Renormalization-Group Equation for the Bottom-Meson Soft Function

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B-meson Soft Functions

- Definition of two-particle B-meson LCDAs [Beneke,Feldmann,2001;Beneke,Braun,Ji,Wei,2018]:

$$\langle 0|\bar{q}(x)\,\Gamma[x,0]\,h_v(0)|\bar{B}(v)\rangle = -\frac{i\tilde{f}_B(\mu)m_B}{4}\,\mathrm{Tr}\Big[\frac{1+\not{v}}{2}\Big\{2\,\Big(\Phi_B^+(v\cdot x)+x^2G_B^+(v\cdot x)\Big) - \frac{\not{x}}{v\cdot x}\Big[\Big(\Phi_B^+(v\cdot x)-\Phi_B^-(v\cdot x)\Big)+x^2\,\Big(G_B^+(v\cdot x)-G_B^-(v\cdot x)\Big)\Big]\Big\}\gamma_5\Gamma\Big].$$

- Definition of three-particle B-meson LCDAs [Kawamura,Kodaira,Qiao,Tanaka,2001; Braun,Ji,Manashov,2017]:

$$\begin{aligned}&\langle 0|\bar{q}_\alpha(z_1\bar{n})g_sG_{\mu\nu}(z_2\bar{n})h_{\nu\beta}(0)|\bar{B}(v)\rangle\\ &=\frac{\tilde{f}_B(\mu)m_B}{4}\Big[(1+\not{v})\Big\{(v_\mu\gamma_\nu-v_\nu\gamma_\mu)\left[\Psi_A(z_1,z_2)-\Psi_V(z_1,z_2)\right]-i\sigma_{\mu\nu}\Psi_V(z_1,z_2)\\ &-(\bar{n}_\mu v_\nu-\bar{n}_\nu v_\mu)X_A(z_1,z_2)+(\bar{n}_\mu\gamma_\nu-\bar{n}_\nu\gamma_\mu)\left[W(z_1,z_2)+Y_A(z_1,z_2)\right]\\ &+i\epsilon_{\mu\nu\alpha\beta}\bar{n}^\alpha v^\beta\gamma_5\tilde{X}_A(z_1,z_2)-i\epsilon_{\mu\nu\alpha\beta}\bar{n}^\alpha\gamma^\beta\gamma_5\tilde{Y}_A(z_1,z_2)\\ &-(\bar{n}_\mu v_\nu-\bar{n}_\nu v_\mu)\,\not{\bar{n}}\,W(z_1,z_2)+(\bar{n}_\mu\gamma_\nu-\bar{n}_\nu\gamma_\mu)\,\not{\bar{n}}\,Z(z_1,z_2)\Big\}\gamma_5\Big]_{\beta\alpha}.\end{aligned}$$

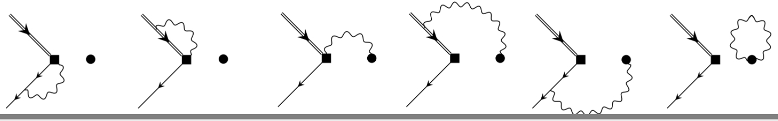
  - Significant in precision calculations for the B-meson decays.

B-meson Soft Functions

- Soft function in $B \to \mu^+\mu^-$ [Beneke, Bobeth, Szafron,2019]:

$$\frac{1}{R_c R_{\bar{c}}} \langle 0|\bar{q}_s(tn)[tn,0]\not{n}\gamma_5 h_v(0)(S_{\bar{n}}^\dagger S_n)(0)|\bar{B}\rangle$$

  - ▷ The soft Wilson lines are decoupling with (anti-)collinear lepton in QED.
  - ▷ Non-universal, structure dependent corrections.



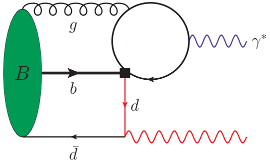
Anomalous Dimension [Beneke, *etc*,2022]:

$$\gamma_{-+}(\omega,\omega') = \frac{\alpha_s C_F}{4\pi}\left[\left(\ln\frac{\mu}{\omega - i0} - \frac{1}{2}\right)\delta(\omega-\omega') - H_+(\omega,\omega')\right]$$
$$- \frac{\alpha_{\rm em}}{\pi}\left[\left(\frac{5}{9}\ln\frac{\mu}{\omega - i0} + \frac{5}{36} + \frac{2\pi i}{3}\right)\delta(\omega-\omega') + \frac{1}{9}H_+(\omega,\omega') - \frac{1}{3}H_-(\omega,\omega')\right].$$

▷ Imaginary part; different support properties: $\omega \in (-\infty, +\infty)$.

B-meson Soft Functions

- ∇ B-meson Soft Functions generated by charm-loop effect.
  - ▷ How important are the soft gluons emitted from the c-quark loop and violating the factorization?
  - ▷ The validity of the approximation "c-quark-loop plus corrections" at large q^2, approaching the charmonium resonance region.

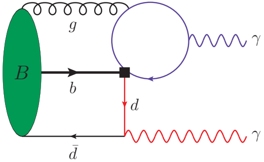


- The hard c-quark loop with the emitted soft gluon generates the soft function for $B \to K^{(*)}\ell^+\ell^-$ decays.[Khodjamirian,Mannel,Pivovarov,Wang,2010]

$$\langle 0|\bar{d}(y)\,\delta[\omega - (in \cdot D)/2]\,G_{\mu\nu}(0)h_v(0)|\bar{B}\rangle.$$

  - ▷ Parameterized by the standard three-particle B-meson LCDAs $(\Psi_A, \Psi_V, X_A, Y_A)$.

B-meson Soft Functions



- The factorization formula of the long-distance penguin contribution to the $B_{d,s} \to \gamma\gamma$ decays has been expressed as [Qin,Shen,Wang,Wang,2022]

$$\frac{Q_q\, f_B}{m_B} H \int d\omega_1\, \frac{J(\omega_1)}{\omega_1} \int d\omega_2 \frac{\bar{J}(\omega_2)}{\omega_2}\, \Phi_{\rm G}(\omega_1, \omega_2, \mu)\,.$$

  - ▷ The anti-hard-collinear c-quark loop with the emitted soft gluon generates the subleading distribution amplitude of the B-meson,

$$2F_B\, m_B \Phi_G(\omega_1, \omega_2, \mu) = \int_{-\infty}^{\infty} d\tau_1\, d\tau_2\, \text{Exp}\left[i\omega_1\, \tau_1 + i\omega_2\, \tau_2\right]$$
$$\langle 0|\, \bar{q}_s(\tau_1\, n)\, [\tau_1\, n,\, 0]\, [0,\, \tau_2 \bar{n}] g_s\, G_{\mu\nu}(\tau_2\, \bar{n})\, [\tau_2\, \bar{n},\, 0]\, \bar{n}^\nu\, \not{n}\, \gamma_\perp^\mu\, \gamma_5\, h_v(0)\, |\bar{B}_v\rangle\,.$$

B-meson Soft Functions

- The hard-collinear c-quark loop with the emitted soft gluon generates the subleading distribution amplitude of the B-meson,

$$2F_B\, m_B \Phi_{\mathrm{G}}(\omega_1, \omega_2, \mu) = \int_{-\infty}^{\infty} d\tau_1\, d\tau_2\, \mathrm{Exp}\left[i\omega_1\, \tau_1 + i\omega_2\, \tau_2\right]$$
$$\langle 0|\, \bar{q}_s(\tau_1\, n)\, [\tau_1\, n,\, 0]\, [0,\, \tau_2 \bar{n}] g_s\, G_{\mu\nu}(\tau_2\, \bar{n})\, [\tau_2\, \bar{n},\, 0]\, \bar{n}^\nu\, \not{n}\, \gamma_\perp^\mu\, \gamma_5\, h_v(0)\, |\bar{B}_v\rangle\, .$$

  - ▷ The non-local operator in HQET with quark and gluon localized on the different light-cone directions, is different with the traditional three-particle LCDAs.
  - ▷ The analogous subleading shape function is also needed in inclusive $B \to X_s \gamma$ decay. [Benzke, Lee, Neubert, Paz,2010]

$$\langle \bar{B}|\bar{h}(tn)\, [tn, 0]\, [0, r\bar{n}]\, G_{\mu\nu}(r\bar{n})[r\bar{n}, 0]\, \bar{n}^\nu \bar{\not{n}} \gamma_\perp^\mu (1 - \gamma_5)\, h(0)|\bar{B}\rangle\, .$$

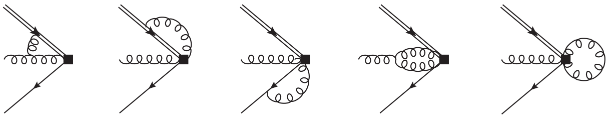
Motivation

Renormalization-Group Equation for $\Phi_{\rm G}$:

$$\frac{d}{d\ln\mu}\,\Phi_{\rm G}(\omega_1,\omega_2,\mu) = \int_{-\infty}^{\infty} d\omega_1'\, d\omega_2'\, \Gamma_{\rm G}(\omega_1,\omega_2;\omega_1',\omega_2')\, \Phi_{\rm G}(\omega_1',\omega_2',\mu)\,.$$

- Why do we compute the RGE of the soft function $\Phi_{\rm G}$?
  - To understand the infrared structure for the exclusive heavy-hadron decay amplitude at next-to-leading power.
  - To pin down and reduce the theory uncertainties.
  - It is necessary to establish the factorization formulae of the long-distance penguin contributions to the exclusive $b \to q\ell^+\ell^-$ and $b \to q\gamma$ decays.

RGE for Soft Function



Anomalous Dimension for $\Phi_{\rm G}$

$$\Gamma_{\rm G}(\omega_1,\omega_2;\omega_1',\omega_2') = \frac{\alpha_s}{\pi}\Bigg[\frac{C_A}{4\pi i}[H_+(\omega_1,\omega_1') - H_-(\omega_1,\omega_1') - 2i\pi\delta(\omega_1-\omega_1')]$$
$$\times [H_+(\omega_2,\omega_2') - H_-(\omega_2,\omega_2') - 2i\pi\delta(\omega_2-\omega_2')]$$
$$+ C_F H_+(\omega_1,\omega_1')\delta(\omega_2-\omega_2') + C_A[H_+(\omega_2,\omega_2') + \Delta H(\omega_2,\omega_2')]\delta(\omega_1-\omega_1')$$
$$- \delta(\omega_1-\omega_1')\delta(\omega_2-\omega_2')\left[C_F \ln\frac{\mu}{\omega_1 - i0} + C_A \ln\frac{\mu}{\omega_2 - i0} - \frac{1}{2}C_F + \frac{i\pi}{2}C_A\right]\Bigg].$$

- The soft function $\Phi_{\rm G}$ is in general complex-valued function.
- The soft function acquires support for $\omega_i \in (-\infty, +\infty)$, arising from the interaction of the Wilson lines localized on two distinct light-cone directions (diagram (e)).

Solution In Laplace Space

- In Laplace space, the soft function and RGE are defined as :

$$\tilde{\Phi}^{>(<)>(<)}(\eta_1, \eta_2, \mu) = \int_0^\infty d\omega_1\, d\omega_2 \frac{1}{\omega_1\, \omega_2} \left(\frac{\mu}{\omega_1}\right)^{\eta_1} \left(\frac{\mu}{\omega_2}\right)^{\eta_2} \Phi_G(\pm\omega_1, \pm\omega_2, \mu)\,;$$

$$\left(\frac{d}{d\ln\mu} - \eta_1 - \eta_2\right) \tilde{\Phi}^i(\eta_1, \eta_2, \mu) = \tilde{\Gamma}^{ij}(\eta_1, \eta_2) \tilde{\Phi}^j(\eta_1, \eta_2, \mu)\,.$$

- The Anomalous Dimension Matrix for the four soft function in Laplace space is non-diagonal due to the mixing between the region $\omega_i > 0$ and $\omega_i < 0$.

$$H_+ \Rightarrow \begin{pmatrix} H_\eta + H_{-\eta} & \Gamma(\eta)\Gamma(1-\eta) \\ 0 & H_{-1-\eta} + H_{-\eta} \end{pmatrix},$$

$$H_- \Rightarrow \begin{pmatrix} H_{-1-\eta} + H_{-\eta} + i\pi & 0 \\ \Gamma(\eta)\Gamma(1-\eta) & H_\eta + H_{-\eta} + i\pi \end{pmatrix}.$$

  - To solve the system of partial differential equations.

Solution In Laplace Space

- Both transformations are related linearly to the function:

$$\tilde{\Phi}_{\text{dia}}^{\pm\pm}(\eta_1,\eta_2,\mu) = \int_{-\infty}^{\infty} \frac{d\omega_1\, d\omega_2}{\mu^2} \left(\frac{\mu}{\omega_1 \pm i0}\right)^{\eta_1+1} \left(\frac{\mu}{\omega_2 \pm i0}\right)^{\eta_2+1} \Phi_{\text{G}}(\omega_1,\omega_2,\mu)\,.$$

$$\tilde{\Phi}_{\text{dia}}^{\pm\pm} = \tilde{\Phi}^{>>} - e^{\mp i\pi\eta_2}\,\tilde{\Phi}^{><} - e^{\mp i\pi\eta_1}\,\tilde{\Phi}^{<>} + e^{\mp i\pi\eta_2 \mp i\pi\eta_1}\,\tilde{\Phi}^{<<}\,.$$

  - Consitant analyticity for ω_i and ω_i'

$$\left(\frac{\mu}{\omega_1 \pm i0}\right)^{\eta_1+1} \left(\frac{\mu}{\omega_2 \pm i0}\right)^{\eta_2+1} \otimes \Gamma_{\text{G}} = \left(\frac{\mu}{\omega_1' \pm i0}\right)^{\eta_1+1} \left(\frac{\mu}{\omega_2' \pm i0}\right)^{\eta_2+1} \times \tilde{\Gamma}^{\pm\pm}\,.$$

- The Anomalous Dimension Matrix is diagonal.

$$\begin{aligned}\tilde{\Gamma}_{\text{dia}}(\eta_1,\eta_2,\mu) &= -\frac{i\pi}{2} C_A\, \text{Dia}\{1\,,1\,,1\,,-1\} - \text{I}_{4\times 4} \\ &\times \left[C_A\left(\partial_{\eta_2} + H_{\eta_2} + H_{-\eta_2} - \frac{1}{\eta_2 - 1}\right) + C_F\left(\partial_{\eta_1} + H_{\eta_1} + H_{-\eta_1} - \frac{1}{2}\right)\right].\end{aligned}$$

  - To solve four independent partial differential equations.

Solution In Laplace Space

- The solution of soft function $\tilde{\Phi}^{i}_{\rm dia}(i=++,+-,-+,--)$ in Laplace space is

$$\tilde{\Phi}^{i}_{\rm dia}(\eta_1,\eta_2,\mu) = \hat{U}_{ij}(\eta_1,\eta_2,\mu,\mu_0)\tilde{\Phi}^{j}_{\rm dia}(\eta_1+a_1,\eta_2+a_2,\mu_0)$$

$$\hat{U}(\eta_1,\eta_2,\mu,\mu_0) = e^{V}\prod_{k=1}^{2} e^{2\gamma_E a_k}\left[\left(\frac{\mu}{\mu_0}\right)^{\eta_k}\frac{\Gamma(k-\eta_k)\Gamma(1+\eta_k+a_k)}{\Gamma(1+\eta_k)\Gamma(k-\eta_k-a_k)}\right];$$

$$\times\,{\rm diag}\{1\,,1\,,1\,,e^{-i\pi a_2}\}\,.$$

The solution in momentum space can be obtained by performing linear transformation and inverse Laplace transformation.

Numerical analysis

- Take the exponential model at $\mu_0 = 1.0\,\mathrm{GeV}$,

$$\Phi_{\rm G}(\omega_1,\omega_2,\mu_0) = \theta(\omega_1)\theta(\omega_2)\,\frac{\lambda_E^2+\lambda_H^2}{3}\,\frac{\omega_1\,\omega_2^2}{2\omega_0^5}\,e^{-\frac{\omega_1+\omega_2}{\omega_0}}\,;$$

$$\Phi_{\rm G}(\omega_1,\omega_2,\mu) = \frac{\lambda_E^2+\lambda_H^2}{6}\,\frac{e^V}{\omega_0^2}\left(\frac{\mu_0\,e^{2\gamma_E}}{\omega_0}\right)^{a_1+a_2}\left[\prod_{k=1}^{2}\theta(\omega_k)\,G_{1,2}^{1,1}\left(\begin{smallmatrix}-a_k\\k,0\end{smallmatrix}\middle|\frac{\omega_k}{\omega_0}\right)\right.$$
$$\left.+\frac{\left[1-e^{-i\pi a_2}\right]}{4\pi^2}\prod_{k=1}^{2}\left[G_{1,2}^{2,1}\left(\begin{smallmatrix}-a_k\\k,0\end{smallmatrix}\middle|-\frac{\omega_k}{\omega_0}\right)+2i\pi\theta(\omega_k)G_{1,2}^{1,1}\left(\begin{smallmatrix}-a_k\\k,0\end{smallmatrix}\middle|\frac{\omega_k}{\omega_0}\right)\right]\right].$$

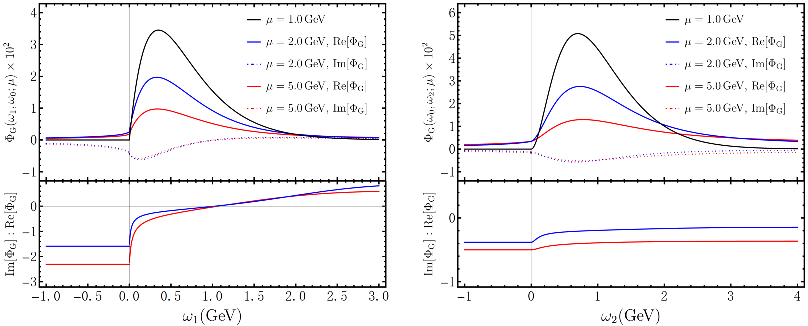
  - ▷ The soft function acquires a constant value in the limit $\omega_{1,2} \to 0$

$$\Phi_{\rm G}(\omega_1\to 0,\omega_2\to 0,\mu) = \frac{\lambda_E^2+\lambda_H^2}{6}\,\frac{e^V}{\omega_0^2}\left(\frac{\mu_0\,e^{2\gamma_E}}{\omega_0}\right)^{a_1+a_2}\frac{1-e^{-i\pi a_2}}{4\pi^2}\Gamma(1+a_1)\Gamma(1+a_2)\,.$$

  - ▷ The large momentum behaviour turns out to be analogous to the behaviour of the leading-twist B-meson LCDA ϕ_B^+

$$\Phi_{\rm G}(\omega_1\to\pm\infty,\omega_2\to\pm\infty,\mu) \propto \left(\frac{\omega_0}{\pm\omega_1}\right)^{1+a_1}\left(\frac{\omega_0}{\pm\omega_2}\right)^{1+a_2}.$$

Numerical analysis



- ▷ Taking into account the one-loop QCD evolution can bring about the $\mathcal{O}(50\ \%)$ reduction in the peak region at $\mu = 2.0$ GeV.
- ▷ In particular it can generate the imaginary part: approximately $\mathcal{O}(20\ \%)$ of the corresponding real part.

Summary

- We have derived the RGE of the generalized B-meson distribution amplitude, defined with partonic fields localized on two distinct light-cone directions.
- The UV renormalization kernel consists in the novel pattern of mixing positive into negative support, thus extending the support region to $-\infty < \omega_i < \infty$.
- The RG evolution is numerically important, bring about the $\mathcal{O}(50\%)$ reduction for the real part and generating the imaginary part about $\mathcal{O}(20\%)$ of the real part.
- Extending our RG analysis to the generic soft functions will be beneficial for evaluating the long-distance charming penguin contributions.

Thank you for your attention!