形状因子及其应用

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第六届重味物理与量子色动力学研讨会
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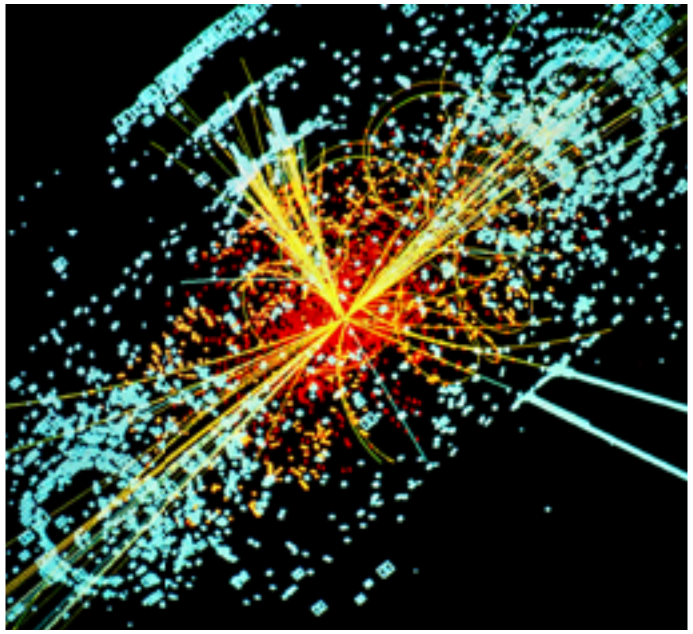
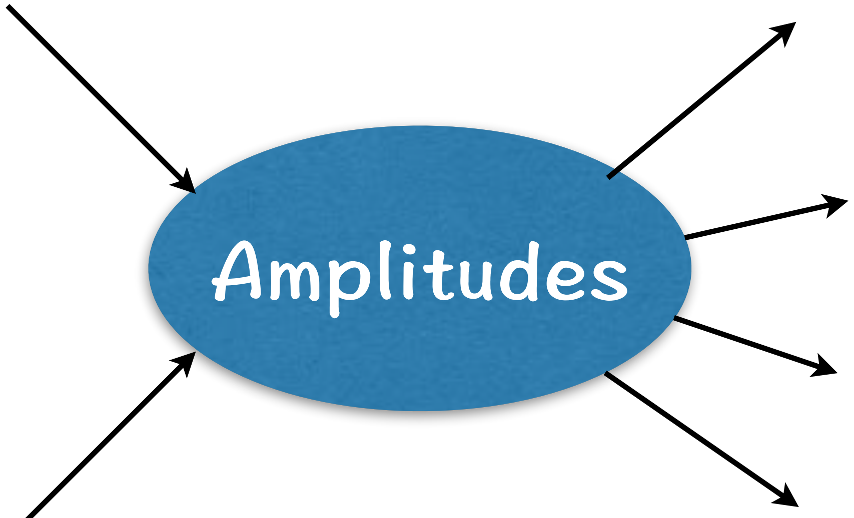
Scattering amplitudes



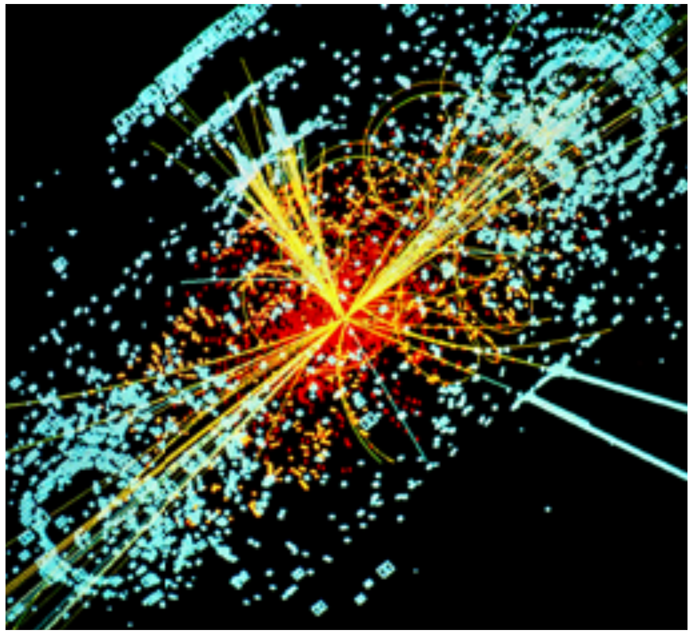


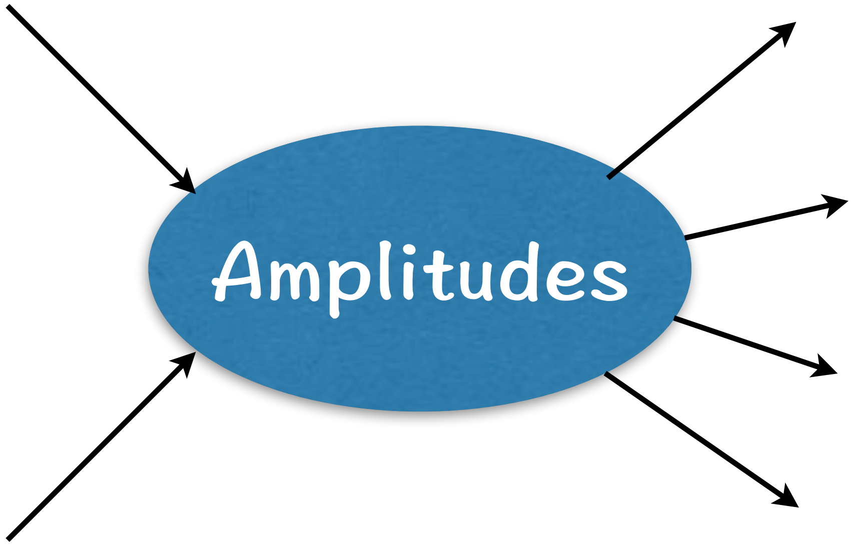
In past over 30 years, significant progress has been made in the studies of scattering amplitudes.

[Parke, Taylor, 1986]

$$A_n^{\text{tree}}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

Scattering amplitudes





In past over 30 years, significant progress has been made in the studies of scattering amplitudes.

New methods **Hidden structures**

Amplitudes

↓

Form Factors

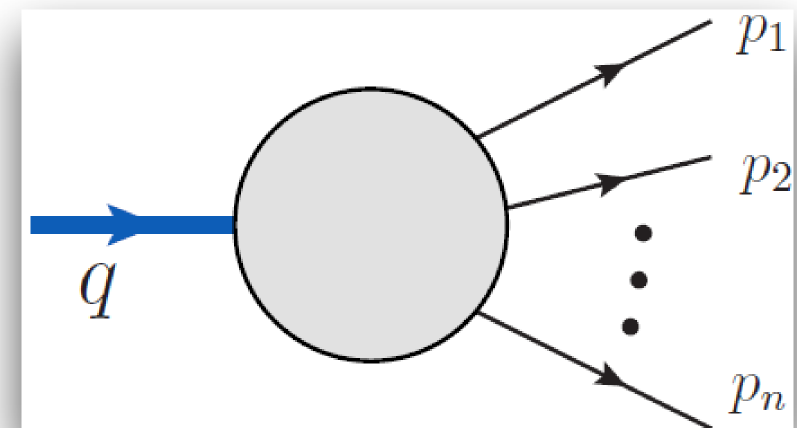
Form Factors

Matrix element of on-shell states and a local operators:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x}\, \langle p_1 \ldots p_n|\mathcal{O}(x)|0\rangle$$

$$= \delta^{(4)}\Big(\sum_{i=1}^{n} p_i - q\Big)\, \langle p_1 \ldots p_n|\mathcal{O}(0)|0\rangle$$

(work in momentum space)



$$q = \sum_i p_i, \qquad q^2 \neq 0$$

$$\langle p_1 p_2 \ldots p_n|0\rangle$$

Amplitudes

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n\rangle$$

Correlation functions

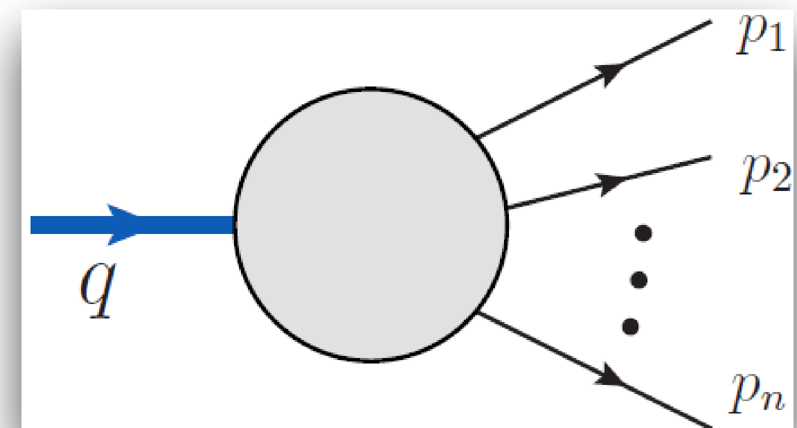
Form Factors

Matrix element of on-shell states and a local operators:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x}\, \langle p_1 \ldots p_n | \mathcal{O}(x) | 0 \rangle$$

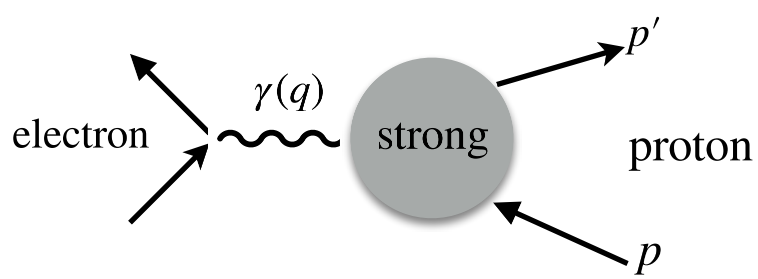
$$= \delta^{(4)}\Big(\sum_{i=1}^{n} p_i - q\Big)\, \langle p_1 \ldots p_n | \mathcal{O}(0) | 0 \rangle$$

(work in momentum space)



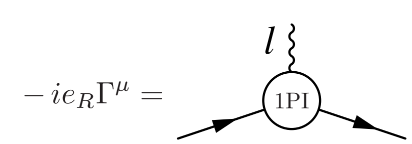
$$q = \sum_i p_i, \qquad q^2 \neq 0$$

- Nuclear form factor



Hofstadter 1956

- Sudakov form factor



$$\Gamma_\sigma(p,q;l) = \gamma_\sigma \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right)^n$$
$$= \gamma_\sigma \exp\left\{-\frac{e^2}{2\pi}\ln\left|\frac{l^2}{p^2}\right|\ln\left|\frac{l^2}{q^2}\right|\right\}.$$

Sudakov 1954

Operators

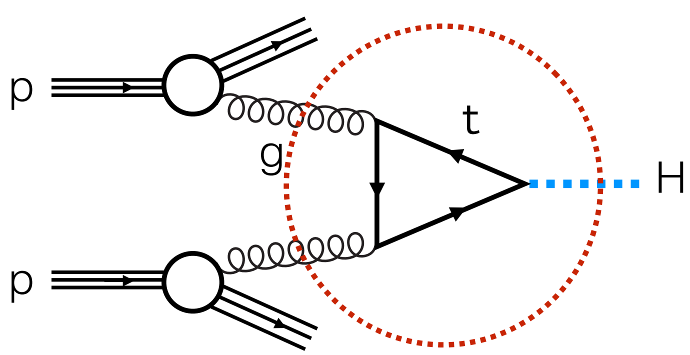
Gauge invariant operators are important in QFT.

- Anomalous dimensions (spectrum of hadrons, RG, OPE, ….)
- Correlation functions (e.g., EEC)

Local operators also appear as vertices in EFT Lagrangian.

For example: Higgs EFT obtained by integrating Top quark loop:

Wilczek, 1977; Shifman et.al., 1979, ….



$m_t \to \infty$

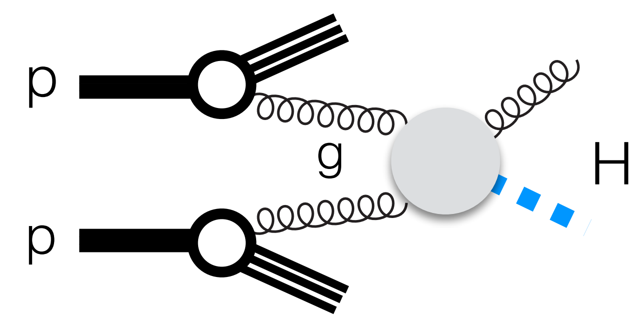
$$\mathcal{L}_{\rm eff} = \hat{C}_0 H \mathcal{O}_{4;0} + \sum_{k=1}^{\infty} \frac{1}{m_{\rm t}^{2k}} \sum_i \hat{C}_i H \mathcal{O}_{4+2k;i}$$

Higgs+gluons scattering

Higgs plus jet production

$$A(q^H, 1^g, 2^g, \ldots, n^g) = F_{\mathcal{O}=\mathrm{tr}(F^2)}(1^g, 2^g, \ldots, n^g)$$

Boughezal, Caola, Melnikov, Petriello, Schulze 2013; Chen, Gehrmann, Glover, Jaquier 2014; Boughezal, Focke, Giele, Liu, Petriello 2015; Harlander, Liebler, Mantler 2016; Anastasiou, Duhr, Dulat, Furlan, Gehrmann, Herzog, Lazopoulos, Mistlberger 2016; Lindert, Kudashkin, Melnikov, Wever 2018; Jones, Kerner, Luisoni 2018; Neumann 2018; …



$p_T \sim 2m_t$ → High-dimension operators become important.

$$\mathcal{L}_{\mathrm{eff}} = C_0 O_0 + \frac{1}{m_{\mathrm{t}}^2} \sum_{i=1}^{4} C_i O_i + \mathcal{O}\left(\frac{1}{m_{\mathrm{t}}^4}\right)$$

Dimension-5 operator

$$O_0 = H\mathrm{tr}(F_{\mu\nu} F^{\mu\nu})$$

2-loop: Gehrmann, Jaquier, Glover, Koukoutsakis 2011

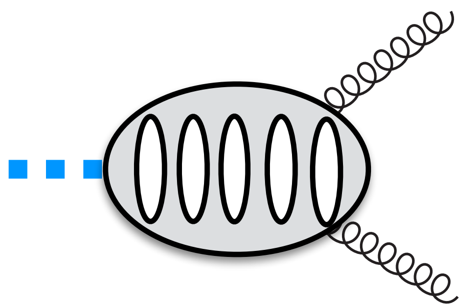
Dimension-7 operators

$$O_1 = H\mathrm{tr}(F_\mu{}^\nu F_\nu{}^\rho F_\rho{}^\mu),$$
$$O_2 = H\mathrm{tr}(D_\rho F_{\mu\nu} D^\rho F^{\mu\nu}),$$
$$O_3 = H\mathrm{tr}(D^\rho F_{\rho\mu} D_\sigma F^{\sigma\mu}),$$
$$O_4 = H\mathrm{tr}(F_{\mu\rho} D^\rho D_\sigma F^{\sigma\mu}).$$

1-loop: Dawson, Lewis, Zeng 2014
2-loop: Jin, GY 2019

Progress in N=4 SYM

$$\mathcal{F}_n = \int d^4x\, e^{-iq\cdot x} \langle p_1, \ldots, p_n \,|\, \mathrm{tr}(F^2)(x) \,|\, 0 \rangle$$



Full-color integrand up to 5 loops

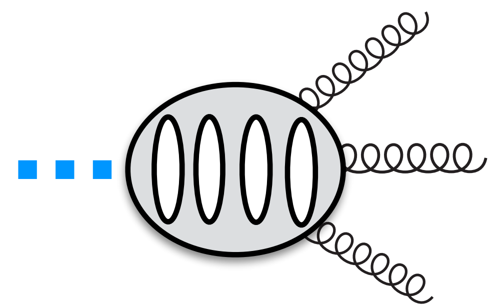
Boels, Kniehl, Tarasov, GY 2012

GY, 2016

Integrated results at 4 loops

Boels, Huber, GY 2017

Huber, von Manteuffel, Panzer, Schabinger, GY 2020



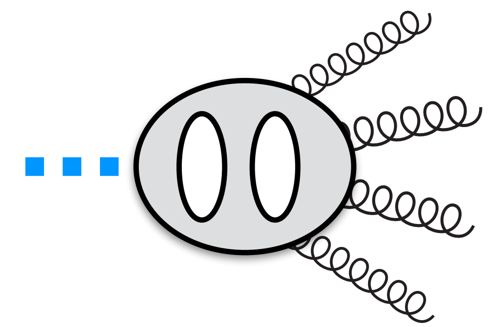
Full-color integrand up to 4 loops

Lin, GY, Zhang, 2021

Integrated results at 3 loops

Lin, GY, Zhang, 2021

Guan, Lin, Liu, Ma, GY 2023



Integrated results at 2 loops

Guo, Wang, GY, 2022

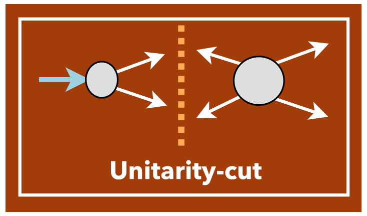
Guo, Wang, GY, Yin to appear

See also: Dixon, Gurdogan, Liu, McLeod, Wilhelm 2021, 2022

Computational tools

- On-shell unitarity method Bern, Dixon, Durban, Kosower 1994; Britto, Cachazo, Feng, 2003

  Simple tree blocks -> Higher loop results



- Color-kinematics duality Bern, Carrasco, Johansson 2008

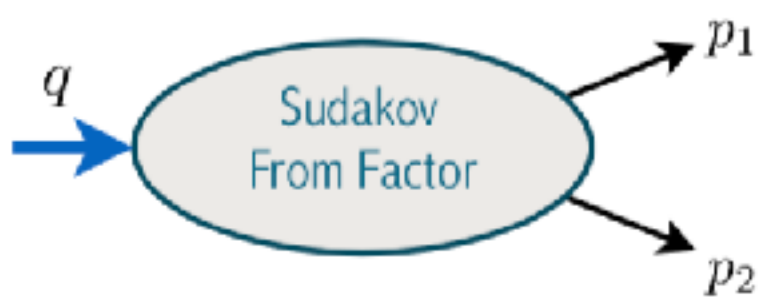
  Large number of diagrams -> Very few “master” diagrams

- Master-integral bootstrap Guo, Wang, GY 2021

  Construct final results directly using physical constraints

Sudakov form factor and Casimir scaling conjecture

Sudakov form factor



$$p_1^2 = p_2^2 = 0$$

$$q^2 = (p_1 + p_2)^2$$

Logarithm behavior is well-understood:

For dim-reg representation, see:
Magnea and Sterman 1990;
Sterman and Tejeda-Yeomans 2002
Bern, Dixon, Smirnov 2005

$$\log F_2(1,2) \simeq -\sum_{l=1}^{\infty} g^{2l}\left(\frac{\gamma_{\rm cusp}^{(l)}}{\epsilon^2} + \frac{\mathcal{G}_{\rm coll}^{(l)}}{\epsilon}\right)(-q^2)^{-l\epsilon} + \mathcal{O}(\epsilon^0)$$

Leading IR singularity -> Cusp anomalous dimension

Color structure

Up to three loops, only quadratic Casimir appears:

At four-loop, there is a new quartic Casimir which contains non-planar part

L-loop	L=1	L=2	L=3	L=4
Color Factor	C_A	C_A^2	C_A^3	C_A^4, d_{44}

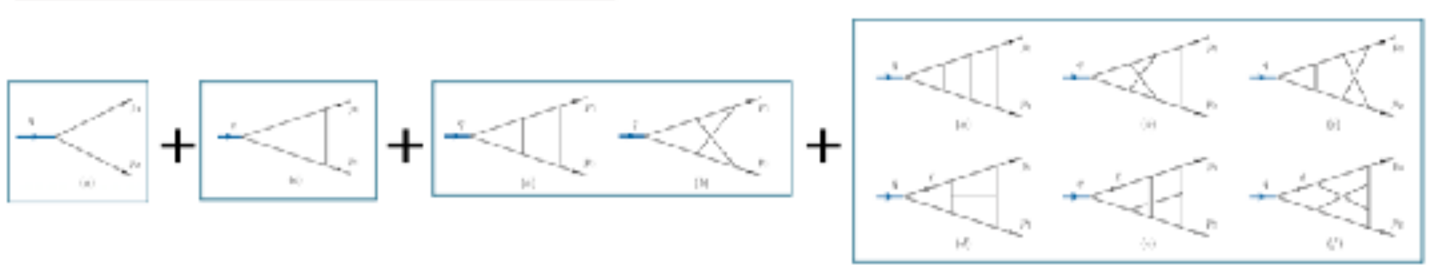
For $SU(N)$:

$C_A = N$

$$d_{44} = \frac{N^2(N^2+36)}{24}$$

Diagram-expansion
up to 3 loops

$$\mathscr{F}^{(l)} = \mathscr{F}^{\text{tree}} \sum_{l=1}^{\infty} g^{2l} (-q^2)^{-l\epsilon} F^{(l)}$$



Casimir scaling conjecture

In "On the Structure of Infrared Singularities of Gauge- Theory Amplitudes",
JHEP 0906, 081 (2009)

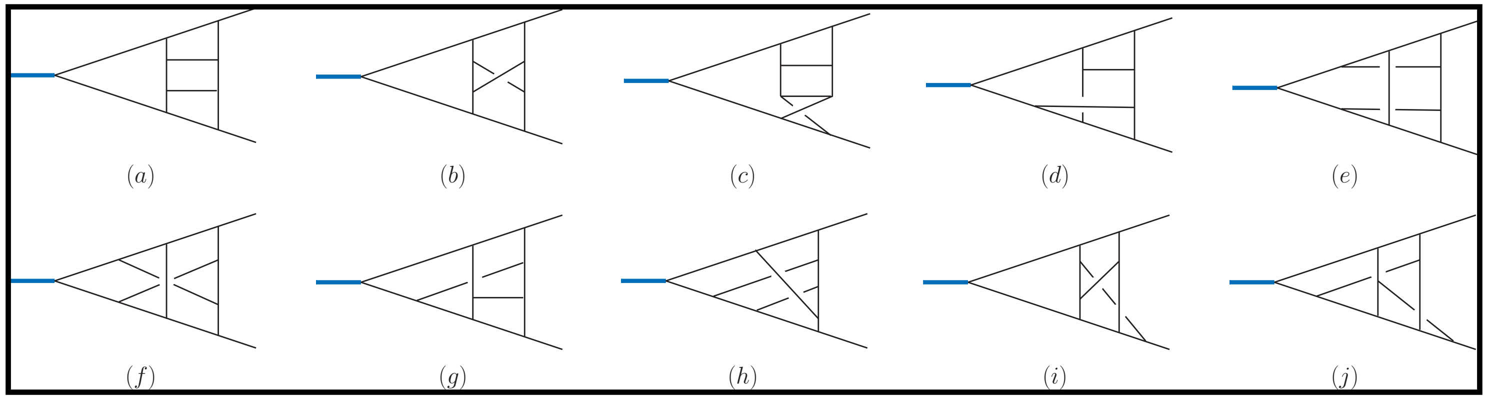
Thomas Becher and Matthias Neubert conjectured that:

"Our formula predicts Casimir scaling of the cusp anomalous dimension to all orders in perturbation theory, and we explicitly check that the constraints exclude the appearance of higher Casimir invariants at four loops."

An explicit four-loop computation is needed.

Four-loop Sudakov form factor

- Integrand: unitarity + color-kinematics duality Boels, Kniehl, Tarasov, GY 2012



- Numerical integration: Boels, Huber, GY 2017

$$\gamma^{(4)}_{\text{cusp, NP}} = -3072 \times (1.60 \pm 0.19) \frac{1}{N_c^2}$$

Finding Uniform Transcendental (UT) basis is the key

- Analytic integration: Huber, von Manteuffel, Panzer, Schabinger, GY 2020
(See also: Henn, Korchemsky, Mistlberger 2020)

$$\gamma^{(4)}_{\text{cusp,NP}} = -3072 \times (\frac{3}{8}\zeta_3^2 + \frac{31}{140}\zeta_2^3)\frac{1}{N_c^2} = -3072 \times 1.52 \frac{1}{N_c^2}$$

Casimir scaling conjecture is incorrect.

Spectrum of YM operators

High dimensional YM operators

Gauge invariant operators:

$$\mathcal{O} \sim \quad c(a_1, ..., a_n)\big(D_{\mu_{11}}...D_{\mu_{1m_1}}F_{\nu_1\rho_1}\big)^{a_1} \cdots \big(D_{\mu_{n1}}...D_{\mu_{nm_n}}F_{\nu_n\rho_n}\big)^{a_n}.$$

$$D_\mu \star = \partial_\mu + ig[A_\mu, \star], \qquad [D_\mu, D_\nu] \star = ig[F_{\mu\nu}, \star] \qquad\qquad F_{\mu\nu} = F^a_{\mu\nu}T^a, \qquad [T^a, T^b] = if^{abc}T^c$$

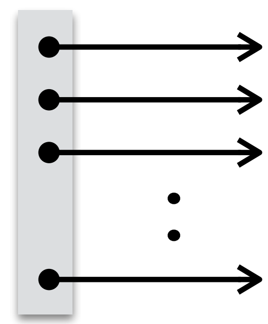
D-dimensional on-shell methods using form factor formalism:

$$\mathcal{F}_n = \int d^4x\, e^{-iq\cdot x}\langle p_1, \ldots, p_n | \mathcal{O}(x) | 0\rangle$$

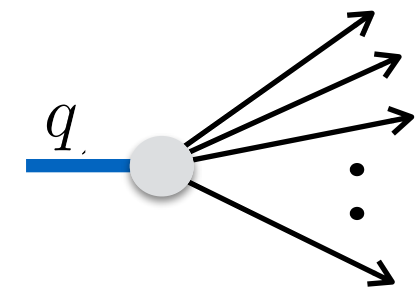
- Operator basis
- Two-loop renormalization and spectrum (e.g dim-16)
- Two-loop EFT amplitudes

1804.04653, 1904.07260, 1910.09384, with Jin; 2011.02494 with Jin, Ren;
2202.08285, 2208.08976, 2301.01786, 2312.08445 with Jin, Ren, Yu

Minimal tree form factors



$$\mathcal{O}_L \;\Leftrightarrow\; \mathcal{F}_{\mathcal{O}_L,L}(1,\ldots,L)$$



Dictionary for YM operators:

operator	$D_{\dot\alpha\alpha}$	$f_{\alpha\beta}$	$\bar f_{\dot\alpha\dot\beta}$
spinor	$\tilde\lambda_{\dot\alpha}\lambda_\alpha$	$\lambda_\alpha\lambda_\beta$	$-\tilde\lambda_{\dot\alpha}\tilde\lambda_{\dot\beta}$

4-dim

$$F_{\mu\nu} \to F_{\alpha\dot\alpha\beta\dot\beta} = \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} f_{\alpha\beta}$$

Used in N=4 SYM: Zwiebel 2011, Wilhelm 2014

operator	D_μ	$F_{\mu\nu}$
kinematics	p_μ	$p_\mu\varepsilon_\nu - p_\nu\varepsilon_\mu$

D-dim

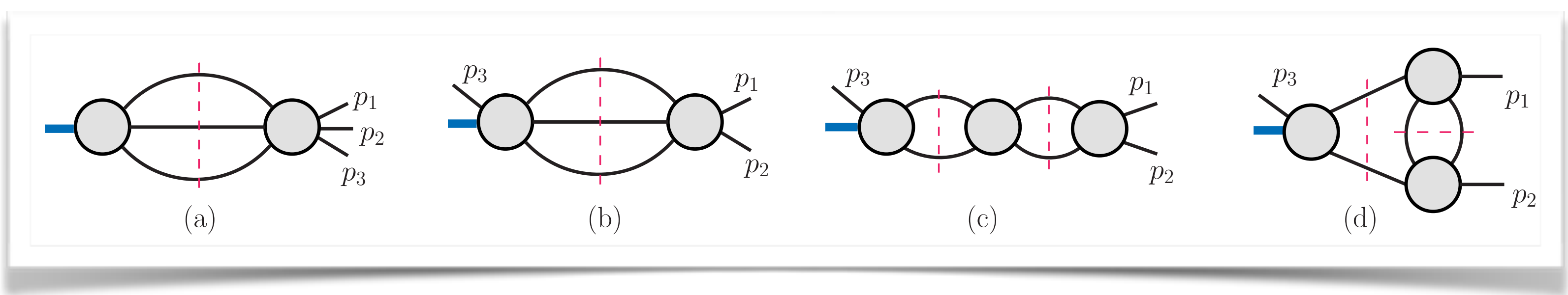
Important for capturing
"Evanescent operators"

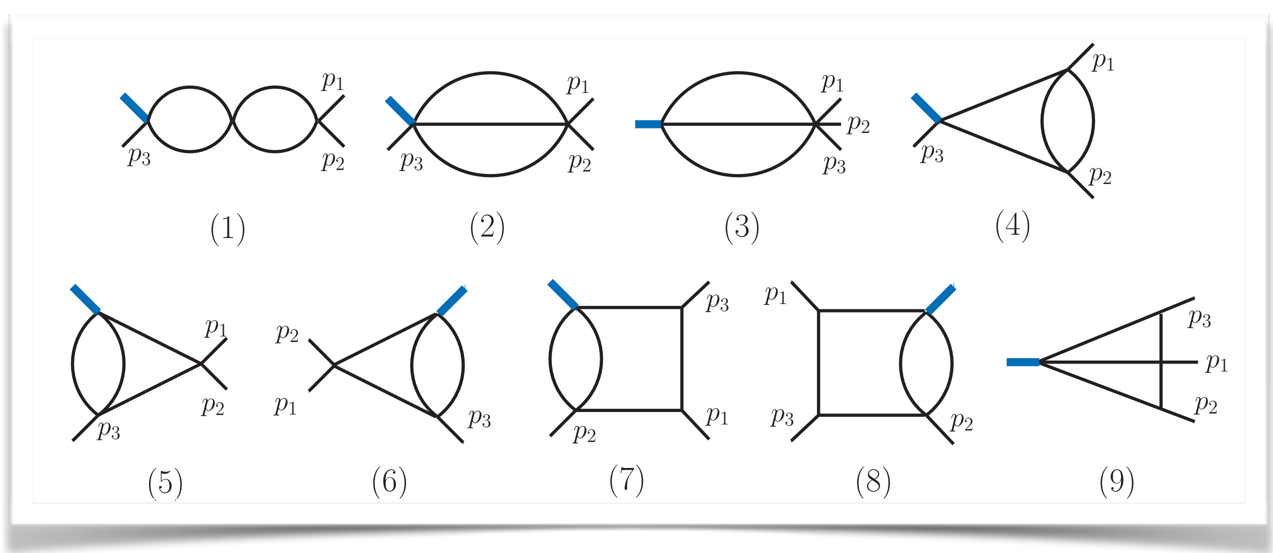
One can translate any local operator into "on-shell" kinematics.

Loop form factor computation

On-shell unitarity-IBP method:

$$\mathcal{F}^{(l)}\Big|_{\rm cut} = \prod (\text{Tree blocks}) = \text{Cut integrand} \xrightarrow{\text{IBP with cuts}} \sum_i c_i \left(I_i \big|_{\rm cut} \right)$$





Evanescent operators

Evanescent operator ("倏逝算符")：
Vanishing in 4 dimension but non-zero in $d = 4 - 2\epsilon$

$$\mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, n\geq L}\Big|_{4\text{-dim}} = 0\,, \qquad \mathbf{F}^{(0)}_{\mathcal{O}^{\rm e}_L, L}\Big|_{d\text{-dim}} \neq 0\,.$$

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Four-fermion dimension-6 operators:

$$\mathcal{O}^{(n)}_{\text{4-ferm}} = \bar{\psi}\gamma^{[\mu_1}\ldots\gamma^{\mu_n]}\psi\bar{\psi}\gamma_{[\mu_1}\ldots\gamma_{\mu_n]}\psi\,, \qquad n \geq 5\,.$$

Buras, Weisz 1990; Dugan, Grinstein 1991; Herrlich and U. Nierste 1994

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Gluonic evanescent operators (start to appear at dimension 10):

$$\mathcal{O}_{\rm e} = \frac{1}{16}\delta^{\mu_1\mu_2\mu_3\mu_4\mu_5}_{\nu_1\,\nu_2\,\nu_3\,\nu_4\,\nu_5}\mathrm{tr}(D_{\nu_5}F_{\mu_1\mu_2}F_{\mu_3\mu_4}D_{\mu_5}F_{\nu_1\nu_2}F_{\nu_3\nu_4}) \qquad \delta^{\mu_1..\mu_n}_{\nu_1...\nu_n} = \det(\delta^{\mu}_{\nu}) = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_1}_{\nu_n} \\ \vdots & & \vdots \\ \delta^{\mu_n}_{\nu_1} & \ldots & \delta^{\mu_n}_{\nu_n} \end{vmatrix}$$

Systematic classification and renormalization at two-loop order.

Jin, Ren, GY, Yu, 2022

Evanescent operators

- *Is Yang-Mills Theory Unitary in Fractional Spacetime Dimension?*

  The answer is NO.

  YM theory is non-unitary in non-integer spacetime dimensions, due to the existence of evanescent operators.

Evanescent operators

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$$
\begin{aligned}
&\partial_\nu\partial_\rho\Big[\delta^{12456\nu}_{3789\mu\rho}\Big(\mathrm{tr}(D_1F_{23}F_{45}D_6F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big],\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2356\nu}_{789\mu\rho}\Big(\mathrm{tr}(D_1F_{23}F_{45}D_6F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2356\nu}_{789\mu\rho}\Big(\mathrm{tr}(D_1F_{23}D_4F_{56}F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2357\nu}_{689\mu\rho}\Big(\mathrm{tr}(D_1F_{23}D_4F_{56}F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2367\nu}_{589\mu\rho}\Big(\mathrm{tr}(D_1F_{23}F_{45}D_6F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{4}\delta^{2378\nu}_{569\mu\rho}\Big(\mathrm{tr}(D_1F_{23}D_4F_{56}F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{1}_{5}\delta^{2347\nu}_{689\mu\rho}\Big(\mathrm{tr}(D_1F_{23}D_4F_{56}F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]\\
&\partial_\nu\partial_\rho\Big[\delta^{2}_{4}\delta^{1567\nu}_{389\mu\rho}\Big(\mathrm{tr}(D_1F_{23}F_{45}D_6F_{78}F_{9\mu})+\mathrm{Rev.}\Big)\Big]
\end{aligned}
$$

Dim-12 evanescent operators

$$
\begin{pmatrix}
-\frac{38}{3\epsilon} & \frac{2}{\epsilon} & -\frac{13}{12\epsilon} & 0 & \frac{14}{3\epsilon} & 0 & \frac{14}{3\epsilon} & \frac{28}{3\epsilon}\\
-\frac{1}{2\epsilon} & -\frac{85}{6\epsilon} & \frac{2}{\epsilon} & \frac{5}{6\epsilon} & -\frac{2}{3\epsilon} & -\frac{5}{12\epsilon} & -\frac{7}{3\epsilon} & -\frac{16}{3\epsilon}\\
0 & -\frac{4}{\epsilon} & -\frac{22}{3\epsilon} & \frac{16}{3\epsilon} & 0 & -\frac{4}{3\epsilon} & 0 & \frac{16}{3\epsilon}\\
0 & -\frac{4}{3\epsilon} & \frac{7}{3\epsilon} & -\frac{34}{3\epsilon} & 0 & -\frac{4}{3\epsilon} & 0 & 0\\
\frac{1}{12\epsilon} & -\frac{1}{12\epsilon} & -\frac{3}{8\epsilon} & \frac{1}{12\epsilon} & -\frac{44}{3\epsilon} & \frac{5}{8\epsilon} & \frac{1}{2\epsilon} & \frac{2}{\epsilon}\\
0 & \frac{4}{3\epsilon} & \frac{2}{3\epsilon} & 0 & 0 & -\frac{18}{\epsilon} & 0 & -\frac{16}{3\epsilon}\\
\frac{1}{6\epsilon} & \frac{3}{2\epsilon} & \frac{9}{16\epsilon} & -\frac{1}{2\epsilon} & \frac{29}{6\epsilon} & -\frac{5}{12\epsilon} & -\frac{49}{6\epsilon} & \frac{13}{3\epsilon}\\
-\frac{5}{6\epsilon} & -\frac{1}{3\epsilon} & \frac{13}{32\epsilon} & -\frac{5}{6\epsilon} & \frac{3}{4\epsilon} & \frac{1}{4\epsilon} & \frac{5}{12\epsilon} & -\frac{91}{6\epsilon}
\end{pmatrix}
$$

One-loop mixing matrix

A pair of complex eigenvalues:

$$1.90386 \pm 0.181142\,\mathrm{i}\,.$$

Jin, Ren, GY, Yu, 2023

“Color” evanescent operators

- *How about Yang-Mills Theory with fractional Nc (rank of gauge group)?*

Mathematically we make analytical continuation for Nc and consider AD as a function of Nc.

In this case, there are also complex AD due to the existence of “color” evanescent operators.

$$\delta^{i_1 i_2 i_3}_{j_1 j_2 j_3} T^{a_1}_{i_1 j_1} T^{a_2}_{i_2 j_2} T^{a_3}_{i_3 j_3} = \mathrm{tr}(123) + \mathrm{tr}(132) = d^{a_1 a_2 a_3}$$

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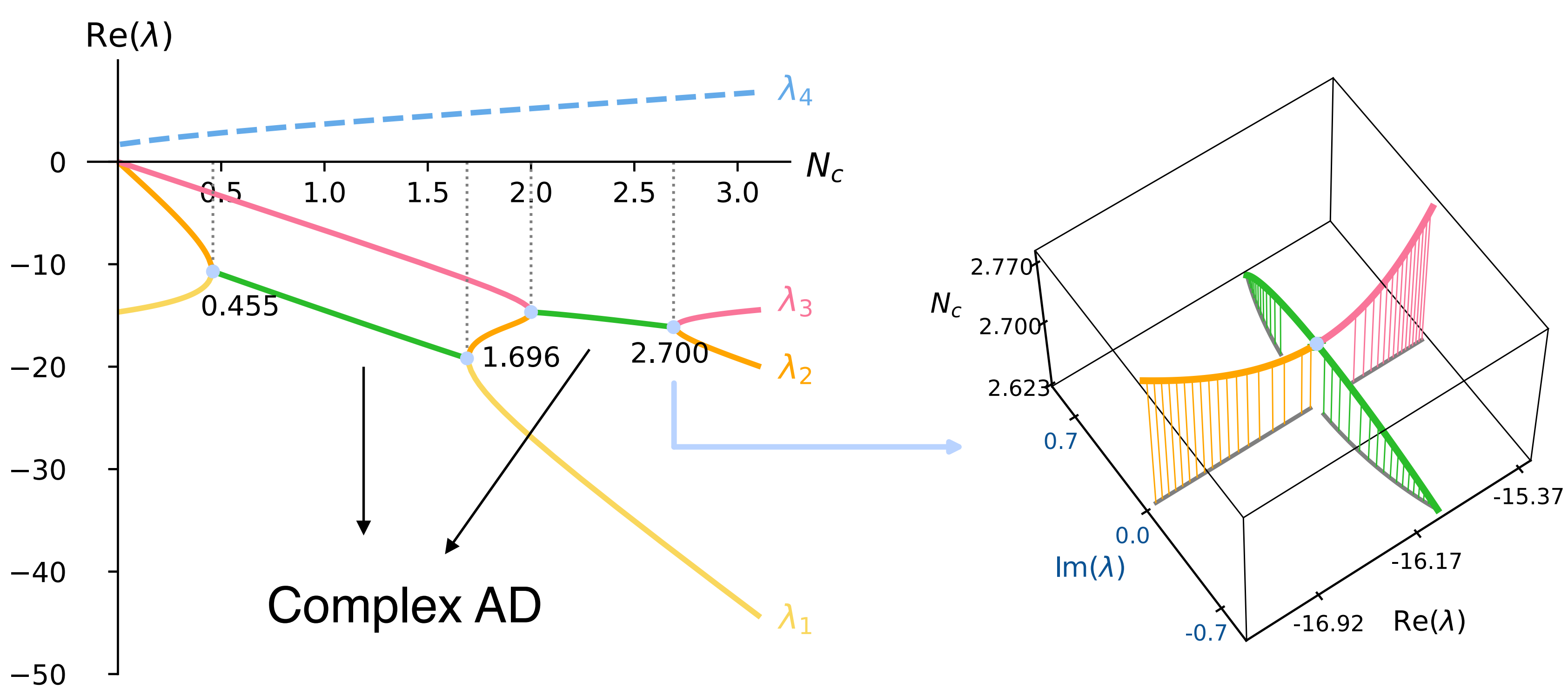

A sector of dim-8 operators

Jin, Ren, GY, Yu, to appear

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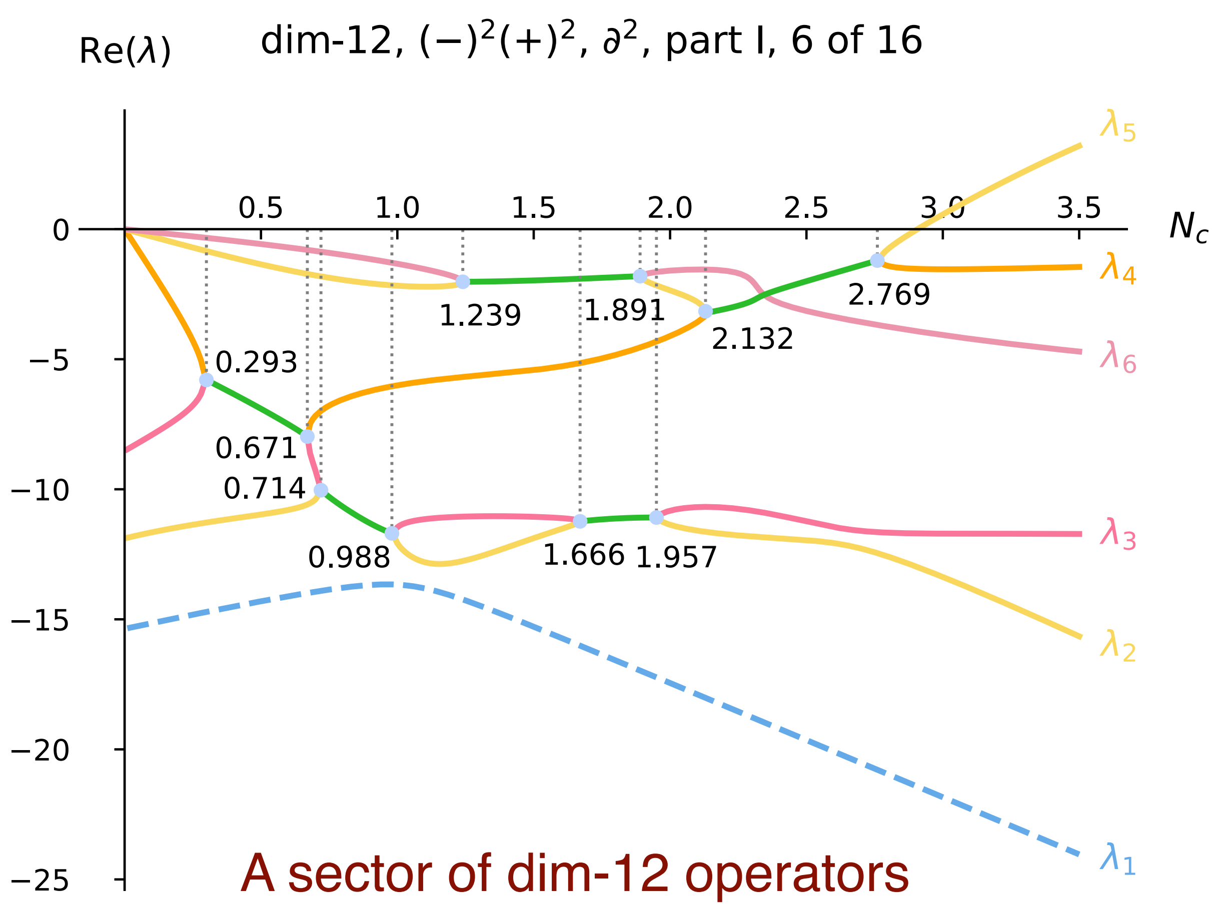


A sector of dim-12 operators

Rich structure for higher dimensional operators

Jin, Ren, GY, Yu, to appear

Summary

On-shell Amplitudes

Form Factors ⟷

Off-shell Operators

- Efficient new methods
- New higher-loop results
- Novel structures uncovered

- More to be explored !

Thank you for your attention!

Thank you for your attention!

Backup slides

Evanescent operators

Evanescent operators are important for renormalization beyond one-loop order.

$$\begin{pmatrix} Z_{\rm pp}^{(1)} & Z_{\rm pe}^{(1)} \\ 0 & Z_{\rm ee}^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} Z_{\rm pp}^{(l)} & Z_{\rm pe}^{(l)} \\ Z_{\rm ep}^{(l)} & Z_{\rm ee}^{(l)} \end{pmatrix}, \quad l \geq 2$$

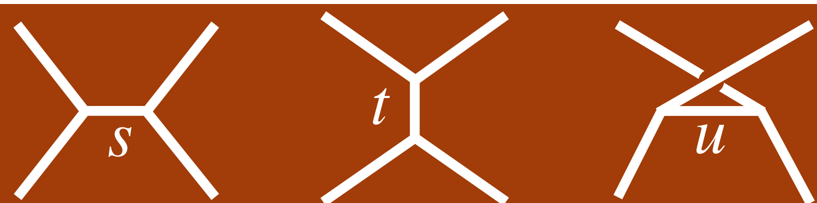
One can use finite renormalization scheme such that

$$\begin{pmatrix} \hat{\mathcal{D}}_{\rm pp}^{(l)} & \hat{\mathcal{D}}_{\rm pe}^{(l)} \\ 0 & \hat{\mathcal{D}}_{\rm ee}^{(l)} \end{pmatrix}$$

but the lower-loop evanescent operator result are needed.

For example, $\hat{\mathcal{D}}_{\rm pp}^{(2)}$ contains $(-2\epsilon \hat{Z}_{\rm pe}^{(1)} \hat{Z}_{\rm ep}^{(1)})$

Color-kinematics duality



$$A_4 = \frac{C_s N_s}{s} + \frac{C_t N_t}{t} + \frac{C_u N_u}{u}$$

Color factors

$$C_s = C_t + C_u$$

Jacobian identity

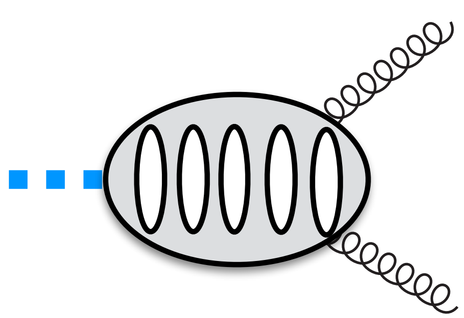
→

Momentum factors

$$N_s = N_t + N_u$$

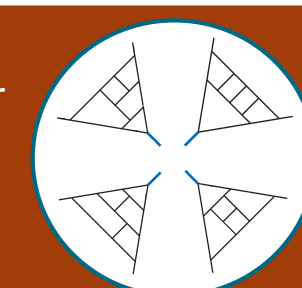
dual Jacobian relation

Large number of diagrams —CK-duality→ Very few “master” diagrams



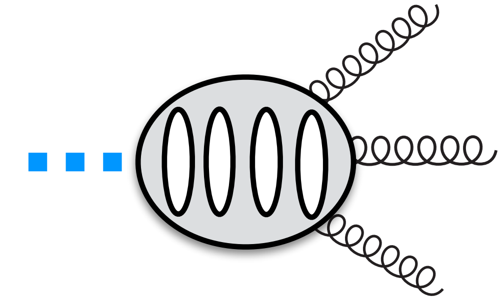
L-loop	*L=1*	*L=2*	*L=3*	*L=4*	*L=5*
# of topologies	*1*	*2*	*6*	*34*	*306*
# of masters	*1*	*1*	*1*	*2*	*4*

Four master graphs @ 5-loop:

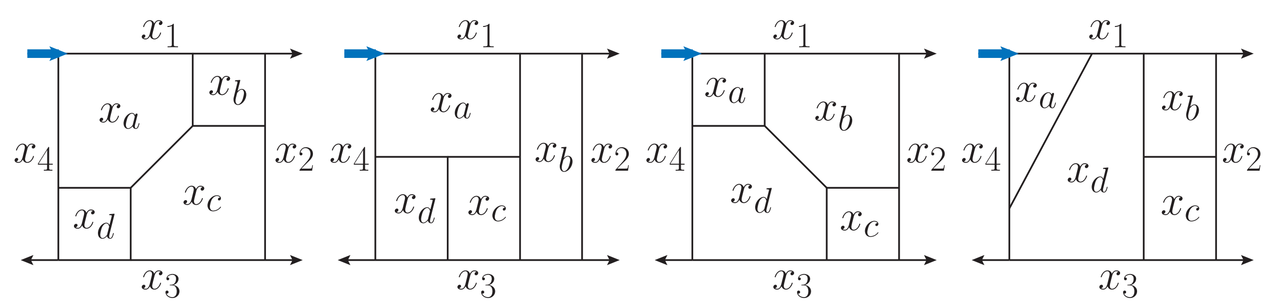


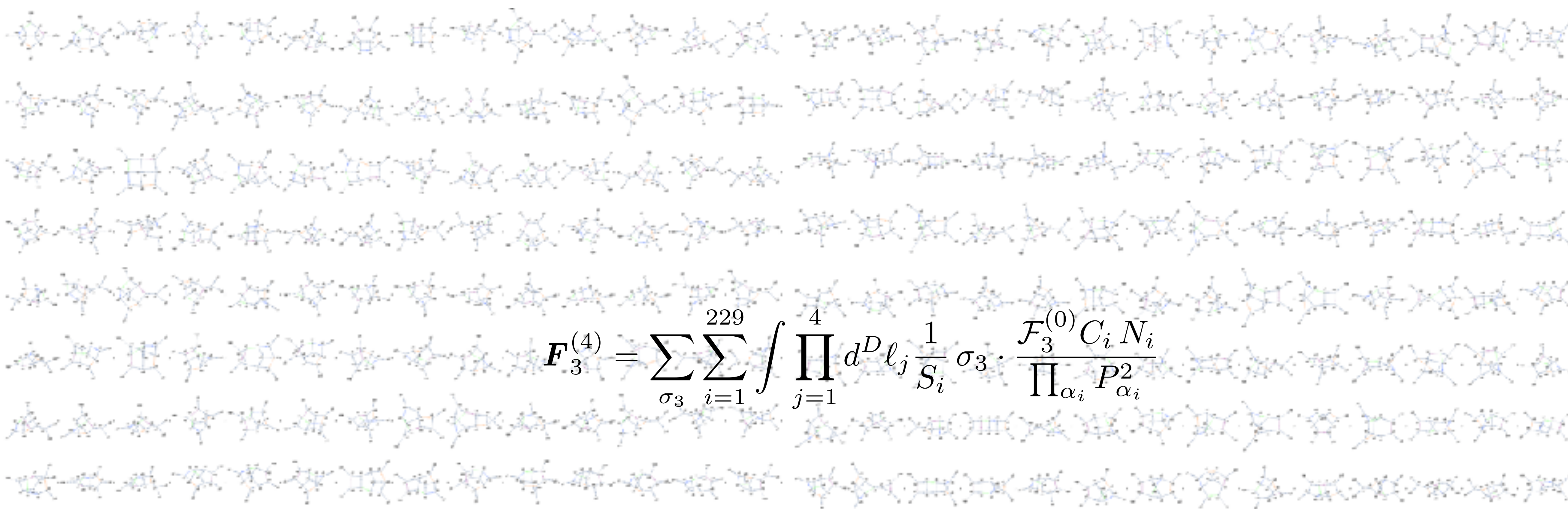
GY, PRL 2016

Three-point form factors



Master graphs

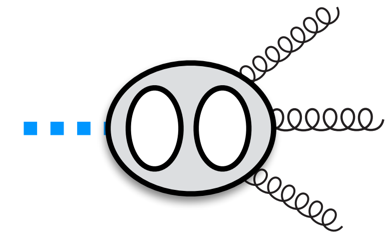




$$\boldsymbol{F}_3^{(4)} = \sum_{\sigma_3} \sum_{i=1}^{229} \int \prod_{j=1}^{4} d^D \ell_j \, \frac{1}{S_i} \, \sigma_3 \cdot \frac{\mathcal{F}_3^{(0)} C_i \, N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

Maximal transcendentality part

N=4 SYM three-point form factor



$$-2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]$$
$$-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)+\frac{\log^2 u_i}{2!}\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4$$

Brandhuber, Travaglini, GY 2012

The same !

$$-2G(0,0,1,0,u)+G(0,0,1-v,1-v,u)+2G(0,0,-v,1-v,u)-G(0,1,0,1-v,u)+4G(0,1,1,0,u)-G(0,1,1-v,0,u)+G(0,1-v,0,1-v,u)$$
$$+G(0,1-v,1-v,0,u)-G(0,1-v,-v,1-v,u)+2G(0,-v,0,1-v,u)+2G(0,-v,1-v,0,u)-2G(0,-v,1-v,1-v,u)-2G(1,0,0,1-v,u)$$
$$-2G(1,0,1-v,0,u)+4G(1,1,0,0,u)-4G(1,1,1,0,u)-2G(1,1-v,0,0,u)+G(1-v,0,0,1-v,u)-G(1-v,0,1,0,u)-2G(-v,1-v,1-v,u)H(0,v)$$
$$-2G(1-v,1,0,0,u)+2G(1-v,1,0,1-v,u)+2G(1-v,1,1-v,0,u)+G(1-v,1-v,0,0,u)+2G(1-v,1-v,1,0,u)-2G(1-v,1-v,-v,1-v,u)$$
$$-G(1-v,-v,1-v,0,u)+4G(1-v,-v,-v,1-v,u)-2G(-v,0,1-v,1-v,u)-2G(-v,1-v,0,1-v,u)-2G(-v,1-v,1-v,0,u)+4G(1,0,1,0,u)$$
$$+4G(-v,-v,1-v,1-v,u)-4G(-v,-v,-v,1-v,u)-G(0,0,1-v,u)H(0,v)-G(0,1,0,u)H(0,v)-G(0,1-v,0,u)H(0,v)+G(0,1-v,1-v,u)H(0,v)$$
$$-G(0,-v,1-v,u)H(0,v)-2G(1,0,0,u)H(0,v)+G(1,0,1-v,u)H(0,v)+G(1,1-v,0,u)H(0,v)+G(1-v,0,0,u)H(0,v)-G(1-v,0,1-v,u)H(0,v)$$
$$-G(1-v,1,0,u)H(0,v)-G(1-v,1-v,0,u)H(0,v)-G(1-v,-v,1-v,u)H(0,v)+G(-v,0,1-v,u)H(0,v)+G(-v,1-v,0,u)H(0,v)+H(1,0,0,1,v)$$
$$-G(0,0,1-v,u)H(1,v)-G(0,0,-v,u)H(1,v)+G(0,1,0,u)H(1,v)-G(0,1-v,0,u)H(1,v)+G(0,1-v,-v,u)H(1,v)-2G(0,-v,0,u)H(1,v)$$
$$+2G(0,-v,1-v,u)H(1,v)+2G(1,0,0,u)H(1,v)-G(1-v,0,0,u)H(1,v)+G(1-v,0,-v,u)H(1,v)-2G(1-v,1,0,u)H(1,v)-G(1-v,0,-v,1-v,u)$$

$+G(1-v,$ … $-v,1-v,-v,u)H(1,v)$

$-4G(-v,-$ … $H(0,0,v)+H(1,0,1,0,v)$

$-G(0,0,u)$ … $,v)-3G(1-v,-v,u)H(0,1,v)$

$-G(-v,0,$ … $)-G(1,0,u)H(1,0,v)$

$+2G(1-v$ … $v,u)H(1,0,v)+G(0,0,u)H(1,1,v)$

$-2G(0,-v$ … $v)+4G(-v,u)H(0,0,1,v)$

$$+G(0,u)H(0,1,0,v)+G(1-v,u)H(0,1,0,v)-G(0,u)H(0,1,1,v)+2G(-v,u)H(0,1,1,v)+G(0,u)H(1,0,0,v)+G(1-v,u)H(1,0,0,v)+H(1,1,0,0,v)$$
$$-G(0,u)H(1,0,1,v)+2G(-v,u)H(1,0,1,v)-G(0,u)H(1,1,0,v)+4G(1-v,u)H(1,1,0,v)-2G(-v,u)H(1,1,0,v)+H(0,0,1,1,v)+H(0,1,0,1,v)$$
$$+G(1-v,1-v,u)H(0,0,v)+2G(1-v,1-v,-v,u)H(1,v)-G(1-v,-v,0,1-v,u)+H(0,1,1,0,v)+G(1-v,0,1-v,0,u)-G(0,1-v,1,0,u)$$
$$+4G(-v,1-v,-v,1-v,u)$$

Maximally transcendental part of QCD Higgs-3-gluon amplitude

Gehrmann, Jaquier, Glover, Koukoutsakis 2011