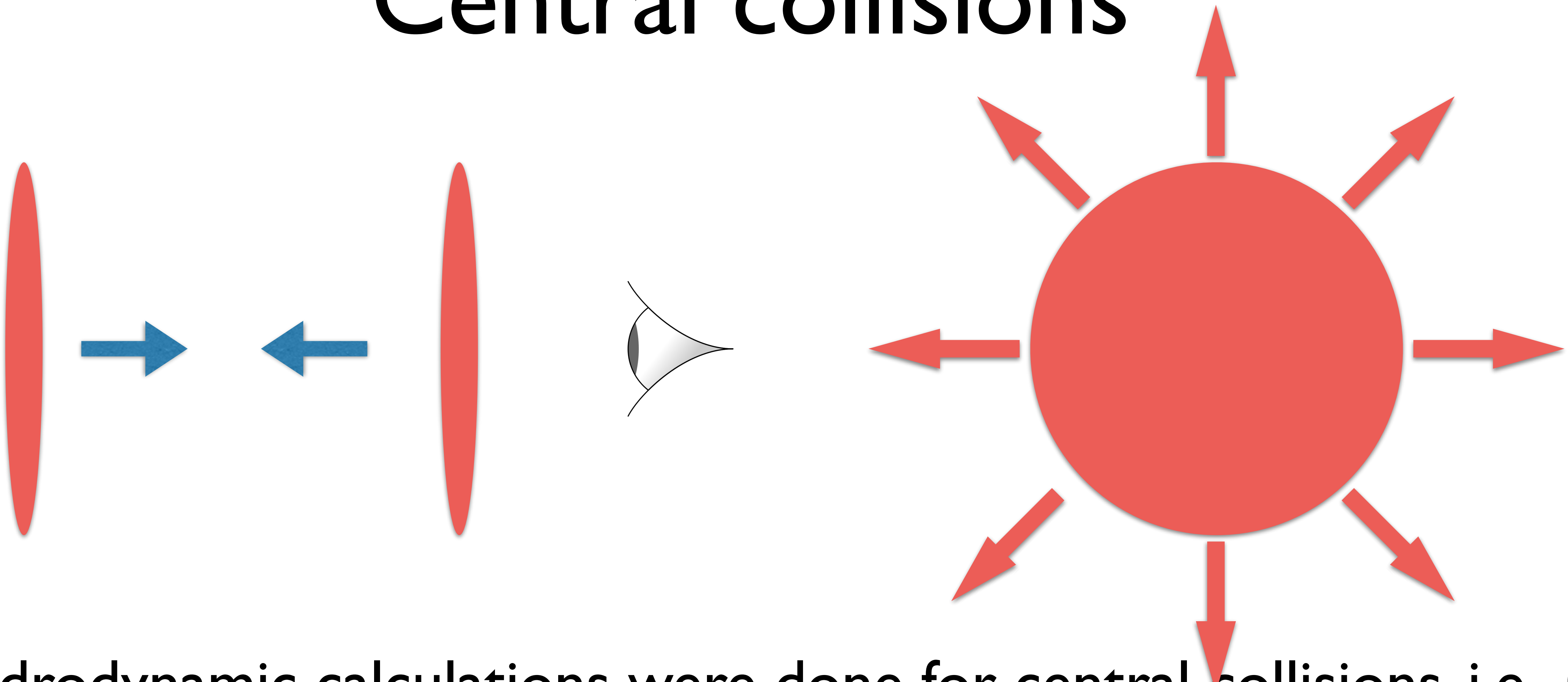
The thermalization hypothesis

1. First experimental data from fixed-target experiments at Brookhaven and CERN were compatible with a nucleus-nucleus collision being just a superposition of independent collisions between the protons and neutrons forming the nuclei.

2. But data could also be accounted for by « thermal » models.

3. I had a hard time believing that thermalization could be reached in such small systems. I tried to devise an analysis that could falsify the hydrodynamic description of, equivalently, the thermalization hypothesis.

Central collisions

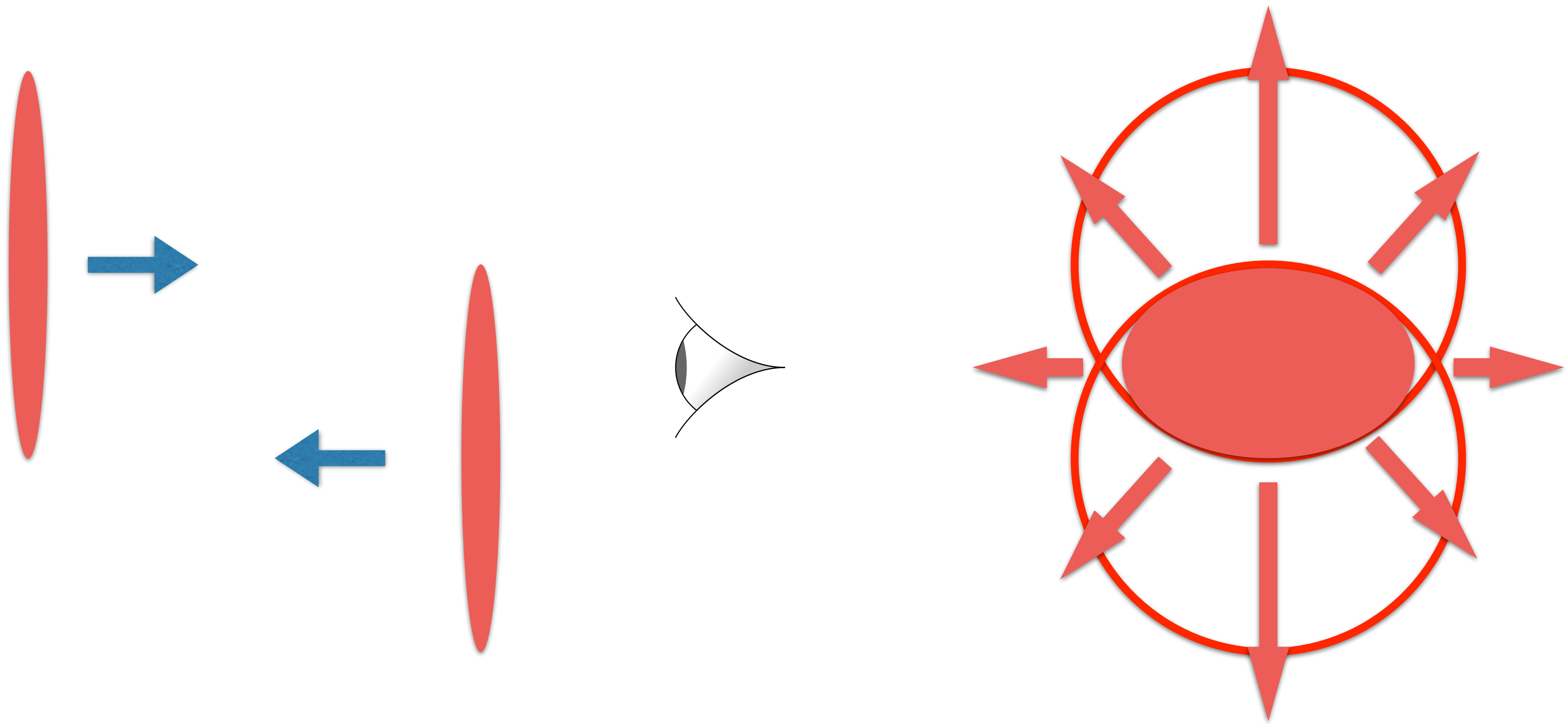


First hydrodynamic calculations were done for central collisions, i.e., 0 impact parameter, for two reasons:

1. The system is larger and more likely to thermalize
2. Hydrodynamic expansion has circular (azimuthal) symmetry which simplifies the calculation (1+1d: computers were slow back then).

However, first experimental data were mostly about how observables depend on impact parameter. It was important to model this.

Non-central collisions



In 1992, I predicted that if thermalization occurs, then pressure gradients break isotropy in non-central collisions due to the almond shape of the interaction region. Emission of particles is larger along the direction of impact parameter. This phenomenon was later called elliptic flow (also known as v_2).

VOLUME 86, NUMBER 3 PHYSICAL REVIEW LETTERS 15 JANUARY 2001

Elliptic Flow in Au + Au Collisions at $\sqrt{s_{NN}} = 130$ GeV

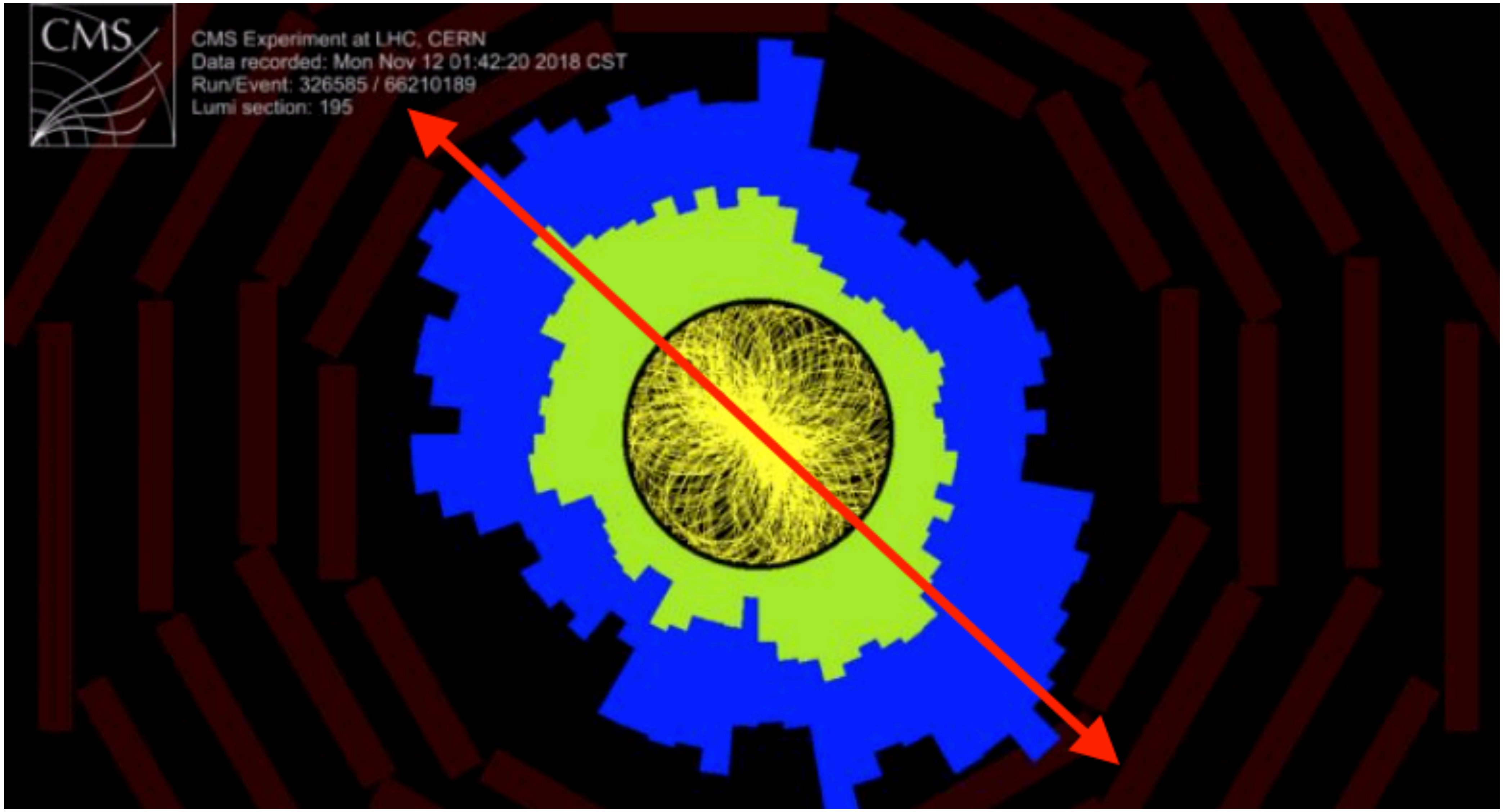
(STAR Collaboration)

Elliptic flow from nuclear collisions is a hadronic observable sensitive to the early stages of system evolution. We report first results on elliptic flow of charged particles at midrapidity in Au + Au collisions at $\sqrt{s_{NN}} = 130$ GeV using the STAR Time Projection Chamber at the Relativistic Heavy Ion Collider. The elliptic flow signal, v_2, averaged over transverse momentum, reaches values of about 6% for relatively peripheral collisions and decreases for the more central collisions. This can be interpreted as the observation of a higher degree of thermalization than at lower collision energies. Pseudorapidity and transverse momentum dependence of elliptic flow are also presented.

The discovery of elliptic flow was the first important physics result from the RHIC collider, obtained with the data collected on the very first day, in June 2000.

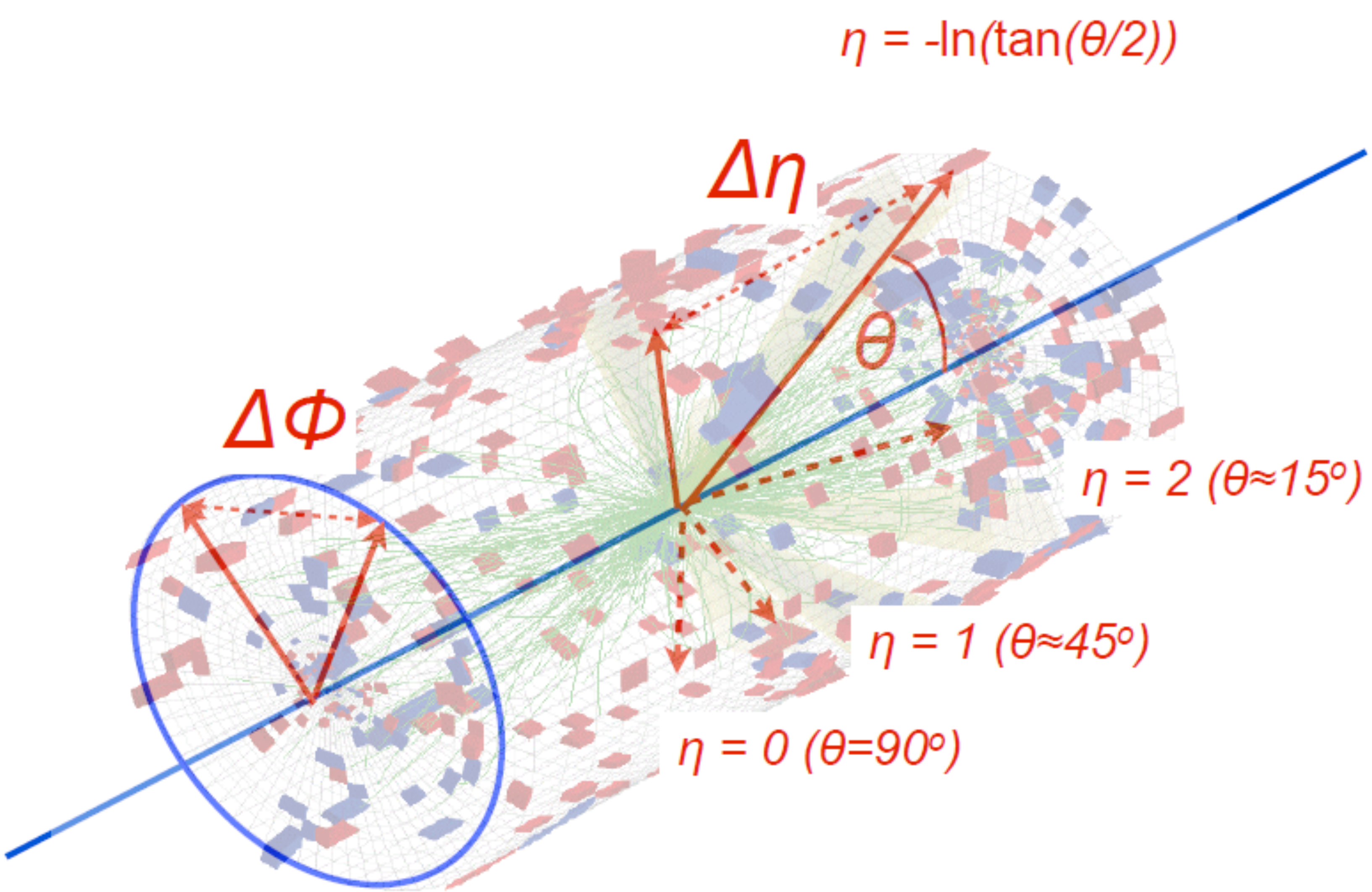
This discovery showed that some thermalization was achieved in nucleus-nucleus collisions. As a consequence, hydrodynamics soon became the standard tool for modeling the expansion of matter produced in nucleus-nucleus collisions.

Elliptic flow at the LHC



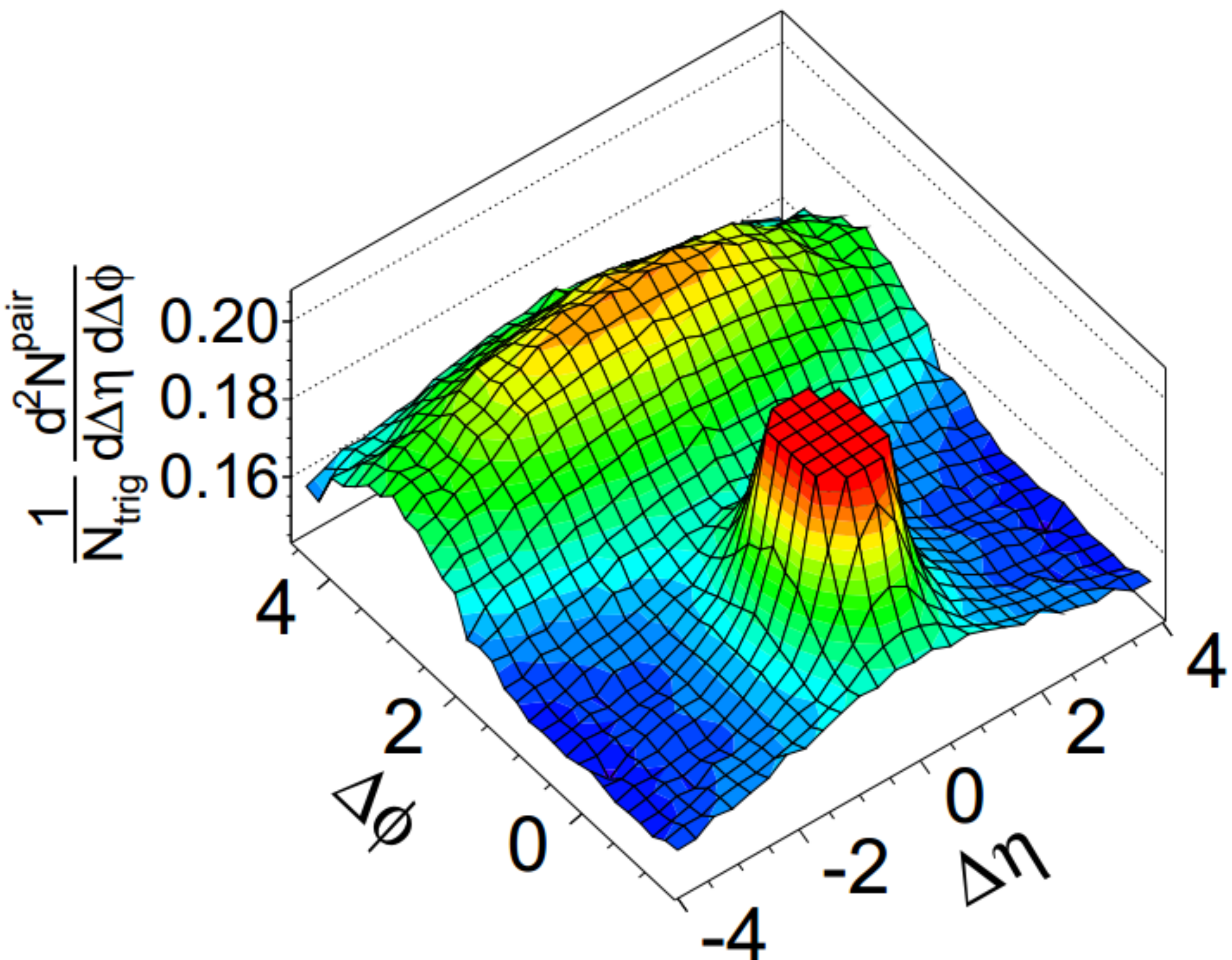
Elliptic flow is now easily seen « by eye » on event displays of non-central collisions: collective motion of particles parallel to the direction of impact parameter, tentatively indicated as an arrow.

Pair correlations



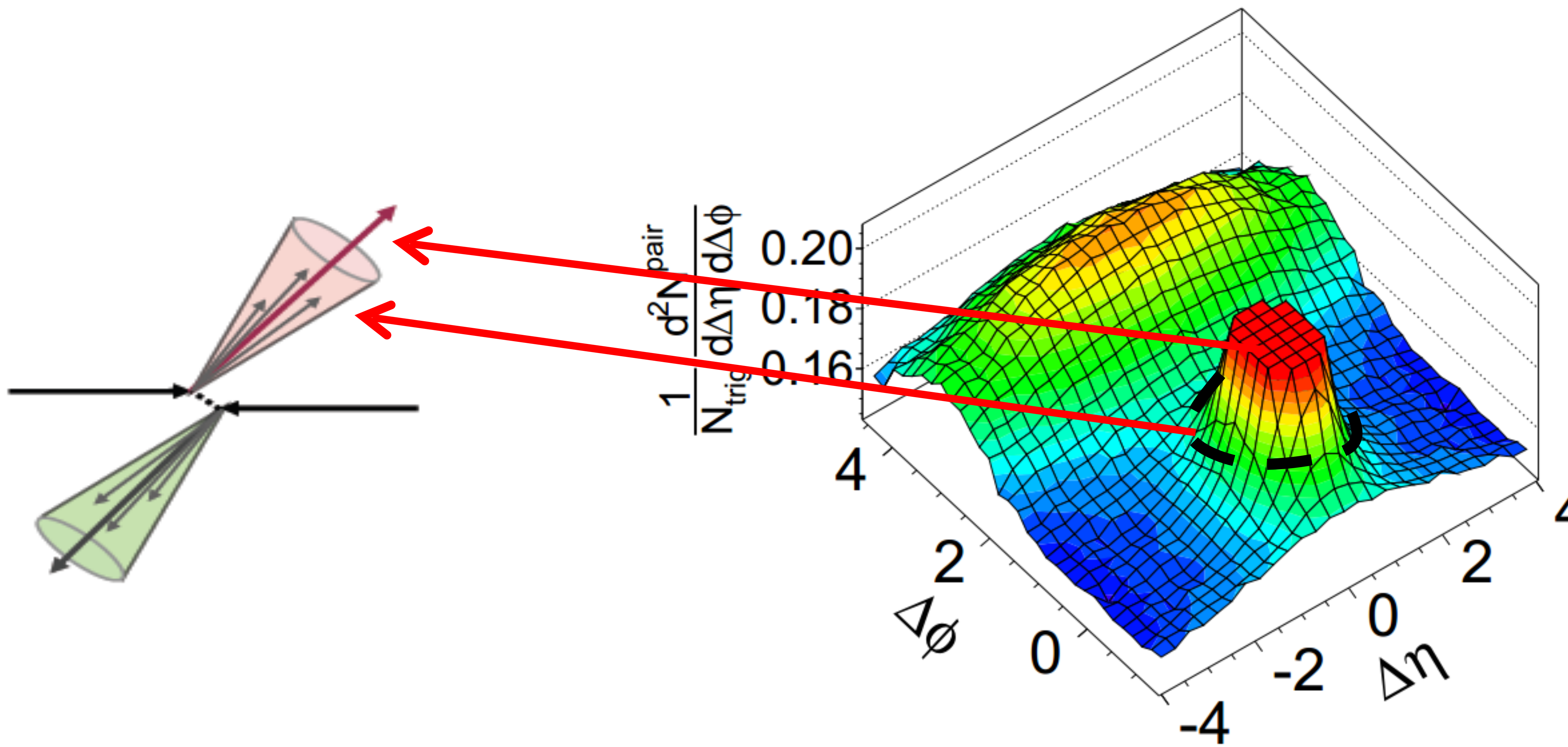
In every event, count pairs of particles as a function of relative azimuthal angle $\Delta\varphi=\varphi_1-\varphi_2$, and relative pseudorapidity: $\Delta\eta=\eta_1-\eta_2$ (~relative longitudinal velocity)

Correlations in proton-proton collisions



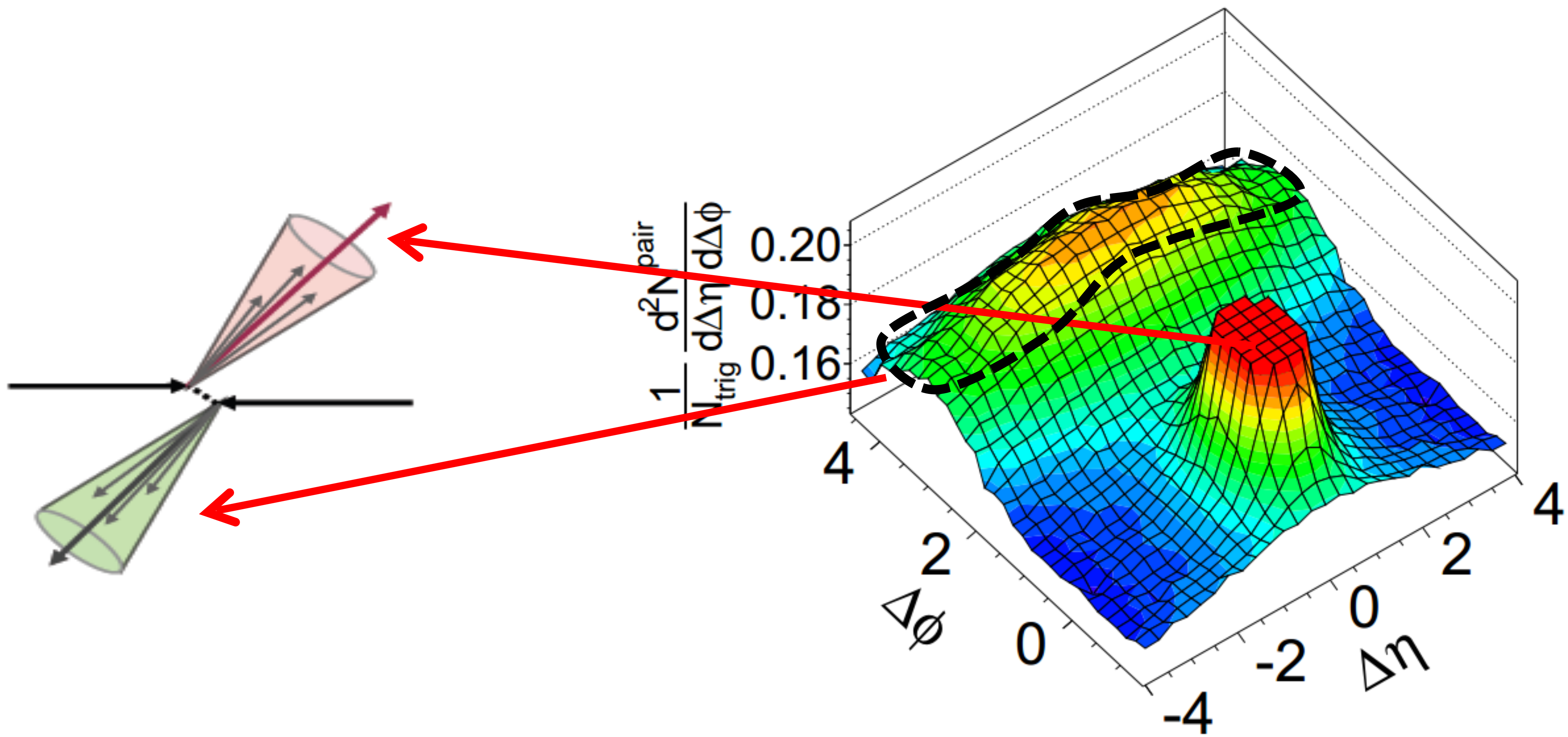
A complex structure, determined by elementary processes

Correlations in proton-proton collisions



Large positive correlation for collinear particles, typically coming from the same jet.

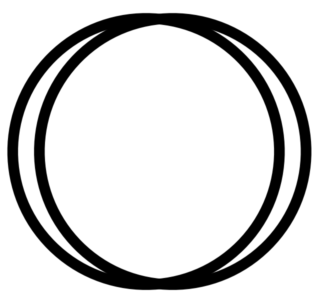
Correlations in proton-proton collisions

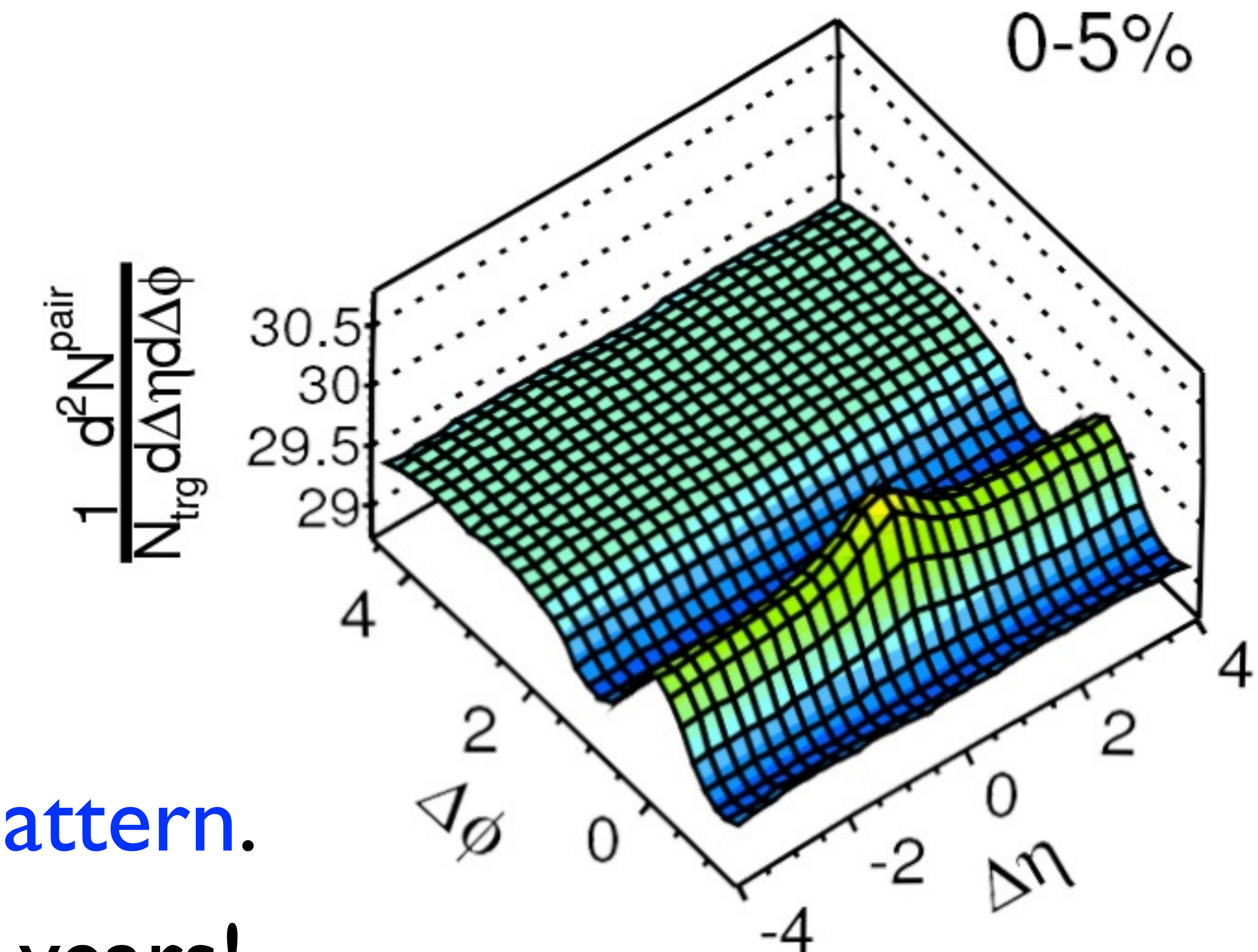


Positive away-side correlation ($\Delta\varphi \sim \pi$) corresponding to one particle in each jet

Correlations in Pb+Pb collisions

Central collisions





A simple, regular wave pattern.

Major puzzle for several years!
(~2005-2010)

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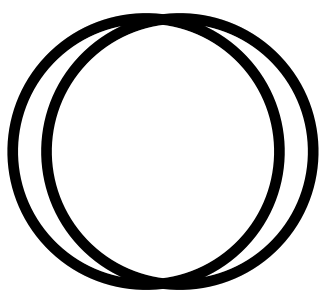
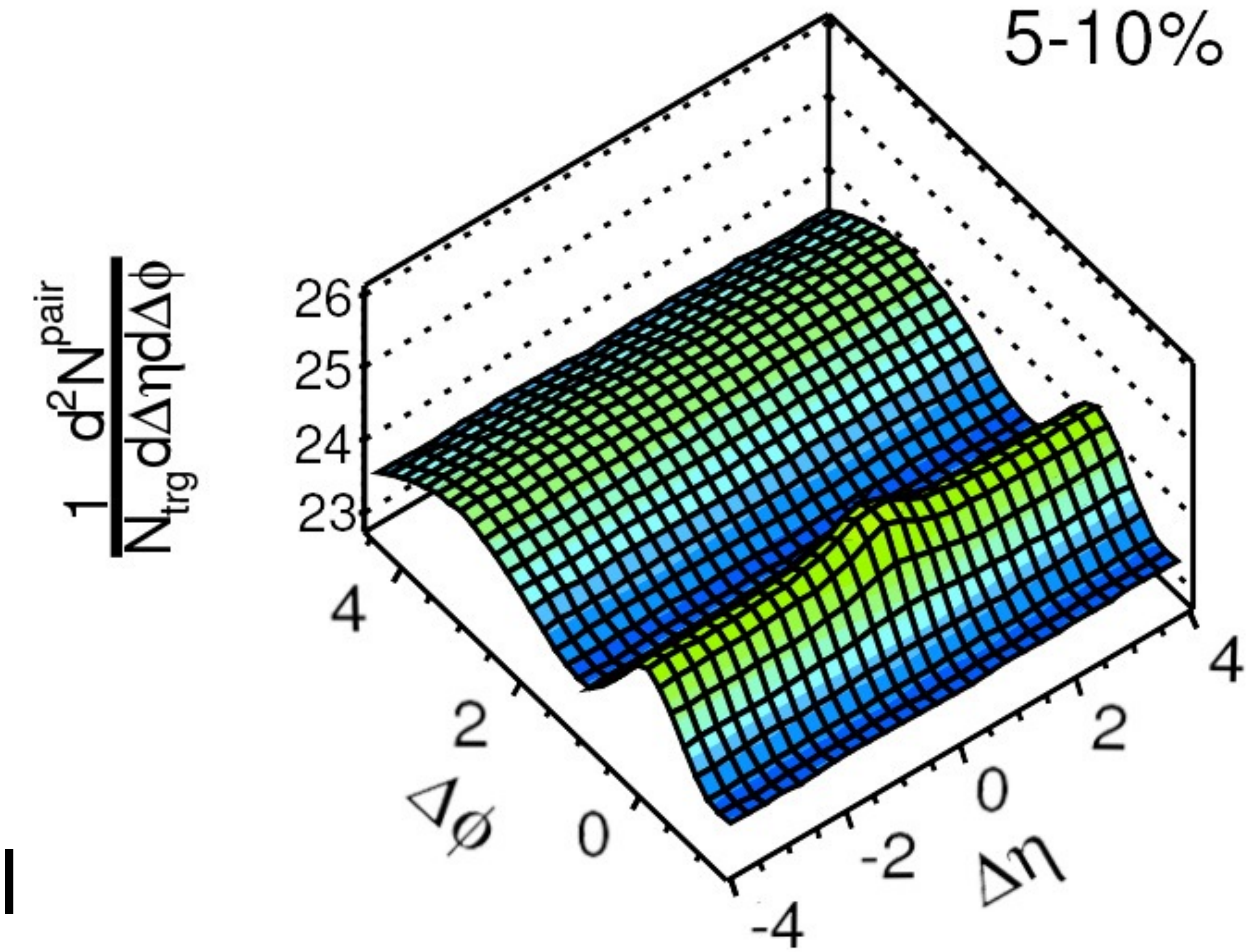
Correlations in Pb+Pb collisions





The wave is present at all centralities, but its shape evolves

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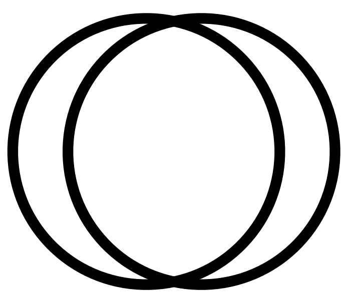
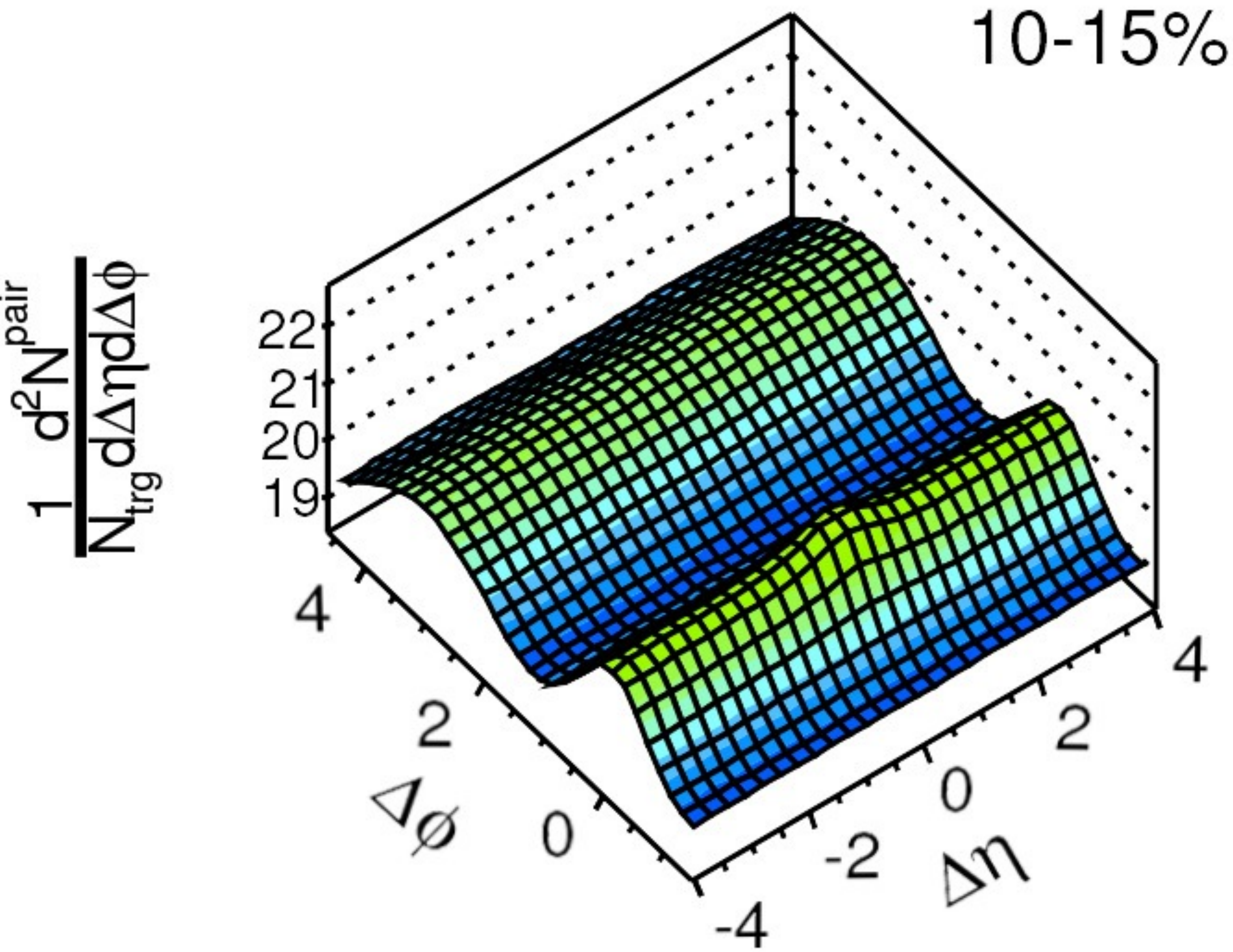
Correlations in Pb+Pb collisions





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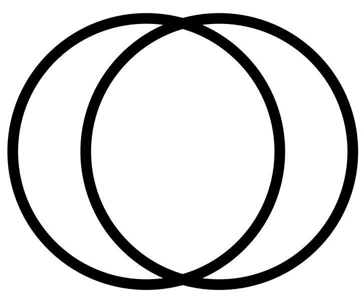

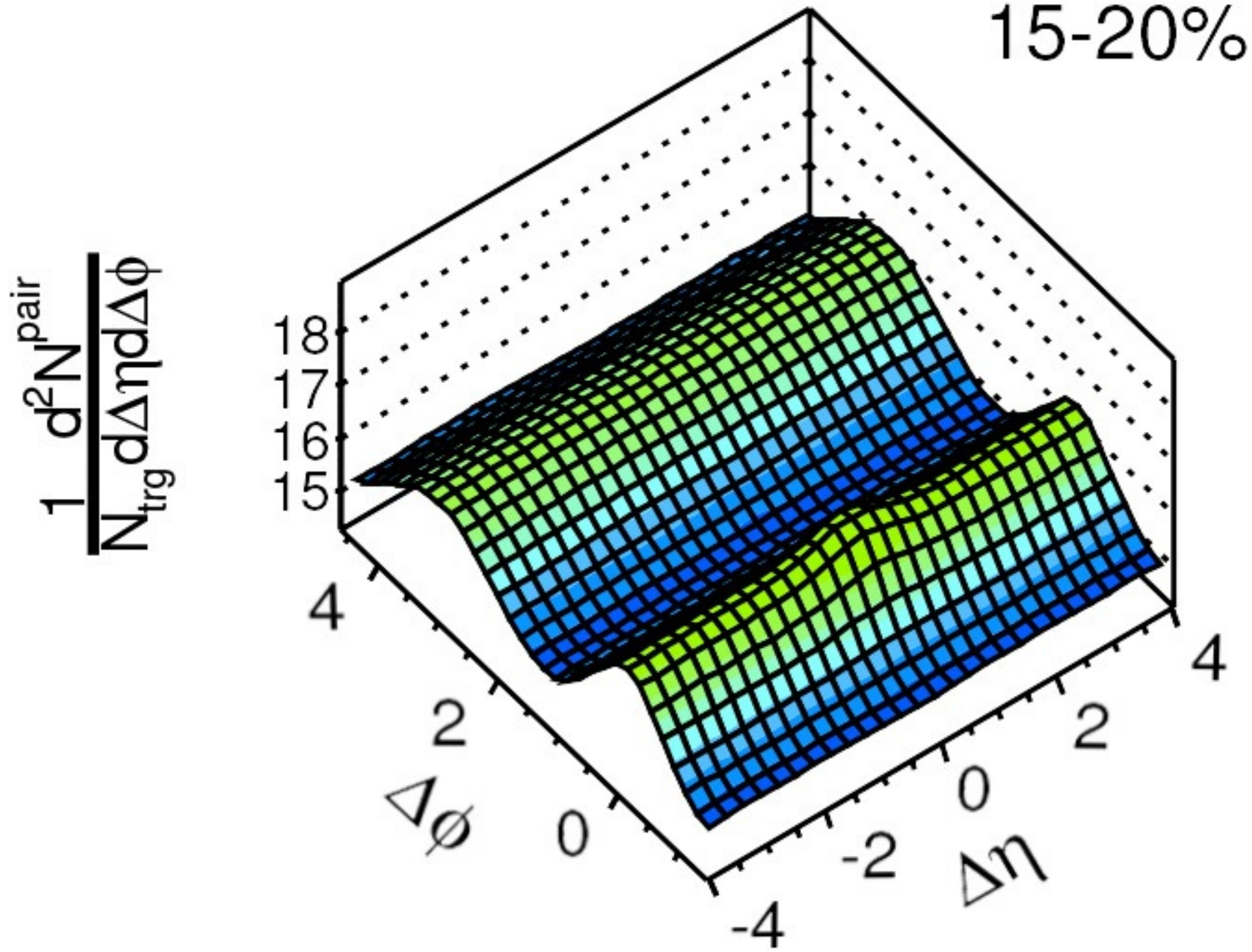

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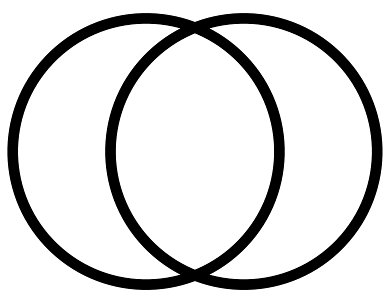

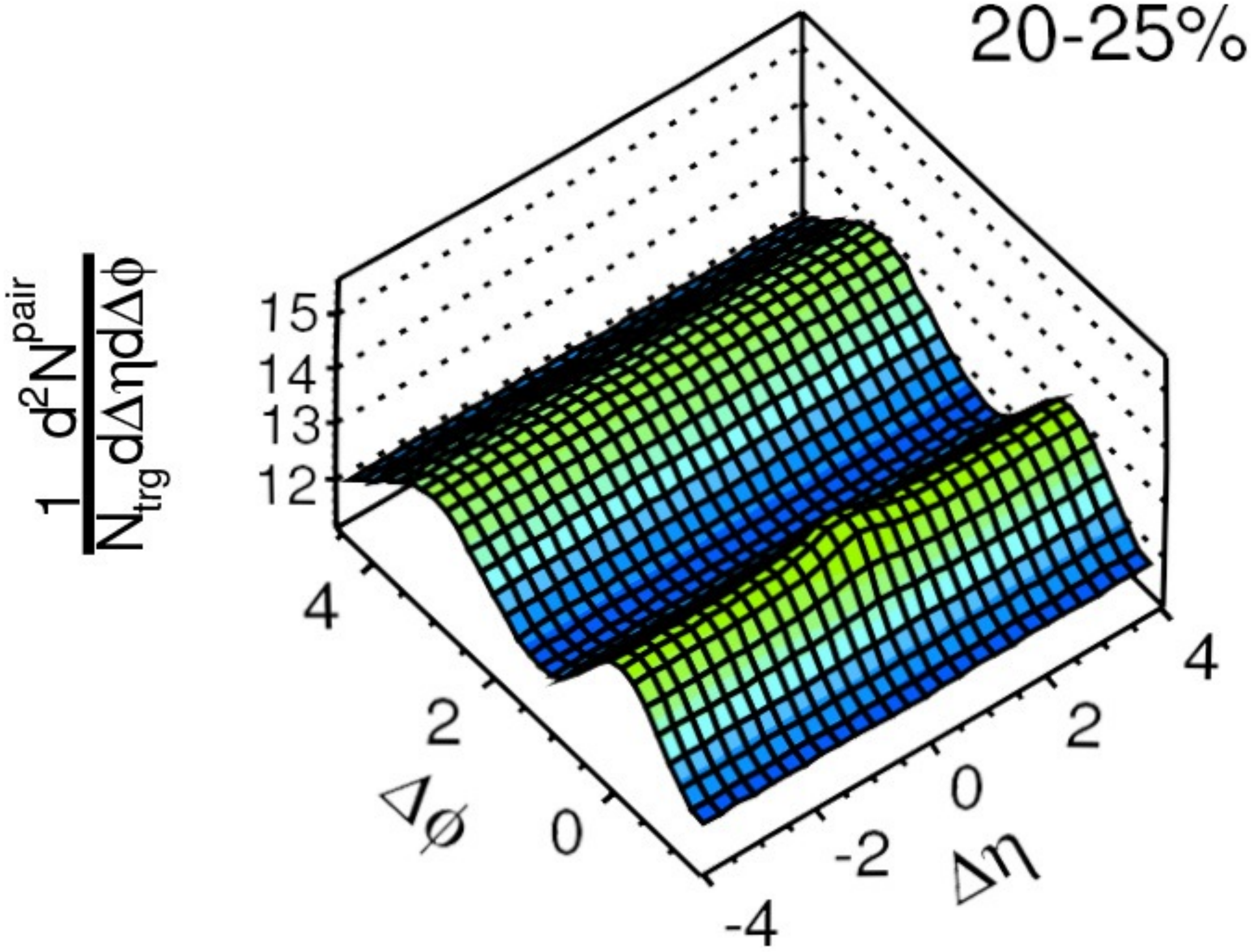

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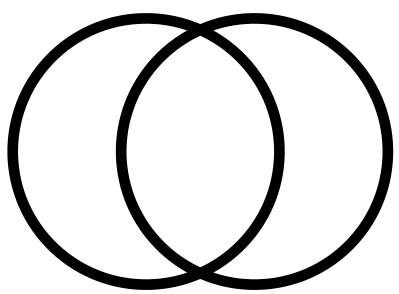
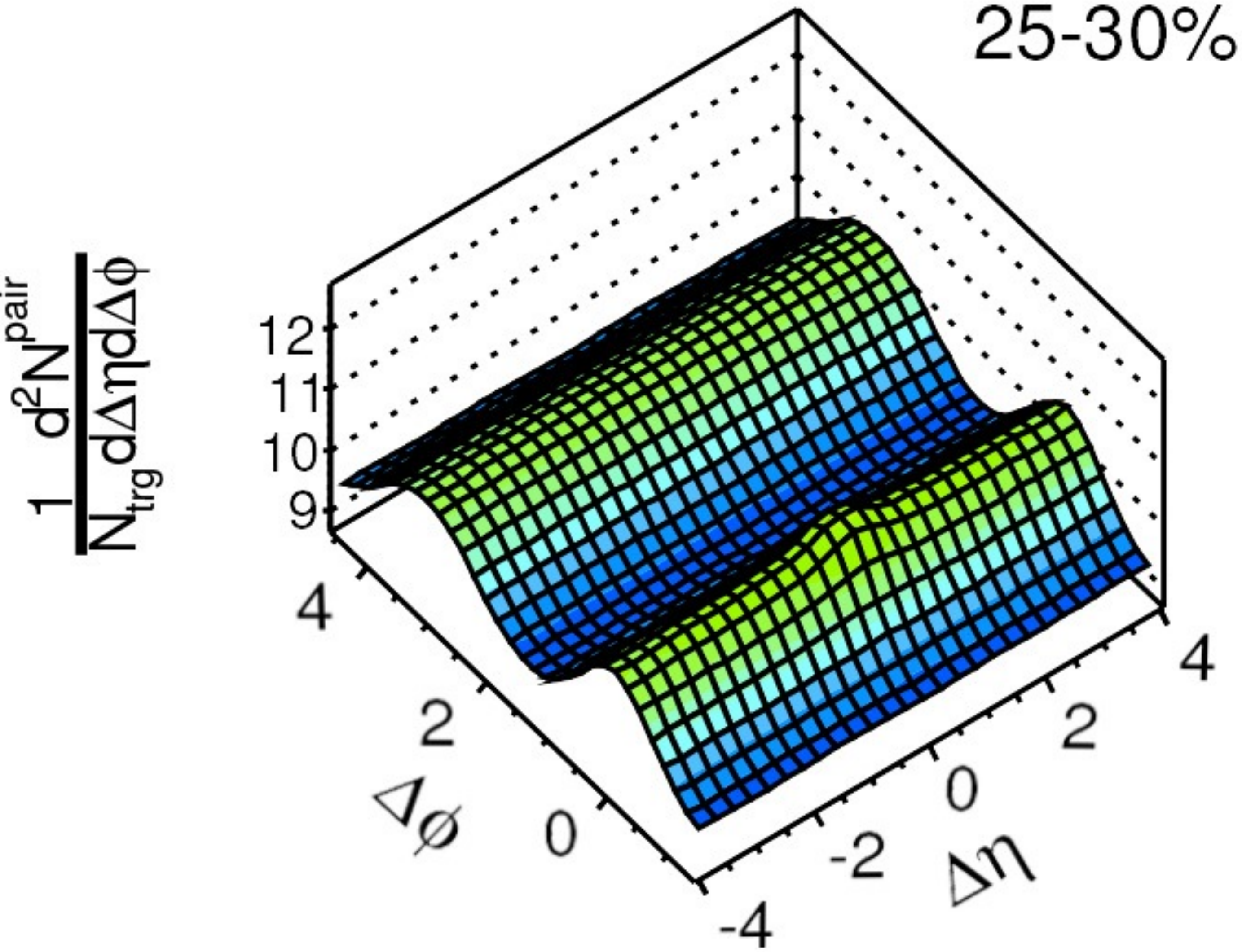
Correlations in Pb+Pb collisions





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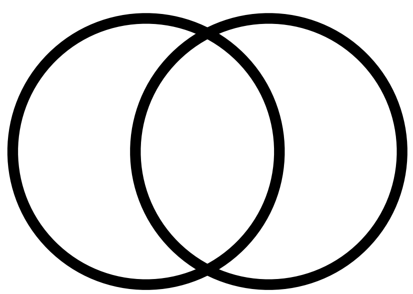

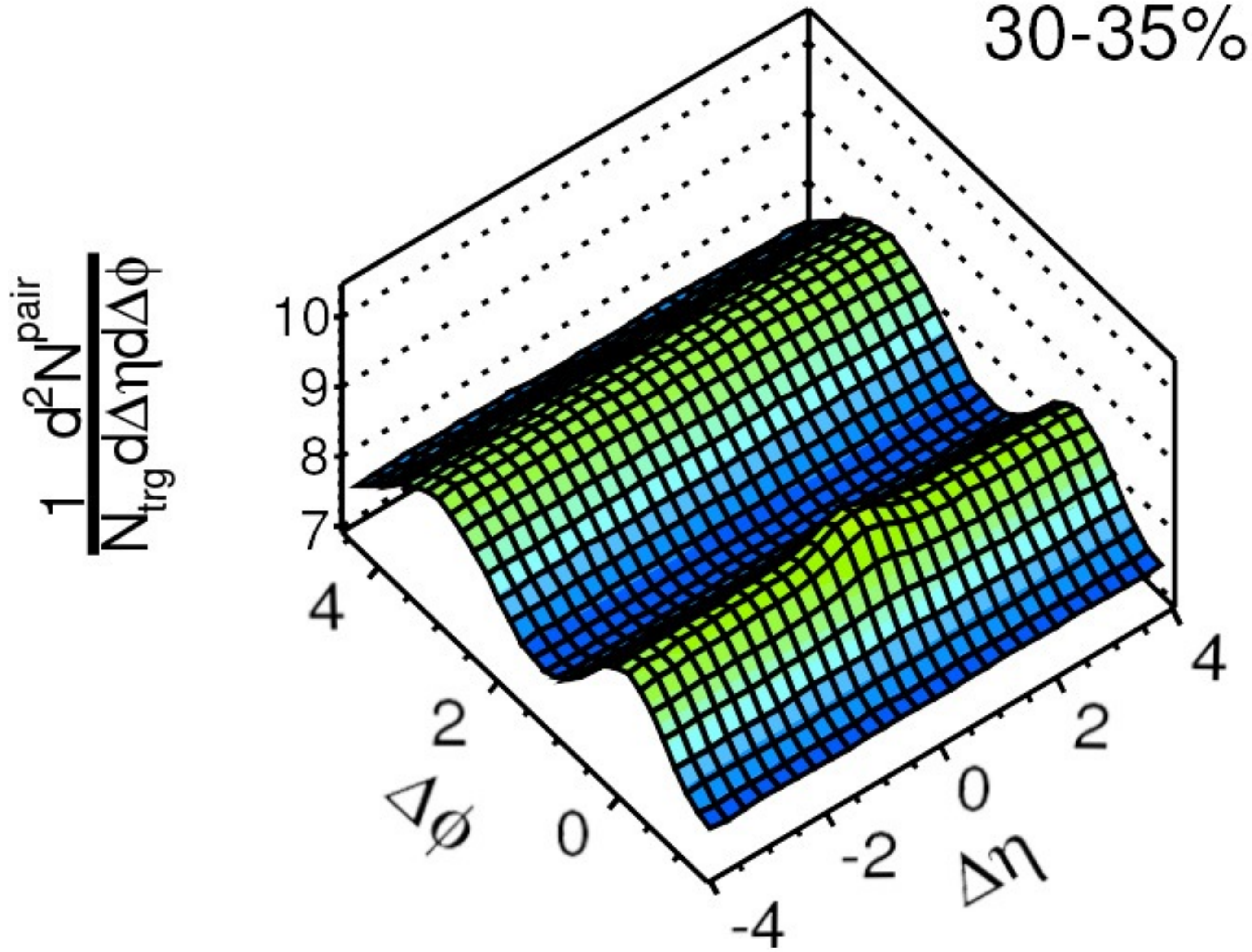

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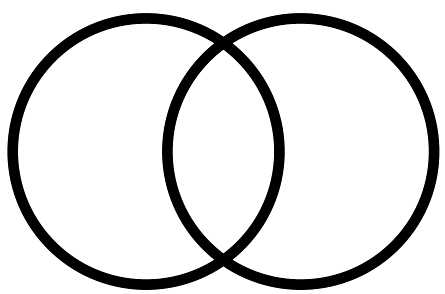
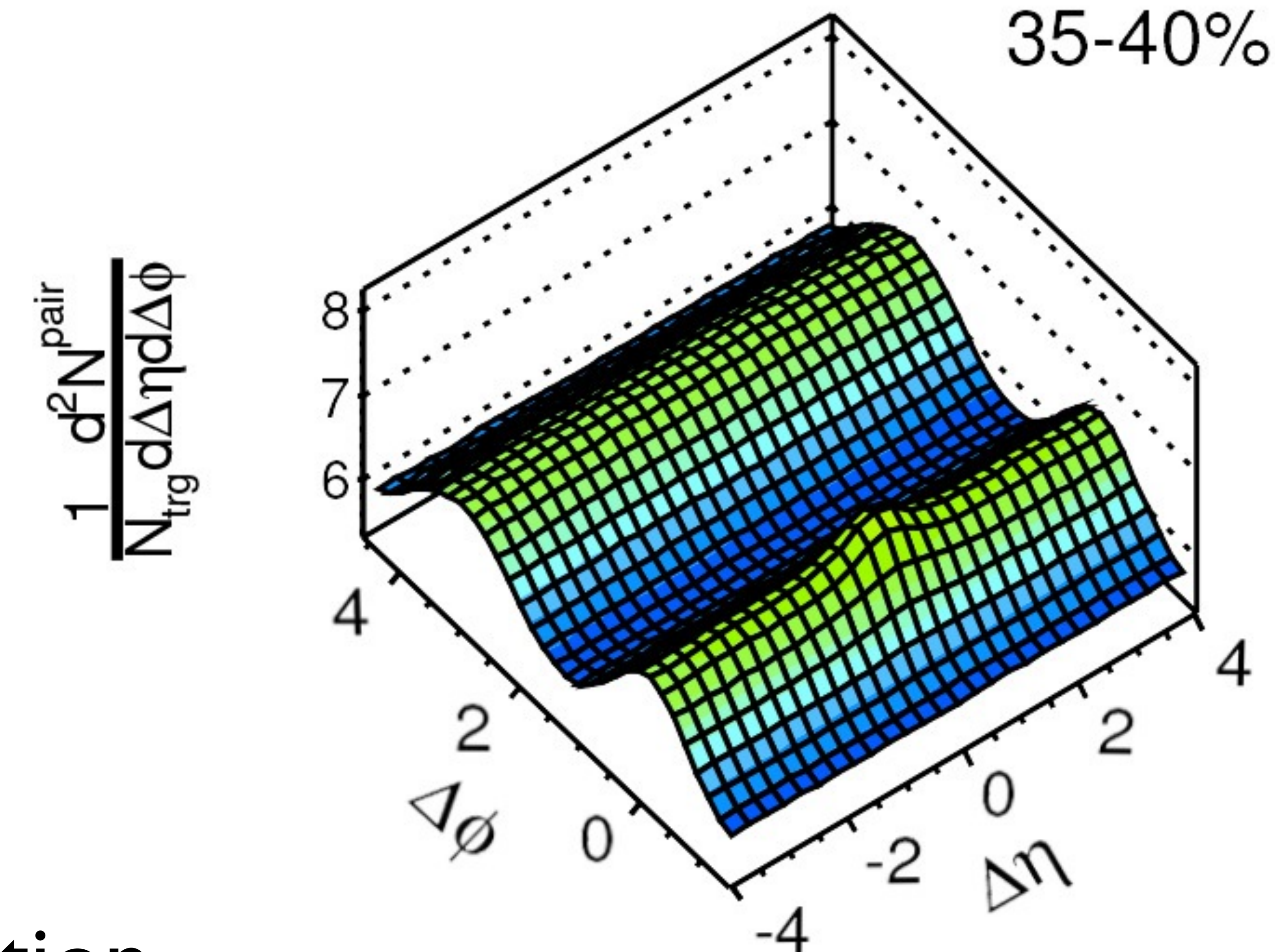




This was understood:

The cos(2Δφ) modulation originates from elliptic flow

CMS 1201.3158

Explaining the ridge

Alver Roland 1003.0194 Luzum 1107.0592

The ridge is naturally explained by hydrodynamics.

1. We have seen that in a hydrodynamic event, the momentum distribution depends on azimuthal angle φ, not on rapidity, and can be *any function* of φ. Write as Fourier series:

$$dN/d\varphi d\eta = f(\varphi) = \sum_n V_n \exp(-in\varphi)$$

$f(\varphi)$ is the φ distribution in a hydrodynamic event.
It fluctuates event-to-event due to different initial conditions.

2. No correlation between particles. They are emitted independently from the freeze-out hypersurface.

Independent particle emission

The distribution of the relative angle $\Delta\varphi=\varphi_2-\varphi_1$ is

$dN_{pair}/d\Delta\varphi = \int f(\varphi_2-\Delta\varphi)\, f(\varphi_2)\, d\varphi_2$

i.e. the convolution of $f(\varphi)$ with $f(-\varphi)$.

Fourier transform of convolution = prod. of Fourier transforms, i.e. V_n for $f(\varphi)$, V_n^* for $f(-\varphi)$:

$$dN_{pair}/d\Delta\varphi=\sum_n |V_n|^2 \exp(-in\Delta\varphi)$$
$$=\sum_n |V_n|^2 \cos(n\Delta\varphi)$$

All Fourier coefficients > 0 !

Independent particle emission

A very simple yet predictive model.

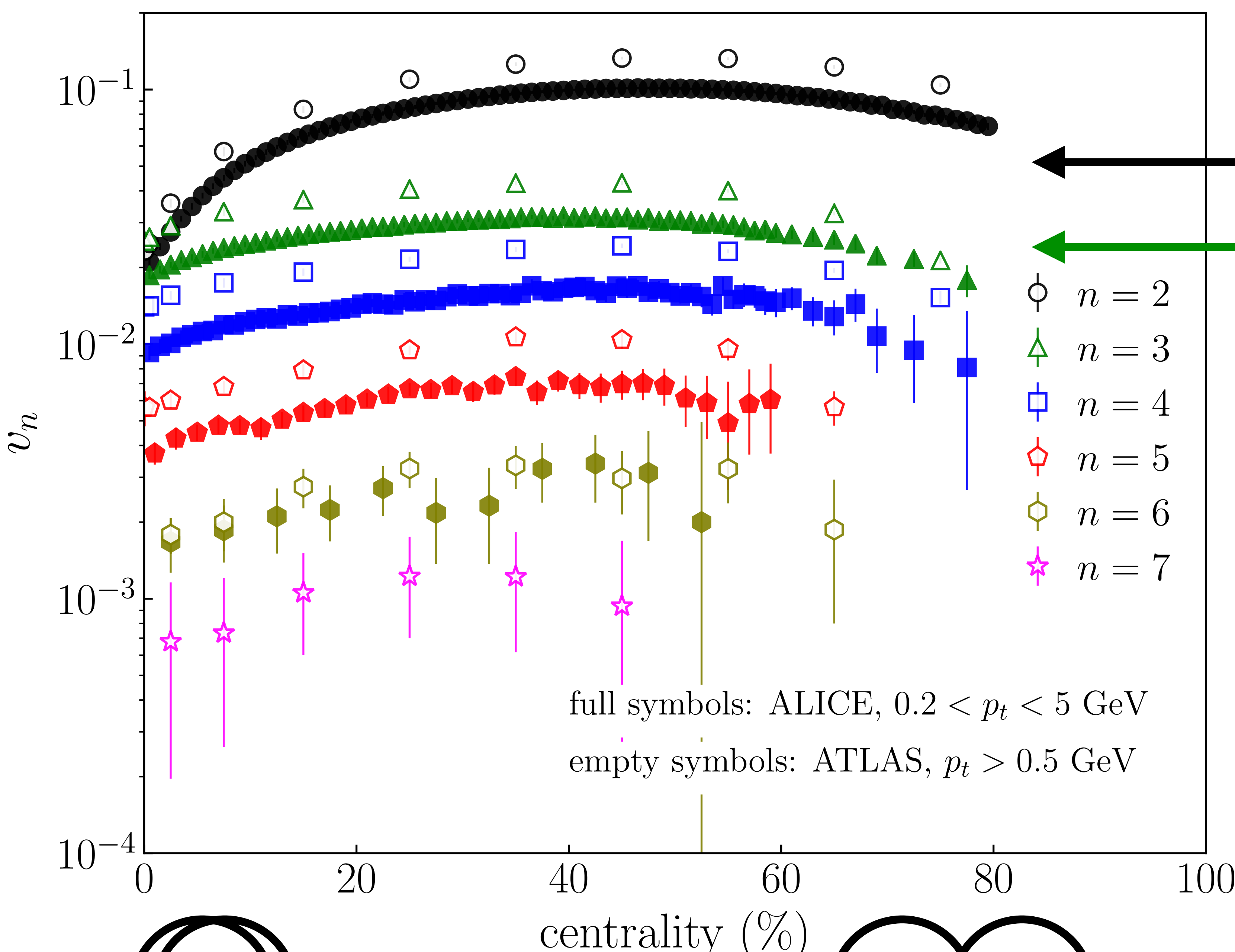
- Naturally explains the regular structure seen in data.
- Predicts that the absolute maximum of the pair distribution is at $\Delta\varphi=0$ (*near-side ridge*).
  This is the most difficult feature, that other models typically don't reproduce.

Measuring anisotropic flow

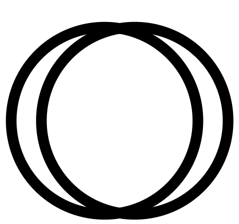
- One does not measure V_n in a single collision event.
- The pair correlation gives $|V_n|^2$ averaged over collision events = average value of $\cos(n\Delta\varphi)$
- This is the simplest, most common measure of anisotropic flow, denoted by $v_n\{2\}$:

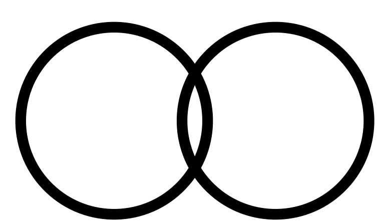
$$v_n\{2\} \equiv \langle \cos(n\Delta\varphi) \rangle^{1/2}$$

rms v_n in Pb+Pb collisions at the LHC



Elliptic and Triangular dominate

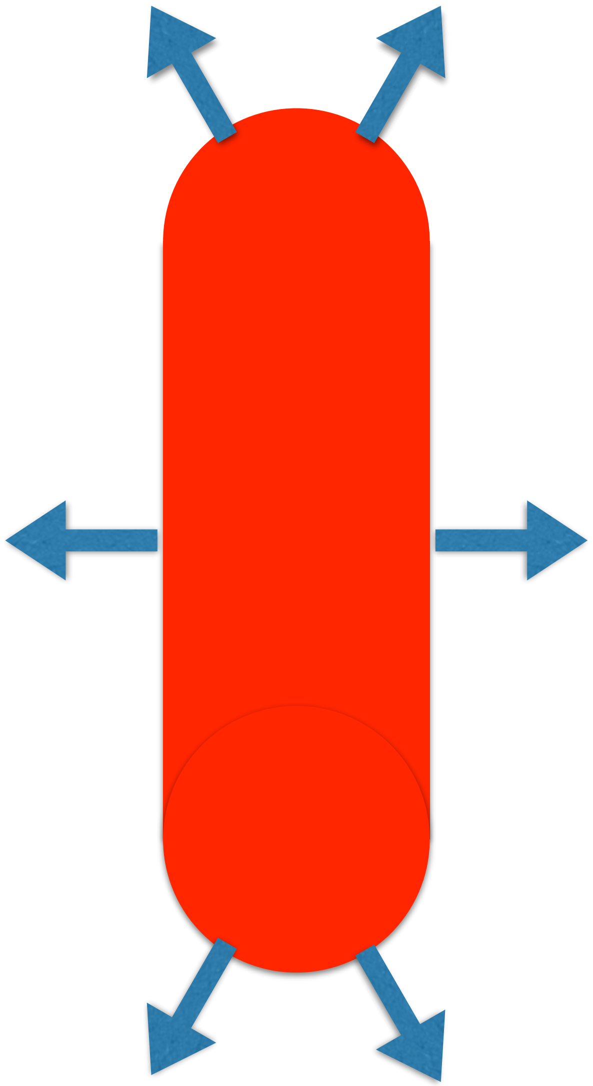




Part 3:

Scale invariance of ideal hydrodynamics

Scale invariance



Equations of hydrodynamics

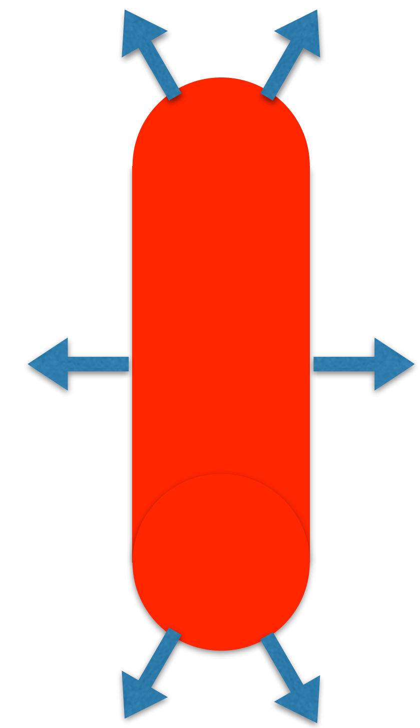
$de/dt=-(e+P)\, \nabla . \mathbf{v}$

$(e+P)d\mathbf{v}/dt=-\nabla P$

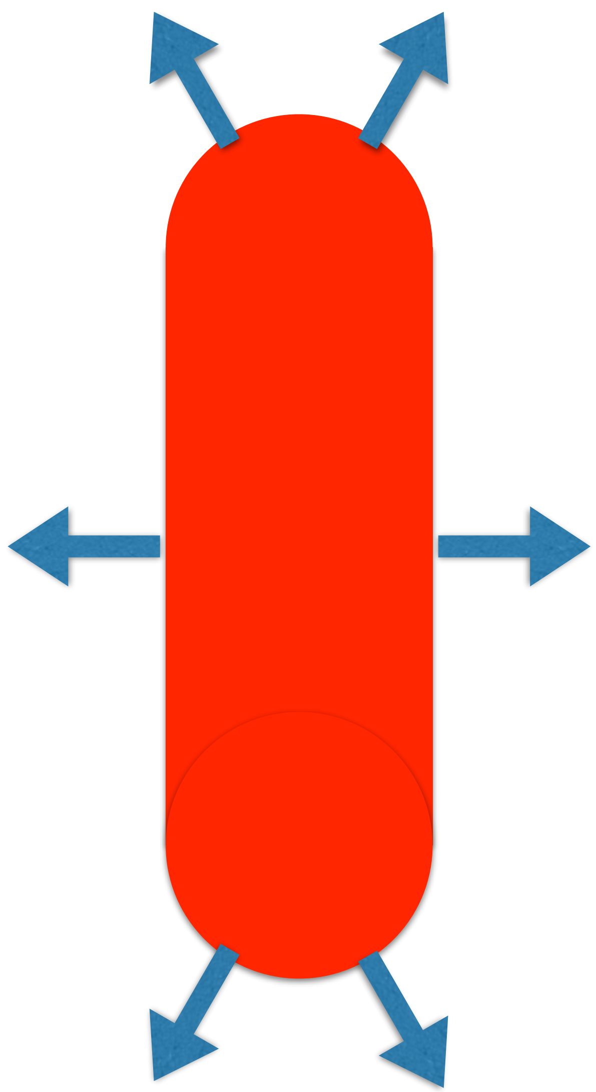
linear in space time derivatives

if v($\mathbf{x}$,t), e($\mathbf{x}$,t) is a solution,

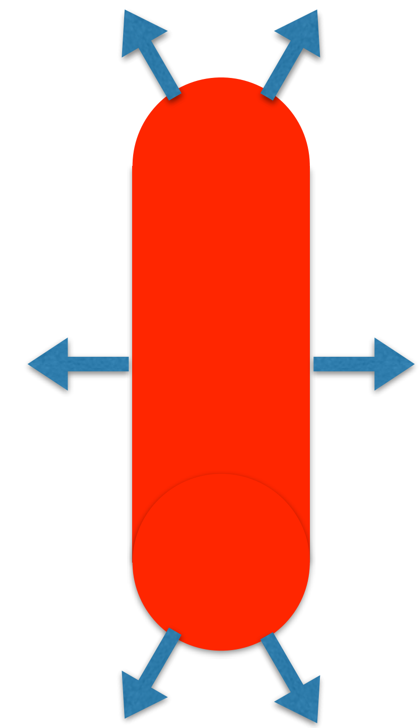
v($\lambda\mathbf{x}$,λt), e($\lambda\mathbf{x}$,λt) is also a solution for any λ.



Scale invariance



Exact same momentum distribution of outgoing hadrons, up to overall normalization λ^{-3}

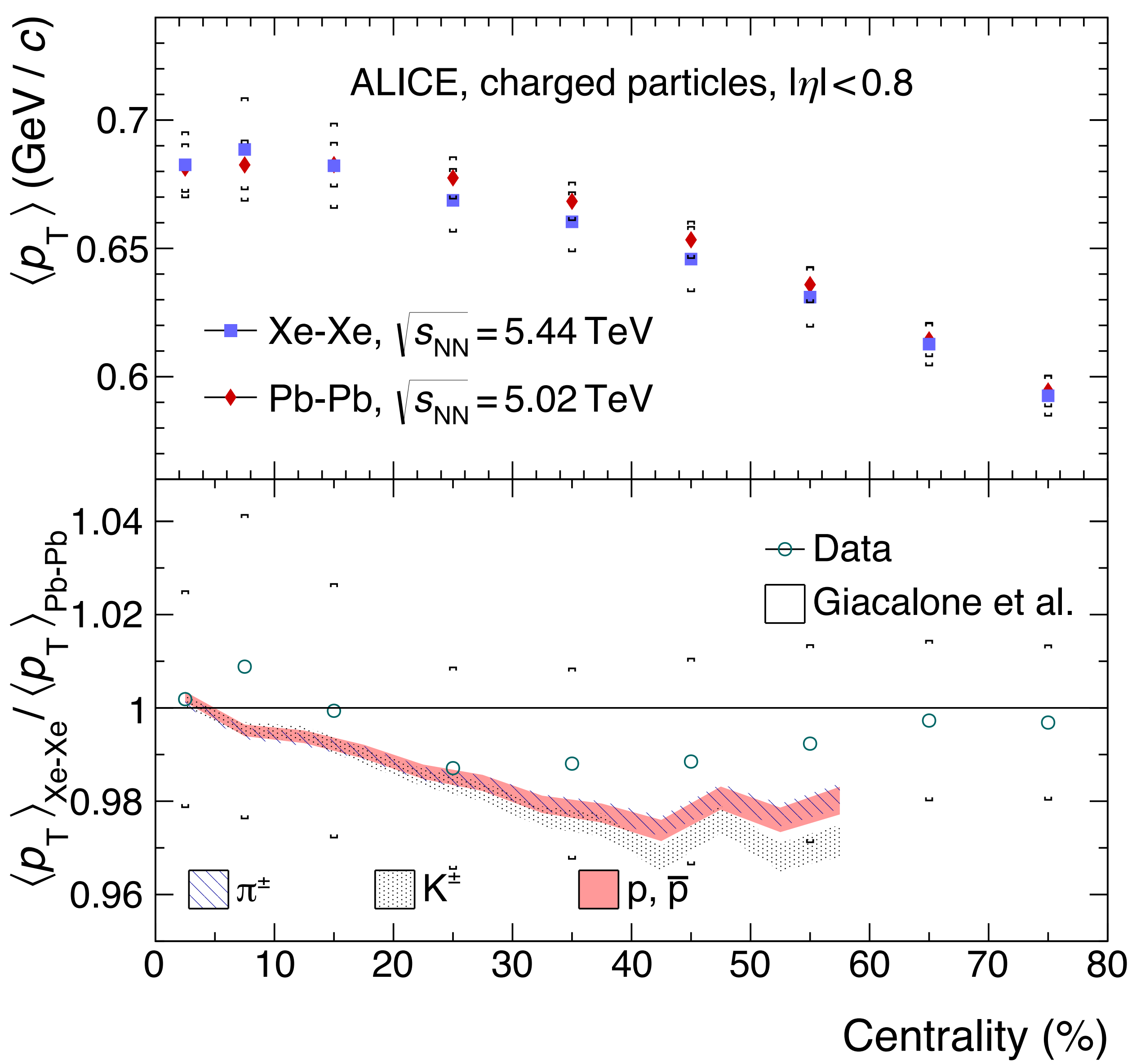


Changing system size (Pb+Pb versus Xe+Xe) = global rescaling

- Nuclear volume roughly proportional to mass number A
- Particle multiplicity in A+A collision: also proportional to A.
- Put into a hydro calculation, it implies that $\langle p_t \rangle$ is independent of A. This is is a robust prediction of hydrodynamics. If it was not seen in data, hydro would be ruled out!

Giacalone Noronha-Hostler Luzum Ollitrault 1711.08499

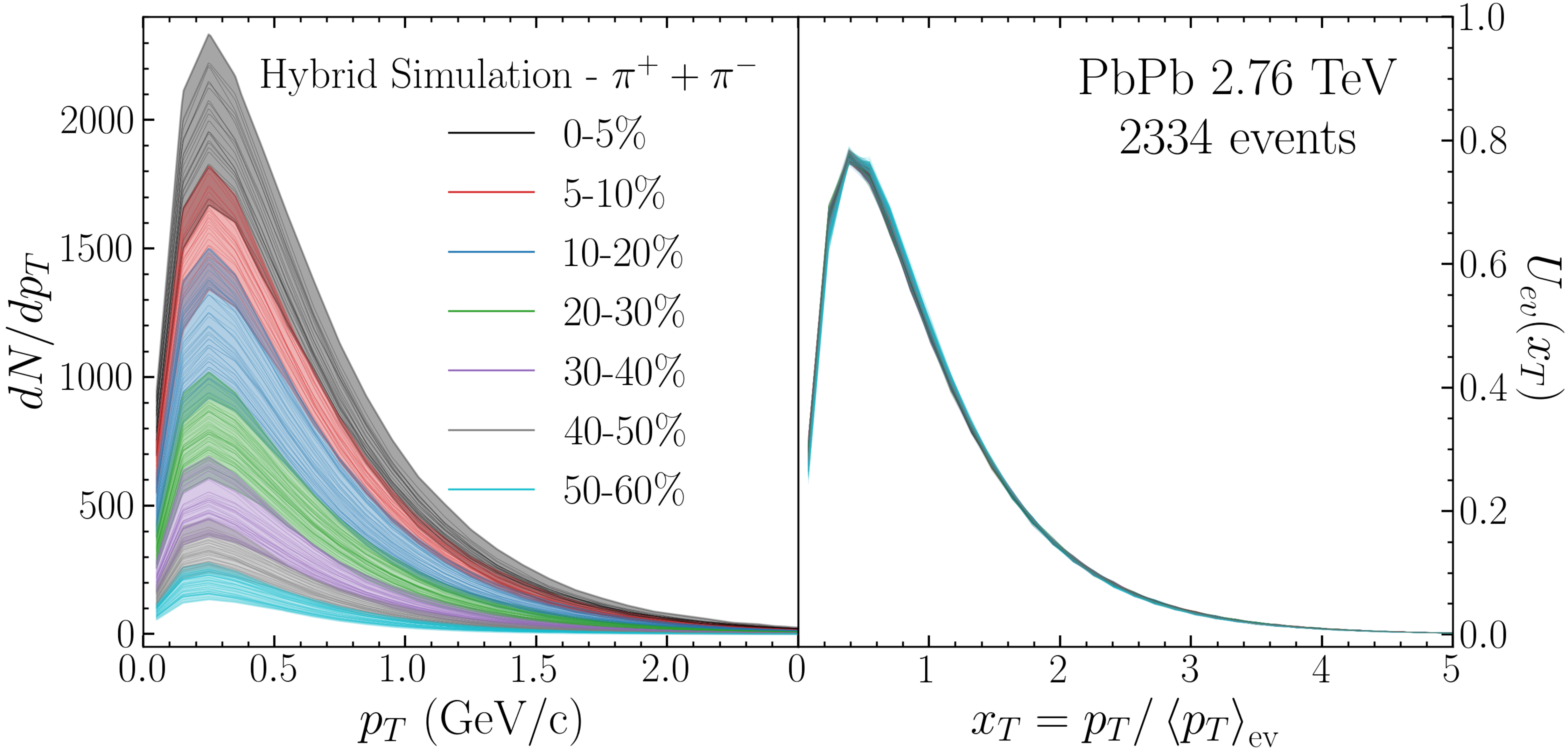
Scale invariance seen in data



ALICE collaboration 1805.04399

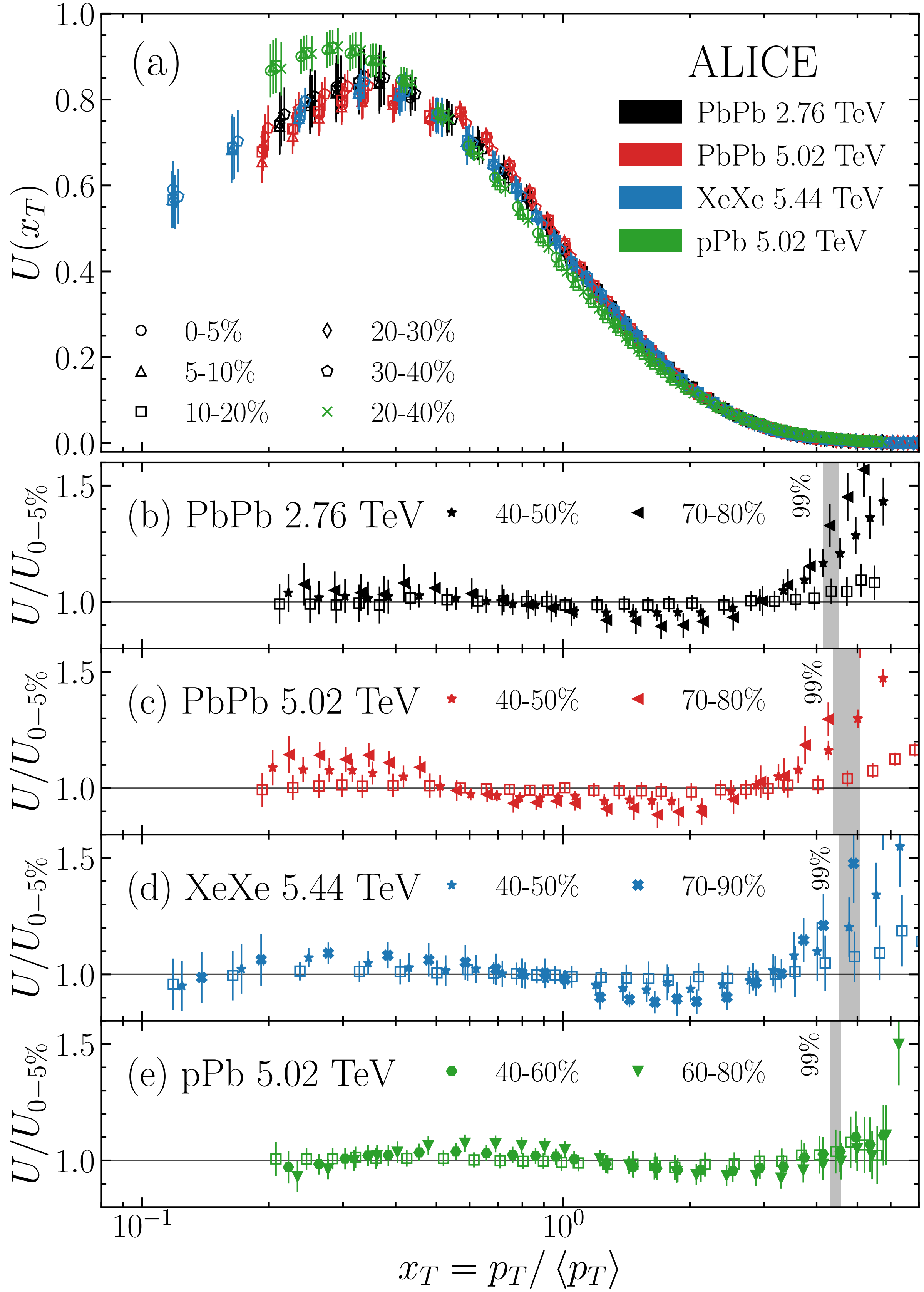
- Hydro prediction that <p_t> in Xe+Xe and Pb+Pb differ by less than 2% confirmed by experiment
- Note also that <p_t> depends very mildly on centrality.
- A change in system size or centrality, at a given $\sqrt{s}$, amounts to a rescaling of space-time coordinates, at the same temperature.

Scale invariance in p_t spectra



ExTrEMe collaboration 2406.15208

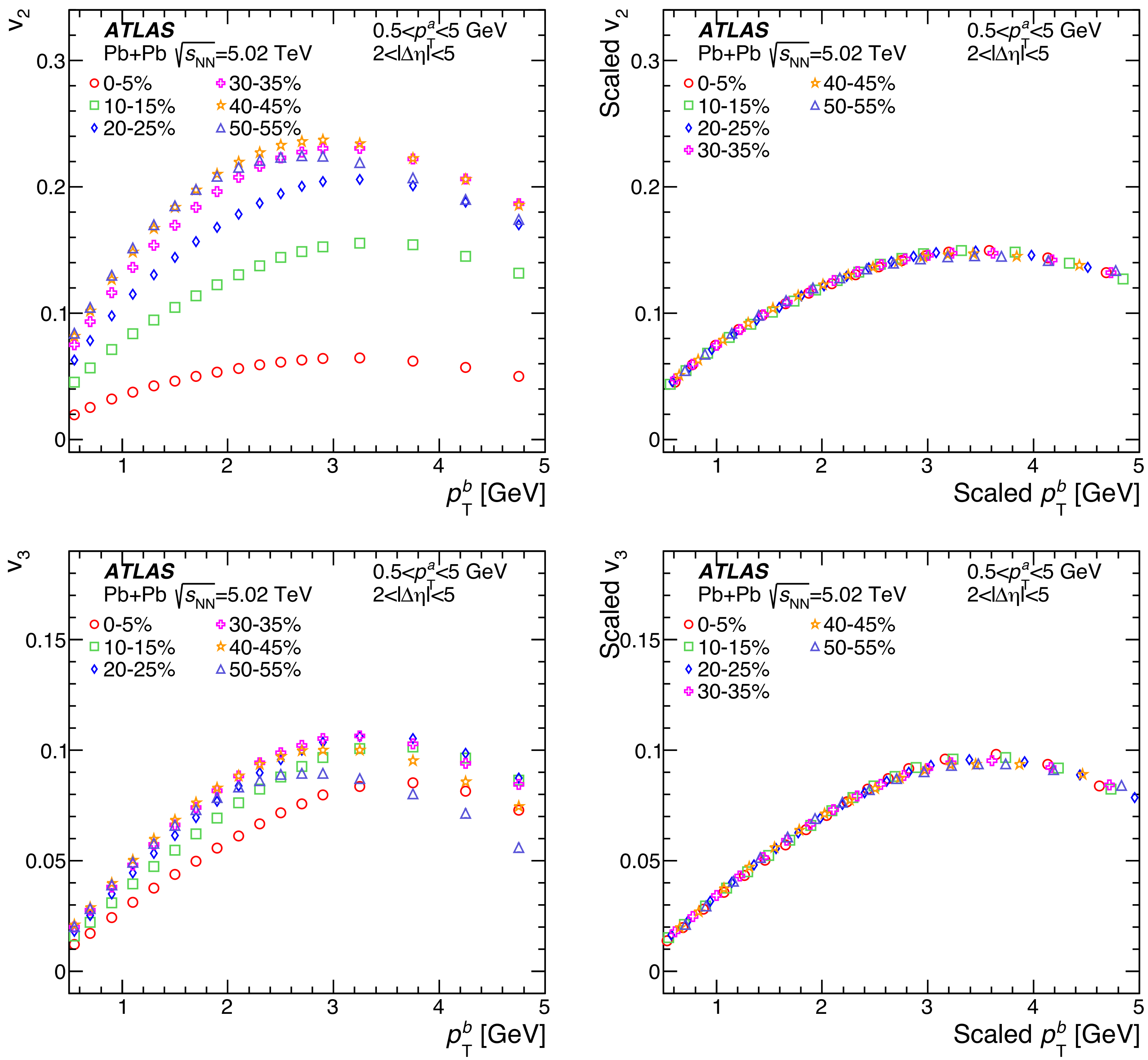
Scale invariance in p_t spectra



This prediction of hydro is again confirmed by experiment:
spectra in Pb+Pb and Xe+Xe collisions at all centralities are essentially identical.

Note that they differ in p+Pb collisions

Scale invariance of anisotropic flow



A similar phenomenon: $v_n(p_t)$ is the product of an initial anisotropy, times a hydrodynamic response which is independent of centrality

ATLAS collaboration 1805.04399

Part 4:

Viscous corrections

Finite-size corrections to ideal hydro

Local thermal equilibrium = first-order approximation.

The momentum equation

$(e+P)dv_k/dt=-\partial_k P$

can be more generally written as

$(e+P)dv_k/dt=-\partial_j P_{jk}$, where P_{jk} is the pressure tensor.

- Ideal hydro: $P_{jk} = P\ \delta_{jk}$
- Viscous corrections:
  Additional contributions to P_{jk}, proportional to gradients of the fluid velocity $\partial_j v_k$

Bulk and shear viscosities

P_{jk} is symmetric. Any symmetric tensor can be decomposed into an isotropic part, and a traceless symmetric part.

$$P_{jk} = (P - \varsigma\, \partial_i v_i)\, \delta_{jk} - \eta\, (\partial_j v_k + \partial_k v_j - (2/3)\delta_{jk} \partial_i v_i)$$

coefficients: bulk viscosity ς and shear viscosity η.
Like pressure P, ς and η are functions of temperature.
Unlike pressure, they are not yet calculated in lattice QCD.

Exercise 5: show that, if $\partial_i v_i = 0$, the momentum equation becomes $(e+P)d\mathbf{v}/dt = -\nabla P + \eta\, \Delta \mathbf{v}$

(Usual Navier-Stokes equation)

Dimensional analysis

$(e+P)d\mathbf{v}/dt = -\nabla P + \eta\, \Delta \mathbf{v}$

if L is a typical space-time scale of the fluid,

the relative magnitude of viscous correction
(Reynolds number)$^{-1}$

$= (1/L)\ \ \eta/(e+P)$ thermodynamic identity: $e+P=Ts$

$= (1/LT)\ \ \eta/s$

$T \sim 200$ MeV ~ 1 fm^{-1}, $L \sim$ few fm

Global theory-to-data comparisons favor $\eta/s<0.2$:
small correction as expected

Nijs van der Schee et al 2010.15130

JETSCAPE coll. 2011.01430

Quantitative estimates

Sensitivity to viscosity is *increased* for v_n by an additional factor $\sim n^2$ from the Δ operator in Navier-Stokes eqs.

Gubser Yarom 1012.1314

Teaney Yan 1206.1905

Confirmed by numerical simulations of central Pb+Pb collisions at the LHC which give a relative viscous suppression

= 1.3 η/s for elliptic flow

= 2.3 η/s for triangular flow

<1 : Ideal fluid still in the ballpark

Gardim & JYO 2207.08692