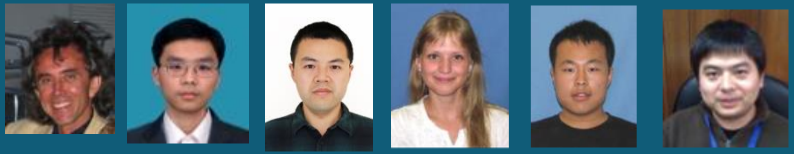
# 3D Theory of Microscopic Instabilities Driven by Space-Charge Forces

Vladimir N Litvinenko, Yichao Jing, Jun Ma, Irina Petrushina, Kai Shih, Gang Wang



Stony Brook University and Brookhaven National Laboratory

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# What the talk is about

- Microbunching - or short-wavelength - instabilities are well-known for drastic reduction of the beam quality, its filamentation and strong amplification of the noise in a beam. Space charge and coherent synchrotron radiation (CSR) are the leading causes for such instability.
- In this talk I present rigorous 3D theory of such instabilities driven by the space-charge forces.
- I will define the condition when our theory is applicable for an arbitrary accelerator system with 3D coupling. Finally, I will show derivation of a linear integral equation describing such instability and identify conditions it can be reduced to an ordinary second order differential equation
- I will also discuss challenges/limitation of the current theory and will discuss how it can be further improved

# Content

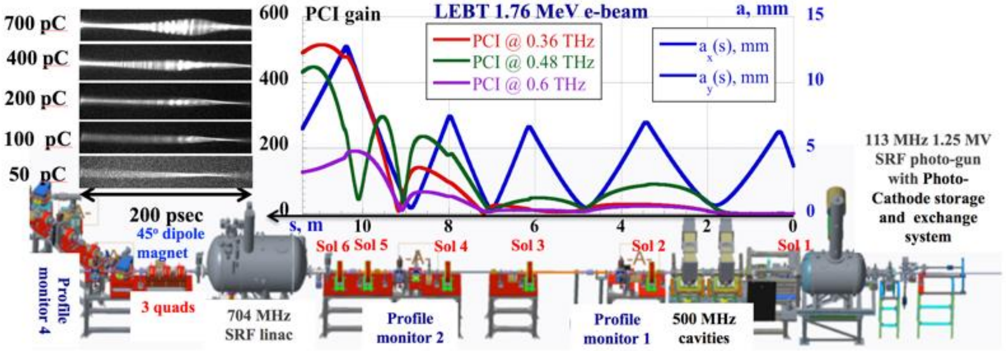
- Motivation for 3D theory of space-charge driven microscopic instability
    - *Example of space charge failing predicting plasma-cascade instability*
- Theoretical approach and key approximations
- Steps in derivations
- Final 3D integral equation and relation to 1D case
- Examples
- When 3D integral equation cand be reduced to 2$^{nd}$ order ODE
- Possible complications and extension of the theory
- Conclusions

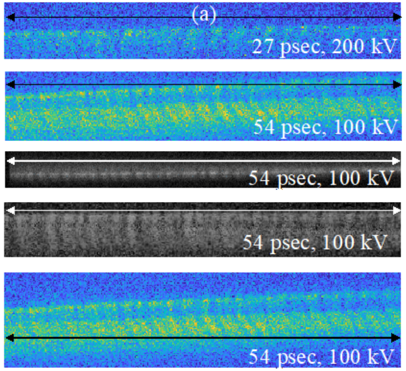
# Motivation for 3D theory of microscopic instabilities

- Accurate self-consistent – not just kick/drift – theory of microscopic instabilities is neither trivial or intuitive. This is especially true if one trying to extend it to 3D (6D phase space)
- Our motivation to get into this challenging endeavor was driven by shocking experience discovering experimentally that microscopic plasma-cascade instability is heating our electron, when the most sophisticated beam dimness (for example Impact-T) failed to predict it
- Probably the best emotional reaction to this discovery was the title of Irina Petrushina's PhD thesis "The Chilling Recount of an Unexpected Discovery: First Observations of the Plasma-Cascade Instability in the Coherent Electron Cooling Experiment"
- It was only after developing **theory** of Plasma-Cascade Instability we were able to reproduce it in simulations. Needless to say, such demonstration it required fine tuning of code's settings and advices from the code's authors on how to avoid suppressing artifacts that dominated the results…
- The moral of this story is the following: does not matter how good and how well-bench-marked are numerical codes, there is always a corner (the questions how big or how small? Or if it is ML or AI?) of parameters where they will become unreliable or, simply saying, just failing..
- This is why theory, does not matter how inconvenient or imperfect, is necessary to illuminate our understanding of physics phenomena which are beyond grasp of current computer codes

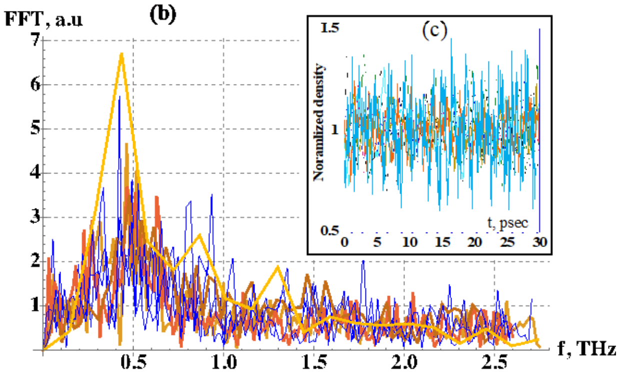
# Experimental observation of plasma cascade instability in 1.75 MeV e-beam in CeC linac

## Uncompressed bunch: simulations and experiment in September 2018



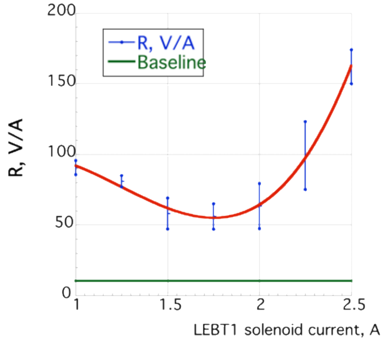



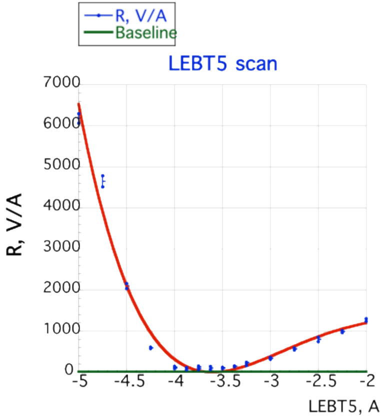

Measured time profiles of 1.75 MeV electron bunches with 0.45 nC to 0.7 nC;




Seven measured overlapping spectra and PCI spectrum simulated by SPACE (slightly elevated yellow line). Clip shows a 30-psec fragment of seven measured relative density modulations.

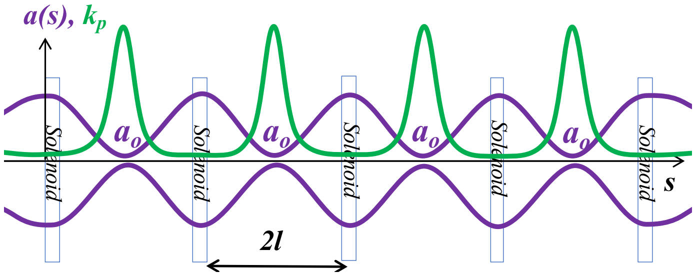
Noise power in the e-beam as function of focusing by two solenoids

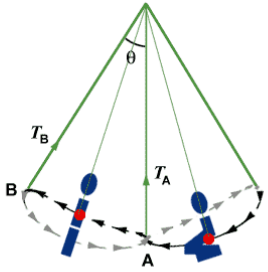


# What is Plasma-Cascade Instability?

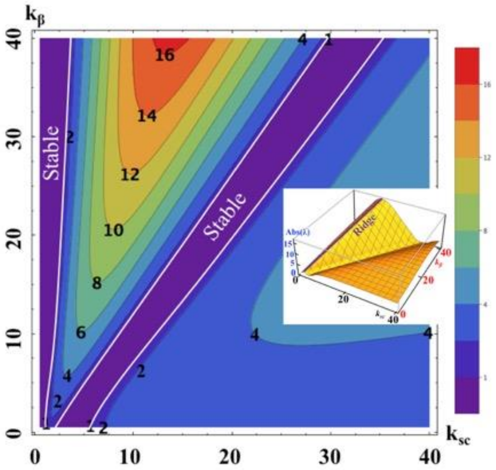
- It is an exponentially growing parametric instability driven by variation of the plasma frequency and driven by the variation of the transverse electron beam size
- We do it by creating dramatic variations of plasma density using modulation of the transverse beam size
- Important questions – when exponential growth occurs and how fast it is? Hence, we developed a self-consistent 3D theory and simulations




$$\hat{a}'' = k_{sc}^2 \hat{a}^{-1} + k_b^2 \hat{a}^{-3}$$

$$k_{sc}^2 = \frac{2}{\beta^3\gamma^3}\frac{I_o}{I_A}\frac{l^2}{a_o^2}; k_b = \frac{\varepsilon l}{a_o^2} = \frac{l}{\beta^*} \qquad \frac{d^2\tilde{n}}{ds^2} + 2k_{sc}^2\left(\frac{a_o}{a(s)}\right)^2 \tilde{n} = 0;$$

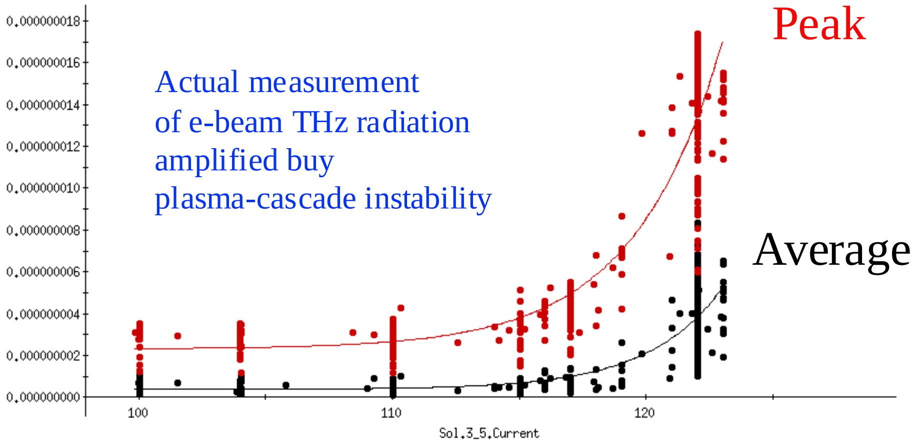



- *Plasma-Cascade micro-bunching Amplifier and Coherent electron Cooling of a Hadron Beams*, arXiv:1802.08677, 2018
- *3D Theory of Microscopic Instabilities Driven by Space-Charge Forces*, Physical Review Accelerators and Beams 26, No.5, 054402, 2023

# Few more words of motivation…

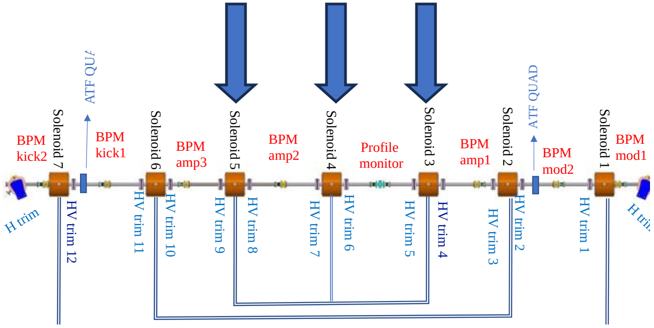
- ✓ When I presented experimental results that shown that a small (just few percents) changes is on the strength of focusing can result in exponential growth of the instability, a review panel "expert" told me his guts telling him that is simply impossible….
- ✓ I think that this exactly the problem between science and "gut feelings" – exponential growth is something inherent for instabilities
- ✓ Theoretical estimation and theory-based numerical simulations may open our eyes to previously unknown phenomena

Exponential growth of the IR signal at the bolometer as function of current in PCA solenoids: e-fold increase each 3 A (2.4%)



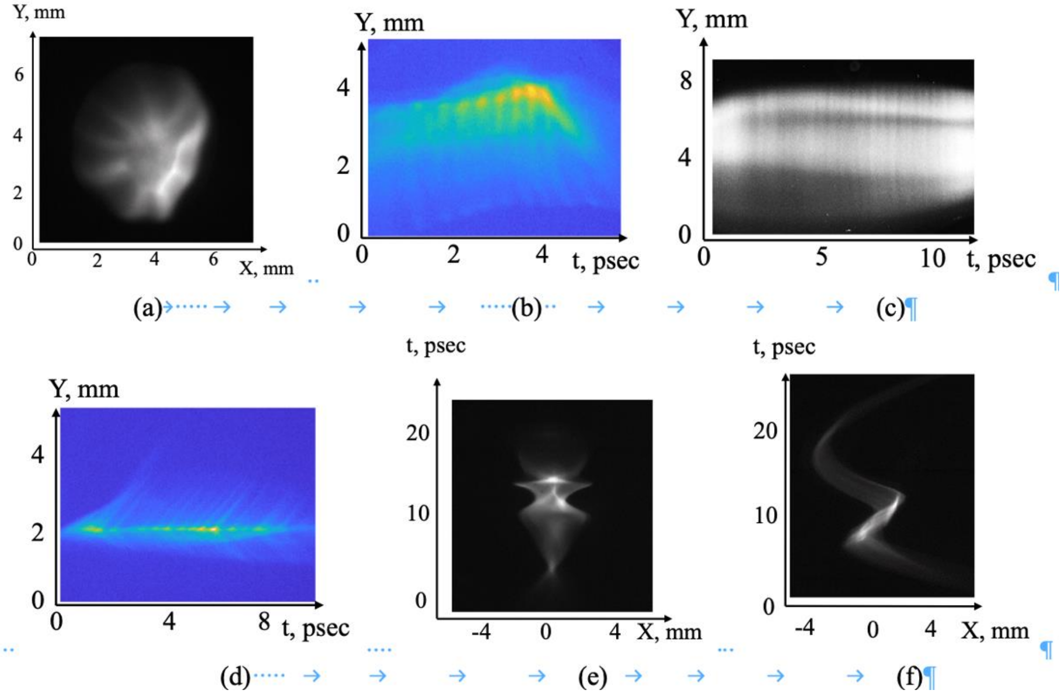

100 110 120 125

Currents in three central PCA solenoids 3-4-5, A



# Why is the 3D treatment?

Because experimentally we observing coupling between degrees of freedom




Samples of measured electron beam distributions in the CeC accelerator, illustrating various aspects of 3D coupling in space-charge dominated electron beam. (a) Coupling between radial and axial modes in SC-driven instability; (b) Coupled 0.5 THz longitudinal density modulation. (c) 2.5 THz density modulation and vertical filamentation. (d) Feather-like coupling between vertical and longitudinal SC-driven instabilities; (e) Time dependent beam envelope and filamentation; (f) An example of coupling between time, energy and horizontal position

# Method used

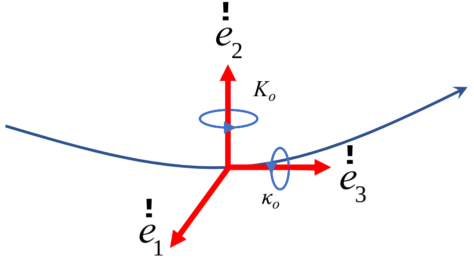
- The classical plasma physics methods, modified for accelerator lingo, for deriving 3D evolution equations of microscopic perturbations in space-charge dominated beam
- We considered accelerators without any limitation on its components, acceleration, deceleration, compression, focusing, coupling, or its 3D beam trajectory.
- We use the length along the reference trajectory, *s*, as independent variable. Particle motion is described as evolution of full set of 6 canonical variables driven by the Hamiltonian, which includes macroscopic space-charge forces. We consider that the beam transport map is evaluated as function of *s* for the unperturbed Hamiltonian and is known.
- **Effects of microscopic instability can be treated as a perturbation**.
- We use Canonical transformation to the initial condition to remove macroscopic components and arrive to the linearized Vlasov equation.
- We identify range when and where our microscopic approach is applicable and derive equation for perturbation Hamiltonian.
- We use local linearization of the transport map with symplectic 6x6 matrix in Alex Dragt's notation. Using this notation allows to clearly identify roles of 3x3 matrix blocks in the evolution of the beam and perturbation parameters.
- We apply Fourier transform and arrive to explicit form if linear integral equation describing evolution of the microscopic perturbations.
- Finally, we identify conditions when the linear integral equation can be reduced to an ordinary second order differential equation for the electron beam density perturbation

# Arbitrary 3D orbit and EM field (including self-fields)

$$\vec{r} = \vec{r}_o(s) + q_1 \cdot \vec{n}(s) + q_2 \cdot \vec{b}(s);$$

$$\vec{\tau} = \frac{d\vec{r}_o}{ds};\; \vec{n} = -\frac{\vec{r}_o\,'}{\left|\vec{r}_o\,'\right|};\vec{b} = \left[\vec{n}\times\vec{\tau}\right];$$

$$\vec{\tau}\,' = -K_o(s)\cdot\vec{n};\; \vec{n}\,' = K_o(s)\cdot\vec{\tau} - \kappa_o(s)\cdot\vec{b};\; \vec{b}\,' = \kappa_o(s)\cdot\vec{n};$$




$$H^* = -(1+K_o q_1)\sqrt{\frac{(H-ej)^2}{c^2} - m^2c^2 - \left(p_1 - \frac{e}{c}A_1\right)^2 - \left(p_2 - \frac{e}{c}A_2\right)^2}$$

$$-\frac{e}{c}A_3 + \kappa_o q_1\left(p_2 - \frac{e}{c}A_2\right) - \kappa_o q_2\left(p_1 - \frac{e}{c}A_1\right);\; A_3 = (1+K_o q_1)A_s + \kappa\left(q_2A_1 - q_1A_2\right);$$

$$q^T = [q_1, q_2, q_3],\; P^T = [P_1, P_2, P_3],\; x^T = [q^T, P^T],$$

## Initial conditions

$$\underline{q} \equiv q(s=0),\; \underline{P} \equiv P(s=0),\; \underline{x} \equiv x(s=0)$$

$$q = q(\underline{q}, \underline{P}, s),\; P = P(\underline{q}, \underline{P}, s),\;\; x = x(\underline{x}, s),$$

## Alex Dragt's notations

$$\frac{dx_i}{ds} = S_{ij}\cdot\frac{\partial h}{\partial x_j} \Leftrightarrow \frac{dx}{ds} = \mathbf{S}\cdot\frac{\partial h}{\partial x};$$

$$\mathbf{S} \equiv \left[S_{ij}\right] = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{3\times3} \\ -\mathbf{I}_{3\times3} & \mathbf{0} \end{bmatrix};\qquad \mathbf{I}_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\qquad \mathbf{S}^2 = -\mathbf{I}_{6\times6};$$

## Canonical transformation to initial conditions

$$\underline{q} = \underline{q}(q, P, s),\qquad \underline{P} = \underline{P}(q, P, s),\qquad \underline{x} = \underline{x}(x, s),$$

$$\underline{h}(\underline{q}, \underline{P}, s) = 0$$

# Variables

- Regular – $q$, $P$, $\xi$ , $k$, $f$ – at arbitrary $s$
- Underscored – $\underline{q}$, $\underline{P}$, $\underline{\xi}$ , $\underline{k}$, $\underline{f}$ – at $s=0$ *(initial values)*

## Vlasov equation with $\zeta=\{q,p\}$

$$\frac{\partial f_o(x,s)}{\partial s} + S_{ik} \cdot \frac{\partial f_o(x,s)}{\partial x_i} \cdot \frac{\partial h_o(x,s)}{\partial x_k} = 0;$$

$$f(\xi,s) = f_o(\xi,s) + \tilde{f}(\xi,s); \qquad \left|\tilde{f}(\xi,s)\right| << \left|f_o(\xi,s)\right|$$

$$\underline{f}_o(\underline{x}) \equiv \underline{f}_o(\underline{x}, s=0) \quad \Rightarrow \quad f_o(x,s) = \underline{f}_o(\underline{x}(x,s)).$$

$$h(\xi,s) = h_o(\xi,s) + \tilde{h}(\xi,s); \qquad \tilde{h}(\xi,s) = O(\varepsilon)$$

## Vlasov equation with variable initial conditions $\zeta=\{q,p\}$

$$\underline{h}(\underline{q},\underline{P},s)=0$$

$$\underline{h}(\underline{\xi},s) = \tilde{\underline{h}}(\underline{\xi},s) \equiv \tilde{h}(\xi(\underline{\xi},s),s),$$

$$\underline{f}(\underline{\xi},s) = \underline{f}_o(\underline{\xi}) + \tilde{\underline{f}}(\underline{\xi},s); \qquad \tilde{f}(\xi,s) \equiv \tilde{\underline{f}}(\underline{\xi}(\xi,s),s);$$

$$\frac{\partial \tilde{\underline{f}}}{\partial s} + S_{ik} \cdot \frac{\partial \underline{f}_o}{\partial \underline{\xi}_i} \cdot \frac{\partial \tilde{\underline{h}}}{\partial \underline{\xi}_k} + S_{ik} \cdot \frac{\partial \tilde{\underline{f}}}{\partial \underline{\xi}_i} \cdot \frac{\partial \tilde{\underline{h}}}{\partial \underline{\xi}_k} = 0.$$

$O(e^2)$

## Linearization

$$\frac{\partial \tilde{\underline{f}}}{\partial s} + S_{ik} \cdot \frac{\partial \underline{f}_o}{\partial \underline{\xi}_i} \cdot \frac{\partial \tilde{\underline{h}}}{\partial \underline{\xi}_k} = -S_{ik} \cdot \frac{\partial \tilde{\underline{f}}}{\partial \underline{\xi}_i} \cdot \frac{\partial \tilde{\underline{h}}}{\partial \underline{\xi}_k} = O(\varepsilon^2) \to 0;$$

$$\frac{\partial \tilde{\underline{f}}}{\partial s} + \frac{\partial \underline{f}_o}{\partial \underline{q}_i} \cdot \frac{\partial \tilde{\underline{h}}}{\partial \underline{P}_i} - \frac{\partial \underline{f}_o}{\partial \underline{P}_i} \cdot \frac{\partial \tilde{\underline{h}}}{\partial \underline{q}_i} \cong 0.$$

# What we can call "microscopic"?

## Introducing scales of uniformity

$$\left|\frac{\partial f_o}{\partial q_i}\right| \propto \frac{f_o}{a_i}$$

Fields of Fourie harmonics $e^{ik^T q} = e^{i\vec{k}\cdot\vec{q}}$ $\quad k^T = [k_1, k_2, k_3]$ do not mix when

$$a_{1,2} \times \sqrt{\left(b_o g_o\right)^2 \times \left(k_1^2 + k_2^2\right) + k_3^2} >> b_o g_o; \; a_3 \times \sqrt{\left(b_o g_o\right)^2 \times \left(k_1^2 + k_2^2\right) + k_3^2} >> 1$$

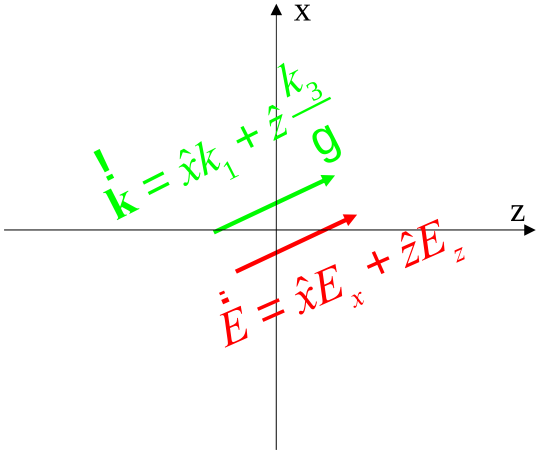
Otherwise, the Fourie harmonics mix and there is no way to reduce even linearized Vlasov equation to a solvable case:

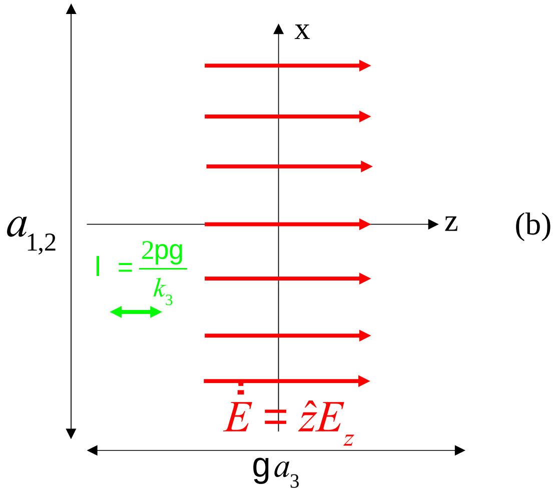
$$\int dQ^3 \cdot e^{-i\vec{k}\cdot\vec{Q}} \cdot \left( \frac{\partial \tilde{f}}{\partial s} + \frac{\partial f_o}{\partial \vec{q}} \cdot \frac{\partial \tilde{h}}{\partial \vec{P}} - \frac{\partial f}{\partial \vec{P}} \cdot \frac{\partial \tilde{h}}{\partial \vec{q}} \right) =$$

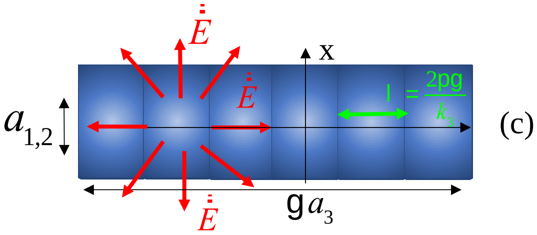
$$\frac{\partial \tilde{f}_{\vec{k}}}{\partial t} + i \int d\mathbf{k}^3 \left\{ f_{o\mathbf{k}} \cdot \left( \mathbf{k} \cdot \frac{\partial \tilde{h}_{\vec{k}-\mathbf{k}}}{\partial \vec{P}} \right) - \tilde{h}_{\vec{k}-\mathbf{k}} \cdot \left( \left(\vec{k} - \mathbf{k}\right) \cdot \frac{\partial f_{o\mathbf{k}}}{\partial \vec{P}} \right) \right\},$$

# Graphic representation co-moving frame

$$\vec{E} = 4\pi \cdot \rho_{\vec{k}}\left(\vec{r}\right) \cdot \frac{\vec{\mathbf{k}}}{i\vec{\mathbf{k}}^{2}} e^{i\vec{\kappa}\cdot\vec{r}} + \delta\vec{E}.$$



# Perturbation Hamiltonian

Simple calculation and transfer to the lab frame gives us

$$\tilde{h} = \frac{4\pi e^2}{c} \cdot \int \frac{\tilde{\rho}_{\vec{k}} \cdot e^{i\vec{k}\cdot\vec{q}} \cdot dk^3}{\gamma_o^{\,2} \cdot \beta_o^{\,2} \cdot \vec{k}_\perp^{\,2} + k_3^{\,2}};$$

$$\delta\left(\frac{d\vec{P}}{ds}\right) = -\frac{\partial \tilde{h}}{\partial \vec{q}} = -\frac{4\pi e^2}{c} \cdot \int \frac{i\vec{k} \cdot \tilde{\rho}_{\vec{k}} \cdot e^{i\vec{k}\cdot\vec{q}} \cdot dk^3}{\gamma_o^{\,2} \cdot \beta_o^{\,2} \cdot \vec{k}_\perp^{\,2} + k_3^{\,2}},$$

$$\rho(q,s) = e\int_{-\infty}^{\infty} dP^3 \cdot f(q,P) = \int_{-\infty}^{\infty} dP^3 \cdot f(\underline{q}(q,P), \underline{P}(q,P))$$

$$X_o(s) = \mathcal{M}(s) : \underline{X}_o$$

$$W(s) = \mathcal{M}(s) : \underline{W}$$

Local linearization of the transfer map in a phase space around a reference trajectory :

$$X = \underline{X}_o + \Delta \underline{X}, \qquad \mathbf{M}_{\underline{X}_o}(s) = \left.\frac{\partial X}{\partial \underline{X}}\right|_{\underline{X}=\underline{X}_o} \equiv \frac{\partial}{\partial \underline{X}}\left(\mathcal{M}(s) : \underline{X}\right)\Big|_{\underline{X}=\underline{X}_o};$$

$$X(s) = X_o(s) + \Delta X(s) = \mathcal{M}(s) : \underline{X}_o + \mathbf{M}_{\underline{X}_o}(s) \cdot \Delta \underline{X} + O\left(|\Delta \underline{X}|^2\right),$$

$$\delta q = \mathcal{M}_q(s) : (\underline{\xi}_o + \Delta\underline{\xi}) - \mathcal{M}_q(s) : \underline{\xi}_o - \mathbf{A}\cdot\Delta\underline{q} - \mathbf{B}\cdot\Delta\underline{P};$$

$$|\vec{k}\cdot\delta\vec{q}| << 1; \qquad \{\Delta\underline{q}, \Delta\underline{P}\} \in \Omega,$$

$$\begin{bmatrix} q(s) \\ P(s) \end{bmatrix} = \begin{bmatrix} \mathcal{M}_q(s) : \underline{X}_o \\ \mathcal{M}_P(s) : \underline{X}_o \end{bmatrix} + \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \cdot \begin{bmatrix} \Delta \underline{q} \\ \Delta \underline{P} \end{bmatrix} + O\left(|\Delta\underline{q}|^2, |\Delta\underline{P}|^2\right),$$

Removing Δ for compactness and using symplecticity of the map:

$$X = \mathbf{M}(s)\cdot \underline{X};\quad \underline{X} = \mathbf{M}^{-1}(s)\cdot X;\quad \mathbf{M}(0) = \mathbf{I}_{6\times 6};$$

$$\mathbf{M}^T \cdot \mathbf{S} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{S} \cdot \mathbf{M}^T = \mathbf{S} \quad \Rightarrow \quad \det \mathbf{M} = 1; \quad \mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^T\cdot\mathbf{S},$$

$$\begin{bmatrix} q \\ P \end{bmatrix} = \mathbf{M}(s) \cdot \begin{bmatrix} \underline{q} \\ \underline{P} \end{bmatrix}; \qquad \begin{bmatrix} \underline{q} \\ \underline{P} \end{bmatrix} = \mathbf{M}^{-1}(s) \cdot \begin{bmatrix} q \\ P \end{bmatrix};$$

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$$q = \mathbf{A}\cdot\underline{q} + \mathbf{B}\cdot\underline{P};\quad P = \mathbf{C}\cdot\underline{q} + \mathbf{D}\cdot\underline{P};$$

$$\underline{q} = \mathbf{D}^T\cdot q - \mathbf{B}^T\cdot P;\quad \underline{P} = -\mathbf{C}^T\cdot q + \mathbf{A}^T\cdot P.$$

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$$\delta q = \mathcal{M}_q(s) : (\underline{\xi}_o + \Delta\underline{\xi}) - \mathcal{M}_q(s) : \underline{\xi}_o - \mathbf{A}\cdot\Delta\underline{q} - \mathbf{B}\cdot\Delta\underline{P};$$

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$$\mathbf{M}^T \cdot \mathbf{S} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{S} \cdot \mathbf{M}^T = \mathbf{S} \quad \Rightarrow \quad \det \mathbf{M} = 1; \quad \mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^T\cdot\mathbf{S},$$

$$\begin{bmatrix} q \\ P \end{bmatrix} = \mathbf{M}(s) \cdot \begin{bmatrix} \underline{q} \\ \underline{P} \end{bmatrix}; \qquad \begin{bmatrix} \underline{q} \\ \underline{P} \end{bmatrix} = \mathbf{M}^{-1}(s) \cdot \begin{bmatrix} q \\ P \end{bmatrix};$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}; \qquad \mathbf{M}^{-1} = -\mathbf{S}\cdot\mathbf{M}^T\cdot\mathbf{S} = \begin{bmatrix} \mathbf{D}^T & -\mathbf{B}^T \\ -\mathbf{C}^T & \mathbf{A}^T \end{bmatrix};$$

$$q = \mathbf{A}\cdot\underline{q} + \mathbf{B}\cdot\underline{P}; \quad P = \mathbf{C}\cdot\underline{q} + \mathbf{D}\cdot\underline{P};$$

$$\underline{q} = \mathbf{D}^T\cdot q - \mathbf{B}^T\cdot P; \quad \underline{P} = -\mathbf{C}^T\cdot q + \mathbf{A}^T\cdot P.$$

# *Continued…*

Important assumption it that initial momenta spread is limited and that linear expansion on $\underline{P}$ can be used in integral for *fixed q*:

$$\underline{P} = -\mathbf{C}^T \cdot q + \mathbf{A}^T \cdot P \;\Rightarrow\; P = (\mathbf{A}^T)^{-1} \cdot (\underline{P} + \mathbf{C}^T q) \quad\Rightarrow\quad dP^3\Big|_{q=const} = \frac{1}{\det\mathbf{A}} \cdot d\underline{P}^3\Big|_{q=const};$$

$$q = \mathbf{A} \cdot \underline{q} + \mathbf{B} \cdot \underline{P} = const \Rightarrow \underline{q} = \mathbf{A}^{-1}(q - \mathbf{B} \cdot \underline{P}) \;\Rightarrow f\left(q, (\mathbf{A}^T)^{-1}(\underline{P} + \mathbf{C}^T \cdot q)\right) = \underline{f}\left(\mathbf{A}^{-1}(q - \mathbf{B} \cdot \underline{P}), \underline{P}\right).$$

$$\rho(q,s) = e\int_{-\infty}^{\infty} dP^3 \cdot f(q,P) \;\Rightarrow\; \frac{e}{\det\mathbf{A}}\int_{-\infty}^{\infty} d\underline{P}^3 \cdot \underline{f}\left(\mathbf{A}^{-1}(q - \mathbf{B} \cdot \underline{P}), \underline{P}\right)$$

This allows to go take local Fourier transform of the density

$$\tilde{\rho}_{\vec{k}} \equiv \tilde{\rho}(s,\vec{k}) = \frac{1}{(2\pi)^3}\iint \tilde{f}(q,P,s) \cdot e^{-i\vec{k}\cdot\vec{q}} \cdot dq^3 \cdot dP^3 =$$

$$\frac{1}{(2\pi)^3}\iint \underline{\tilde{f}}(\underline{q},\underline{P},s)\, e^{-i\vec{k}\cdot\vec{q}(\underline{\xi},t)} \cdot d\underline{q}^3 \cdot d\underline{P}^3 = \frac{1}{(2\pi)^3}\iint \underline{\tilde{f}}(\underline{q},\underline{P},s) \cdot e^{-i\vec{k}\cdot\left(\vec{\mathbf{A}}\cdot\vec{\underline{q}} + \vec{\mathbf{B}}\cdot\vec{\underline{P}}\right)} \cdot d\underline{q}^3 \cdot d\underline{P}^3,$$

$$\vec{q} = \vec{\mathbf{A}} \cdot \vec{\underline{q}} + \vec{\mathbf{B}} \cdot \vec{\underline{P}}$$

$$q = \mathbf{A} \times \underline{q} + \mathbf{B} \times \underline{P}$$

for *k*-"vector" evolving with *s* $\quad \underline{k} = k(s=0);\; \underline{k}^T = k^T(s) \times \mathbf{A}(s) \quad$ Þ $\quad k^T(s) = \underline{k}^T \times \mathbf{A}^{-1}(s)$

$$\vec{k}(s) = \vec{\underline{k}} \cdot \vec{\mathbf{A}}^{-1}(s);\; \vec{k}(s) \cdot \vec{q} = \vec{\underline{k}} \cdot \left(\vec{\underline{q}} + \overleftrightarrow{\mathbf{A}^{-1}\mathbf{B}} \cdot \vec{\underline{P}}\right) \qquad k^T(s) = [k_1(s), k_2(s), k_3(s)]; \qquad \underline{k} = \mathbf{A}^T k(s) \;\Leftrightarrow\; k(s) = (\mathbf{A}^T)^{-1}\underline{k}$$

# *Back to Vlasov equation*

We can describe local initial distribution function as function of momenta: $f_o \Rightarrow n_o \cdot f_o\left(\underline{P}\right); \quad \int_{-\infty}^{\infty} f_o\left(\underline{P}\right) \cdot d\underline{P}^3 = 1; \quad n_o = \frac{j_o}{ec}$

$$\delta\left(\frac{dq_j}{ds}\right) = \frac{\partial \tilde{h}(\xi,s)}{\partial P_j} = 0 \;\Rightarrow\; \begin{array}{l} \underline{P} = \mathbf{A}^T \cdot P - \mathbf{C}^T \cdot q; \qquad d\underline{P}_i = A_{ji} \cdot dP_j - C_{ji} \cdot dq_j; \\ \frac{\partial \tilde{f}}{\partial s} = -n_o \cdot \frac{\partial f_o}{\partial \underline{P}_i} \cdot \frac{\partial \underline{P}_i}{\partial P_j} \cdot \delta\left(\frac{dP_j}{ds}\right) - n_o \cdot \frac{\partial f_o}{\partial \underline{P}_i} \cdot \frac{\partial \underline{P}_i}{\partial q_j} \cdot \delta\left(\frac{dq_j}{ds}\right) = n_o \cdot \frac{\partial f_o}{\partial \underline{P}_i} \cdot A_{ji} \cdot \frac{\partial \tilde{h}}{\partial q_j}, \end{array}$$

$$\Rightarrow\; \frac{\partial \tilde{f}}{\partial s} = n_o \cdot \frac{\partial f_o}{\partial \underline{P}_i} \cdot A_{ji}(s) \cdot \mathbf{F}_j(q,s);$$

$$\mathbf{F}_j(q,s) = \frac{\partial \tilde{h}}{\partial q_j} = \frac{4\pi e^2}{c} \int \frac{ik_j \cdot \tilde{\rho}_{\vec{k}} \cdot e^{i\vec{k}\cdot\vec{q}} \cdot dk^3}{\gamma_o^{\,2} \cdot \beta_o^{\,2} \cdot \vec{k}_\perp^2 + k_3^{\,2}}$$

And taking Fourie transform of the Vlasov equation $\quad \frac{\partial \tilde{f}_{\vec{k}}}{\partial s} = \frac{n_o}{(2\pi)^3} \cdot \frac{\partial f_o}{\partial \underline{P}_i} \cdot A_{ji}(s) \cdot \int d\underline{q}^3 \cdot e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot \mathbf{F}_j(q,s)$

we get to r.h.s.

$$\mathbf{F}_{\underline{\vec{k}}} = \frac{1}{(2\pi)^3} \cdot \int e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot \mathbf{F}(q,s) \cdot d\underline{q}^3 \Big|_{\underline{P}=const} = \frac{4\pi e^2}{c} \cdot \int \frac{i\vec{k} \cdot \tilde{\rho}_{\vec{k}} \cdot dk^3}{\gamma_o^{\,2} \cdot \beta_o^{\,2} \cdot \vec{k}_\perp^2 + k_3^{\,2}} \cdot \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{q}} e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot d\underline{q}^3;$$

$$\frac{1}{(2\pi)^3} \cdot \int e^{i\vec{k}\cdot\vec{q}} e^{-i\vec{\underline{k}}\cdot\vec{\underline{q}}} \cdot d\underline{q}^3 = \frac{e^{i\vec{k}\cdot\overleftrightarrow{\mathbf{B}}\cdot\vec{\underline{P}}}}{(2\pi)^3} \int e^{i\left(\vec{k}\cdot\overleftrightarrow{\mathbf{A}}-\vec{\underline{k}}\right)\cdot\vec{\underline{q}}} \cdot d\underline{q}^3 = e^{i\vec{k}\cdot\overleftrightarrow{\mathbf{B}}\cdot\vec{\underline{P}}}\,\delta\left(\vec{k}\cdot\overleftrightarrow{\mathbf{A}} - \vec{\underline{k}}\right) = \frac{e^{i\vec{\underline{k}}\cdot\overleftrightarrow{\mathbf{A}^{-1}\cdot\mathbf{B}}\cdot\vec{\underline{P}}}}{\det \mathbf{A}}\,\delta\left(\vec{k} - \vec{\underline{k}}\cdot\overleftrightarrow{\mathbf{A}}^{-1}\right),$$

Resulting in

$$\frac{\partial \tilde{f}_{\vec{\underline{k}}}\left(\underline{P},s\right)}{\partial s} = \frac{4\pi n_o e^2}{c} \cdot \frac{\tilde{\rho}\left(s,\vec{k}(s)\right)}{\gamma_o(s)^2 \cdot \beta_o(s)^2 \cdot \vec{k}_\perp(s)^2 + k_3(s)^2} \frac{e^{i\vec{\underline{k}}\cdot\overleftrightarrow{\mathbf{A}^{-1}(s)\cdot\mathbf{B}(s)}\cdot\vec{\underline{P}}}}{\det \mathbf{A}(s)} \left(i\underline{k}_i \cdot \frac{\partial f_o}{\partial \underline{P}_i}\right)$$

# Continued..

Using easily established ratio $\tilde{\rho}\left(s,\vec{k}(s)\right)=\int \tilde{f}_{\underline{k}}\left(\underline{P},s\right)\cdot e^{-i\underline{k}\cdot\overleftrightarrow{\mathbf{A}(s)^{-1}\mathbf{B}(s)}\cdot\underline{P}}\,d\underline{P}^3,$

we arrive to a solvable linear integral equation

$$\tilde{\rho}\left(s,\vec{k}(s)\right)=\tilde{\rho}_{o\underline{k}}(s)+\frac{4\pi i\cdot e^2\cdot n_o}{c}\cdot\int_o^s\frac{\tilde{\rho}\left(\zeta,\vec{k}(\zeta)\right)\cdot d\zeta}{\det\mathbf{A}(\zeta)}\int\frac{e^{i\left(\vec{k}(\zeta)\cdot\vec{\mathbf{B}}(\zeta)-\vec{k}(s)\cdot\vec{\mathbf{B}}(s)\right)\cdot\underline{P}}}{\gamma_o(\zeta)^2\cdot\beta_o(\zeta)^2\cdot\vec{k}_\perp^2(\zeta)+k_3{}^2(\zeta)}\cdot\underline{k}_i\cdot\frac{\partial f_o}{\partial\underline{P}_i}\cdot d\underline{P}^3;$$

$$\tilde{\rho}_{o\underline{k}}(s)=\int e^{-i\vec{k}(s)\cdot\vec{\mathbf{B}}(s)\cdot\underline{P}}\cdot\tilde{f}_{\underline{k}}\left(\underline{P},0\right)\cdot d\underline{P}^3.$$

which can be further "improved" using partial integration $\int\frac{\partial f_o}{\partial\underline{P}_i}\cdot\varphi\cdot d\underline{P}_i=f_o\cdot\varphi\Big|_{P_i=-\infty}^{P_i=\infty}-\int f_o\cdot\frac{\partial\varphi}{\partial\underline{P}_i}\cdot d\underline{P}_i$ $\quad\boxed{f_o\left(\underline{P}_i=\pm\infty\right)=0}$

$$\sum_{i=1}^{3}\underline{k}_i\cdot\frac{\partial}{\partial\underline{P}_i}e^{i\underline{k}\cdot\left(\vec{\mathbf{U}}(\zeta)-\vec{\mathbf{U}}(s)\right)\cdot\underline{P}}=-i\cdot\left(u(s)-u(\zeta)\right)\cdot e^{i\underline{k}\cdot\left(\vec{\mathbf{U}}(\zeta)-\vec{\mathbf{U}}(s)\right)\cdot\underline{P}};$$

$$u(\zeta)=\vec{k}\cdot\vec{\mathbf{B}}(\zeta)\cdot\underline{k}\equiv\sum_{i,j}\mathbf{B}_{ij}(\zeta)\cdot k_i\cdot\underline{k}_j=\sum_{i,j}\left[\mathbf{A}(\zeta)^{-1}\mathbf{B}(\zeta)\right]_{ij}\cdot\underline{k}_i\cdot\underline{k}_j$$

we get to our final form of linear integral equation with identifiable kernel and Landau damping terms

$$\tilde{\rho}\left(s,\vec{k}(s)\right)=-\int_o^s\tilde{\rho}\left(\zeta,\vec{k}(\zeta)\right)\cdot K(\zeta)\cdot\left[u(s)-u(\zeta)\right]\cdot L_d(s,\zeta)\cdot d\zeta+\tilde{\rho}_{o\underline{k}}(s);$$

$$K(\zeta)=\frac{4\pi\cdot n_o\cdot e^2}{c\cdot\det\mathbf{A}(\zeta)\cdot\upsilon(\zeta)};\qquad L_d\left(\underline{k},s,\zeta\right)=\int e^{i\left(\vec{k}(\zeta)\cdot\vec{\mathbf{B}}(\zeta)-\vec{k}(s)\cdot\vec{\mathbf{B}}(s)\right)\cdot\underline{P}}\cdot f_o\left(\underline{P}\right)\cdot d\underline{P}^3;$$

$$u(\zeta)=\vec{k}(\zeta)\cdot\vec{\mathbf{B}}(\zeta)\cdot\underline{k}\equiv\underline{k}(\zeta)\cdot\vec{\mathbf{U}}(\zeta)\cdot\underline{k};\qquad\upsilon(s)=\gamma_o(s)^2\cdot\beta_o(s)^2\cdot\vec{k}_\perp(s)^2+k_3(s)^2;$$

$$\mathbf{U}=\mathbf{A}^{-1}\cdot\mathbf{B}$$

$$\frac{u(s)-u(\zeta)}{\upsilon(\zeta)}=\frac{\underline{\vartheta}\cdot\left(\vec{\mathbf{U}}(\zeta)-\vec{\mathbf{U}}(s)\right)\cdot\underline{\vartheta}}{\gamma_o(s)^2\cdot\beta_o(s)^2\cdot\vec{\vartheta}_\perp(s)^2+\vec{\vartheta}_3(s)^2};$$

$$\vec{\vartheta}(s)=\frac{\vec{k}(s)}{\left|\underline{\vec{k}}\right|};\qquad\underline{\vartheta}=\vec{\vartheta}(0)=\frac{\underline{\vec{k}}}{\left|\underline{\vec{k}}\right|}.$$

# *Examples..*

## Landau term for Gaussian distribution

$$\underline{f}_o(P) = \prod_{i=1}^{3} \frac{1}{\sqrt{2p} \cdot s_i} \cdot \exp\left( - \frac{P_i^2}{2 \cdot s_i^{\,2}} \right) \Rightarrow L_d = \int e^{i(\vec{\eta}(\zeta)-\vec{\eta}(s))\cdot\vec{P}} \cdot F_o(P) \cdot dP^3 = \prod_{i=1}^{3} \exp\left( -\frac{\sigma_i^{\,2} \cdot (\eta_i(\zeta)-\eta_i(s))^2}{2} \right); \qquad \vec{\eta}(\zeta) = \vec{k}(\zeta) \cdot \vec{\mathbf{B}}(\zeta)$$

## Longitudinal equation for uncoupled motion (matrix **A** is diagonal)

$$\tilde{\rho}(s,k(s)) = -\frac{4\pi \cdot n_o \cdot e^2}{c} \int_0^s \frac{\tilde{\rho}(\zeta,k(\zeta))}{A_{11}(\zeta)A_{22}(\zeta)} \cdot \left( \frac{A_{33}(\zeta)}{A_{33}(s)} \cdot B_{33}(s) - B_{33}(\zeta) \right) \cdot L_d(s,\zeta) \cdot d\zeta + \tilde{\rho}_{o\underline{k}}(s); \qquad k(s) = \underline{k} / A_{33}$$

$$L_d(s,\zeta) = \int e^{i(k(\zeta)\cdot B_{33}(\zeta) - k(s)\cdot B_{33}(s))\cdot \underline{P}} \underline{f}_o(\underline{P}) d\underline{P}; \qquad \tilde{\rho}_{o\underline{k}}(s) = \int e^{-ik(s)\cdot B_{33}(s)\cdot \underline{P}} \cdot \tilde{\underline{f}}_{\underline{k}}(\underline{P},0) \cdot d\underline{P},$$

Conventional definition of transport matrix

$$R_{11} = A_{11};\ R_{33} = A_{22};\ R_{55} = A_{33};\ R_{56} = B_{33};\ r_c = \frac{e^2}{mc^2};$$

$$\underline{P} = \delta \cdot E_o / c; \quad E_o = \gamma_o mc^2; \quad \tilde{\underline{f}}_o(\delta) = \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left( -\frac{\delta^2}{2} \right).$$

$$\tilde{\rho}_k(s) = -\frac{4\pi \cdot n_o \cdot r_c}{\gamma_o} \int_0^s \frac{\tilde{\rho}_k(\zeta)}{R_{11}(\zeta)R_{33}(\zeta)} \cdot \left( R_{56}(s) - R_{56}(\zeta) \right) \cdot e^{-\frac{k^2 \cdot (R_{56}(\zeta)-R_{56}(s))^2 \cdot \sigma_\delta^{\,2}}{2}} \cdot d\zeta + \tilde{\rho}_{ok}(s),$$

# *Reducibility to 2nr order ODE*

I personally prefer linear integral equation to a 2nd order ordinary differential equation, but in some cases it can be useful for theoretical developments. Hence, here is a short summary (see our paper for details):

When the Landau damping term allows separation of variables s and $\xi$ $L_d(s,z) = \Lambda(z)\cdot\Lambda^{-1}(s)$; $\Lambda(s) = e^{-f(s)}$;

dt is reducible to an ODE: $\tilde{q}'' - \alpha'\cdot\tilde{q}' + K\cdot u'\cdot\tilde{q} = \tilde{q}_o'' - \alpha'\cdot\tilde{q}_o'$; $\tilde{q}(s) = e^{\phi(s)}\cdot\tilde{\rho}(s,\vec{k}(s))$

$$\tilde{q}_o(s) = e^{\phi(s)}\cdot\tilde{\rho}_{\vec{k}0}(s);\quad \alpha = \ln\frac{u'}{u'_o};\quad u'_o = u'(0),\quad u' > 0$$

or even to inhomogeneous Hill's equation and possibility of using linear matrix relations

$$\hat{q}'' + \hat{K}(s)\cdot\hat{q} = \varsigma(s);\qquad \hat{K}(s) = K(s)\cdot u'(s) - \frac{\alpha'(s)^2}{4} + \frac{\alpha''(s)}{2};$$

$$\hat{q} = e^{-\frac{\alpha(s)}{2}}\tilde{q} \equiv \sqrt{\frac{u'_o}{u'(s)}}\cdot e^{\phi(s)}\cdot\tilde{\rho}(s,\vec{k}s);\qquad \varsigma(s) = e^{-\frac{\alpha(s)}{2}}\cdot\left(\tilde{q}''_o(s) - \tilde{q}'_o(s)\cdot\alpha'(s)\right).$$

$$\begin{bmatrix}\hat{q}(s)\\ \hat{q}'(s)\end{bmatrix} = \mathbf{R}(s)\cdot\begin{bmatrix}\hat{q}(0)\\ \hat{q}'(0)\end{bmatrix};\quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12}\\ r_{21} & r_{22}\end{bmatrix};$$

$$\mathbf{R}' = \begin{bmatrix} 0 & 1\\ -\hat{K}(s) & 0\end{bmatrix}\cdot\mathbf{R};\quad \det\mathbf{R} = 1;$$

Just to mention few cases that can be treated like this: cold beam (plasma) $f_0(P) = \delta(P_1)\cdot\delta(P_2)\cdot\delta(P_3)$

and some cases with Lorentzian initial momenta distribution $F_0(P) = F_{k-1}(P) = \frac{1}{\pi^3}\prod_{i=1}^{3}\frac{\sigma_i}{\sigma_i^2 + P_i^2},$

# Discussion

- Matrix **A** plays a special role in the evolution of the microscopic perturbation in beams with strong SC, specifically that beam density $n(s) = n_o \det \mathbf{A}^{-1}$ became infinite when det**A**=0. It is easy to see that this is the result of "infinite plasma" assumption and it not the case for finite beams. It is resolved by replacing beam density by that found in macroscopic simulations

$$\tilde{\rho}\left(s,\vec{k}(s)\right) = -\int_o^s \frac{4\pi \cdot n(\zeta) \cdot e^2}{c \cdot \upsilon(\zeta)} \tilde{\rho}\left(\zeta,\vec{k}(\zeta)\right) \cdot \left(u(s)-u(\zeta)\right) \cdot L_d(s,\zeta) \cdot d\zeta + \tilde{\rho}_{\vec{k}_o}(s);$$

- Second complication related to with k-vector transformation $k^T(z) = \underline{k}^T \times \mathbf{A}^{-1}(z)$: $\left|k\right| \underset{\det \mathbf{A} \to 0}{\to} \infty$
This is not a real problem since for any finite momenta spread, the corresponding portion of the integral is diminishing to zero:

$$L_d = \int e^{i\left(\vec{k}(\zeta)\cdot\vec{\mathbf{B}}(\zeta) - \vec{k}(s)\cdot\vec{\mathbf{B}}(s)\right)\cdot\vec{P}} \cdot \underline{f}_o(P) \cdot dP^3 \underset{\left|\vec{k}(\zeta)\cdot\vec{\mathbf{B}}(\zeta)\right| + \left|\vec{k}(s)\cdot\vec{\mathbf{B}}(s)\right| \to \infty}{\Longrightarrow} 0$$

- The most non-trivial complication arise in case when the transfer map can not be linearized over all initial spread of the beam, which was very important assumption necessary for derivation of our linear integral equation:

$$q = \mathbf{A} \times \underline{q} + \mathbf{B} \times \underline{P};\ P = \mathbf{C} \times \underline{q} + \mathbf{D} \times \underline{P};$$
$$\underline{q} = \mathbf{D}^T \times q - \mathbf{B}^T \times P;\ \underline{P} = -\mathbf{C}^T \times q + \mathbf{A}^T \times P.$$

# Discussion

- It is understandable that accurate linearization of a ”generally speaking” nonlinear map is possible if we limit the area of the phase space where our conditions applicability of separation for Fourier harmonics is valid

$$\delta q = \mathrm{M}_q(s):\left(\underline{\xi}_o + \Delta\underline{\xi}\right) - \mathrm{M}_q(s):\underline{\xi}_o - \mathbf{A}\cdot\Delta\underline{q} - \mathbf{B}\cdot\Delta\underline{P};$$
$$\left|\vec{k}\cdot\delta\vec{q}\right| << 1; \qquad \left\{\Delta\underline{q},\Delta\underline{P}\right\}\in\Omega,$$

- In this case, evaluation of the Fourier harmonics would require summation over the phase space volumes and evaluation of each step of the history. I currently writing a paper with detailed description of the method, but it is too involved to add in this presentation..

$$\tilde{\rho}_{\vec{k}} \equiv \tilde{\rho}(s,\vec{k}) = \frac{1}{(2\pi)^3}\int\int \tilde{f}(q,P,s)\cdot e^{-i\vec{k}\cdot\vec{q}}\cdot dq^3\cdot dP^3 =$$
$$\frac{1}{(2\pi)^3}\int\int \tilde{\underline{f}}(\underline{q},\underline{P},s)\, e^{-i\vec{k}\cdot\vec{q}(\underline{q},\underline{P},s)}\cdot d\underline{q}^3\cdot d\underline{P}^3 = \sum_{\Omega}\oiint \tilde{\underline{f}}(\underline{q},\underline{P},s)\cdot e^{-i\vec{k}\cdot\left(\vec{q}_{\Omega}+\overleftrightarrow{\mathrm{A}}_{\Omega}\cdot\Delta\underline{\vec{q}}+\overleftrightarrow{\mathrm{B}}_{\Omega}\cdot\Delta\underline{\vec{P}}\right)}\cdot d\Delta\underline{q}^3\cdot d\Delta\underline{P}^3$$

- The method is numerically tractable, but it requires Ω-matrix of density values traced back in *s* with varying *k*-vector. It is definitely more complex than evaluation of a single linear integral equation, but does not suffer from possible violation of applicablity

# Conclusions

- Since we discovered unknown microbunching plasma-cascade instability, which was simply missed by so-call self-consistent PIC space charge codes, we are looking for an alternative ways – either theoretical or theory-numerical or alternative numerical methods to identify and characterize inutilities
- We were also puzzled by observing strong 2D and 3D coupling for microscopic instabilities and understanding that equations of motion in one dimension can cause instability in the other – hence a desire to have a description which is not limited to 1D
- Using several approximations, we derive linear integral equation that describes evolution of 3D microscopic instabilities driven by space charge and identified conditions then it is equivalent to a second order ordinary differential equation
- There is one assumption, typical for high brightness sources, that momenta spread at the source is very small (so called laminar flow approximation) which I would like to eliminate as the next step towards even more rigorous 3D theory of such instabilities.

# Thank you for attention

# Conditions for applicability of the short period (microscopic) perturbations: co-moving frame

$$\vec{k}_{cm} \equiv \vec{\mathbf{k}} = \hat{z}\cdot\mathbf{k}_z + \vec{\mathbf{k}}_\perp; \qquad \omega_{cm} = 0; \qquad \vec{k}_{lab} \equiv \vec{k} = \hat{z}\cdot k_z + \vec{k}_\perp;$$

$$\vec{k}_\perp = \vec{\mathbf{k}}_\perp; \qquad k_z = \gamma_o \cdot \left(\mathbf{k}_z + \beta_o \cdot \frac{\omega_{cm}}{c}\right) = \gamma_o \mathbf{k}_z; \qquad \omega_{lab} = \gamma_o \cdot \left(\omega_{cm} + \mathrm{v}_o \cdot \mathbf{k}_z\right) = \mathrm{v}_o \cdot k_z;$$

$$k_1 = k_x = \mathbf{k}_x; \qquad k_2 = k_y = \mathbf{k}_y; \qquad k_3 = b_o \times k_z = g_o \times b_o \times \mathbf{k}_z;$$

$$a_1 = a_x; \qquad a_2 = a_y; \qquad a_3 = a_z / b_o.$$

$$div\vec{E} = 4\pi\cdot\rho; \qquad curl\vec{E} = 0.$$

$$\mathbf{k}_{x,y} \times a_{x,y} = k_{x,y} \times a_{x,y} >> 2p; \qquad g\mathbf{k}_z \times a_z = k_z \times a_z >> 2p.$$

$$\vec{E} = 4\pi\cdot\rho_{\vec{\mathbf{k}}}\left(\vec{r}\right)\cdot\frac{\vec{\mathbf{k}}}{i\vec{\mathbf{k}}^2}e^{i\vec{\kappa}\cdot\vec{r}} + \delta\vec{E}.$$

$$div\vec{E} = 4\pi\cdot\rho_{\vec{\mathbf{k}}}\left(\vec{r}\right)e^{i\vec{\mathbf{k}}\cdot\vec{r}} + 4\pi\cdot\frac{\left(\vec{\mathbf{k}}\cdot\vec{\nabla}\rho_{\vec{\mathbf{k}}}\left(\vec{r}\right)\right)}{\vec{\mathbf{k}}^2}e^{i\vec{\mathbf{k}}\cdot\vec{r}} + div\delta\vec{E} = 4\pi\cdot\rho_{\vec{\mathbf{k}}}\left(\vec{r}\right)\cdot e^{i\vec{\mathbf{k}}\cdot\vec{r}};$$

$$curl\vec{E} = 4\pi\cdot\frac{\left(\vec{\mathbf{k}}\times\vec{\nabla}\rho_{\vec{\mathbf{k}}}\left(\vec{r}\right)\right)}{\vec{\mathbf{k}}^2}e^{i\cdot\vec{\mathbf{k}}\vec{r}} + curl\delta\vec{E} = 0;$$

$$div\delta\vec{E} = -4\pi\frac{\left(\vec{\mathbf{k}}\cdot\vec{\nabla}\rho_{\vec{\mathbf{k}}}\left(\vec{r}\right)\right)}{\vec{\mathbf{k}}^2}e^{i\vec{\mathbf{k}}\cdot\vec{r}} \sim \left|\vec{\mathbf{k}}\right|\cdot\left|\delta\vec{E}\right|$$

$$curl\delta\vec{E} = -4\pi\frac{\left(\vec{\mathbf{k}}\times\vec{\nabla}\rho_{\vec{\mathbf{k}}}\left(\vec{r}\right)\right)}{\vec{\mathbf{k}}^2}e^{i\vec{\mathbf{k}}\cdot\vec{r}} \sim \left|\vec{\mathbf{k}}\right|\cdot\left|\delta\vec{E}\right|$$

$$\left|\delta E_y\right| \sim \left|\vec{E}\right|\cdot\left(\frac{1}{\gamma\cdot\left|\vec{\mathbf{k}}\right|\cdot a_z}\cdot\frac{\left|\mathbf{k}_y\right|}{\left|\mathbf{k}_z\right|} + \frac{1}{\left|\vec{\mathbf{k}}\right|\cdot_y}\right); \left|\delta E_y\right| \sim \left|\vec{E}\right|\cdot\left(\frac{1}{\left|\vec{\mathbf{k}}\right|\cdot a_y} + \frac{1}{\left|\vec{\mathbf{k}}\right|\cdot a_x}\cdot\frac{\left|\mathbf{k}_y\right|}{\left|\mathbf{k}_x\right|}\right);$$

$$\left|\delta E_z\right| \sim \left|\vec{E}\right|\cdot\left(\frac{1}{\gamma\cdot\left|\vec{\mathbf{k}}\right|\cdot a_z} + \frac{1}{\left|\vec{\mathbf{k}}\right|\cdot a_y}\cdot\frac{\left|\mathbf{k}_z\right|}{\left|\mathbf{k}_y\right|}\right); \left|\delta E_z\right| \sim \left|\vec{E}\right|\cdot\left(\frac{1}{\gamma\left|\vec{\mathbf{k}}\right|a_z} + \frac{1}{\left|\vec{\mathbf{k}}\right|a_x}\frac{\left|\mathbf{k}_z\right|}{\left|\mathbf{k}_x\right|}\right);$$

$$\left|\delta E_x\right| \sim \left|\vec{E}\right|\cdot\left(\frac{1}{\gamma\cdot\left|\vec{\mathbf{k}}\right|\cdot a_z}\cdot\frac{\left|\mathbf{k}_x\right|}{\left|\mathbf{k}_z\right|} + \frac{1}{\left|\vec{\mathbf{k}}\right|\cdot a_x}\right); \left|\delta E_x\right| \sim \left|\vec{E}\right|\cdot\left(\frac{1}{\left|\vec{\mathbf{k}}\right|\cdot a_x} + \frac{1}{\left|\vec{\mathbf{k}}\right|\cdot a_y}\cdot\frac{\left|\mathbf{k}_x\right|}{\left|\mathbf{k}_y\right|}\right).$$

# Result

$$\varepsilon_x = \frac{1}{\left|\vec{\mathbf{k}}\right| \cdot a_x}; \qquad \varepsilon_y = \frac{1}{\left|\vec{\mathbf{k}}\right| \cdot a_y}; \qquad \varepsilon_z = \frac{1}{\gamma \cdot \left|\vec{\mathbf{k}}\right| \cdot a_z};$$

$$r_{xy} = \frac{\left|\mathbf{k}_x\right|}{\left|\mathbf{k}_y\right|}; \qquad r_{xz} = \frac{\left|\mathbf{k}_x\right|}{\left|\mathbf{k}_z\right|}; \qquad r_{yz} = \frac{\left|\mathbf{k}_y\right|}{\left|\mathbf{k}_z\right|};$$

$$\left|\delta E_x\right| \sim \left|\vec{E}\right| \cdot \min\left( \varepsilon_x + \frac{\varepsilon_y}{r_{xy}} + \frac{\varepsilon_z}{r_{xz}}, \varepsilon_x + \varepsilon_z \cdot r_{xz}, \varepsilon_x + \varepsilon_y \cdot r_{xy} \right) \le \varepsilon_x + \varepsilon_y + \varepsilon_z;$$

$$\left|\delta E_y\right| \sim \left|\vec{E}\right| \min\left( \varepsilon_x \cdot r_{xy} + \varepsilon_y + \frac{\varepsilon_z}{r_{yz}}, \varepsilon_y + \varepsilon_z \cdot r_{yz}, \frac{\varepsilon_x}{r_{xy}} + \varepsilon_y \right) \le \varepsilon_x + \varepsilon_y + \varepsilon_z;$$

$$\left|\delta E_z\right| \sim \left|\vec{E}\right| \min\left( \varepsilon_x \cdot r_{xz} + \varepsilon_y \cdot r_{yz} + \varepsilon_z, \frac{\varepsilon_y}{r_{yz}} + \varepsilon_z, \frac{\varepsilon_x}{r_{xz}} + \varepsilon_z \right) \le \varepsilon_x + \varepsilon_y + \varepsilon_z;$$

$$\Leftrightarrow$$

$$\left|\vec{\mathbf{k}}\right| \cdot a_x >> 1 \quad ; \left|\vec{\mathbf{k}}\right| \cdot a_y >> 1; \quad \gamma_o \cdot \left|\vec{\mathbf{k}}\right| \cdot a_z >> 1;$$

$$a_{1,2} \cdot \sqrt{\left(\beta_o \cdot \gamma_o\right)^2 \cdot \left(k_1^{\ 2} + k_2^{\ 2}\right) + k_3^{\ 2}} >> \beta_o \cdot \gamma_o;$$

$$a_3 \cdot \sqrt{\left(\beta_o \cdot \gamma_o\right)^2 \left(k_1^{\ 2} + k_2^{\ 2}\right) + k_3^{\ 2}} >> 1.$$