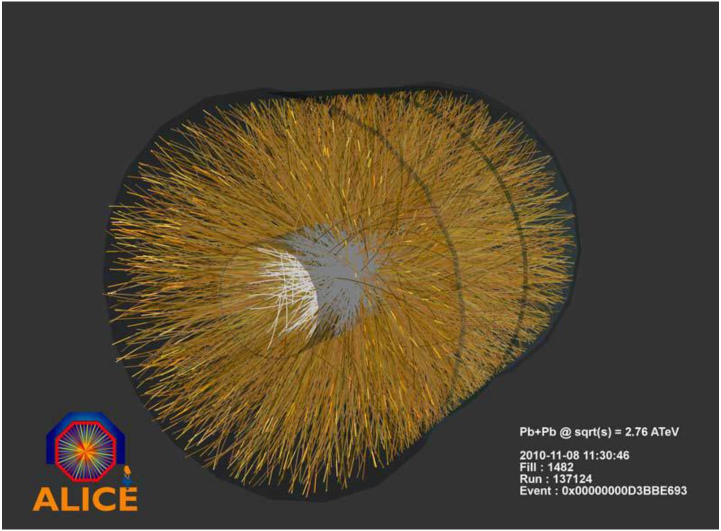
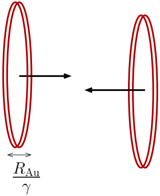
Hydrodynamics of the chiral phase transition

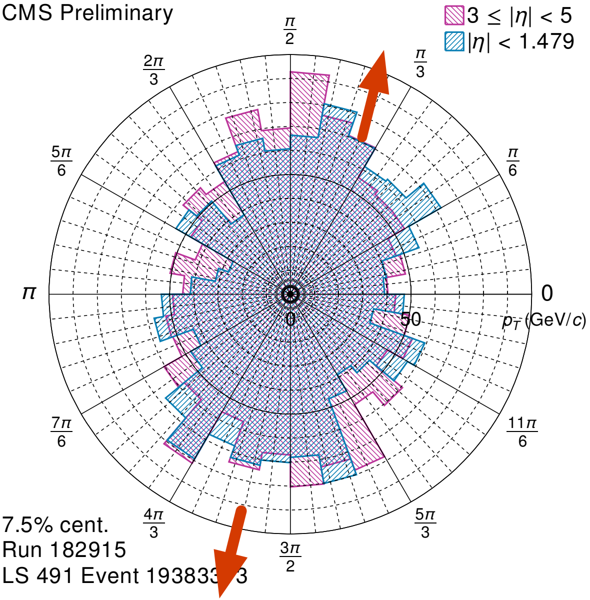
Derek Teaney
Stony Brook University

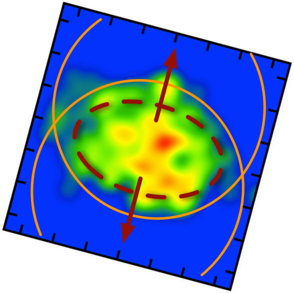
Colliding Nuclei and Creating Plasma of Quarks and Gluons (QGP)



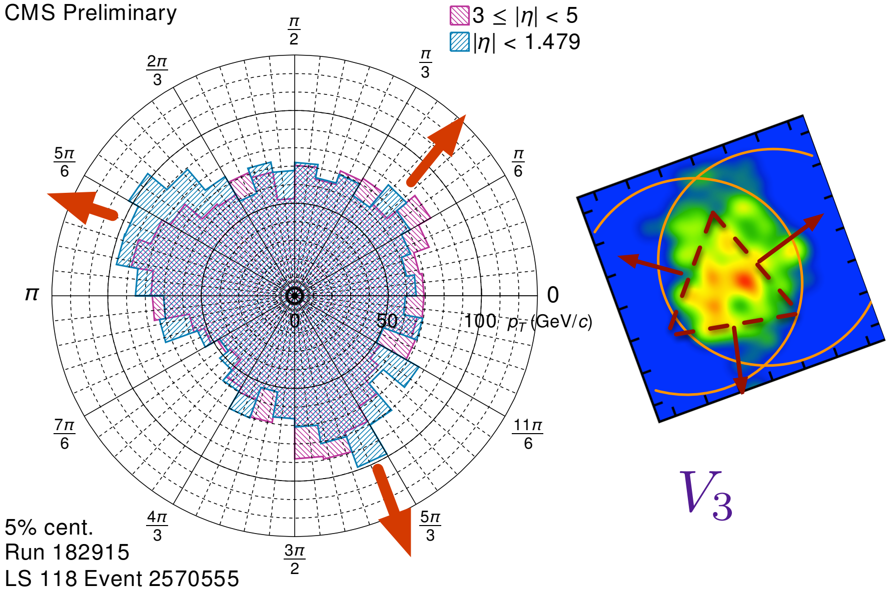


Measuring the hydrodynamics of the plasma





V_2



CMS Preliminary
3 ≤ |η| < 5
|η| < 1.479
π/2
π/3
π/6
2π/3
5π/6
π
0
7π/6
4π/3
3π/2
5π/3
11π/6
0
50
100
p_T (GeV/c)
5% cent.
Run 182915
LS 118 Event 2570555
V_3

Amazing Success: the “Standard” Hydro Model

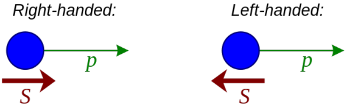
1. $V_1 \ldots V_6$
2. Momentum dependence $V_n(p)$
3. Probabilities $P(|V_n|^2)$
4. Covariances between harmonics: $\langle V_2 V_3 V_5^* \rangle$
5. Full covariance matrix: $\langle V_2(p_1) V_2^*(p_2) \rangle$

Uses the equation of state from lattice QCD and

$$\partial_\mu T^{\mu\nu} = 0$$

but we want more...

QCD and Chiral Symmetry



u_R, d_R $\qquad$ u_L, d_L

$$\begin{pmatrix} u'_L \\ d'_L \end{pmatrix} = U_L \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

and ditto for right

QCD is (almost) symmetric between, left and right, and up and down:

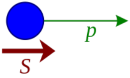
$$\mathcal{L}_{QCD} = \sum_{q=u,d} \bar{q}_L(i\not{D})q_L + \bar{q}_R(i\not{D})q_R \; - \; \underbrace{m_q\left(\bar{q}_L q_R + \bar{q}_R q_L\right)}_{\text{small}}$$

Then one would expect four approx. conservation laws, u_L, d_L, u_R, d_R:

n_B :	$(u_L + d_L) + (u_R + d_R)$	Baryon number
n_{anom} :	$(u_L - u_R) + (d_L - d_R)$	Anomalous: not consv.

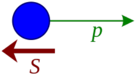
QCD and Chiral Symmetry

Right-handed:



u_R, d_R

Left-handed:



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Then one would expect four approx. conservation laws, u_L, d_L, u_R, d_R:

$\vec{n}_V$:	$(u_L + u_R) - (d_L + d_R)$	Isovector charge
$\vec{n}_A$:	$(u_L - u_R) - (d_L - d_R)$	Isoaxial vect. charge

Details of $SU_L(2) \times SU_R(2)$

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad q_L \longrightarrow q_L' = e^{i\vec{\theta}_L \cdot \vec{\tau}_L/2} \begin{pmatrix} u_L \\ d_L \end{pmatrix} \equiv U_L\, q_L$$

$$q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix} \qquad q_R \longrightarrow q_R' = e^{i\vec{\theta}_R \cdot \vec{\tau}_R/2} \begin{pmatrix} u_R \\ d_R \end{pmatrix} \equiv U_R\, q_R$$

with $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ pauli matrices $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The $U(1)$ transformations lead to the conserved currents

$$n_B = \bar{q}_L \gamma^0 \mathbb{1}\, q_L + \bar{q}_R \gamma^0 \mathbb{1}\, q_R$$

= Baryon Number = Left + Right

$$n_A^{U(1)} = \bar{q}_L \gamma^0 \mathbb{1}\, q_L - \bar{q}_R \gamma^0 \mathbb{1}\, q_R$$

= AXIAL Current = Left − Right

Anomalous and not conserved!

The currents that are arising from the SU(2) character are:

$$\vec{n}_V = \bar{q}_L \gamma^0 \vec{\tau} q_L + \bar{q}_R \gamma^0 \vec{\tau} q_R = \text{isovector charge}$$

$u - d$

So

$$n_{V,3} = \bar{u} \gamma^0 u - \bar{d} \gamma^0 d = \text{"isospin"}$$

Then there is the isovector-axial charge

$$\vec{n}_A = \bar{q}_L \gamma^0 \vec{\tau} q_L - \bar{q}_R \gamma^0 \vec{\tau} q_R = \text{iso-axial vector charge}$$

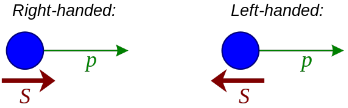
$u_L - d_L - (u_R - d_R)$

So

$$n_{A,3} = \bar{u}_L \gamma^0 u_L - \bar{d}_L \gamma^0 d_L - (\bar{u}_R \gamma^0 u_R - \bar{d}_R \gamma^0 d_R)$$

These are conserved in the heavy ion collision!

Recap: Chiral Symmetry



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One has the approx. conservation laws of u_L, d_L, u_R, d_R:

$\vec{n}_V$:	$(u_L + u_R) - (d_L + d_R)$	Isovector charge
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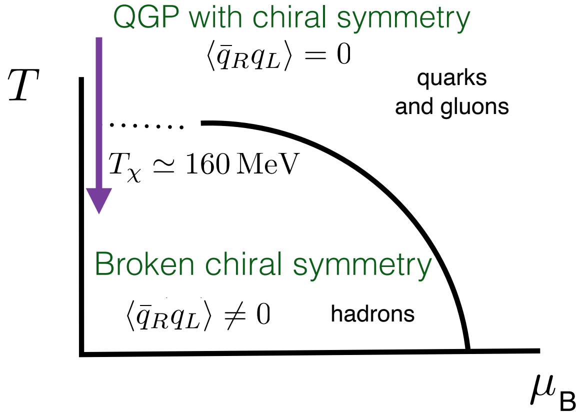
Pisarski, Wilczek



For two massless quarks the chiral symmetry group is

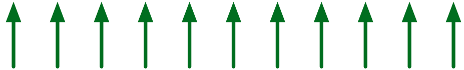
$$SU_L(2) \times SU_R(2) \simeq O(4)$$

This is broken, and the transition is 2nd order.

The mass smooths the transition to a crossover, like a magnetic field in the Ising model

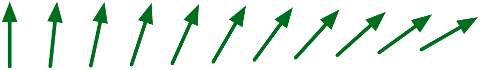
Chiral symmetry plays no role in the "Standard Hydro Model" ...

Our cold world: T< Tcritical



$$\langle \bar{q}_R q_L \rangle = \bar{\sigma}\, \mathbb{I}_{2\times 2}$$

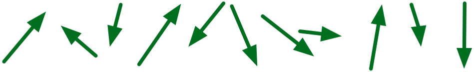
Order parameter $\langle \bar{q}_R q_L \rangle$ is like the magnetization. $q = u, d$



$$\bar{q}_R q_L = \bar{\sigma}\, e^{-i\vec{\tau}\cdot\vec{\varphi}(x)}$$

The slow modulation of the $SU_A(2)$ phase of $\bar{q}_R q_L$ is a pion, $\vec{\pi}{=}\bar{\sigma}\vec{\varphi}$

The hot world: T> Tcritical



$$\langle \bar{q}_R q_L \rangle = 0$$

State is disordered: pion propagation is frustrated

We will describe pion propagation during the $O(4)$ phase transition
Transition is strongly analogous to a normal-fluid/superfluid transition

The Chiral Condensate

$$\langle \bar{q}_R \, q_L \rangle \propto \bar{\sigma} \, (\mathbb{1})_{2\times 2} \qquad i = u, d$$

matrix

Under a constant axial phase rotation with $\vec{\theta}_R = -\vec{\theta}_L = \vec{\varphi}$

$$\bar{q}_R q_L \longrightarrow \bar{q}_R' q_L' = e^{-i\vec{\varphi}\cdot\vec{\tau}/2} \, \bar{q}_R q_L \, e^{-i\vec{\varphi}\cdot\vec{\tau}/2}$$

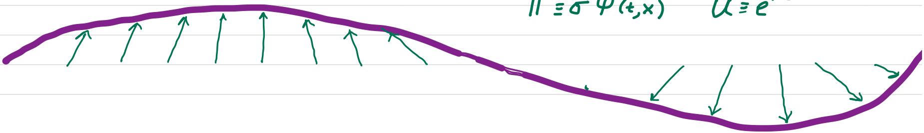
So the effecitive field (the chiral condensate) is rotated:

$$\sigma \mathbb{1} \longrightarrow \Sigma = \bar{\sigma} e^{-i\vec{\varphi}\cdot\vec{\tau}} = \bar{\sigma} U$$

matrix

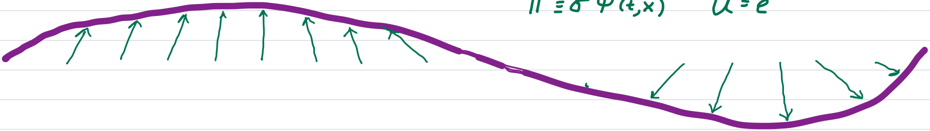
Now if the angle φ is constant this describes an equivalent physical system. If the angle changes slowly in space and time, this effective field costs very little energy.

$$\vec{\pi} \equiv \bar{\sigma} \vec{\varphi}(t,x) \qquad U \equiv e^{i\vec{\tau}\cdot\vec{\varphi}}$$



Effective Field Theory

$\vec{\pi} = \vec{\sigma} \cdot \vec{\varphi}(t,x) \qquad U = e^{i \vec{\tau} \cdot \vec{\varphi}}$



For $T = 0$

$$S = \int d^4x \, \frac{f^2}{4} \, Tr\left[\partial_\mu U \, \partial^\mu U^+\right]$$

← Weinberg Lagrangian

$$\simeq \int d^4x \, \frac{f^2}{2} \, \partial_\mu \vec{\varphi} \cdot \partial^\mu \vec{\varphi}$$

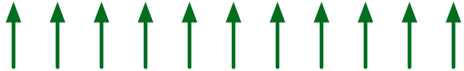
describing massless wave

When the quark mass is non zero

$$S = \int d^4x \, \frac{f^2}{4} \, tr\left[\partial_\mu U \, \partial^\mu U^+\right] + \frac{f^2 m^2}{4} \, Tr\left[U + U^+\right]$$

correction for finite quark mass

Our cold world: T< Tcritical



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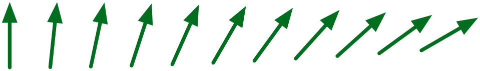

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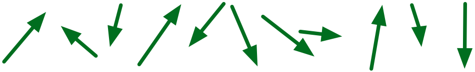

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Transition is strongly analogous to a normal-fluid/superfluid transition

Ising O(4) Model

magnetization $\vec{M}$

magnetic field $\vec{H}$

$$\mathcal{H} = \int d^3x \, \vec{H} \cdot \vec{M}$$

QCD

$$\bar{q}_L q_R = \sigma e^{-i\vec{\tau}\cdot\vec{\varphi}}$$ condensate

m_q or H quark mass

$$\mathcal{H} = \int d^3x \, m_q \left(\bar{q}_R q_L + \bar{q}_L q_R\right)$$

$\vec{\tau}$ are Pauli matrices for the SU(2) order parameter

Real World QCD and Progress from Lattice

- There are three flavors of quarks u, d, s which are massive
  - This changes structure phase diagram
- We will assume the real world is "close" to the $O(4)$ critical point.
  - The u and d quarks should be approximately massless

HotQCD 2019, 2020,
Cuteri, Philipsen, Sciara 2021
Kotov, Lombardo, Trunin, 2021

See review Phillipsen, 2021

Strong evidence for the $O(4)$ phase transition at physical s-quark mass,
And the u, d mass are light enough for the critical-dynamics.

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magnetic field $\vec{H}$

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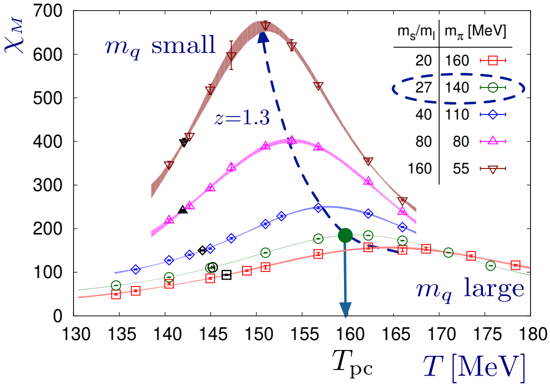
Real world lattice QCD and the $O(4)$ critical point:

Hot QCD, PRL 2019

Fluctuations of order parameter, $\sigma \propto \bar{u}u + \bar{d}d$, vs temperature and m_q

$$\chi_M = \langle \sigma^2 \rangle - \langle \sigma \rangle^2$$



O(4) Ising model predicts how the fluctuations grow for small quark mass

The QCD lattice knows about the $O(4)$ critical point! Hydro should too!

Ising O(4) Model

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magnetic field $\vec{H}$

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$\vec{\tau}$ are Pauli matrices for the SU(2) order parameter

Math: $SU_L(2) \otimes SU_R(2)$ and $O(4)$

- The field $\Sigma = \sigma e^{-i\vec{\varphi}\cdot\vec{\tau}}$ is characterized by four real numbers ϕ_a

$$\phi_a = (\phi_0, \vec{\phi}) \simeq (\sigma, \vec{\pi}) \quad \Leftarrow \text{vector under } O(4)$$

since for small angles and $\vec{\pi} \equiv \sigma\vec{\varphi}$

$$\Sigma = \sigma e^{-i\vec{\tau}\cdot\vec{\varphi}} \simeq \sigma - i\vec{\tau}\cdot\vec{\pi}$$

- In general we define four Clifford algebra matrices $\hat{\tau}_a \equiv (\mathbb{1}, -i\vec{\tau})$:

$$\Sigma \equiv \phi_a\,\hat{\tau}_a = \phi_0\,\mathbb{1} - i\vec{\tau}\cdot\vec{\phi}$$

- The chiral transformation is equivalent to a four rotation of ϕ_a

$$\Sigma' = U_L \Sigma U_R^\dagger \qquad \text{means that} \qquad \phi'_a = \underbrace{\Lambda_{ab}}_{\text{rotation matrix}} \phi_b$$

More Math: $O(4)$ or Chiral Rotation Matrices

- The infinitesimal rotation matrices are parameterized by the angles $\vec{\theta}_V = (\vec{\theta}_L + \vec{\theta}_R)/2$ and $\vec{\theta}_A = (\vec{\theta}_L - \vec{\theta}_R)/2$

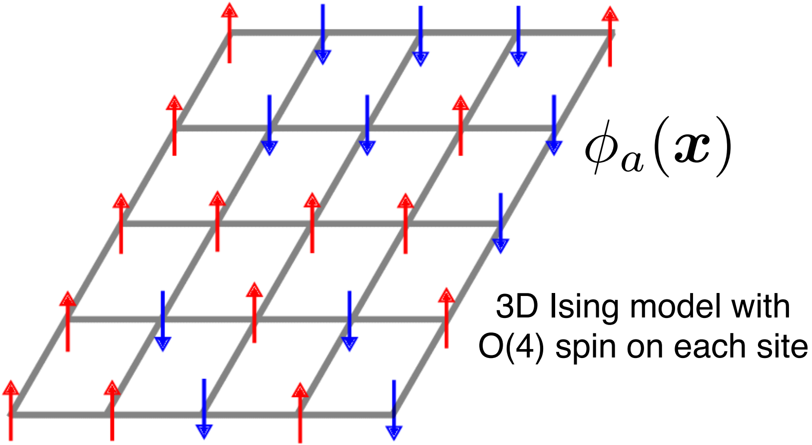
$$\underbrace{\Lambda}_{\text{Rotation Matrix}} \simeq \mathbb{1} + \begin{pmatrix} 0 & \theta_A^1 & \theta_A^2 & \theta_A^3 \\ -\theta_A^1 & 0 & \theta_V^3 & -\theta_V^2 \\ -\theta_A^2 & -\theta_V^3 & 0 & \theta_V^1 \\ -\theta_A^3 & \theta_V^2 & -\theta_V^1 & 0 \end{pmatrix}$$

$$= \mathbb{1} + \frac{1}{2}\theta_{ab}\,\mathcal{J}_{ab}$$

where for example :

$$\theta_{01} = \theta_A^1 \qquad \mathcal{J}_{01} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

or with schooling, $(\mathcal{J}_{ab})_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$


$\phi_a(\boldsymbol{x})$
3D Ising model with
O(4) spin on each site

Static Universality and the $O(4)$ Ising Model

- The Landau Ginzburg function for the $O(4)$ order parameter is: $\phi^2 \equiv \phi_a \phi_a$

$$\mathcal{H} = \int d^3\boldsymbol{x} \, \frac{1}{2}\nabla\phi_a \cdot \nabla\phi_a + \frac{1}{2}m_0^2\,\phi^2 + \frac{\lambda}{4}\,\phi^4$$

- Sample field configurations according to the statistical weight

$$P[\phi] \propto e^{-\beta_c \mathcal{H}[\phi]}$$

- The model has a critical mass parameter $m_c^2(\lambda) < 0$ which you must find

$$\frac{m_0^2 - m_c^2}{m_c^2} \propto t_{\rm r} \equiv \frac{T - T_c}{T_c}$$

The critical model makes a universal prediction for the susceptibility:

$$\chi_M \equiv \left\langle \phi_0^2 \right\rangle - \left\langle \phi_0 \right\rangle^2$$

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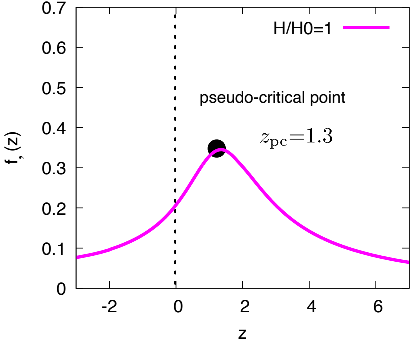
Scaling predictions from the $O(4)$ model

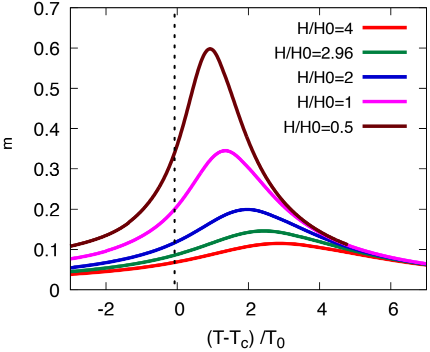
Simulations at different magnetic field are related to each other

$$\chi_M = h^{1/\delta - 1} f_\chi(z) \qquad z = z_0\, t_{\rm r} h^{-1/\beta\delta}$$

Here $h = H/H_0$ and $t_{\rm r} \propto (T - T_C)$ are the reduced field and temperature



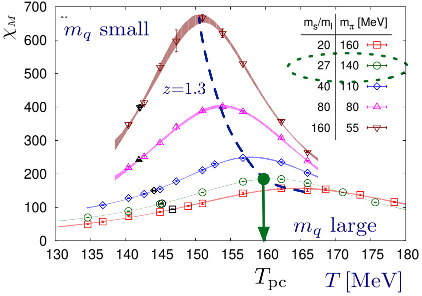
numerical data from
Engels, Seniuch, Fromme, Karsch

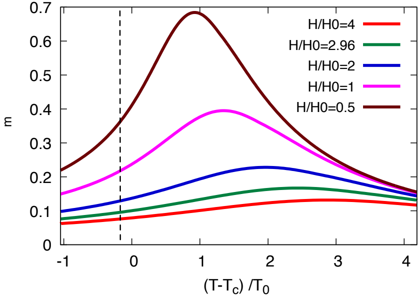


Scaling predictions and QCD

Hot QCD, 2019

$$\chi_M = \langle \sigma^2 \rangle - \langle \sigma \rangle^2$$





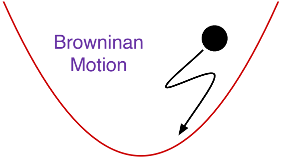
Scaling predictions reasonably describe how the peak rises and shifts.

$$\chi_M \propto m_q^{1/\delta - 1} f_\chi(z) \qquad z = z_0 \left(\frac{T - T_C}{T_C} \right) m_q^{-1/\beta\delta}$$

Part I: From Thermodynamics to Hydrodynamics

Dissipative Processes

Mathematical Structure of Dissipative Processes: Brownian Motion



Free Energy

$$\mathcal{H} = \frac{p^2}{2m} + V(q)$$

The stochastic equations of motion with noise $\langle \xi(t)\xi(t') \rangle = 2T\eta\,\delta_{tt'}$:

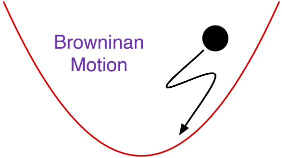
$$\underbrace{\begin{aligned} \partial_t q + \{q, \mathcal{H}\} &= 0 \\ \partial_t p + \{p, \mathcal{H}\} &= 0 \end{aligned}}_{\text{Hamiltonian Dynamics}}$$

The probability distribution $P(t, q, p)$ evolves to equilibrium: $P_{\rm eq} = e^{-\beta H}$

$$\partial_t P + \{\mathcal{H}, P\} = T\eta\,\nabla_p \Big(\underbrace{\beta \nabla_p \mathcal{H}\, P}_{-\nabla_p P_{\rm eq}} + \nabla_p P \Big)$$

A unique mathematical structure which reaches equilibrium

Mathematical Structure of Dissipative Processes: Brownian Motion



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$$\underbrace{\partial_t q + \{q,\mathcal{H}\}}_{} = 0$$

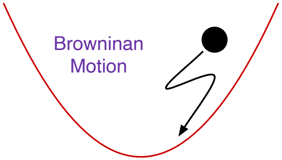
$$\underbrace{\partial_t p + \{p,\mathcal{H}\}}_{\text{Hamiltonian Dynamics}} = -\eta \underbrace{\left(\frac{\partial \mathcal{H}}{\partial p}\right)}_{\text{velocity } p/m}$$

The probability distribution $P(t,q,p)$ evolves to equilibrium: $P_{\rm eq} = e^{-\beta H}$

$$\partial_t P + \{\mathcal{H}, P\} = T\eta\,\nabla_p\Big(\underbrace{\beta\nabla_p\mathcal{H}\,P}_{-\nabla_p P_{\rm eq}} + \nabla_p P\Big)$$

A unique mathematical structure which reaches equilibrium

Mathematical Structure of Dissipative Processes: Brownian Motion



Free Energy

$$\mathcal{H} = \frac{p^2}{2m} + V(q)$$

The stochastic equations of motion with noise $\langle \xi(t)\xi(t') \rangle = 2T\eta\, \delta_{tt'}$:

$$\partial_t q + \{q, \mathcal{H}\} = 0$$

$$\underbrace{\partial_t p + \{p, \mathcal{H}\}}_{\text{Hamiltonian Dynamics}} = -\eta \underbrace{\left(\frac{\partial \mathcal{H}}{\partial p}\right)}_{\text{velocity } p/m} + \underbrace{\xi}_{\text{noise}}$$

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A unique mathematical structure which reaches equilibrium

Dissipative Dynamics From Metropolis Updates

$$\partial_t p = \underbrace{-\eta \left(\frac{\partial \mathcal{H}}{\partial p}\right) + \xi}_{\text{drag + noise}} \quad \text{with} \quad \left\langle \xi(t)\xi(t') \right\rangle = 2T\eta\, \delta_{tt'}$$

- Make a proposal with the right variance

$$p \to p + \Delta p \qquad \text{with} \quad \left\langle \Delta p^2 \right\rangle = 2T\eta\Delta t$$

- Compute the change in free energy

$$\Delta\mathcal{H} = \mathcal{H}(p + \Delta p) - \mathcal{H}(p) \simeq \left(\frac{\partial \mathcal{H}}{\partial p}\right) \Delta p$$

- If $\Delta\mathcal{H} < 0$ accept proposal. If $\Delta\mathcal{H} > 0$ accept with probability:

$$P_{\rm up} = e^{-\beta\Delta\mathcal{H}}$$

The *accepted* proposals reproduce the dissipation and variance

$$\langle \Delta p \rangle = -\eta \left(\frac{\partial \mathcal{H}}{\partial p}\right) \Delta t \quad \text{and} \quad \left\langle (\Delta p)^2 \right\rangle = 2T\eta\Delta t$$

Application of dissipative process to the $O(4)$ critical point

1. We generate configurations distributed with the Boltzmann weight

$$P(p) = e^{-\beta \mathcal{H}(p)} \qquad \mathcal{H}(p) \equiv \frac{p^2}{2m}$$

   by evolving the Langevin equations with Metropolis steps:

$$\partial_t p = -\eta \left(\frac{\partial \mathcal{H}}{\partial p} \right) + \xi \quad \text{with} \quad \left\langle \xi(t)\xi(t') \right\rangle = 2T\eta\delta_{tt'}$$

2. Can generate field configurations with Landau-Ginzburg weight

$$\mathcal{H}[\phi] = \int d^3x \, \frac{1}{2}\nabla\phi_a \cdot \nabla\phi_a + \frac{1}{2}m_0^2(T)\phi^2 + \frac{\lambda}{4}\phi^4 - H\phi_0$$

   by evolving the Langevin equation with Metropolis Steps

$$\partial_t \phi_a = -\Gamma \left(\frac{\delta \mathcal{H}}{\delta \phi_a} \right) + \xi \qquad \text{with} \qquad \left\langle \xi(x)\xi(x') \right\rangle = 2T\Gamma\delta^4_{xx'}$$

Comments on the Metropolis Approach

All dissipative processes need to conform to this model!

Other recent examples from Hot QCD:

- Simulation of QCD Critical point: Chattopadhyay, Ott, Schaefer, and Skokov
- Relativistic viscous hydrodynamics: Basar, Bhambure, Singh, Teaney

Part II: From Thermodynamics to Hydrodynamics

Hydro $=$ Hamiltonian Dynamics $+$ Dissipation

Hydrodynamics of the $O(4)$ transition:

Rajagopal and Wilczek '92, Son '99, Son and Stephanov '01, and finally us, arxiv:2101.10847.

1. The order parameter

$$\phi_a = (\sigma, \vec{\pi})$$

2. The approximately conserved charges quantities:

$$\vec{n}_V = \underbrace{\bar{\psi}\gamma^0\vec{\tau}\psi}_{\text{isovect chrg}} \qquad \textit{and} \quad \vec{n}_A = \underbrace{\bar{\psi}\gamma^0\gamma^5\vec{\tau}\psi}_{\text{isoaxial-vect chrg}}$$

which are combined into an anti-symmetric $O(4)$ tensor of charges

$$n_{ab} = \begin{pmatrix} 0 & n_A^1 & n_A^2 & n_A^3 \\ -n_A^1 & 0 & n_V^3 & -n_V^2 \\ -n_A^2 & -n_V^3 & 0 & n_V^1 \\ -n_A^3 & n_V^2 & -n_V^1 & 0 \end{pmatrix} \qquad N_{ab}(t) = \int_{\mathbf{x}} n_{ab}(t, \boldsymbol{x})$$

Use these fields to construct the Hydrodynamic effective Hamiltonian

Poisson Brackets for effective Hydrodynamic Hamiltonians

Dzyaloshinskii & Volovik '79

- An infinitesimal $O(4)$ rotation is parameterized by

$$\phi'_c = \Lambda_{cd}\,\phi_d \simeq \phi_c + \tfrac{1}{2}\,\theta\cdot\mathcal{J}_{cd}\,\phi_d$$

- In quantum mechanics the charges N_{ab} generate rotations:

$$\phi'_c \simeq \phi_c + \tfrac{i}{2}\,[\theta\cdot N, \phi_c]$$

which specifies the Poisson brackets in classical mechanics

$$\phi'_c = \phi_c + \tfrac{1}{2}\,\{\theta\cdot N, \phi_c\}$$

- Comparing the expressions, we find symmetry dictated Poisson brackets between hydrodynamic variables:

$$\{n_{ab}(\boldsymbol{x}), \phi_c(\boldsymbol{y})\} = (\mathcal{J}_{ab})_{cd}\,\phi_d(\boldsymbol{x})\;\delta(\boldsymbol{x}-\boldsymbol{y})$$

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$$\{n_{ab}(\boldsymbol{x}), \phi_c(\boldsymbol{y})\} = (\delta_{ac}\phi_b(\boldsymbol{x}) - \delta_{bc}\,\phi_a(\boldsymbol{x}))\ \delta(\boldsymbol{x}-\boldsymbol{y})$$

The Landau-Ginzburg Hamiltonian for the $O(4)$ transition:

The Hamiltonian is tuned to the crit. point with $m_0^2(T) < 0$ and $H \propto m_q$:

$$\mathcal{H} = \int d^3x \, \frac{1}{2}\nabla\phi_a \cdot \nabla\phi_a + \frac{1}{2}m_0^2(T)\phi^2 + \frac{\lambda}{4}\phi^4 - H\phi_0 + \frac{n_{ab}^2}{4\chi_0}$$

and gives the equilibrium distribution with the correct critical EOS:

$$Z = \int D\phi \, Dn \, e^{-\mathcal{H}[\phi,n]/T_c}$$

The hydro equations of motion take the form

$$\frac{\partial\phi}{\partial t} + \{\phi, \mathcal{H}\} = 0 + \text{visc. corrections} + \text{noise}$$

$$\frac{\partial n_{ab}}{\partial t} + \{n_{ab}, \mathcal{H}\} = 0 + \text{visc. corrections} + \text{noise}$$

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$$\frac{\partial n_{ab}}{\partial t} + \{n_{ab}, \mathcal{H}\} = \underbrace{\sigma_0 \nabla^2 \frac{\delta\mathcal{H}}{\delta n_{ab}}}_{\text{dissipation}} + \underbrace{\nabla\cdot\xi_{ab}}_{\text{noise}}$$

The equations and the simulations:

see also Schlichting, Smekal

We have a charge diffusion equation coupled to order parameter:

$$\partial_t n_{ab} + \underbrace{\nabla \cdot (\nabla \phi_{[a}\phi_{b]})}_{\text{poisson bracket}} + H_{[a}\phi_{b]} = \underbrace{D_0 \nabla^2 n_{ab}}_{\text{diffusion}} + \underbrace{\nabla \cdot \xi_{ab}}_{\text{noise}}$$

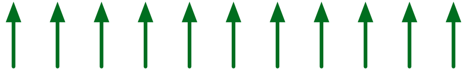
and a rotation of the order parameter induced by the charge:

$$\partial_t \phi_a + \underbrace{\frac{n_{ab}}{\chi_0}\phi_b}_{\text{poisson bracket}} = \underbrace{-\Gamma_0 \frac{\delta \mathcal{H}}{\delta \phi_a}}_{\text{dissipation}} + \underbrace{\xi_a}_{\text{noise}}$$

Numerical scheme based operator splitting:

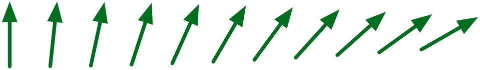
1. Evolve the Hamiltonian evolution with a symplectic stepper
2. Treat the dissipative Langevin steps as Metropolis-Hastings updates

Our cold world: T< Tcritical



$$\langle \bar{q}_R q_L \rangle = \bar{\sigma}\, \mathbb{I}_{2\times 2}$$

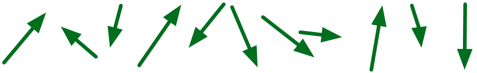
Order parameter $\langle \bar{q}_R q_L \rangle$ is like the magnetization. $q = u, d$



$$\bar{q}_R q_L = \bar{\sigma}\, e^{-i\vec{\tau}\cdot\vec{\varphi}(x)}$$

The slow modulation of the $SU_A(2)$ phase of $\bar{q}_R q_L$ is a pion, $\vec{\pi}{=}\bar{\sigma}\vec{\varphi}$

The hot world: T> Tcritical

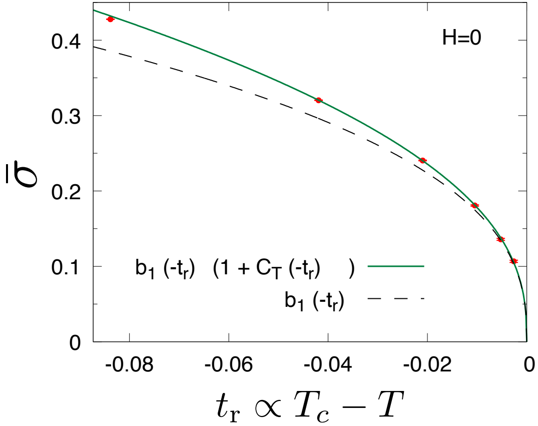


$$\langle \bar{q}_R q_L \rangle = 0$$

State is disordered: pion propagation is frustrated

Statics and the Chiral Condensate

$$M_a(t) \equiv \frac{1}{V}\int_{\boldsymbol{x}} \phi_a(t,\boldsymbol{x}) \qquad \bar{\sigma} \equiv \langle M_0(t)\rangle$$



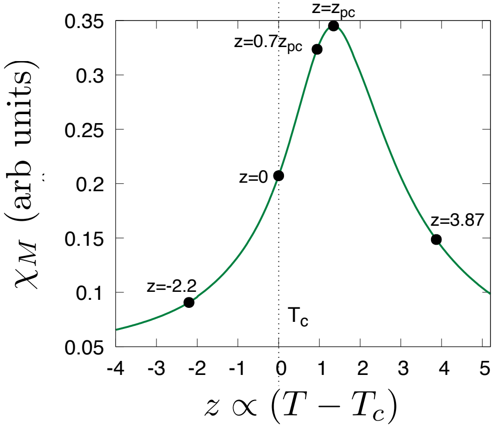
Scan the phase transition:

Engels, Fromme, Seniuch, Karsch; Hasenbuch

We measured the mean order parameter and fluctuations:

$$\left\langle \sigma^2 \right\rangle - \left\langle \sigma \right\rangle^2 = h^{1/\delta - 1} f_\chi(z) \qquad z = t_{\rm r} h^{-1/\beta\delta}$$

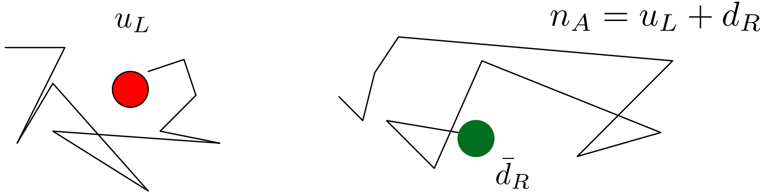
fixing the scaling parameters, $h = H/H_0$, and $t_{\rm r} = (m_0^2 - m_c^2)/\mathfrak{m}^2$



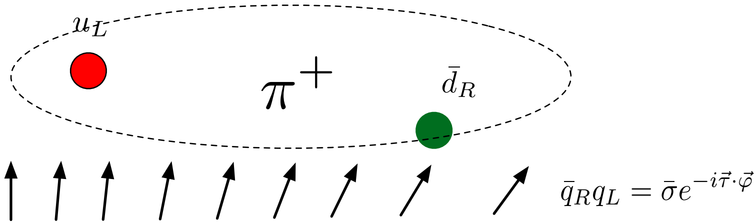
$$\chi_M = \langle \sigma^2 \rangle - \langle \sigma \rangle^2$$

“Artists” conception of the phase transition dynamics

High Temperature: Diffusion of axial charge



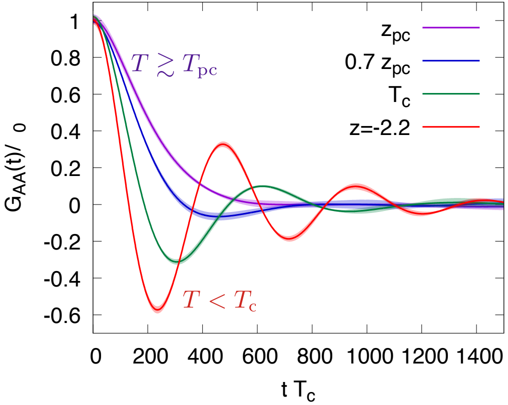
Low Temperature: pion propagation



The phase transition and axial charge correlations:

$$G_{AA}(t) = \int \mathrm{d}^3x \; \langle \vec{n}_A(t, \boldsymbol{x}) \cdot \vec{n}_A(0, \mathbf{0}) \rangle$$

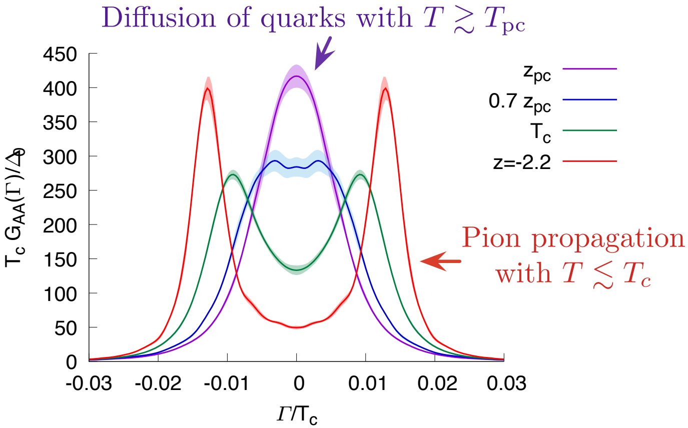
See a change in the dynamics across $T_{\rm pc}$:



Let's take a fourier transform and analyze the transition

Features of the phase transition in the axial charge correlations:

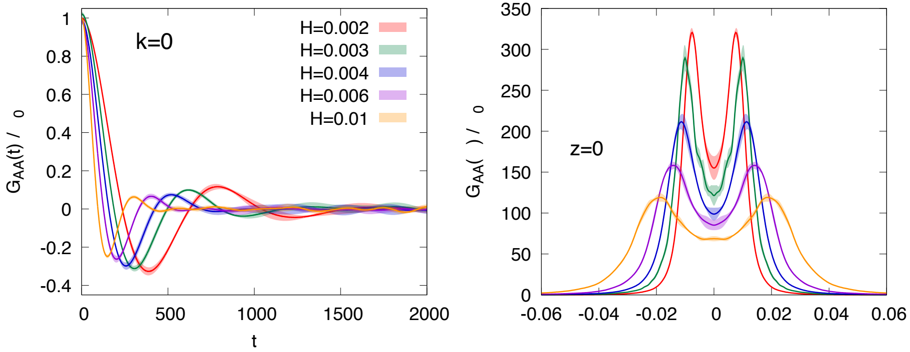
$$G_{AA}(\omega) = \int \mathrm{d}t\,\mathrm{d}^3x\,e^{i\omega t}\,\langle \vec{n}_A(t,\boldsymbol{x}) \cdot \vec{n}_A(0,\mathbf{0})\rangle$$



Can see the transition from diffusion of quarks to propagation of pions...

Scaling of simulations at T_c:

At $T = T_c$, we varied the magnetic field, finding the response functions:



See a time scaling of the real time correlations with quark mass H, which tunes the correlation length.

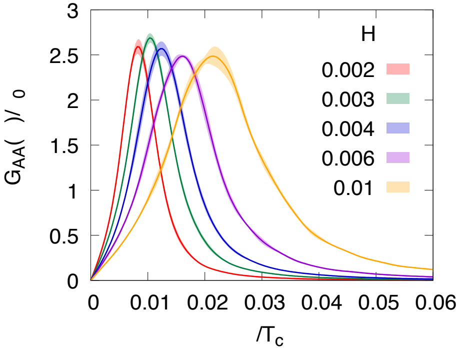
Dynamical critical exponent of the $O(4)$ transition:

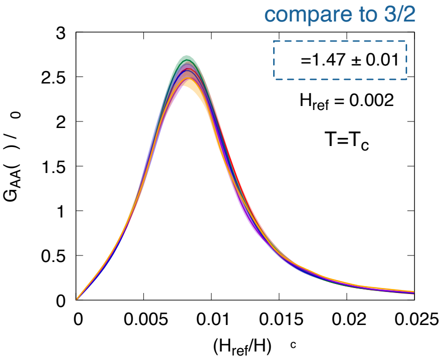
Rajagopal, Wilczek

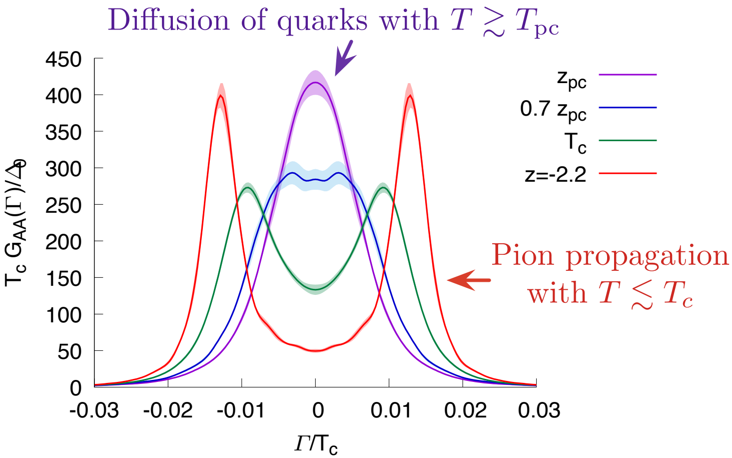
The relaxation time and correlations *scale* with the correlation length ξ:

$$\omega\, G_{AA}(\omega,\xi) = \underbrace{f(\omega\,\tau_R)}_{\text{universal fcn}} \quad \text{with} \quad \underbrace{\tau_R \propto \xi^{\zeta}}_{\text{relaxation time}}$$

The correlation length scales as $\xi \propto H^{-\nu_c}$ and the time as $\tau_R \propto H^{-\zeta\nu_c}$:






Diffusion of quarks with $T \gtrsim T_{pc}$
450
400
350
300
250
200
150
100
50
0
$T_c\, G_{AA}(\Gamma)/\Delta_0$
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
Γ/T_c
z_{pc}
$0.7\ z_{pc}$
T_c
z=-2.2
Pion propagation with $T \lesssim T_c$

The pion hydrodynamic EFT :

Well below the critical point, the $\bar{\sigma}(T)$ is *constant* and $\varphi(t, \boldsymbol{x})$ fluctuates

$$\Sigma = \bar{\sigma}(T)\, e^{-i\vec{\tau}\cdot\vec{\varphi}(t,\boldsymbol{x})} \qquad \phi_a = \bar{\sigma}(T)\,(1\,, \vec{\varphi}(t, \boldsymbol{x}))$$

The Hamilton equations of motion are

$$\partial_t n_A + \{n_A, H\} = 0 \qquad \text{and} \qquad \partial_t \varphi + \{\varphi, H\} = 0$$

and lead to

$$\underbrace{\partial_t n_A + \nabla \cdot \boldsymbol{J_A} = f^2 m^2 \varphi}_{\text{PCAC}} \qquad \text{and} \qquad \underbrace{\partial_t \varphi = \frac{n_A}{\chi}}_{\text{Josephn's constraint}}$$

Here $J_A = f^2 \nabla\varphi$ and the parameters are proptional to $\bar{\sigma}(T)$

$$f^2 \propto \bar{\sigma}^2(T) \qquad \underbrace{f^2 m^2 = H\bar{\sigma}(T)}_{\text{GOR}}$$

Long wavelength pion (superfluid) modes:

Son, Stephanov hep-ph/020422

- Linearizing the equation of motion $\varphi = Ce^{-i\omega t + i\boldsymbol{q}\cdot\boldsymbol{x}}$ one finds

$$\varphi(t,\boldsymbol{q}) = Ce^{-(\Gamma/2)t}e^{-i\omega_q t}$$

- The quasi-particle energy is:

$$\omega_q^2 \equiv v_0^2(q^2+m^2) \qquad\qquad v_0^2(T) \equiv \frac{f^2}{\chi_0} \quad \Leftarrow \text{pion velocity}$$

The parameters scale with the chiral condensate:

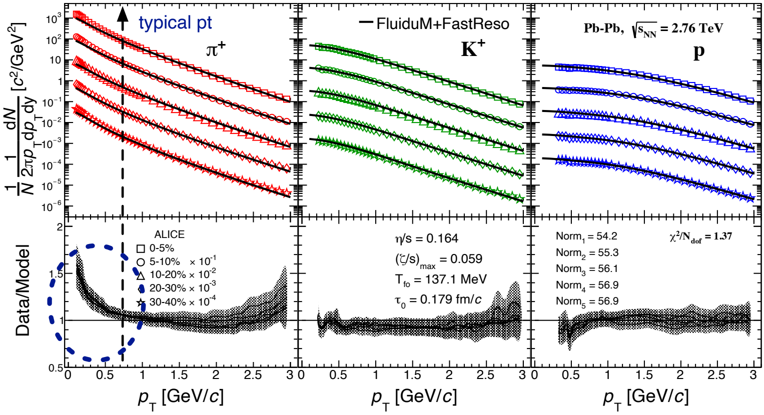
$$v_0^2 \propto \bar{\sigma}(T)^2 \qquad v_0^2 m^2 \propto H\bar{\sigma}(T)$$

And $\bar{\sigma}(T) = \langle\bar{q}q\rangle$ vanishes near T_c, frustrating the propagation. . .

Phenomenology of Soft Pions in Data

Evidence for the chiral crossover in the heavy ion data?

A recent ordinary hydro fit from Devetak et al 1909.10485



See also, Guillen&Ollitrault arXiv:2012.07898; Schee, Gürsoy, Snellings: arXiv:2010.15134

Because the pions are the Goldstones of the transition, I expect an enhancement at low p_T, relative to vanilla hydro

Evidence for the chiral crossover in the heavy ion data?

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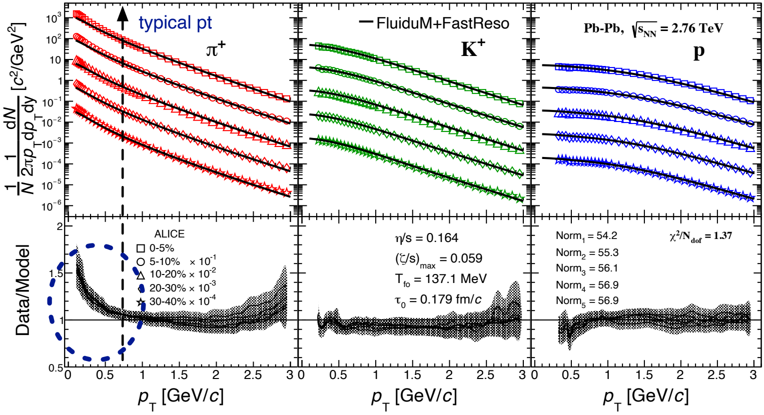


See also, Guillen&Ollitrault arXiv:2012.07898; Schee, Gürsoy, Snellings: arXiv:2010.15134

Expect an enhancement at low p_T

$$n(\omega_q) = \frac{1}{e^{vq/T} - 1} \simeq \frac{T}{vq} \Rightarrow \infty,$$

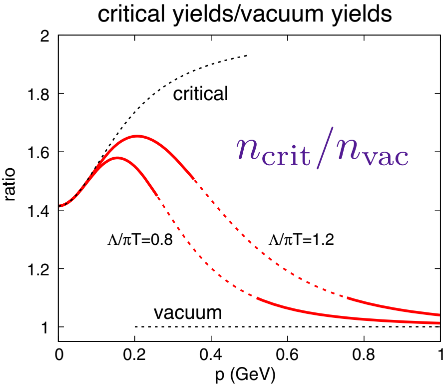
Since at T_c, the velocity $v \Rightarrow 0$!

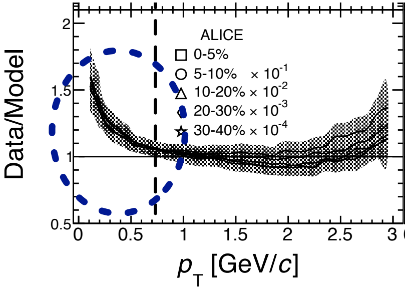
With a modified dispersion curve (relative to vacuum) the yields increase

$$n(\omega(p)) = \frac{1}{e^{\omega(p)/T} - 1} \qquad \omega^2(p) = v^2(T)(p^2 + m^2(T))$$

v goes to zero at T_c

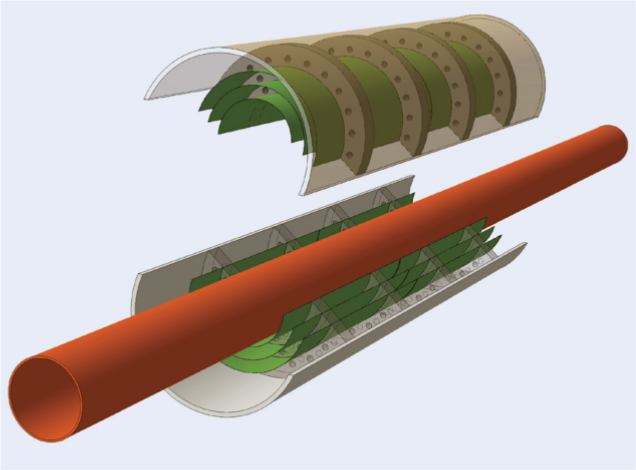
We estimated the drop in $v^2(T)$ and $v^2 m^2(T)$ from lattice data on $\bar{\sigma}(T)$





Encouraging estimate which motivates additional work on critical dynamics

New Detector: ALICE ITS3



Summary and Outlook:

1. We are encouraged by estimates and current measurements.
2. We are simulating the real-time dynamics of the chiral critical point
   - The numerical method may be useful for stochastic hydro generally
3. We reproduced the expected dynamical scaling laws:

$$\tau_R \propto \xi^{\zeta} \qquad \zeta = \frac{d}{2} \simeq 1.47 \pm 0.01$$

4. The pion waves are well calibrated.
5. The next step is to study the expanding case:
   - This will predict soft pions and their correlations with expansion for heavy ion collisions

The hadronization of the pion is the (only) hadronization process that can be studied rigorously, *and only with hydrodynamics!*

Backup

Dynamical scaling of σ correlation functions:

$$G_{\sigma\sigma}(\omega) = \int \mathrm{d}t\, \mathrm{d}^3x\, e^{i\omega t}\ \langle \sigma(t, \boldsymbol{x}) \cdot \sigma(0, \mathbf{0}) \rangle$$