

Analytic third-order QCD corrections to top-quark and semileptonic $b \rightarrow u$ decays

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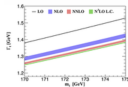
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Editors' Suggestion

Letter

Analytic third-order QCD corrections to top-quark and semileptonic $b \rightarrow u$ decays

Long-Bin Chen, Hai Tao Li, Zhao Li, Jian Wang, Yefan Wang, and Quan-feng Wu
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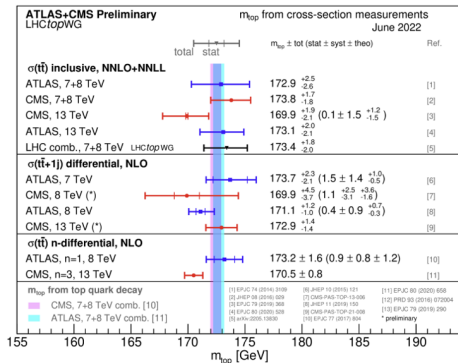
The authors compute the leading color contribution to the third-order QCD correction to the top quark decay width analytically. They additionally obtain the leading color third-order QCD correction to the inclusive semileptonic $b \rightarrow u$ decay.

[Show Abstract +](#)

Motivation

Top-quark mass is one of the fundamental parameters in the Standard Model.

Summary of the top-mass analyses at the LHC.

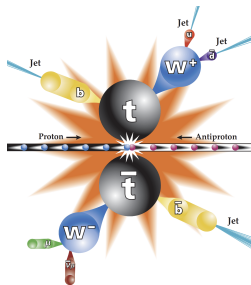


Motivation

Top decay width Γ_t is one of the fundamental properties of top-quark.

Due to its large mass, Γ_t is expected to be very large.

The measurement of Γ_t could hint new physics.



[Denisov, Vellidis 2015]

Motivation

The top-quark decays **almost exclusively to Wb** . $\Gamma_t = \Gamma_t(t \rightarrow Wb)$.

$$|V_{\text{CKM}}| = \begin{pmatrix} 0.97435 \pm 0.00016 & 0.22500 \pm 0.00067 & 0.00369 \pm 0.00011 \\ 0.22486 \pm 0.00067 & 0.97349 \pm 0.00016 & 0.04182^{+0.00085}_{-0.00074} \\ 0.00857^{+0.00020}_{-0.00018} & 0.04110^{+0.00083}_{-0.00072} & 0.999118^{+0.000031}_{-0.000036} \end{pmatrix}$$

[PDG 2022]

The most precise measurement for Γ_t by now is given by CMS

$$\Gamma_t = 1.36 \pm 0.02 \text{ (stat.)}_{-0.11}^{+0.14} \text{ (syst.) GeV [CMS, 2014].}$$

In the future e^+e^- collider, Γ_t can be measured with an uncertainty of 30 MeV

[Martinez, Miquel 2019].

Theoretical Development

NLO QCD corrections [Jezabek, Kuhn 1989, Czarnecki 1990, Li, Oakes, Yuan 1991]

NLO EW corrections [Denner, Sack 1991, Eilam, Mendel, Migneron, Soni 1991]

Numerical result of full NNLO QCD corrections [Gao, Li, Zhu 2013, Brucherseifer, Caola, Melnikov 2013]

Numerical result of full N³LO QCD corrections [Chen, Chen, Guan, Ma 2023]

Analytic results of NNLO and N³LO QCD corrections [Chen, Li, Wang, Wang 2022, Chen, Li, Li, Wang, Wang, Wu 2023]

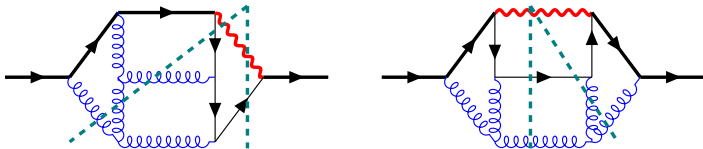
PMC scale settings up to N³LO QCD [Jiang, Wu, Zhou, Li, Shan 2024]

Optical Theorem

The forward scattering amplitudes Σ for $t \rightarrow Wb \rightarrow t$

$$\Gamma_t = \frac{\text{Im}(\Sigma)}{m_t} \quad (1)$$

For N³LO QCD corrections, the self-energy four-loop diagrams are considered. For example,



The imaginary part comes from cut diagrams.

The complicated phase space integration can be avoided.

Calculation Method

For $t \rightarrow Wb \rightarrow t$, b quark is considered as massless. We define $w = m_w^2/m_t^2$.

1. Generating diagrams and amplitudes.
2. Reducing amplitudes to master integrals by FIRE [Smirnov, Chuharev 2020].
3. Analytically calculating the master integrals.

Master Integrals Calculations

Method: [canonical differential equations](#). For example,

$$\frac{\partial F_4(w, \epsilon)}{\partial w} = \frac{\epsilon(F_5 - 2F_4)}{w-1} - \frac{\epsilon(F_4 + F_5)}{w}, \quad w = \frac{m_W^2}{m_t^2}, \quad D = 4 - 2\epsilon \quad (2)$$

Two important ingredients:

1. [Construct canonical form \(\$\epsilon\$ form, \$d \log\$ form\)](#) – [Libra \[Lee 2021\]](#)
2. [Boundary conditions](#) – [AMFlow \[Liu, Ma 2022\]](#) and [PSLQ method \[Ferguson, Beiley, Arno 1992 1999\]](#)

Results: [harmonic polylogarithms \(HPLs\)](#).

Analytic calculations of [four loop](#) master integrals are [non-trivial](#).

Leading Color Contribution at NNLO

QCD corrections of Γ_t .

$$\Gamma(t \rightarrow Wb) = \Gamma_0 \left[X_0 + \frac{\alpha_s}{\pi} X_1 + \left(\frac{\alpha_s}{\pi} \right)^2 X_2 + \left(\frac{\alpha_s}{\pi} \right)^3 X_3 \right], \quad (3)$$

$$\Gamma_0 = \frac{G_F m_t^3 |V_{tb}|^2}{8\sqrt{2}\pi}. \quad (4)$$

Leading Color Contribution at NNLO,

$$\begin{aligned} X_2 &= C_F [C_F X_F + C_A X_A + T_R n_l X_l + T_R n_h X_h] \\ &= C_F \left[N_c \left(X_A + \frac{X_F}{2} \right) + \frac{n_l}{2} X_l - \frac{1}{2N_c} X_F + \frac{n_h}{2} X_h \right] \end{aligned} \quad (5)$$

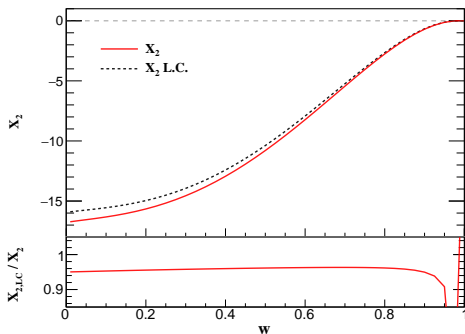
$$C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_A = N_c, \quad T_R = \frac{1}{2},$$

$n_l(n_h)$ = number of **light** (**heavy**) quark flavors.

Leading Color Contribution at NNLO

$$\begin{aligned} X_2 &= C_F [C_F X_F + C_A X_A + T_R n_l X_l + T_R n_h X_h] \\ &= C_F \left[N_c \left(X_A + \frac{X_F}{2} \right) + \frac{n_l}{2} X_l - \frac{1}{2N_c} X_F + \frac{n_h}{2} X_h \right] \end{aligned} \quad (6)$$

In the top quark decay, $N_c = 3, n_l = 5, n_h = 1, w(m_w^2/m_t^2) \approx 0.22, X_{2,L.C.}/X_2 > 95\%$.



Analytic Calculations at N³LO

According to NNLO, the leading color contributions are dominant.

The analytic calculation of full N³LO is quite difficult. We concentrate on the **leading color contributions** at N³LO.

$$X_3 = C_F \left[N_c^2 Y_A + \tilde{Y}_A + \frac{\bar{Y}_A}{N_c^2} + n_l n_h Y_{lh} + n_l \left(N_c Y_l + \frac{\tilde{Y}_l}{N_c} \right) + n_l^2 Y_{l2} \right. \\ \left. + n_h \left(N_c Y_h + \frac{\tilde{Y}_h}{N_c} \right) + n_h^2 Y_{h2} \right]. \quad (7)$$

$$X_{3,\text{L.C}} = C_F \left[N_c^2 Y_A + n_l N_c Y_l + n_l^2 Y_{l2} \right]. \quad (8)$$

In leading color approximation, there are still **408 Feynman diagrams** and **185 master integrals** need to be calculated.

Analytic Results of N³LO

$$X_{3,\text{L.C}} = C_F \left[N_c^2 Y_A + n_l N_c Y_l + n_l^2 Y_{l2} \right].$$

$$\begin{aligned} Y_l = & (8w^3 + 12w^2 + 43w + 8) \left(H(0, 0, 0, 0, 1, w) - H(0, 0, 0, 1, 0, w) \right) \\ & + (-2w^3 + 51w^2 + 86w + 7) H(0, 0, 0, 1, 1, w) + (2w^3 - 15w^2 - 20w - 1) H(0, 0, 1, 0, 1, w) \\ & + (-2w^3 - 45w^2 - 80w - 9) H(0, 0, 1, 1, 0, w) - 2(6w^3 - 33w^2 - 46w - 1) H(0, 0, 1, 1, 1, w) \\ & + (6w^3 + 3w^2 + 14w + 5) H(0, 1, 0, 0, 1, w) + 2(6w^3 + 21w^2 + 50w + 8) H(0, 1, 0, 1, 1, w) \\ & - 2(2w^3 + 27w^2 + 56w + 6) H(0, 1, 1, 0, 1, w) + 2(2w^3 - 27w^2 - 40w - 3) H(0, 1, 1, 1, 0, w) \\ & - (2w + 1)(w - 1)^2 \left(2H(0, 1, 0, 1, 0, w) - H(1, 0, 0, 0, 1, w) + H(1, 0, 0, 1, 0, w) \right. \\ & + 4H(1, 0, 0, 1, 1, w) - 2H(1, 0, 1, 0, 1, w) - 2H(1, 0, 1, 1, 0, w) \\ & \left. - 4H(1, 1, 0, 0, 1, w) + 4H(1, 1, 0, 1, 0, w) \right) + \dots \\ Y_{l2} = & -\frac{(w-1)^2(2w+1)}{18} \left(H(0, 0, 0, 1, w) - H(0, 0, 1, 0, w) + 2H(0, 0, 1, 1, w) \right. \\ & \left. - H(0, 1, 1, 0, w) + 4H(0, 1, 1, 1, w) + 2H(1, 0, 0, 1, w) - 3H(1, 0, 1, 0, w) - 4H(1, 1, 1, 0, w) \right) + \dots \end{aligned}$$

Expansion series near $w = 0$

$$\begin{aligned}
 Y_A = & \left[\frac{203185}{41472} - \frac{12695\pi^2}{1944} - \frac{4525\zeta(3)}{576} - \frac{1109\pi^4}{25920} + \frac{37\pi^2\zeta(3)}{36} + \frac{1145\zeta(5)}{96} + \frac{47\pi^6}{2835} - \frac{3\zeta(3)^2}{4} \right] \\
 & + w \left[-\frac{157939}{2304} + \frac{140863\pi^2}{20736} + \frac{5073\zeta(3)}{64} - \frac{14743\pi^4}{6480} - \frac{169\pi^2\zeta(3)}{72} - \frac{45\zeta(5)}{16} + \frac{3953\pi^6}{22680} \right. \\
 & \left. - \frac{15\zeta(3)^2}{4} \right] + w^2 \left[\log(w) \left(\frac{851099}{27648} - \frac{5875\pi^2}{2304} - \frac{33\zeta(3)}{8} + \frac{\pi^4}{10} \right) - \frac{82610233}{331776} + \frac{799511\pi^2}{27648} \right. \\
 & \left. + \frac{4093\zeta(3)}{32} - \frac{5987\pi^4}{2880} - \frac{91\pi^2\zeta(3)}{16} - \frac{275\zeta(5)}{8} + \frac{347\pi^6}{3024} - \frac{9\zeta(3)^2}{8} \right] + \mathcal{O}(w^3), \\
 Y_{l2} = & \left[-\frac{695}{2592} - \frac{91\pi^2}{972} + \frac{11\zeta(3)}{36} - \frac{2\pi^4}{405} \right] + w \left[\frac{245}{144} - \frac{73\pi^2}{648} - \frac{\zeta(3)}{3} \right] \\
 & + w^2 \left[\log(w) \left(\frac{245}{432} - \frac{\pi^2}{72} \right) - \frac{791}{162} + \frac{85\pi^2}{432} + \frac{3\zeta(3)}{4} + \frac{2\pi^4}{135} \right] + \mathcal{O}(w^3), \tag{10}
 \end{aligned}$$

Numerical Results

Input parameters from [P.D.G 2022]

$$\begin{aligned}m_t &= 172.69 \text{ GeV}, & m_b &= 4.78 \text{ GeV}, \\m_W &= 80.377 \text{ GeV}, & \Gamma_W &= 2.085 \text{ GeV}, \\m_Z &= 91.1876 \text{ GeV}, & G_F &= 1.16638 \times 10^{-5} \text{ GeV}^{-2}, \\|V_{tb}| &= 1, & \alpha_s(m_Z) &= 0.1179.\end{aligned}\tag{11}$$

$\Gamma_t^{(0)} = 1.486 \text{ GeV}$ with $m_b = 0$ and on-shell W .

$$\begin{aligned}\Gamma_t &= \Gamma_t^{(0)} [(1 + \delta_b^{(0)} + \delta_W^{(0)}) \\&+ (\delta_b^{(1)} + \delta_W^{(1)} + \delta_{\text{EW}}^{(1)} + \delta_{\text{QCD}}^{(1)}) \\&+ (\delta_b^{(2)} + \delta_W^{(2)} + \delta_{\text{EW}}^{(2)} + \delta_{\text{QCD}}^{(2)} + \delta_{\text{EW} \times \text{QCD}}^{(2)}) \\&+ (\delta_b^{(3)} + \delta_W^{(3)} + \delta_{\text{EW}}^{(3)} + \delta_{\text{QCD}}^{(3)} + \delta_{\text{EW} \times \text{QCD}}^{(3)})],\end{aligned}\tag{12}$$

Numerical Results

Corrections in percentage (%) normalized by the LO width $\Gamma_t^{(0)} = 1.486$ GeV with $m_b = 0$ and on-shell W .

	$\delta_b^{(i)}$	$\delta_W^{(i)}$	$\delta_{EW}^{(i)}$	$\delta_{QCD}^{(i)}$	Γ_t [GeV]
LO	-0.273	-1.544	—	—	1.459
NLO	0.126	0.132	1.683	-8.575	$1.361_{-0.0130}^{+0.0091}$
NNLO	≈ 0.03	0.030	*	-2.070	$1.331_{-0.0051}^{+0.0055}$
N ³ LO	*	0.009	*	-0.667	$1.321_{-0.0021}^{+0.0025}$

QCD corrections are **dominant**.

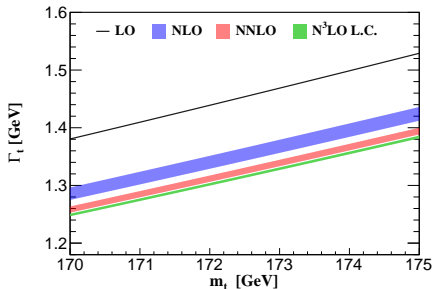
$X_{3,L.C}/X_3 \approx 95\%$ compared with full numerical result. [Chen, Chen, Guan, Ma 2023]

NLO EW correction is 1.683%.

The off-shell W boson effect at NNLO and N3LO are **further suppressed**.

Numerical Results

m_t varies from 170 GeV to 175 GeV. μ varies from $\frac{m_t}{2}$ to $2m_t$.



The scale uncertainty at N³LO is reduced to $\pm 0.2\%$, only half of that at NNLO.

It is very convenient to use fitted function within this range

$$\Gamma_t(m_t) = 0.027037 \times m_t - 3.34801 \text{ GeV} \quad (13)$$

Mathematica program

`TopWidth.m` can be downloaded from <https://github.com/haitaoli1/TopWidth>. The package HPL is required [Maitre 2006]. It has been used in [Jiang, Wu, Zhou, Li, Shan 2024]

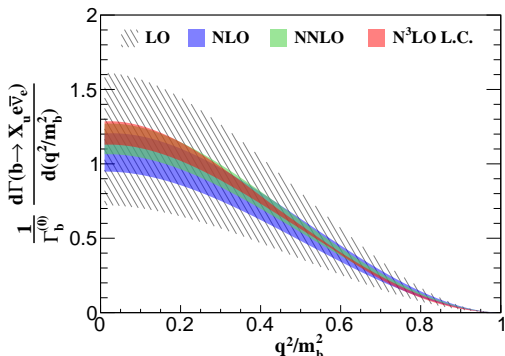
```
<< TopWidth`  
      (***** TopWidth-1.0 *****)  
      Authors: Long-Bin Chen, Hai Tao Li, Jian Wang, YeFan Wang  
      TopWidth[QCOrder, mbCorr, WwidthCorr, EWcorr, mu] is provided for top width calculations  
      Please cite the paper for reference: arXiv:2212.06341  
  
*-*-*-*-* HPL 2.0 *-*-*-*-*  
  
Author: Daniel Maitre, University of Zurich  
  
(* SetParameters[mt,mb,mw,Wwidth, GF] *)  
SetParameters[ $\frac{17\,269}{100}$ ,  $\frac{478}{100}$ , 80 377 / 1000, 2085 / 1000, 911 876 / 10 000, 11 663 788  $\times 10^{-12}$ ];  
TopWidth[3, 1, 1, 1,  $\frac{17\,269}{100}$ ]  
  
1.32073
```

The dilepton invariant mass spectrum for $b \rightarrow X_u e \bar{\nu}_e$ up to N³LO

The dilepton invariant mass spectrum in the on-shell mass scheme

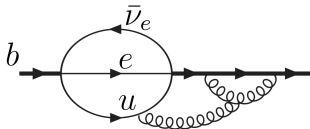
$$\frac{d\Gamma(b \rightarrow X_u e \bar{\nu}_e)}{dq^2} = \Gamma_b^{(0)} \sum_{i=0} \left(\frac{\alpha_s}{\pi}\right)^i X_i \left(\frac{q^2}{m_b^2}\right), \quad (14)$$

with $\Gamma_b^{(0)} = G_F^2 |V_{ub}|^2 m_b^3 / 96 \pi^3$. The scale uncertainties are around $\pm 12\%$, $\pm 9\%$, and $\pm 6\%$ at NLO, NNLO, and N³LO, respectively, at $q^2/m_b^2 = 0.2$.



Relations With Other Process

Integrating w ($w = m_W^2/m_t^2$) from 0 to 1, we obtain N³LO QCD corrections in semileptonic decay $\Gamma(b \rightarrow X_u e \bar{\nu}_e)$. The N³LO correction is about 6% at $\mu = 4.78$ GeV.



$$\Gamma(b \rightarrow X_u e \bar{\nu}_e) = \frac{G_F^2 |V_{ub}|^2 m_b^5}{192\pi^3} \left[1 + \sum_{i=1} \left(\frac{\alpha_s}{\pi} \right)^i b_i \right]. \quad (15)$$

$$\begin{aligned} b_3 = C_F \left[N_c^2 \left(\frac{9651283}{82944} - \frac{1051339\pi^2}{62208} - \frac{67189\zeta(3)}{864} + \frac{4363\pi^4}{6480} + \frac{59\pi^2\zeta(3)}{32} + \frac{3655\zeta(5)}{96} \right. \right. \\ \left. \left. - \frac{109\pi^6}{3780} \right) + n_l N_c \left(-\frac{729695}{27648} + \frac{48403\pi^2}{15552} + \frac{1373\zeta(3)}{108} + \frac{133\pi^4}{1728} - \frac{13\pi^2\zeta(3)}{72} - \frac{125\zeta(5)}{24} \right) \right. \\ \left. + n_l^2 \left(\frac{24763}{20736} - \frac{1417\pi^2}{15552} - \frac{37\zeta(3)}{216} - \frac{121\pi^4}{6480} \right) + \text{subleading color} \right] \\ = (-195.3 \pm 9.8) C_F. \quad (16) \end{aligned}$$

Summary

We provide the first **leading color QCD correction** at N^3LO analytically, which can be used to perform **fast** numerical evaluations.

The NNLO and N^3LO QCD corrections decrease the LO result by -2.07% and -0.667% with $m_t = 172.69$ GeV and $\mu = m_t$

We derive the **analytic N^3LO QCD leading color predictions** for the semileptonic $b \rightarrow u$ decay width.

Thanks !