One-loop integrals, AMFlow, and dimension-changing transformation

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Outline

I. Introduction

II. AMFlow and dimension-changing transformation

III. Examples

IV. Summary and outlook

Quantum field theory

➢ The foundational theoretical framework of physics

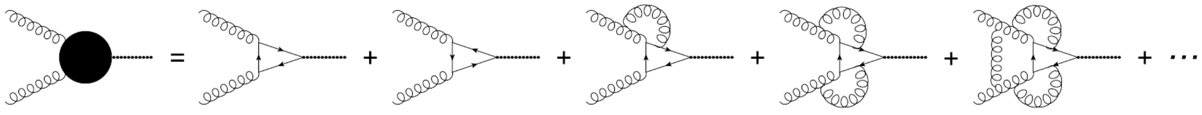
- Particle physics, nuclear physics
- Many-body quantum system, cold atom physics
- Gravitational waves
- …In future

➢ Two ways to solve QFT

- Perturbation theory
- Non-perturbative methods
- Complementary to each other

Perturbative QFT computation

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules 
- Amplitudes: linear combinations of Feynman loop integrals with rational coefficients

2. Calculate Feynman loop integrals (FIs)

3. Perform phase-space integrations

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals via reverse unitarity

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\left(\frac{\mathrm{i}}{p^2+\mathrm{i}0^+} + \frac{-\mathrm{i}}{p^2-\mathrm{i}0^+}\right)$$

Mainstream strategy to compute FIs

➢ Reduce plenty of FIs to much less bases (master integrals)

- A family of FIs form a FINITE-dim. linear space Proved by: Smirnov, Petukhov, 1004.4199

$$I_{\vec{\nu}} = \sum_{i=1}^{M} c_i I_i$$

- Using integration-by-parts identities, powered by Laporta's algorithm Laporta, 0102033
- Public codes: AIR, FIRE, LiteRed, Reduze, Kira, FiniteFlow, NeatIBP, Blade…

➢ Compute Master integrals

- One-loop MIs See Bo Feng's talk for one-loop reduction
- Multi-loop MIs See Wen Chen's talk for multi-loop computation

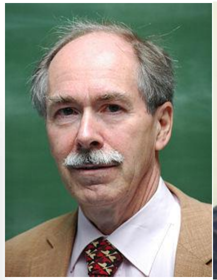
History of one-loop FIs computation

➢ **Analytical results up to 4 points**

Scalar One Loop Integrals

Gerard 't Hooft (Utrecht U.), M.J.G. Veltman (Utrecht U.) (Nov, 1978)

Published in: *Nucl.Phys.B* 153 (1979) 365-401



't Hooft

Veltman

➢ **Numerical stable version, implemented in FF package**

New Algorithms for One Loop Integrals

G.J. van Oldenborgh (NIKHEF, Amsterdam), J.A.M. Vermaseren (NIKHEF, Amsterdam) (Oct, 1989)

Published in: *Z.Phys.C* 46 (1990) 425-438

History of one-loop FIs computation

➢ 5-point up to finite part

Dimensionally regulated pentagon integrals

Zvi Bern (UCLA), Lance J. Dixon (SLAC), David A. Kosower (CERN and Saclay) (May, 1993)

Published in: *Nucl.Phys.B* 412 (1994) 751-816 • e-Print: hep-ph/9306240 [hep-ph]

- **Dimensional recurrence relation**

$$\hat{I}_n = \frac{1}{2N_n}\left[\sum_{i=1}^{n}\gamma_i\,\hat{I}_{n-1}^{(i)} + (n-5+2\epsilon)\,\hat{\Delta}_n\,\hat{I}_n^{D=6-2\epsilon}\right]$$

- **Special case for 5-point**

$$\hat{I}_5 = \frac{1}{2N_5}\sum_{i=1}^{5}\gamma_i\,\hat{I}_4^{(i)} + \mathcal{O}(\epsilon)$$

One-loop MIs: used by hundreds times each year

Automatized one loop calculations in four-dimensions and D-dimensions

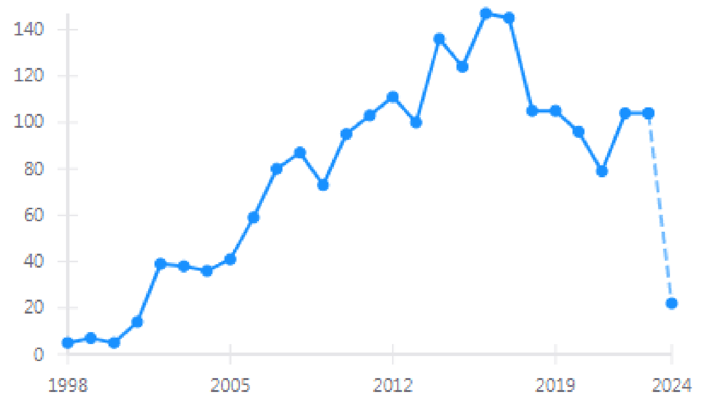
T. Hahn (Karlsruhe U.), M. Perez-Victoria (Granada U., Theor. Phys. Astrophys.)
Jul, 1998

16 pages
Published in: *Comput.Phys.Commun.* 118 (1999) 153-165
e-Print: hep-ph/9807565 [hep-ph]
DOI: 10.1016/S0010-4655(98)00173-8
Report number: UG-FT-87-98, KA-TP-7-1998
View in: ADS Abstract Service

Looptools

pdf   cite   claim   reference search   2,060 citations

Citations per year



Collier: a fortran-based Complex One-Loop LIbrary in Extended Regularizations

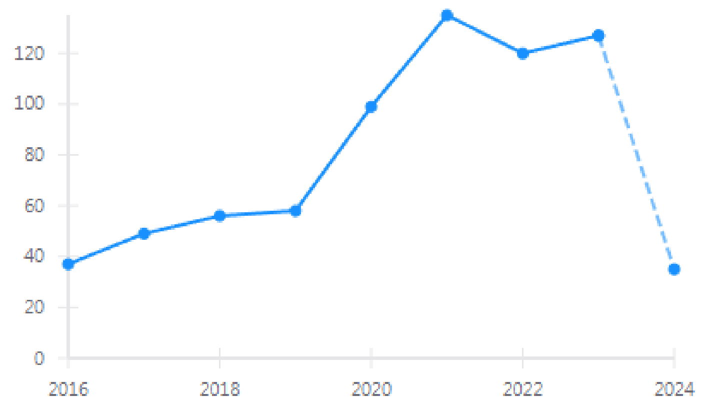
Ansgar Denner (Wurzburg U.), Stefan Dittmaier (Freiburg U.), Lars Hofer (Barcelona U., ECM and ICC, Barcelona U.)
Apr 22, 2016

19 pages
Published in: *Comput.Phys.Commun.* 212 (2017) 220-238
Published: Mar, 2017
e-Print: 1604.06792 [hep-ph]
DOI: 10.1016/j.cpc.2016.10.013
Report number: FR-PHENO-2016-003, ICCUB-16-016
View in: ADS Abstract Service

Collier

pdf   cite   claim   reference search   716 citations

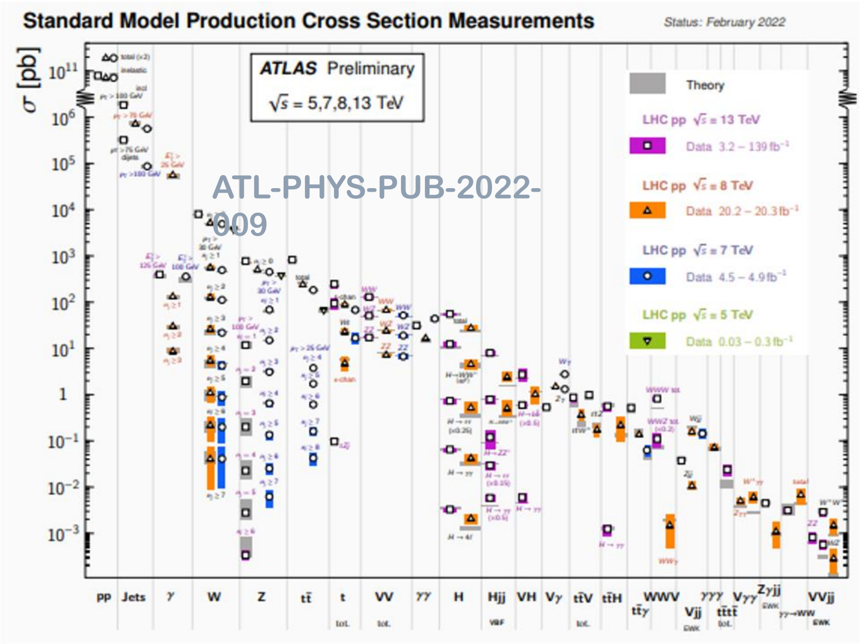
Citations per year



Era of precision physics at the LHC

➢ High-precision data

- Many observables probed at precent-level precision



- At least NNLO QCD corrections generally required (plus NLO EW, parton shower, resummation, etc.)

➢ Higher order computation

- In additional to multiloop MIs, also needs one-loop MIs to higher order in ϵ
- Not available from existed packages

Canonical differential equations method

- Choosing proper basis

$$\frac{\partial}{\partial s_i}\boldsymbol{I}'(\epsilon,\vec{s}\,) = \epsilon A_i'(\vec{s}\,)\boldsymbol{I}'(\epsilon,\vec{s}\,)$$

Henn, 1304.1806

- Solution after expanding ϵ: Multiple Polylogarithms

$$G(a_1,a_2,\cdots,a_n;z) := \int_0^z \frac{\mathrm{d}t}{t-a_1} G(a_2,\cdots,a_n;t),$$

$$G(\overbrace{0,\cdots,0}^{n};z) := \frac{1}{n!}\log^n z, \quad G(;z) := 1.$$

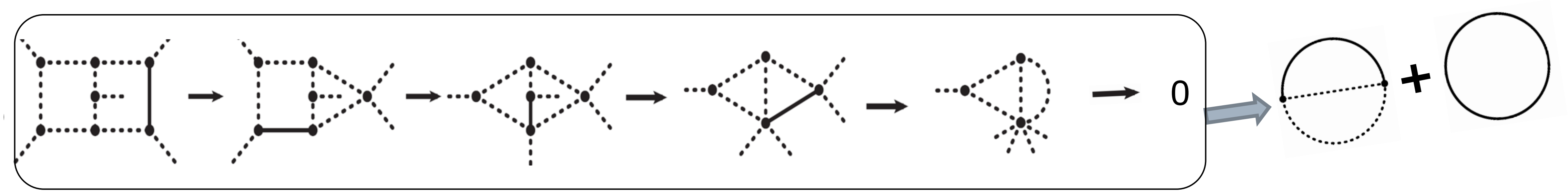
- Advantage: can compute to high order in ϵ
- Disadvantage: process dependent, no existed package for non-expert to use

Auxiliary Mass Flow

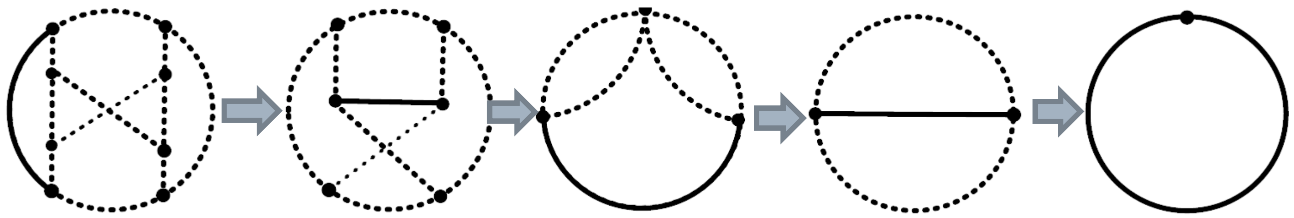
➢ Introducing auxiliary mass and taking it to infinity

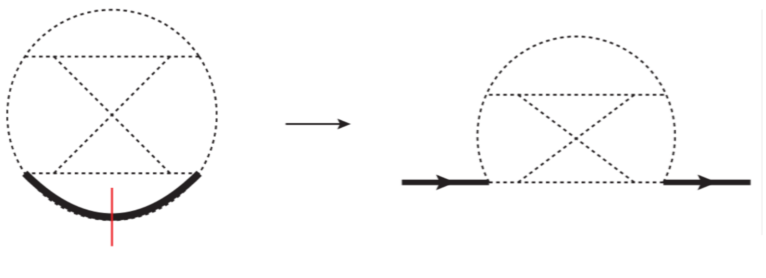
$$I^{\rm aux}_{\vec{\nu}}(D,\vec{s},\eta)=\int\prod_{i=1}^{L}\frac{{\rm d}^D\ell_i}{{\rm i}\pi^{D/2}}\frac{\mathcal{D}_{K+1}^{-\nu_{K+1}}\cdots\mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1-\lambda_1\,\eta+{\rm i}0^+)^{\nu_1}\cdots(\mathcal{D}_K-\lambda_K\,\eta+{\rm i}0^+)^{\nu_K}}$$

Liu, YQM, Wang, 1711.09572
Liu, YQM, 2107.01864



➢ Recursively compute vacuum integrals





Liu, YQM, 2201.11637

Zero input; valid to any loop, any spacetime dimension, any kinematics

Auxiliary Mass Flow

➢ AMFlow package

Liu, YQM, 2201.11669

Link: https://gitlab.com/multiloop-pku/amflow

- The first package that can calculate any FI (with any number of loops, any D and $\vec{s}$) to arbitrary precision, given sufficient resource

➢ Efficiency

- Typically costs minutes to days for each phase-space point
- Computing a few points is fine, but difficult for all phase-space points

Impossible $\overset{2022}{\Longrightarrow}$ possible $\overset{\text{future}}{\Longrightarrow}$ efficiency

➢ Ways to improve efficiency

1. Combine AMFlow with differential equation method (currently widely used)
2. Mathematica -> C++
3. Combining with novel strategy

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AMFlow at 1-loop

➢ Introduction of auxiliary mass

Liu, YQM, Wang, 1711.09572

$$I_{\vec{\nu}}^{D}(\eta) \equiv \int \frac{\mathrm{d}^D l}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha - \eta)^{\nu_\alpha}}$$

$$\mathcal{D}_\alpha \equiv (l + p_\alpha)^2 - m_\alpha^2$$

- Two limits

$$I_{\vec{\nu}}^{D} \equiv \lim_{\eta \to \mathrm{i}0^-} I_{\vec{\nu}}^{D}(\eta)$$

$$I_{\vec{\nu}}^{D}(\eta) \overset{\eta \to \infty}{\longrightarrow} (-1)^{\nu} \frac{\Gamma(\nu - D/2)}{\Gamma(\nu)} \eta^{D/2-\nu}.$$

AMFlow at 1-loop

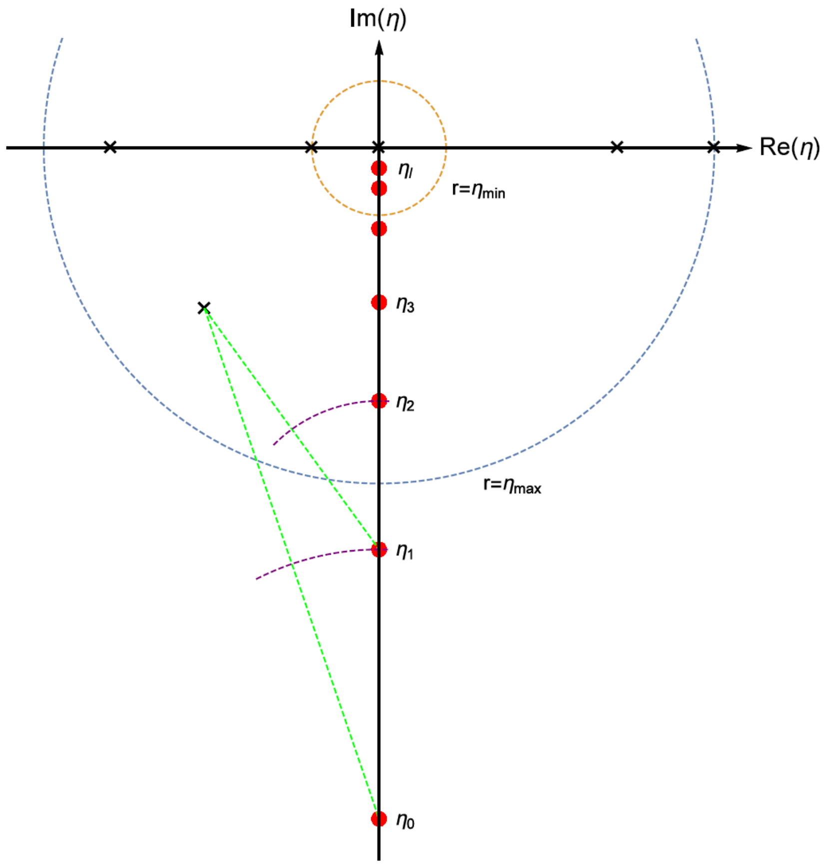
➢ Flow of auxiliary mass

- **Differential equations**

$$(2\eta - C)\frac{\mathrm{d}}{\mathrm{d}\eta} I^D_{\vec{\nu}}(\eta) = (D-1-\nu) I^D_{\vec{\nu}}(\eta) + \sum_{\alpha=1}^{N} z_\alpha I^{D-2}_{\vec{\nu}-\vec{e}_\alpha}(\eta).$$

C, z_α: **calculable using modified Cayley determinant and the Gram determinant**

- **Solving DEs numerically along negative imaginary axis**



Dimension-changing transformation

➢ Split loop momentum

Huang, Jian, YQM, Mu, Wu, In preparation

$$I_{\vec{\nu}}^{D} = \int \frac{\mathrm{d}^{D-D_0} l_\perp}{\pi^{(D-D_0)/2}} \int \frac{\mathrm{d}^{D_0} l_{\parallel}}{\mathrm{i}\pi^{D_0/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha^{\parallel} - l_\perp^2)^{\nu_\alpha}}$$

➢ Dimension-changing transformation

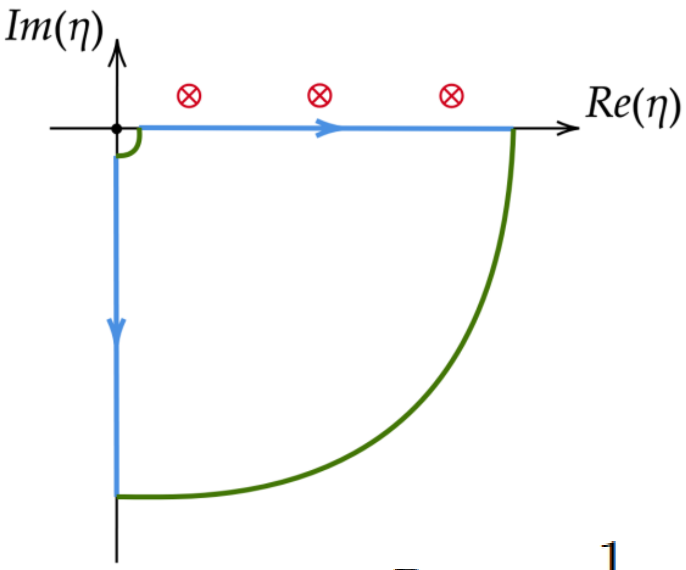
$$\delta = D - D_0$$

$$I_{\vec{\nu}}^{D} = \frac{1}{\Gamma(\delta/2)} \int_0^\infty \mathrm{d}\eta\, \eta^{\delta/2-1} I_{\vec{\nu}}^{D_0}(\eta),$$

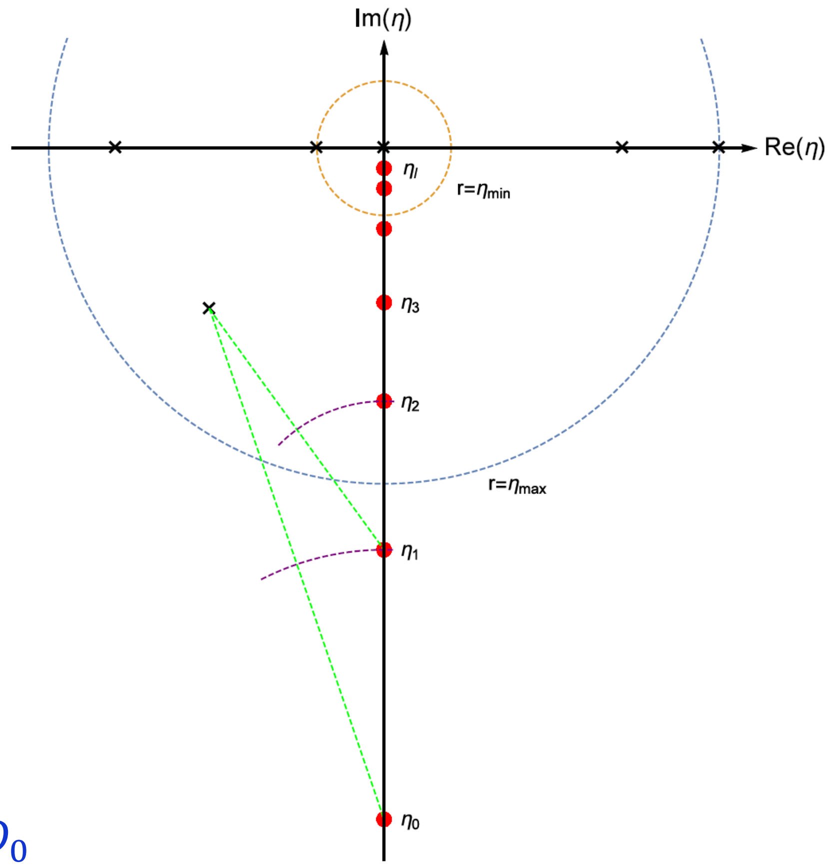
- Relate one-loop FIs in different spacetime dimensions

Relating to AMFlow solver

➢ Deform contour



$$I_{\vec{\nu}}^{D} = \frac{1}{\Gamma(\delta/2)} \int_0^{-i\infty} d\eta\, \eta^{\delta/2-1} I_{\vec{\nu}}^{D_0}(\eta).$$

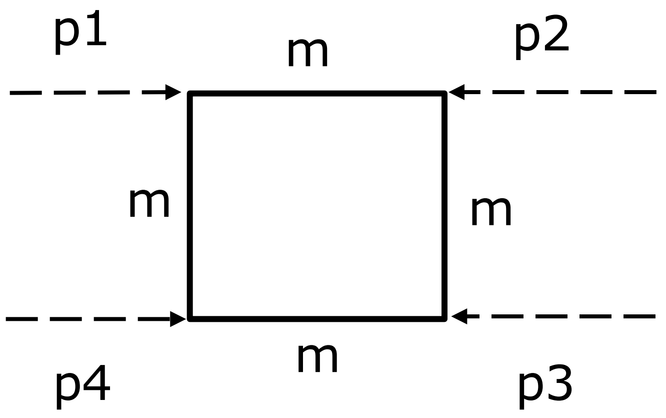


- **Obtain FIs at any D by solving AMFlow once at any given D_0**

Outline

Comparison with LoopTools

Example 1



pi^2=0，s=(p1+p2)^2=10,
t=(p2+p3)^2=-3, varying m^2

11 MIs：
(1,1,1,1)(1,1,1,0)(1,1,0,1)(1,0,1,1)
(0,1,1,1)(1,0,1,0)(0,1,0,1)(1,0,0,0)
(0,1,0,0)(0,0,1,0)(0,0,0,1)

AMFlow (Mathematica version): about 10s

Evaluation time: $t/(10^{-3}s)$ Up to finite part in ϵ

m^2	DCT/AMF	LoopTools
10	0.309	0.252
100	0.271	0.270
10000	0.247	0.274
1000000	0.277	0.257
100000000	0.297	0.253

Relative precision As good as analytical expression!

m^2	DCT/AMF	LoopTools
10	5.08e-18	7.08e-16
100	3.49e-17	2.33e-13
10000	4.78e-17	1.67e-10
1000000	1.50e-16	2.34e-10
100000000	1.49e-16	2.80e-8

Higher order in epsilon

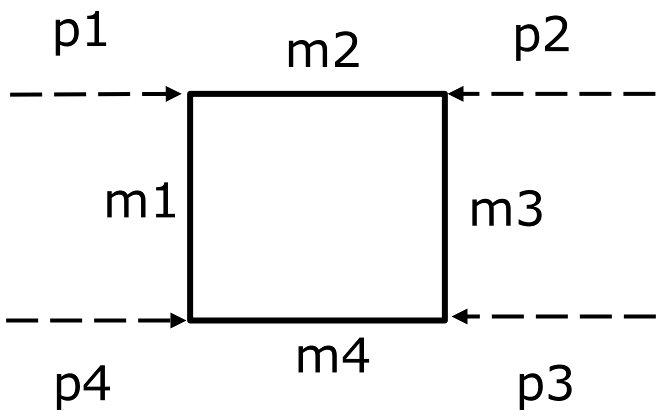
Example 1

ε^{n}	$t/(10^{-3}s)$
0	0.277
1	0.341
2	0.426
3	0.568
4	0.603
5	0.695
6	0.869
7	0.983
8	1.210
9	1.271

- 1.2e-16 precision for every order in ϵ
- DCT: scaling as $O(n)$ easy to compute to high order
- AMF: scaling as $O(n^2)$

More complex example

Example 2

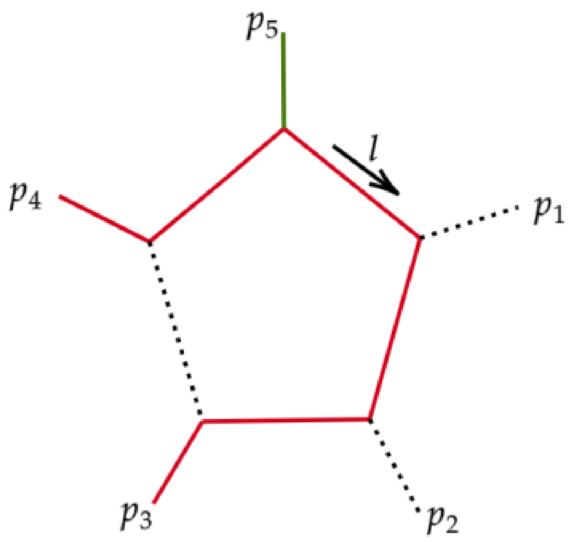


	DCT	LoopTools
$t/(10^{-3}s)$	0.419	0.387
precision	1e-17	1e-15

p1^2=1.01, p2^2=1.02, p3^2=1.03, p4^2=1.04, m1^2=0.99, m2^2=0.98, m3^2=0.97, m4^2=0.96, s=(p1+p2)^2=9.7, t=(p2+p3)^2=-0.1

Pentagon

Example 3 Relevant for g g -> t tbar H See also Li lin Yang's talk



	DCT	LoopTools
$t/(10^{-3}s)$	1.97	2.67
precision	1e-15	1e-15

p1^2=0, p2^2=0, p3^2=1, p4^2=1, p5^2=0.511, m1^2=1, m2^2=1, m3^2=1,m4^2=0, m5^2=1, p1*p2=16.3, p1*p3=-1.20, p1*p4=-5.83, p2*p3= -7.55, p2*p4=-5.04, p3*p4=2.54

A total of 29 MIs：
(1,1,1,1,1)(1,1,1,1,0)(1,1,1,0,1)(1,1,0,1,1)(1,0,1,1,1)(0,1,1,1,1)(1,1,1,0,0) (1,1,0,1,0) (1,0,1,1,0) (0,1,1,1,0) (1,1,0,0,1) (1,0,1,0,1) (0,1,1,0,1) (1,0,0,1,1) (0,1,0,1,1) (0,0,1,1,1) (0,0,1,0,1) (0,0,0,1,1) (0,0,1,1,0) (0,1,0,0,1) (0,1,0,1,0) (1,0,0,0,1) (1,0,0,1,0) (0,0,1,1,1) (0,0,0,0,1) (0,0,0,1,0)(0,0,1,0,0)(0,1,0,0,0)(1,0,0,0,0)

Summary

- One-loop FIs: remaining an important topic
- DCT+AMFlow: fully systematic and extremely efficient, provides an ultimate solution (arbitrary spacetime dimension)
- Future: improving the efficiency of multiloop FIs computation

Thank you!