ITMO faculty of PHYSICS

Dynamics and radiation of vortex electrons in magnetic fields

~~Georgii~~ George Sizykh

In collaboration with: Alisa Chaikovskaia, Dmitrii Grosman, Ilia Pavlov, and Dmitry Karlovets

Vortex states in nuclear and particle physics

Zhuhai, Guangdong, China April 24–28, 2024

Contents

Transit of vortex electrons through magnetic fields

- Vortex electrons in vacuum
- Nonstationary states approach
- Properties of the electron state in the field

Radiation of vortex electrons in magnetic fields

- Evolved-state approach
- Emitted photons' properties

faculty of PHYSICS

Transit of vortex electrons through magnetic fields

*Plots are taken from Knyazev, B. A., & Serbo, V. G. (2018). Beams of photons with nonzero projections of orbital angular momenta: new results. Physics-Uspekhi, 61(5), 449.

ITsMOre than a UNIVERSITY

Scalar cat waves

Plane waves

$\Psi(p_x, p_y, p_z)$

Cylindrical waves

$\Psi(p_z, p_\perp, l)$

Spherical waves

$\Psi(\omega, j, l)$

Vortex electrons in vacuum

faculty of PHYSICS

$\hbar = c = 1$

Nonrelativistic QM

No spin

Schrödinger equation

$$i\frac{\partial \Psi(\mathbf{r},t)}{\partial t} = \hat{\mathcal{H}}\Psi(\mathbf{r},t)$$

OAM eigenstate

$$\hat{l}_z\Psi(\boldsymbol{r},t) = l_z\Psi(\boldsymbol{r},t)$$

Schrödinger equation in cylindrical coordinates

$$i\frac{\partial \Psi(\boldsymbol{r},t)}{\partial t} + \frac{1}{2m}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Psi(\boldsymbol{r},t)}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Psi(\boldsymbol{r},t)}{\partial \varphi^2} + \frac{\partial^2 \Psi(\boldsymbol{r},t)}{\partial z^2}\right] = 0$$

Bessel beam

$$\Psi(\boldsymbol{r},t) = J_l\left(p_\perp \rho\right) e^{-i\varepsilon t + il\varphi}$$

Transmission of VEs through magnetic fields
– **VE in vacuum**
– Nonstationary states approach
– Properties of the electron state in the field

Radiation of VEs in magnetic fields
– Evolved-state approach
– Emitted photons' properties

Paraxial Laguerre-Gaussian packet

Optics

Paraxial Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2ik\frac{\partial u}{\partial z} = 0$$

Nonrelativistic quantum mechanics

Paraxial Schrödinger equation

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + 2ip\frac{\partial \Psi}{\partial z} = 0$$

Paraxial Laguerre-Gaussian (LG) wave packet

$$u(r,\phi,z) = C_{lp}^{LG}\frac{w_0}{w(z)}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}\exp\left(-\frac{r^2}{w^2(z)}\right)L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\times$$
$$\exp\left(-ik\frac{r^2}{2R(z)}\right)\exp(-il\phi)\exp(i\psi(z))$$

Nonstationary LG wave packet

faculty of PHYSICS

Paraxial stationary Schrödinger equation

$$\Delta_\perp \Psi + 2ik\frac{\partial \Psi}{\partial z} = 0$$

Nonstationary Schrödinger equation

$$\Delta \Psi + 2im\frac{\partial \Psi}{\partial t} = 0$$

$$\Psi(s) = N\frac{\rho^{|l|}}{\sigma^{|l|+1}(s)}L_n^{|l|}\left(\frac{\rho^2}{\sigma^2(s)}\right)\exp\left[-\frac{\rho^2}{2\sigma^2(s)}\right]\exp[il\varphi + i\times \mathrm{phase}(s)]$$

$$s = z$$

$$s = t$$

Paraxial stationary solution

Exact nonstationary solution

Comparison of Bessel and LG beams

Bessel vortex electrons

$$\Psi = N\Psi_{\parallel}(z)J_l(p_\perp\rho)e^{-i\varepsilon t+il\varphi}$$

ε – energy
$p_\perp$ – transverse momentum
l – z-projection of OAM

Paraxial LG vortex electrons

$$\Psi = N\frac{\rho^{|l|}}{\sigma^{|l|+1}(z)}L_n^{|l|}\left(\frac{\rho^2}{\sigma^2(z)}\right)\exp\left[-\frac{\rho^2}{2\sigma^2(z)}\right]\exp\left[-i\varepsilon t + il\varphi + i\times\text{phase}(z)\right]$$

Nonstationary LG (NSLG) vortex electrons

$$\Psi = N\Psi_{\parallel}(z)\frac{\rho^{|l|}}{\sigma^{|l|+1}(t)}L_n^{|l|}\left(\frac{\rho^2}{\sigma^2(t)}\right)\exp\left[-\frac{\rho^2}{2\sigma^2(t)}\right]\exp\left[il\varphi + i\times\text{phase}(t)\right]$$

	Bessel	Paraxial LG	NSLG
Definite OAM	☑	☑	☑
Normalizable	☒	☑	☑
Exact solution	☑	☒	☑
Stationary	☑	☑	☒

Transmission of VEs through magnetic fields
– **VE in vacuum**
– Nonstationary states approach
– Properties of the electron state in the field

Radiation of VEs in magnetic fields
– Evolved-state approach
– Emitted photons' properties

NSLG wave packet in free space

Transverse wave function

$$\Psi_{nl}(\boldsymbol{\rho}, t) = N \frac{\rho^{|l|}}{\sigma^{|l|+1}(t)} L_n^{|l|}\left(\frac{\rho^2}{\sigma^2(t)}\right) \exp\left[-\frac{\rho^2}{2\sigma^2(t)}\right] \exp\left[il\varphi + i \times \mathrm{phase}(t)\right]$$

Rms radius and transverse coordinate dispersion

$$\rho(t) \equiv \sqrt{\langle \rho^2 \rangle - \langle \boldsymbol{\rho} \rangle^2} = \sigma(t)\sqrt{2n + |l| + 1}.$$

Rms radius of a free LG packet

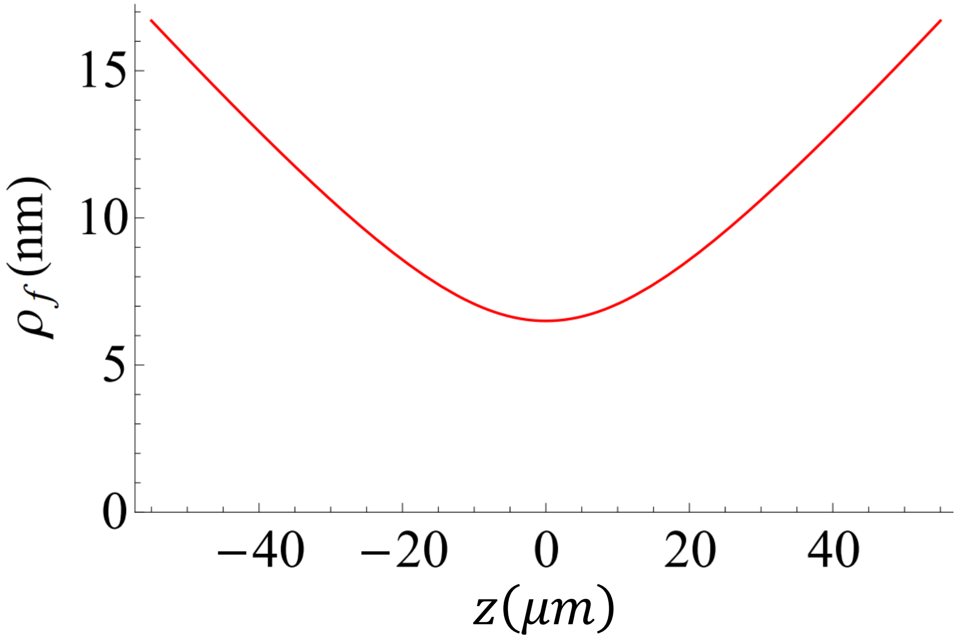
$$\rho_{\mathrm{f}}(t) = \rho_{\mathrm{w}}\sqrt{1 + (t - t_{\mathrm{g}})^2/\tau_{\mathrm{d}}^2}$$

ρ_{w} – rms radius at the waist

t_{g} – time of generation

$\tau_{\mathrm{d}} = m\sigma_{\mathrm{w}}^2$ – diffraction time

$\sigma_{\mathrm{w}} = \rho_{\mathrm{w}}/\sqrt{2n + |l| + 1}$



Rms radius of a free LG vortex electron wave packet, $z = vt$.*

* Parameters are as in the work G. Guzzinati, et al. Phys. Rev. Lett., **110**, 093601, 2013
Observation of the Larmor and Gouy rotations with electron vortex beams

Motivation

faculty of PHYSICS

- Keeping vortex electrons from spreading
- Controlling parameters of vortex-electron wave packets
- Accounting for boundary conditions

Problem statement

Incoming electron state

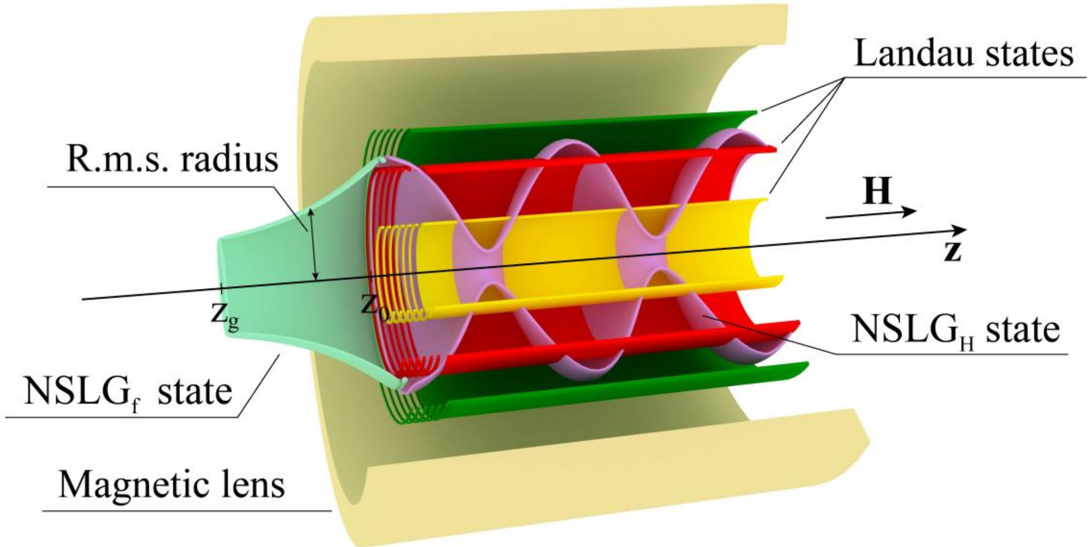
$$\Psi_{l,n}^{(f)}(\boldsymbol{\rho},t) = N\frac{\rho^{|l|}}{\sigma^{|l|+1}(t)}L_n^{|l|}\left(\frac{\rho^2}{\sigma^2(t)}\right)\exp\left(-\frac{\rho^2}{2\sigma^2(t)}\right)\exp\left(il\varphi - i\Phi_G(t) + i\frac{\rho^2}{2\sigma^2(t)}\frac{t}{t_d}\right)$$

Semi-infinite solenoid model

$$\boldsymbol{H} = H\theta\left(z - z_0\right)\boldsymbol{e}_z$$

Target values:

- The state of a VE inside the field
- Rms radius of the VE in the field

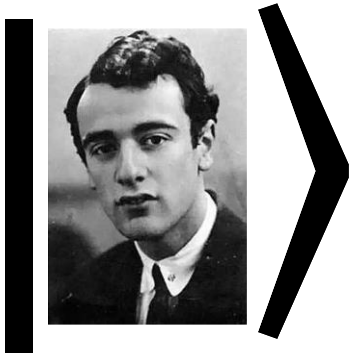


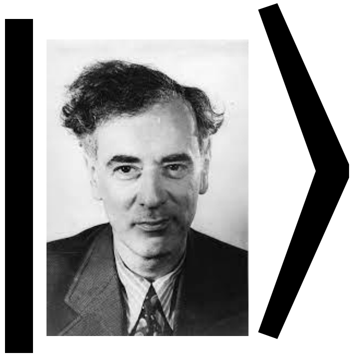
An NSLG vortex electron enters a magnetic field region and propagates inside it, z_g – location of generating device, z_0 – position of the boundary dividing vacuum and the magnetic field region.

Two approaches

Landau states approach

Nonstationary Laguerre-Gaussian approach





Superposition

$$\Psi_{n,l} = \sum_{n',l'} c_{n',l'} \Psi^{(\mathrm{L})}_{n',l'}$$

Single mode

$$\Psi_{n,l} = \Psi^{(\mathrm{NSLG})}_{n,l}$$

Initial condition

$$\Psi_{n,l} = \Psi^{(\mathrm{in})}_{n,l}$$

LG states in magnetic field

Liping Zou, Pengming Zhang, and Alexander J. Silenko
Phys. Rev. A **103**, L010201 – Published 6 January 2021

Paraxial stationary Schrödinger equation in magnetic field

$$\left(\nabla_\perp^2 - ieH\frac{\partial}{\partial\phi} - \frac{e^2H^2r^2}{4} + 2es_zH + 2ik\frac{\partial}{\partial z}\right)\Psi = 0$$

Nonstationary Schrödinger equation in magnetic field

$$\left(\nabla_\perp^2 - ieH\frac{\partial}{\partial\phi} - \frac{e^2H^2r^2}{4} + 2im\frac{\partial}{\partial t}\right)\Psi = 0$$

System of optical equations

NSLG state in magnetic field

$$\Psi = N_{nl}\frac{\rho^{|l|}}{\sigma^{|l|+1}(t)}L_n^{|l|}\left(\frac{\rho^2}{\sigma^2(t)}\right)\exp\left[-\frac{\rho^2}{2\sigma^2(t)}\right]\exp\left[il\varphi - i\Phi_{\rm G}(t) + i\frac{\rho^2}{2\lambda_{\rm C}R(t)}\right]$$

Paraxial stationary Schrödinger equation in magnetic field

$$\left(\nabla_\perp^2 - ieH\frac{\partial}{\partial\phi} - \frac{e^2H^2r^2}{4} + 2im\frac{\partial}{\partial t}\right)\Psi = 0$$

System of optical equations

$$\frac{1}{R(t)} = \frac{\sigma'(t)}{\sigma(t)},$$

$$\frac{1}{\lambda_{\rm C}^2 R^2(t)} + \frac{1}{\lambda_{\rm C}^2}\left[\frac{1}{R(t)}\right]' = \frac{1}{\sigma^4(t)} - \frac{1}{\sigma_{\rm L}^4},$$

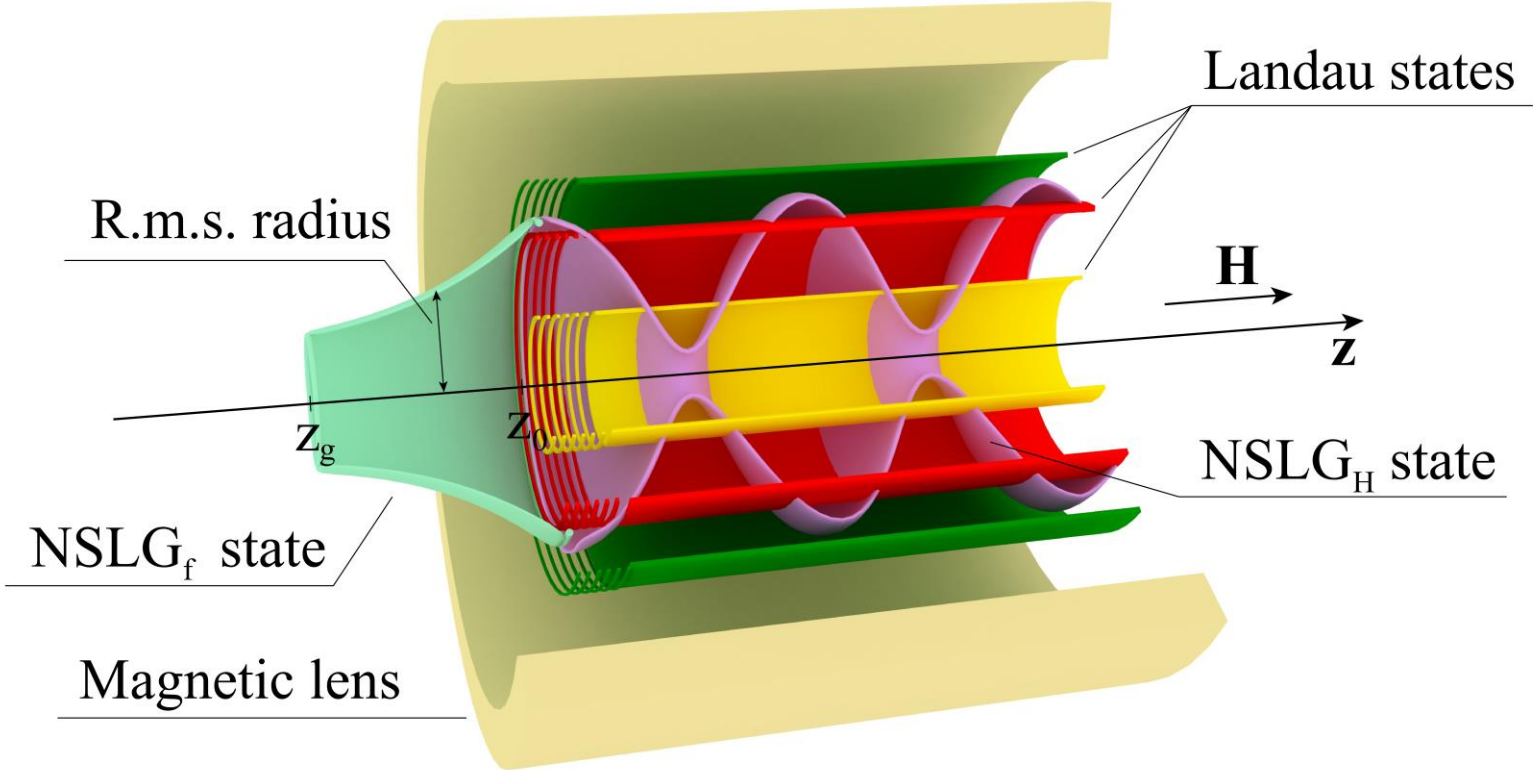
$$\frac{1}{\lambda_{\rm C}}\Phi_{\rm G}'(t) = \frac{l}{\sigma_{\rm L}^2} + \frac{2n+|l|+1}{\sigma^2(t)}.$$

$\lambda_{\rm C} = m^{-1}$ – Compton wavelength

$\sigma_{\rm L} = \sqrt{2/|eH|}$ – magnetic length

Initial conditions

$$\sigma(t_0) = \sigma_{\mathrm{f}}(t_0) = \sigma_0 \qquad \sigma'(t_0) = \sigma'_{\mathrm{f}}(t_0) = \sigma'_0 \qquad \Phi_{\mathrm{G}}(t_0) = \Phi_{\mathrm{f}}(t_0) = \Phi_0$$



An NSLG vortex electron enters a magnetic field region and propagates inside it, z_g – location of generating device, z_0 – position of the boundary dividing vacuum and the magnetic field region.

NSLG state in the field

Dispersion

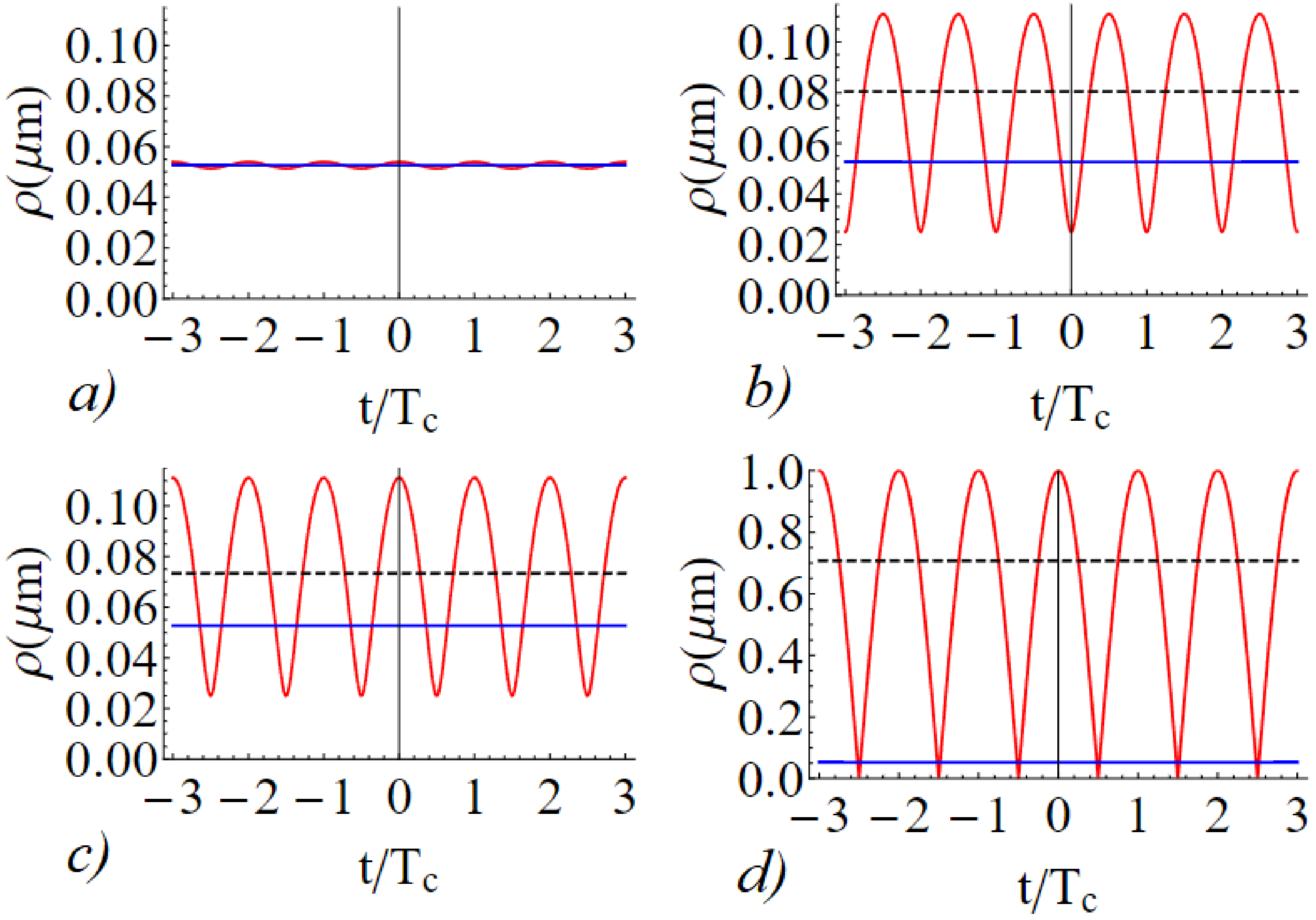
$$\sigma(t) = \sigma_{\mathrm{st}}\sqrt{1+\sqrt{1-\left(\frac{\sigma_{\mathrm{L}}}{\sigma_{\mathrm{st}}}\right)^4}\sin\left[s(\sigma_0,\sigma_0')\omega(t-t_0)-\theta\right]}$$

Rms radius

$$\rho(t) \equiv \sqrt{\langle\rho^2\rangle - \langle\boldsymbol{\rho}\rangle^2} = \sigma(t)\sqrt{2n+|l|+1}$$

$$\sigma_{\mathrm{st}} \geq \sigma_{\mathrm{L}} \longrightarrow \rho_{\mathrm{st}} = \sqrt{(2n+|l|+1)}\sigma_{\mathrm{st}} \geq \rho_{\mathrm{L}}$$

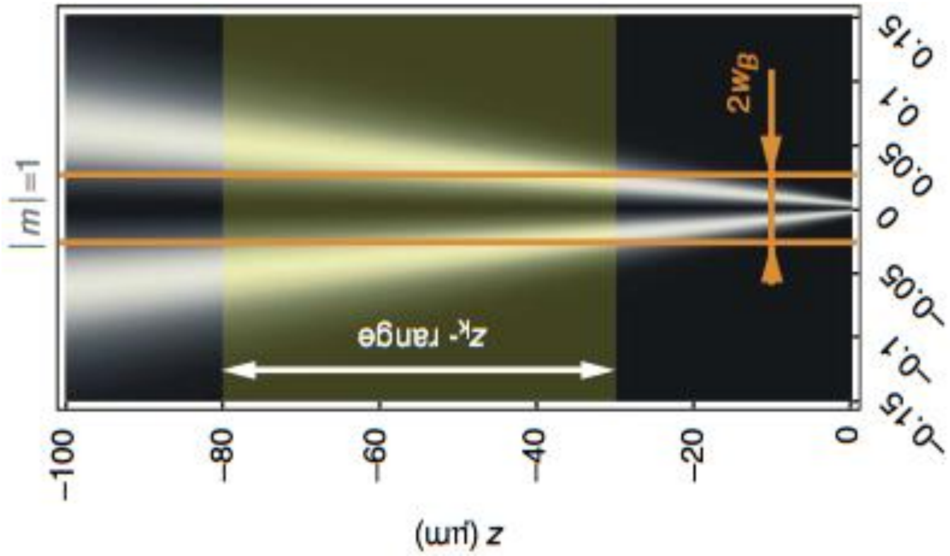
Rms radius oscillations



Rms radius oscillations (in red). $\rho_L = 52.7$ nm (in blue). Black dashed lines correspond to the square of the period-averaged transverse radius squared. Magnetic field strength H=1.9 T, n=0, l=3, $\rho_0' = 0$. a) $\rho_0 = 54$ nm, b) $\rho_0 = 25$ nm, c) $\rho_0 = 111.1$ nm, d) $\rho_0 = 1$ µm.

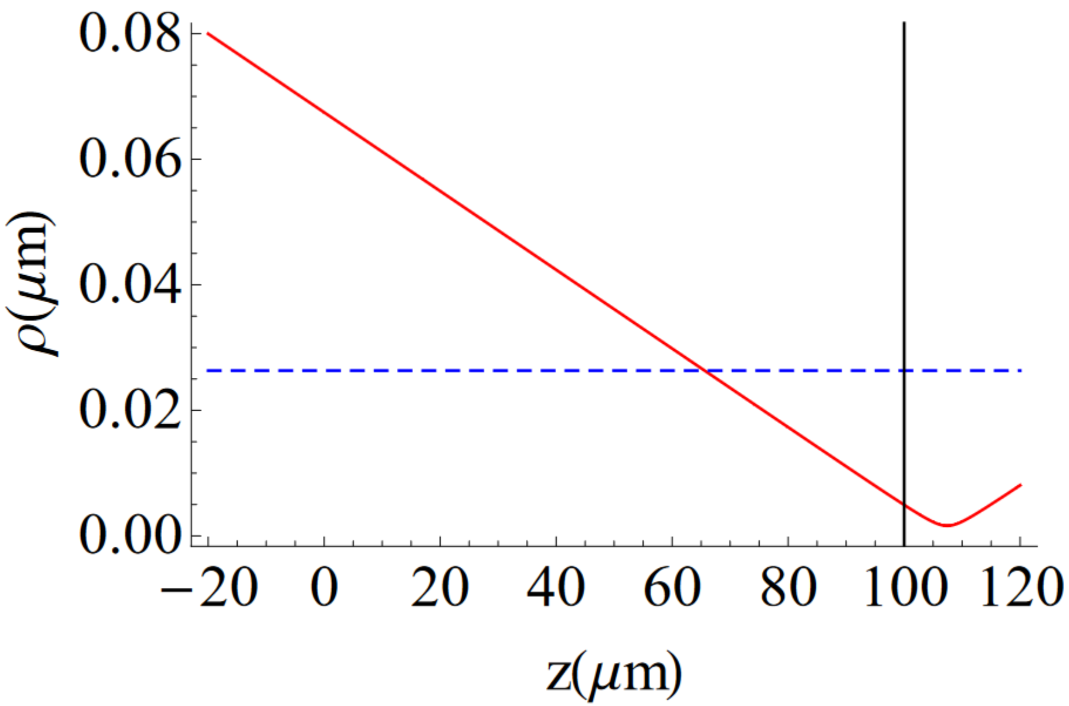
Experiment confirms?

Experiment of Schattschneider and others



Rms radius of a vortex electron in a TEM. *

NSLG calculation



Rms radius of a vortex electron in a TEM (in red). Rms radius of the Landau state (in blue). $E_{\parallel} = 200$ keV, $H = 1.9$ T, $n = 0$, $l = 1$.

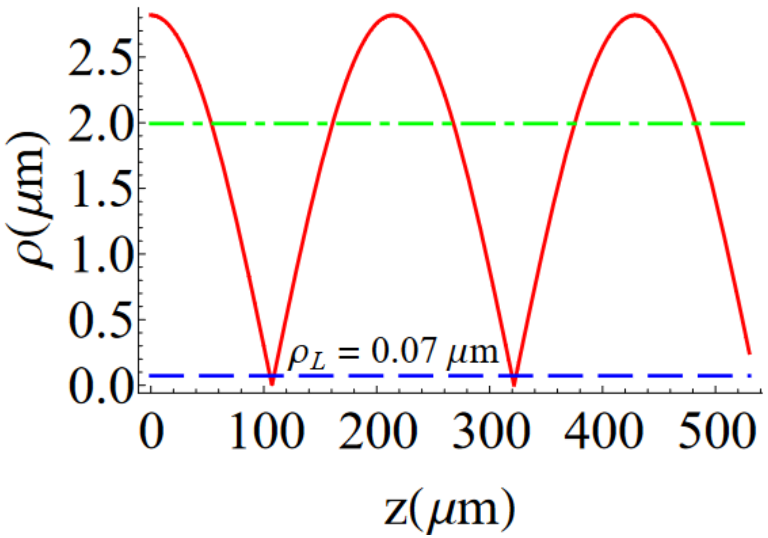
Transmission of VEs through magnetic fields
- VE in vacuum
- Nonstationary states approach
- **Properties of the electron state in the field**

Radiation of VEs in magnetic fields
- Evolved-state approach
- Emitted photons' properties

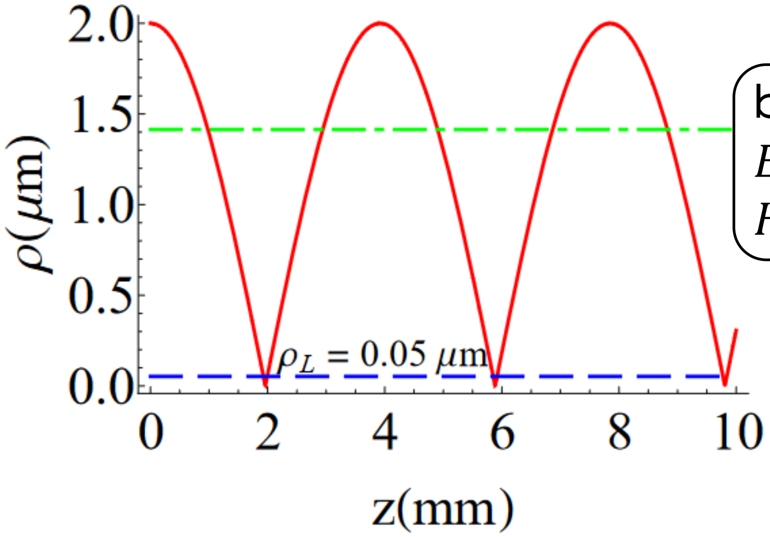
** From the work P. Schattschneider et al., Nature Comm.,* ***5****:4586, 2014*
Imaging the dynamics of free-electron Landau states.

Experimental feasibility

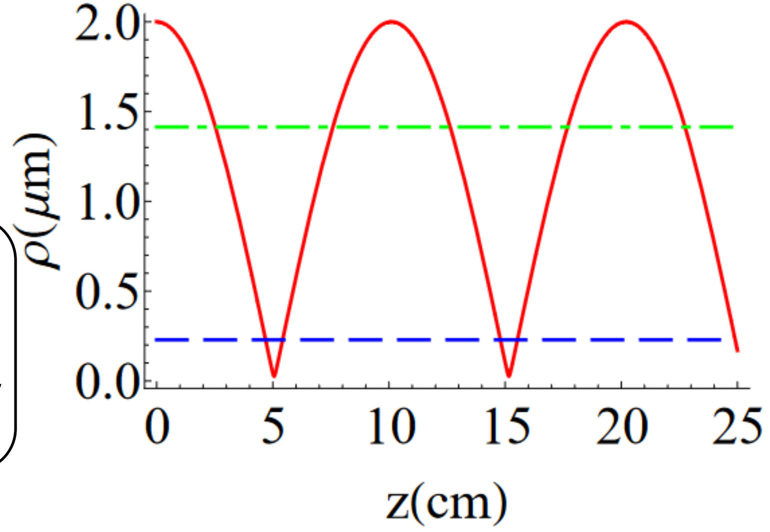
a) SEM
$E_{\parallel} = 1$ keV,
$H = 1$ T



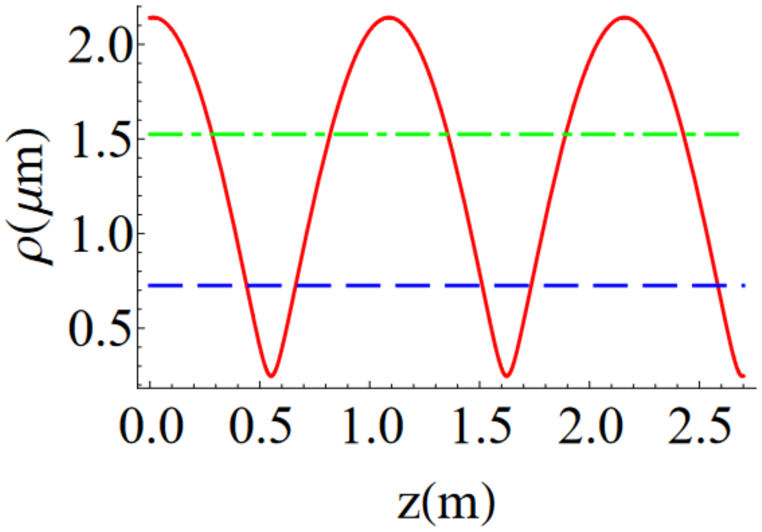
b) TEM
$E_{\parallel} = 200$ keV,
$H = 1.9$ T



c) medical linac
$E_{\parallel} = 1$ MeV,
$H = 0.1$ T



d) Linac
$E_{\parallel} = 1$ GeV,
$H = 0.01$ T

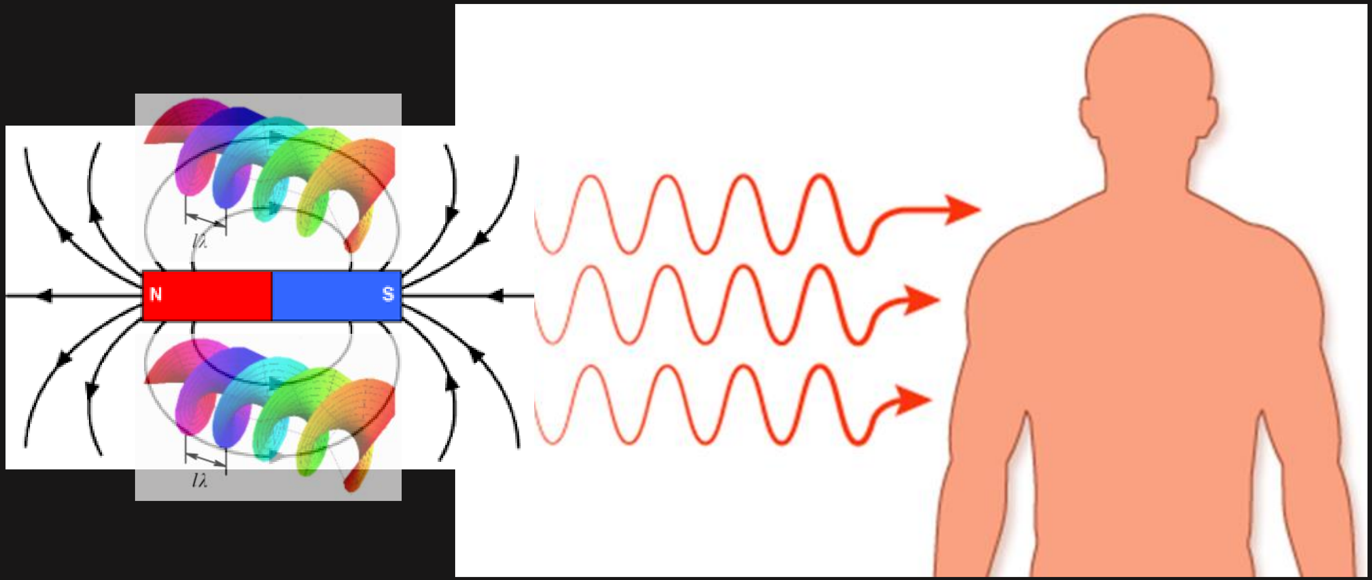


Transmission of VEs through magnetic fields
- VE in vacuum
- Nonstationary states approach
- **Properties of the electron state in the field**

Radiation of VEs in magnetic fields
- Evolved-state approach
- Emitted photons' properties

faculty of PHYSICS

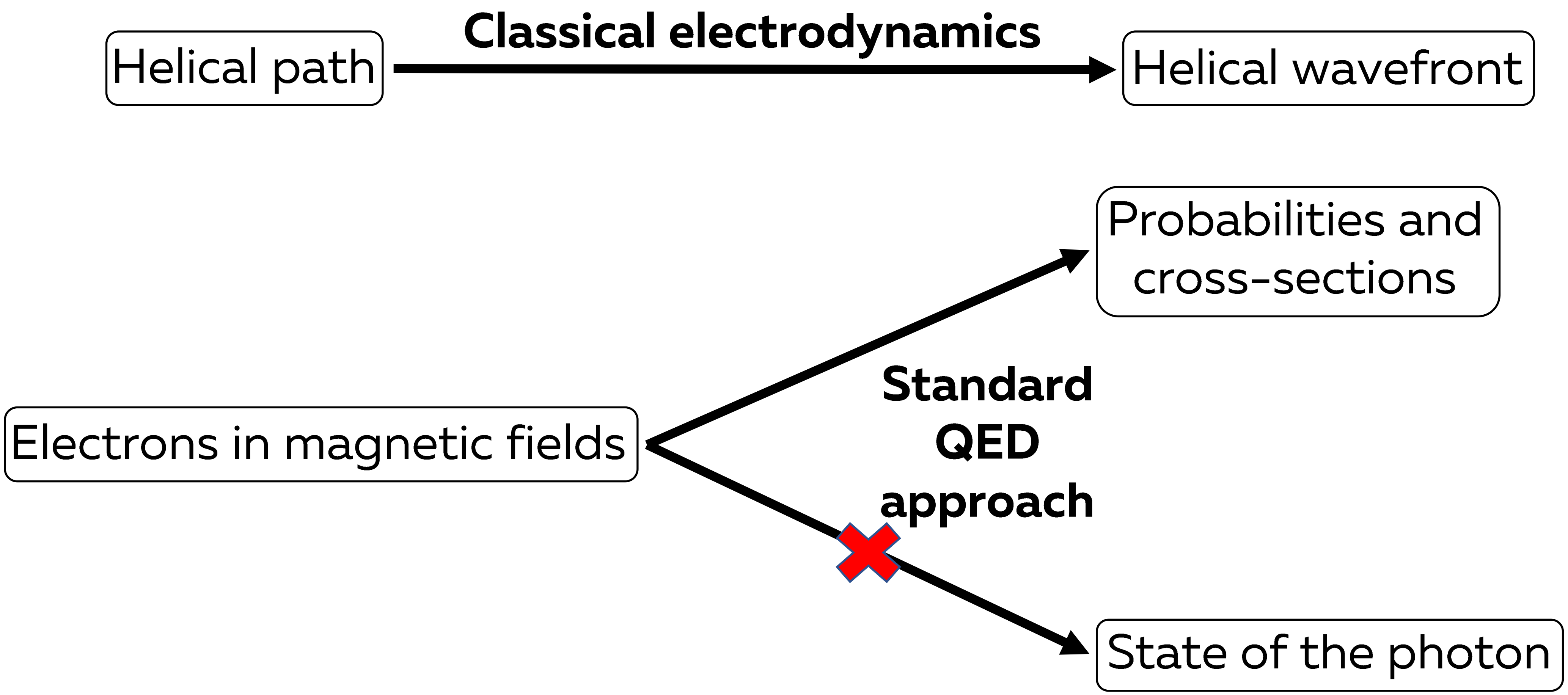
Radiation of vortex electrons in magnetic fields



*Plots are taken from Knyazev, B. A., & Serbo, V. G. (2018). Beams of photons with nonzero projections of orbital angular momenta: new results. Physics-Uspekhi, 61(5), 449.

ITsMOre than a UNIVERSITY

Motivation



Epp V., Guselnikova U., Angular momentum of radiation from a charge in circular and spiral motion, Physics Letters A, 2019
V. A. Bordovitsyn et al., Synchrotron radiation theory and its development: in memory of IM Ternov, vol. 5 (World Scientific, 1999).

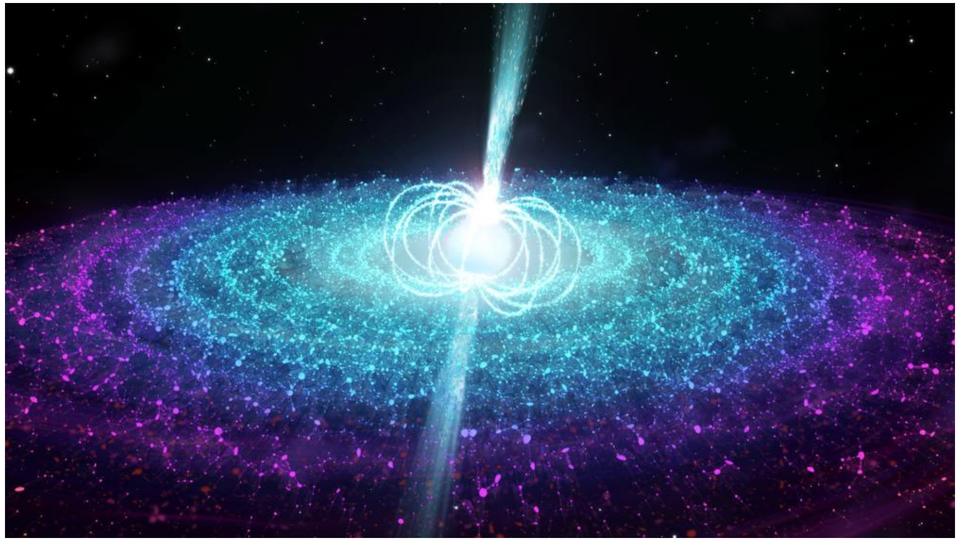
Applications

- Generation of vortex photons in laboratory
- Astrophysics: neutron stars et al. $H \lesssim H_c = 4.4 \times 10^9 T$

A vortex photon in a star is born

↓

The photon interacts with matter (not like a plane-wave photon!):



- Higher Landau states can be excited
- Multipole transitions in atoms become significant
- Vortex cross-section ≠ plane-wave cross-section

State of the system

QED

Ilia Pavlov

Photon emission

$$e \to e' + \gamma$$

$$|f\rangle = \hat{S}^{(1)}|i_e\rangle$$

Entangled final state (electron + photon)

First-order S-matrix element

Initial state (single electron)

Disentangling the final state

Final electron is measured: $|f_e\rangle$

Photon is **not** measured

Disentangled final state

$$|f'\rangle = P(f_e)\hat{S}^{(1)}|i_e\rangle = \sum_{f_\gamma} |f_e, f_\gamma\rangle \underbrace{\langle f_e, f_\gamma|\hat{S}^{(1)}|i_e\rangle}_{S_{fi}^{(1)}}$$

$$P(f_e) = |f_e\rangle\langle f_e| \otimes \hat{1}_\gamma$$

Evolved state of the photon

Final disentangled state

$$|f'\rangle = |f_e\rangle \otimes |\gamma\rangle_{ev}$$

State of the photon

$$|\gamma\rangle_{ev} = \sum_{\lambda=\pm 1} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} S_{fi}^{(1)} |\mathbf{k}, \lambda\rangle$$

Vector potential of the photon

Vector potential of the emitted photon

$$\boldsymbol{A}(t,\boldsymbol{r}) = \langle 0|\hat{\boldsymbol{A}}(t,\boldsymbol{r})|\gamma\rangle_{ev} = \sum_{\lambda}\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega}}\boldsymbol{A}_{\lambda}(t,\boldsymbol{r};\boldsymbol{k})S_{fi}^{(1)}$$

$$\boldsymbol{A}_{\lambda}(t,\boldsymbol{r};\boldsymbol{k}) = \frac{\boldsymbol{e}_{\lambda}(\boldsymbol{k})}{\sqrt{2\omega}}e^{-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})}$$

Vector potential in the momentum representation

$$\boldsymbol{A}(\boldsymbol{k}) = \sum_{\lambda=\pm 1} S_{fi}^{(1)}\boldsymbol{e}_{\lambda}(\boldsymbol{k})$$

S-matrix element

Relativistic Landau state

$$\Psi^{\uparrow}_{s,\ell}(x) = N^{\uparrow} \begin{pmatrix} (m+\varepsilon)\Phi_{s,\ell-1/2}(\rho)e^{-i\varphi/2} \\ 0 \\ p_z\Phi_{s,\ell-1/2}(\rho)e^{-i\varphi/2} \\ -ieH\Phi_{s,\ell+1/2}(\rho)e^{i\varphi/2} \end{pmatrix} e^{-it\varepsilon_{s,\ell}+i\ell\varphi+ip_z z}$$

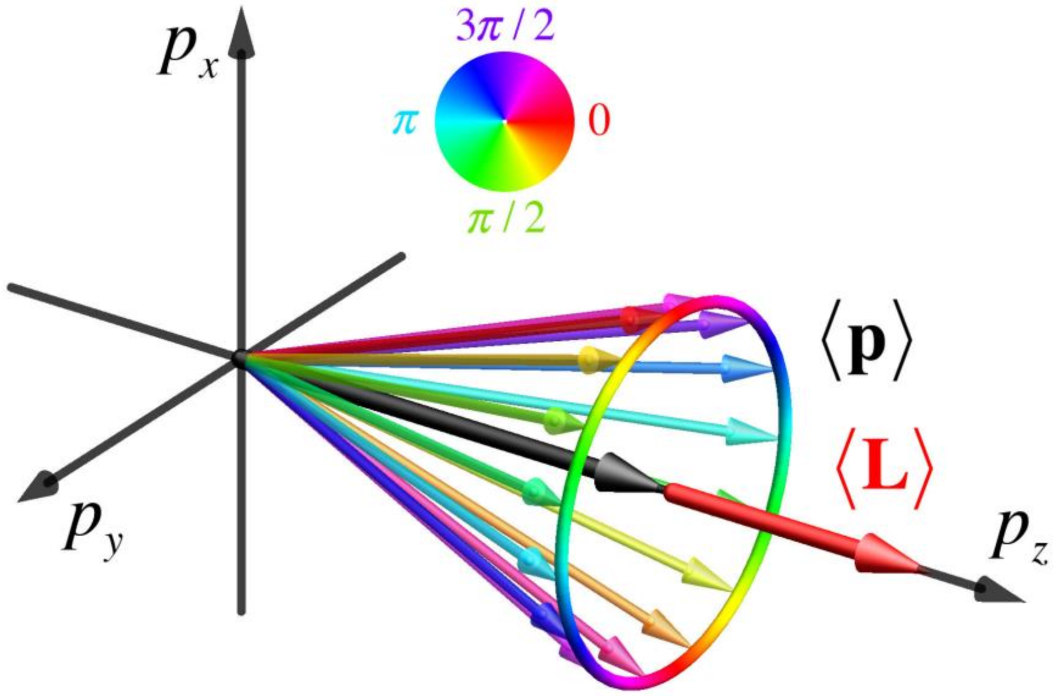
S-matrix element

$$S^{(1)}_{fi} = \delta(k_{\perp} - \kappa)\delta(k_z + p'_z - p_z)e^{i(\ell-\ell')\varphi_k}\mathcal{F}$$

Photon state

Photon is a Bessel beam

$$|\gamma\rangle_{ev} = (\varepsilon - \varepsilon') \sum_{\lambda=\pm 1} \mathcal{F} \int_0^{2\pi} d\varphi_k \, |\boldsymbol{k}, \lambda\rangle \, e^{i(\ell-\ell')\varphi_k}$$



From the work Bliokh, K.Y., et al. (2017). Theory and applications of free-electron vortex states. *Physics Reports*, *690*, 1-70.

Photon TAM

$$\hat{j}_z \mathbf{A}^{(ev)}(\mathbf{k}) = (\ell - \ell')\mathbf{A}^{(ev)}(\mathbf{k})$$

Emission probability and intensity

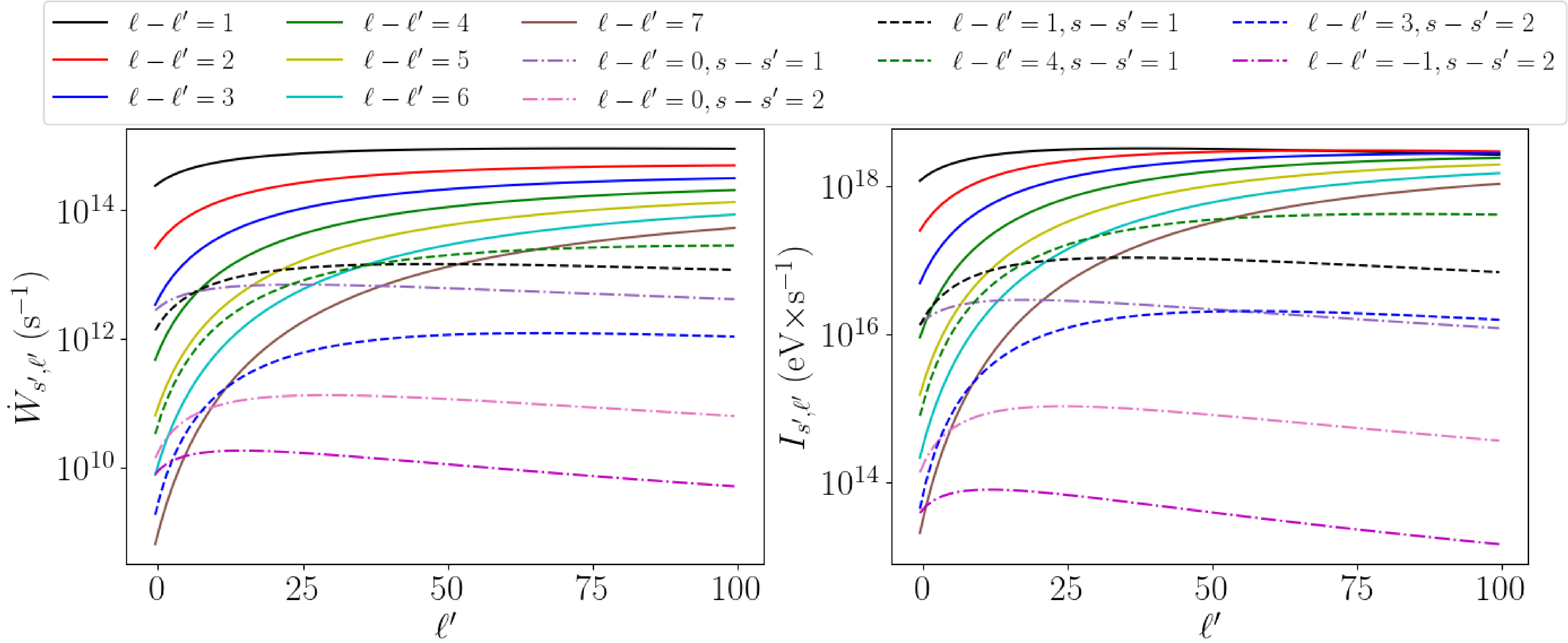
Emission probability

$$\langle\gamma|\gamma\rangle = \sum_{\lambda=\pm1} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left|S_{fi}^{(1)}\right|^2 \equiv W_{s',\ell'}^{(1)}(p'_z)$$

Intensity

$$I_{s',\ell'}^{(1)} \equiv \int L \frac{dp'_z}{2\pi}\, \omega \dot{W}_{s',\ell'}^{(1)}(p'_z)$$

Evolved-state approach



Probability and intensity of emission with definite angular momentum (with no spin-flip).

$H = 10^{-2} H_c,\ s = s' = 5,\ p_z = 10^{-3} m$

$H_c = 4.4 \times 10^9 T$

Conclusion

- NSLG states provide a single-mode description of a vortex electron transfer through a solenoid
- The NSLG states approach enable to easily account for the boundary conditions
- NSLG states highlight the dynamics of vortex electrons on magnetic fields
- Electrons in a constant magnetic field spontaneously emit twisted photons
- Total angular momentum is conserved in this process.
- The majority of the emitted photons turn out to be twisted,

with $\ell - \ell' \gtrsim 1$

- The obtained results can offer a new perspective on the synchrotron radiation, generation of twisted photons and astrophysical processes in extreme magnetic field.

faculty of PHYSICS

Sizykh, G. K., Chaikovskaia, A. D., Grosman, D. V., Pavlov, I. I., & Karlovets, D. V. (2024). Transmission of vortex electrons through a solenoid. *Physical Review A*, *109*(4), L040201.

Pavlov, I., & Karlovets, D. (2024). Emission of twisted photons by a Dirac electron in a strong magnetic field. *Physical Review D*, *109*(3), 036017.

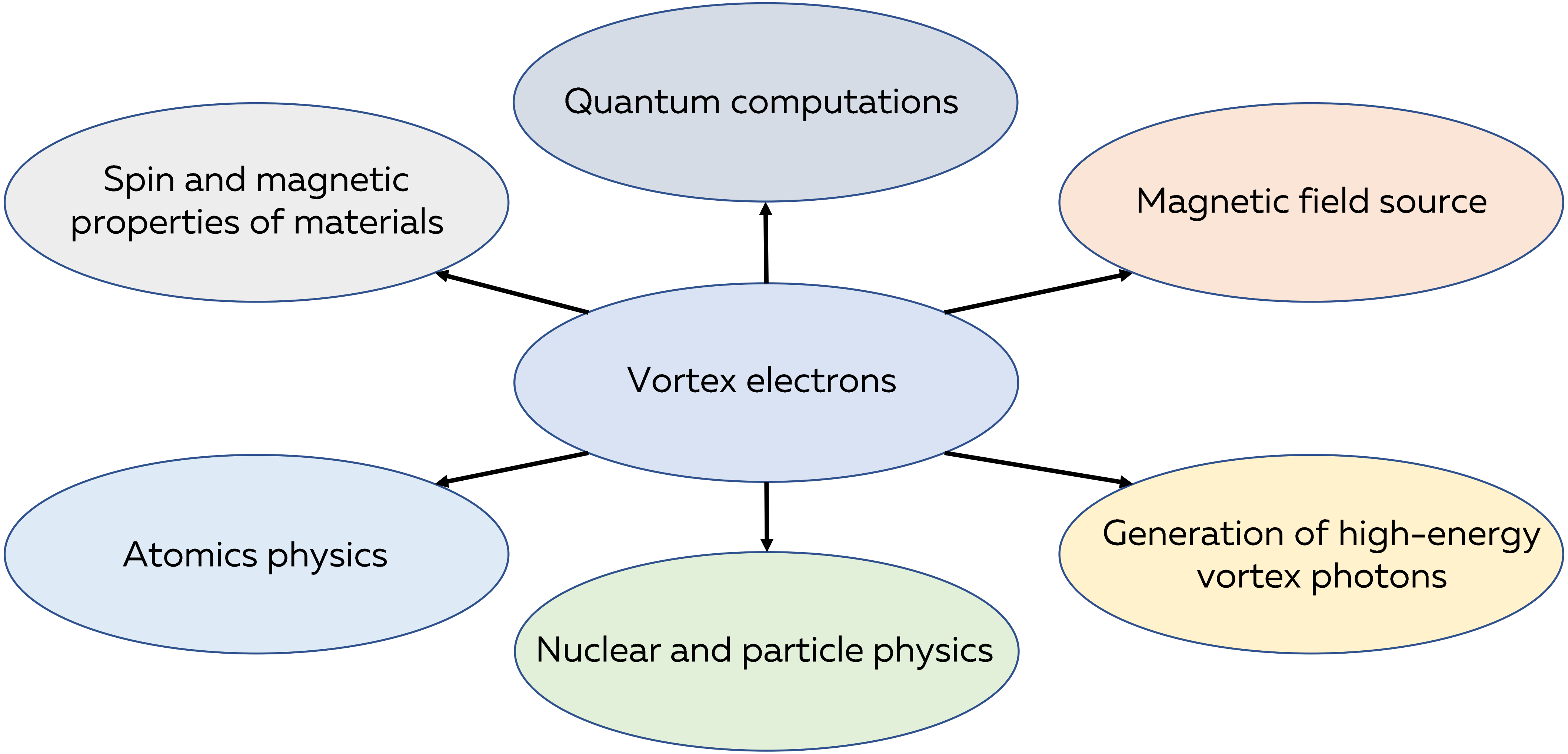
Contacts:

georgii.sizykh@metalab.ifmo.ru

ilya.pavlov@metalab.ifmo.ru

ITsMOre than a UNIVERSITY

Why study vortex electrons?



How to produce VE?

Vortex electrons

Low energies → Phase plates

Low energies → Diffraction gratings

Arbitrary energies → Interaction, Acceleration

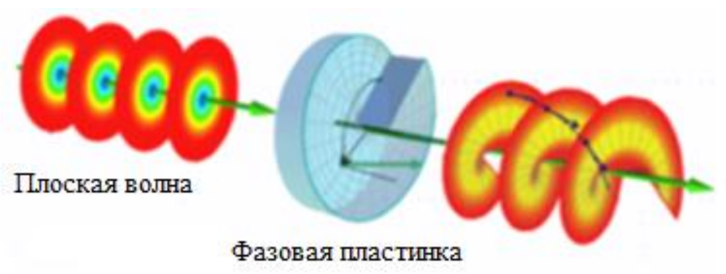


Рис. ??. Generation of vortex electrons via spiral phase plate.*

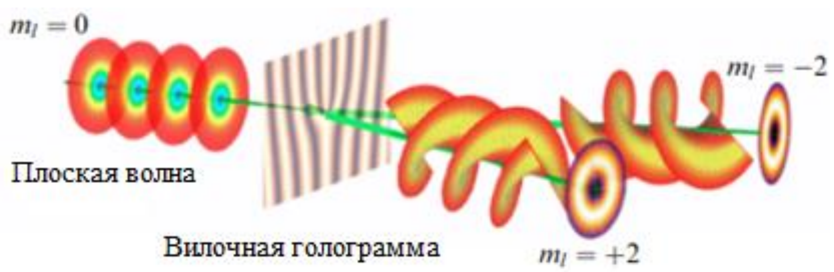


Рис. ??. Generation of vortex electrons with a fork-like computer-generated hologram.*

* From Knyazev, B. A., & Serbo, V. G. (2018). Beams of photons with nonzero projections of orbital angular momenta: new results. *Physics-Uspekhi*, *61*(5), 449.

Gouy phase of a free wave packet

faculty of PHYSICS

$$\Phi_G(t) = (2n + |l| + 1)\,\mathrm{arctan}\left(\frac{t}{t_d}\right)$$

NSLG state in the field

faculty of PHYSICS

General form of an NSLG state

$$\Psi_{n,l}(\boldsymbol{\rho},t) = N\frac{\rho^{|l|}}{\sigma^{|l|+1}(t)}L_n^{|l|}\left(\frac{\rho^2}{\sigma^2(t)}\right)\exp\left[il\varphi - i\Phi_G(t) - \frac{\rho^2}{2\sigma^2(t)}\left(1 - i\frac{\sigma^2(t)}{\lambda_C R(t)}\right)\right]$$

Curvature radius

$$R(t) = \frac{\sigma(t)}{\sigma'(t)}$$

Dispersion

$$\sigma(t) = \sigma_{\mathrm{st}}\sqrt{1 + \sqrt{1 - \left(\frac{\sigma_{\mathrm{L}}}{\sigma_{\mathrm{st}}}\right)^4}\sin\left[s(\sigma_0,\sigma_0')\omega(t-t_0) - \theta\right]}$$

Gouy phase

$$\Phi_{\mathrm{G}}(t) = \Phi_0 + \frac{l\omega(t-t_0)}{2} + (2n+|l|+1)s(\sigma_0,\sigma_0')$$
$$\times\left[\arctan\left(\frac{\sigma_{\mathrm{st}}^2}{\sigma_{\mathrm{L}}^2}\tan\frac{s(\sigma_0,\sigma_0')\omega(t-t_0)+\theta}{2} + \frac{\sigma_{\mathrm{st}}^2}{\sigma_{\mathrm{L}}^2}\sqrt{1-\left(\frac{\sigma_{\mathrm{L}}}{\sigma_{\mathrm{st}}}\right)^4}\right)\right.$$
$$\left. - \arctan\left(\frac{\sigma_{\mathrm{st}}^2}{\sigma_{\mathrm{L}}^2}\tan\frac{\theta}{2} + \frac{\sigma_{\mathrm{st}}^2}{\sigma_{\mathrm{L}}^2}\sqrt{1-\left(\frac{\sigma_{\mathrm{L}}}{\sigma_{\mathrm{st}}}\right)^4}\right)\right].$$

Stationary dispersion

$$\sigma_{\mathrm{st}}^2 = \frac{\sigma_0^2}{2}\left(1 + \left(\frac{\sigma_{\mathrm{L}}}{\sigma_0}\right)^4 + \left(\frac{\sigma_0'\sigma_{\mathrm{L}}^2}{\lambda_{\mathrm{C}}\sigma_0}\right)^2\right)$$

Initial phase

$$\theta = \arcsin\frac{1-(\sigma_0/\sigma_{\mathrm{st}})^2}{\sqrt{1-(\sigma_{\mathrm{L}}/\sigma_{\mathrm{st}})^4}}$$

Sign function

$$s(\sigma_0,\sigma_0') = \begin{cases}\mathrm{sgn}(\sigma_0'), & \sigma_0' \neq 0,\\ \mathrm{sgn}(\sigma_{\mathrm{L}}-\sigma_0), & \sigma_0' = 0,\\ 0, & \sigma_0 = \sigma_{\mathrm{L}} \text{ and } \sigma_0' = 0.\end{cases}$$

Gouy phase

$$\Delta\Phi_G = (2n + |l| + l + 1)\pi$$

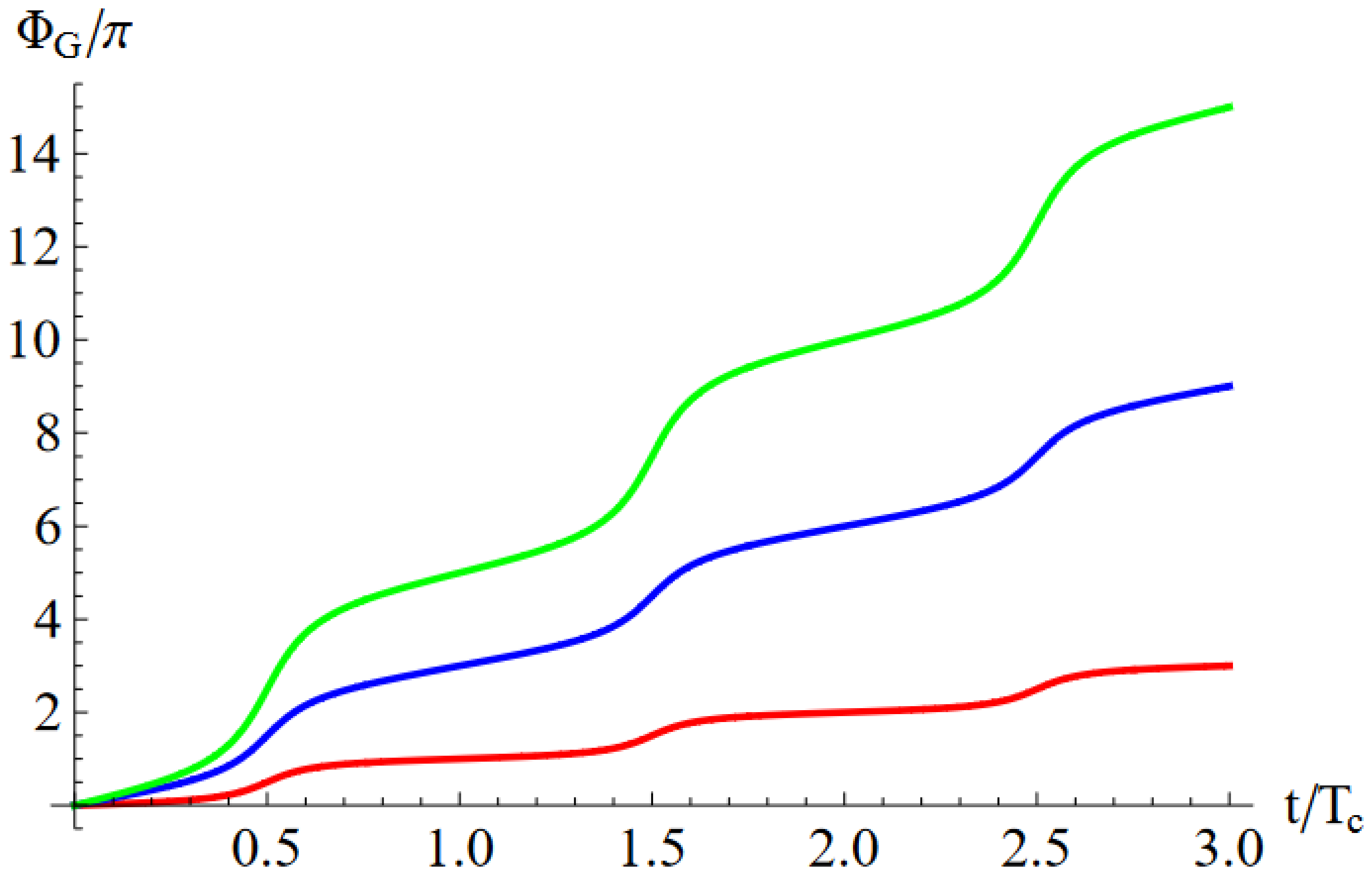
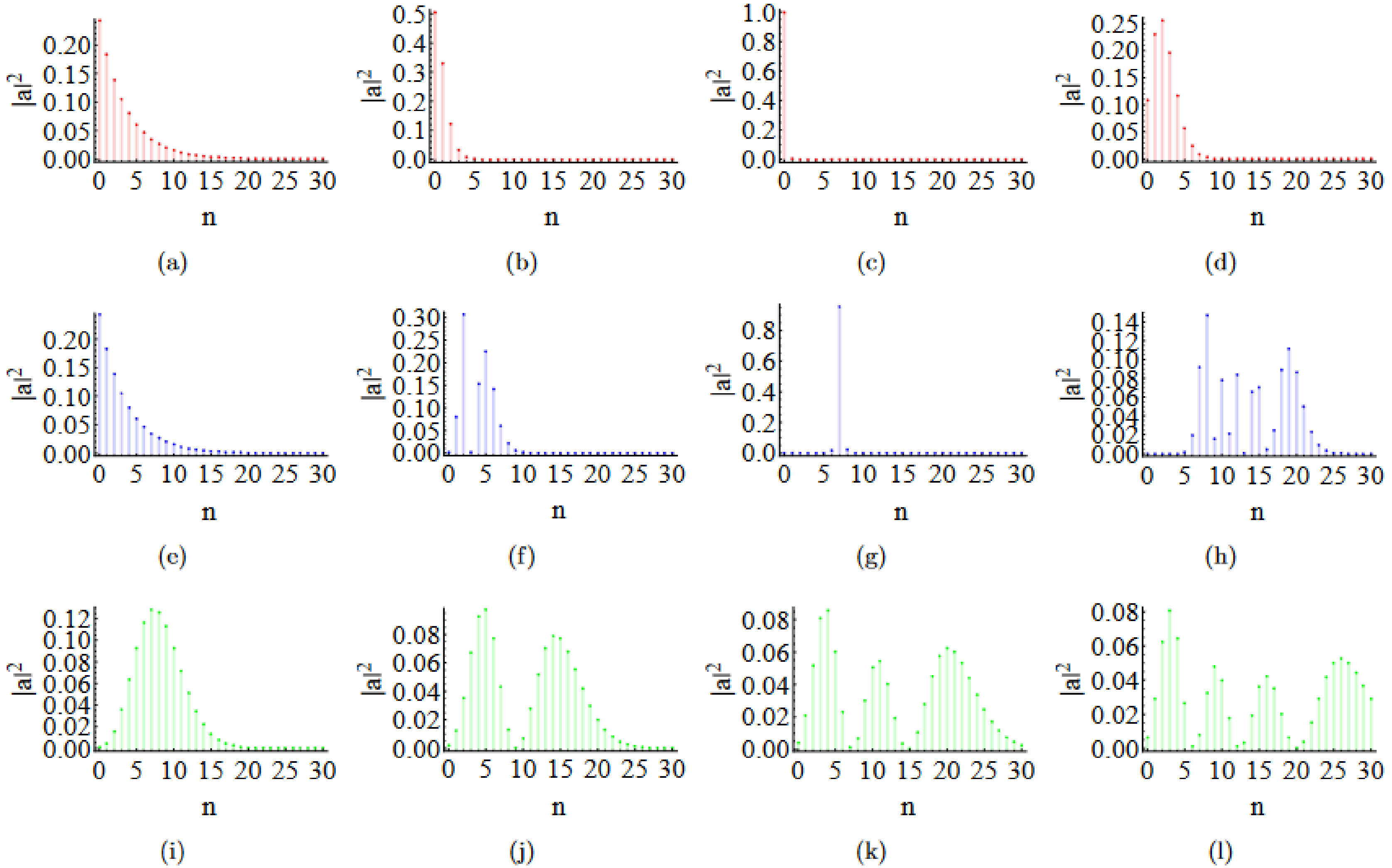


Figure 4: Gouy phase $\Phi_G(t)$ of the $NSLG_H$ wave packet. The field strength $H = 1.9$ T (corresponding $\rho_L \approx 37$ nm), $\rho_0 \approx 71$ nm, $\rho_0' = 0$, $T_c \approx 0.02$ ns, and $\Phi_0 = 0$. The quantum numbers are: $n = 0$, $l = 0$ (red line); $n = 0$, $l = 1$ (blue); and $n = 1$, $l = 1$ (green).

Decomposition into Landau states

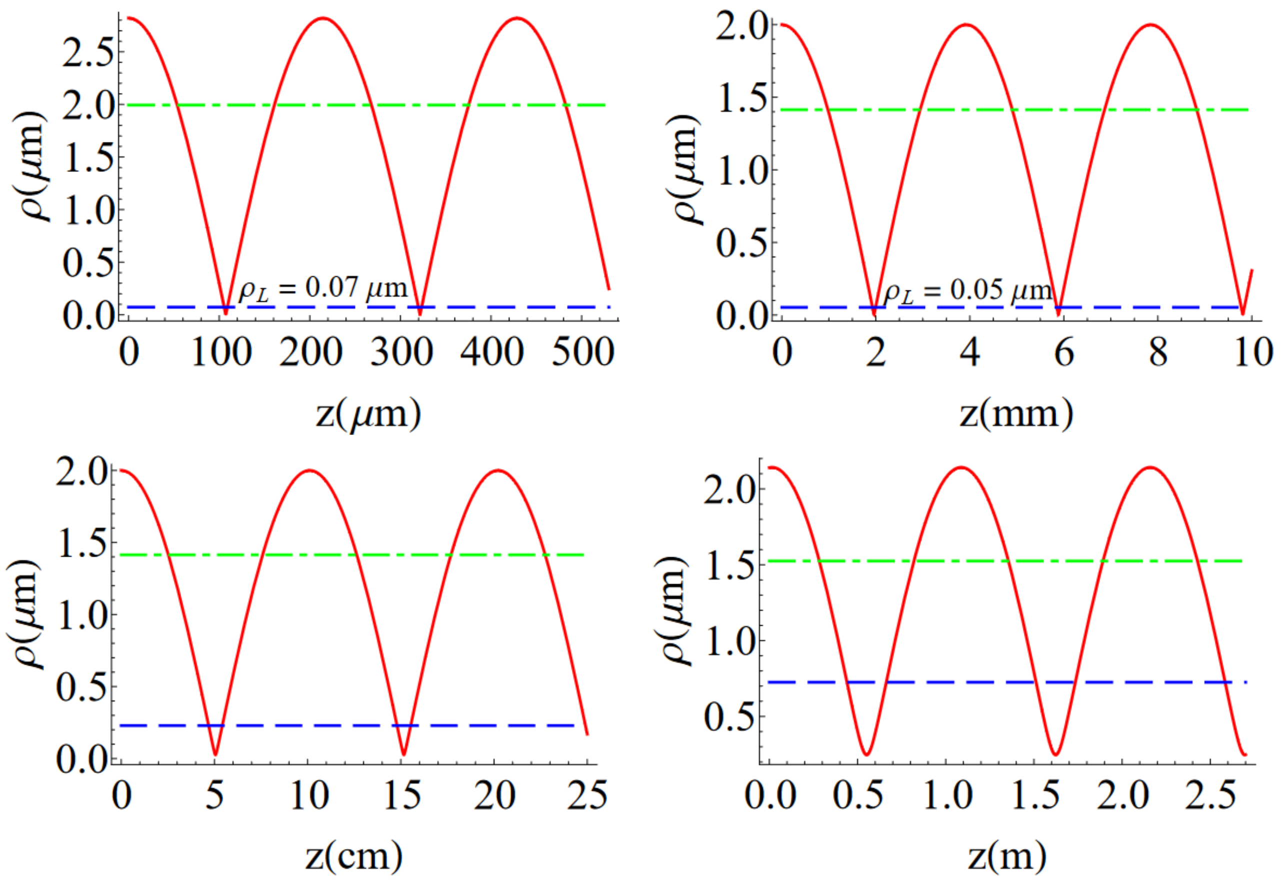


Probability coefficients $|a_{nn'l}|^2$ for an NSLG state with the quantum numbers n, l decomposed into a superposition of the Landau states with the quantum numbers n', l. $\rho_0 = 100$ nm, $\rho_0' = 0$, $B = 1.9$ T. Top panel (in red): (a) $n = 0$, $l = 0$, (b) $n = 0$, $l = 7$, (c) $n = 0$, $l = 13$, (d) $n = 0$, $l = 25$; middle panel (in blue): (e) $n = 0$, $l = 0$, (f) $n = 3$, $l = 0$, (g) $n = 7$, $l = 0$, (h) $n = 12$, $l = 0$; bottom panel (in green): (i) $n = 0$, $l = 35$, (j) $n = 1$, $l = 35$, (k) $n = 2$, $l = 35$, (l) $n = 3$, $l = 35$.

Parameters of Schattschneider et al.

work [17]. The parameters of the NSLG packet are: energy $\varepsilon = 200$ KeV (corresponding velocity $v \approx 0.7c$), $n = 0, l = 1$, $\rho_0 \approx 67.5$ nm, $\rho_0' \approx -4.4 \times 10^{-4}$. Magnetic field strength $B = 1.9$ T (corresponding $\sigma_L \approx 26$ nm).

Oscillations parameters



Setup	$E_\parallel$	v	H	ρ_L	d	z_R	ρ_0	$d\rho/dz\|_{z=z_0}$	ξ_1	ξ_2
SEM	100 eV	$0.02c$	1 T	72.6 nm	5.16 cm	5.16 cm	2.82 μm	27 pm/μm	0.025	6.6×10^{-4}
TEM	200 keV	$0.70c$	1.9 T	52.7 nm	10 cm	179 cm	2 μm	62 pm/mm	0.026	3.9×10^{-5}
Medical linac	1 MeV	$0.94c$	0.1 T	0.23 μm	10 cm	243 cm	2 μm	0.34 nm/cm	0.115	5.5×10^{-4}
Linac	1 GeV	c	0.01 T	0.72 μm	100 cm	258 cm	2.14 μm	0.28 μm/m	0.339	0.045

Transversely relativistic motion

faculty of PHYSICS

Free NSLG: $$2n + |l| + 1 \ll \frac{\rho_{\mathrm{w}}}{\lambda_{\mathrm{C}}}.$$

However, being refocused to a 1 nm waist size, electrons with quantum numbers of the order of 10^3 become transversely relativistic.

Landau: $$\sqrt{(2n + |l| + l + 1)} \ll \frac{\sigma_{\mathrm{L}}}{\lambda_{\mathrm{C}}}.$$

such a state remains nonrelativistic for any attainable values of n and $|l|$. However, for $l > 0$, the relativistic regime cannot be achieved either, as it would require OAM of the order of 10^{10}.

NSLG in a field: $$\sqrt{\left[(2n + |l| + 1)\frac{\sigma_{\mathrm{st}}^2}{\sigma_{\mathrm{L}}^2} + l\right]} \ll \frac{\sigma_{\mathrm{L}}}{\lambda_{\mathrm{C}}} \qquad \sqrt{(2n + |l| + 1)} \ll \frac{\sigma_{\mathrm{L}}}{\lambda_{\mathrm{C}}}\frac{\sigma_{\mathrm{L}}}{\sigma_0}.$$

From here it follows that for wave packets with $\sigma_0/\sigma_{\mathrm{L}} \geq \sigma_{\mathrm{L}}/\lambda_{\mathrm{C}}$, even a Gaussian mode with $n = l = 0$ is relativistic. For a field strength of the order of 1 T, this happens when $\sigma_0 \sim 1$ mm, which can also be decreased if the divergence rate σ_0' in (34) is taken into account.

Slight misalignment

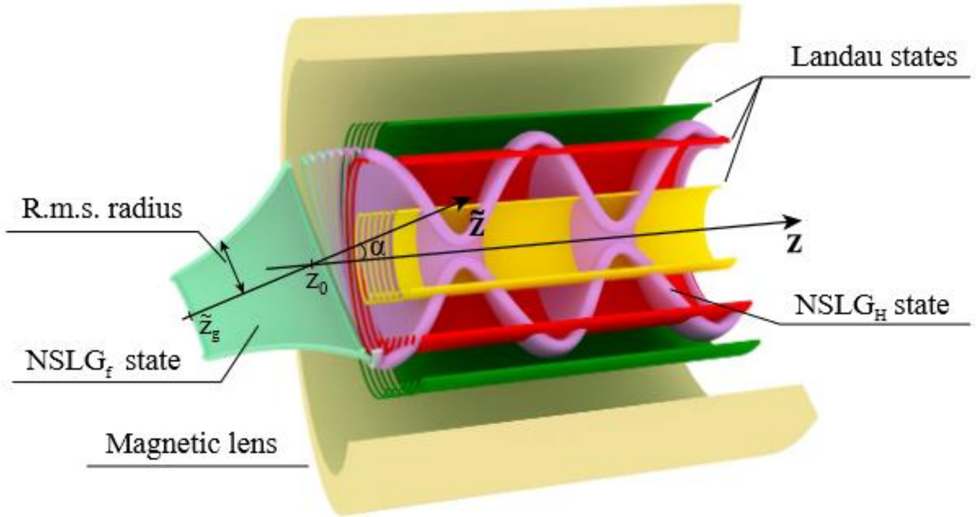


Figure 7: Free twisted electron entering a magnetic lens at a small angle α with respect to the field direction.

$$
\begin{aligned}
|c_{nn'll'}| = & \delta_{n,n'}\delta_{l,l'} + \frac{\alpha \langle p_z \rangle \sigma(t_0)}{4\pi}\delta_{|l'|,|l|-1}\left[\delta_{n',n}\sqrt{n+|l|} + \delta_{n',n+1}\sqrt{n+1}\right] \\
& + \frac{\alpha \langle p_z \rangle \sigma(t_0)}{4\pi}\delta_{|l'|,|l|+1}\left[\delta_{n',n}\sqrt{n+|l|+1} + \delta_{n',n-1}\sqrt{n}\right].
\end{aligned}
$$

From Eq. (44), we see that the actual dimensionless parameter defining the magnitude of the coefficients is $\alpha \langle p_z \rangle \sigma(t_0)$. In real life, the value of $\sigma(t_0)$ is of the order of several μm or less. Provided that currently $n \sim 1$, $l \lesssim 10^4$, even for 10 GeV-electrons with $\langle p_z \rangle \sim 10^{-3}\mu\text{m}^{-1}$, we obtain $|c_{nn'll'}| \lesssim 10^{-2}\alpha$. This means that the off-axis corrections are negligible for any feasible experimental scenario.

Quantum rms emittance

$$\epsilon_i = \sqrt{\langle x_i^2\rangle\langle \hat{v}_i^2\rangle - \langle x_i \hat{v}_i\rangle\langle \hat{v}_i x_i\rangle} = \frac{1}{2}\sqrt{\langle \boldsymbol{\rho}^2\rangle\langle \hat{\boldsymbol{v}}^2\rangle - \langle \boldsymbol{\rho}\cdot\hat{\boldsymbol{v}}\rangle\langle \hat{\boldsymbol{v}}\cdot\boldsymbol{\rho}\rangle} \equiv \frac{\epsilon}{2}$$

$$\epsilon_{\mathrm{H}}(t) = \lambda_{\mathrm{C}}\sqrt{\frac{\epsilon_{\mathrm{f}}^2}{\lambda_{\mathrm{C}}^2} + \left[(2n+|l|+1)\frac{\sigma^2(t)}{\sigma_{\mathrm{L}}^2} + l\right]^2 - l^2}$$

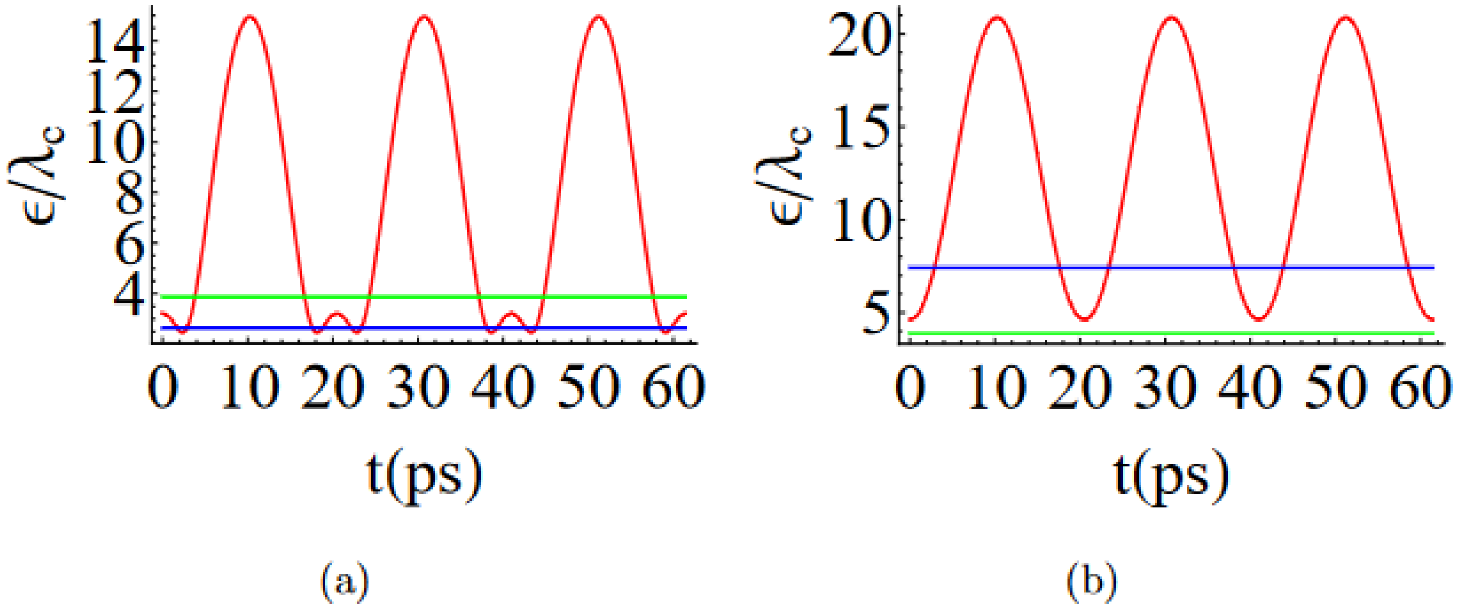


Figure 9: Emittances of the $\mathrm{NSLG_H}$ (in red), the Landau (in blue), and the $\mathrm{NSLG_f}$ (in green) states. $B = 1.9$ T, $n = 0$, $\sigma_0 = 25$ nm, $\sigma' = 0$. (a) $l = -3$, (b) $l = 3$.

Rms radius oscillations

Heisenberg equation

⬇

$$\rho^2(t) = \rho_{\rm st}^2 + \left(\rho_0^2 - \rho_{\rm st}^2\right)\cos\left(\omega\tau\right) + \frac{2\rho_0\rho_0'}{\omega}\sin\left(\omega\tau\right)$$

$$\rho_{\rm st}^2 = 2\lambda_{\rm C}\omega^{-1}\left(2\omega^{-1}\langle\hat{\mathcal{H}}_\perp\rangle + \langle\hat{L}_z\rangle\right)$$

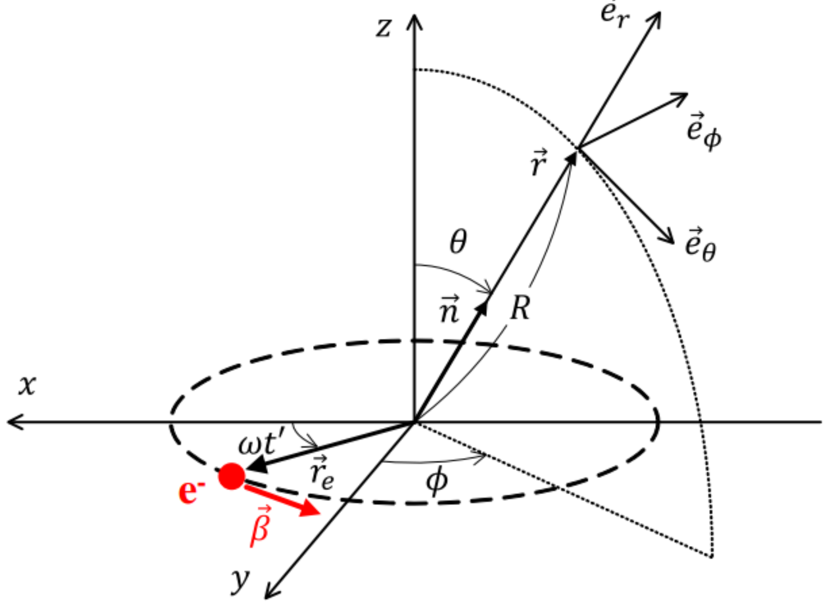
Classical radiation of vortex photons

$$\vec{E}(R_0,\, \theta,\, \varphi,\, \omega t) = \mathrm{Re}\sum_{l=1}^{\infty}\frac{e}{cR} l\omega\{\varepsilon_+^l(\theta)e^{i(l-1)\varphi}\vec{e}_+ + \varepsilon_-^l(\theta)e^{i(l+1)\varphi}\vec{e}_- + i\varepsilon_z^l(\theta)\vec{e}_z e^{il\varphi}\}e^{-il(\omega t-\frac{R_0}{c})}$$

$$\varepsilon_\pm^l(\theta) \equiv \frac{\varepsilon_x^l(\theta) \pm \varepsilon_y^l(\theta)}{\sqrt{2}}$$

$$= \beta J_l'(l\beta\sin\theta) \pm \frac{\cos^2\theta}{\sin\theta}J_l(l\beta\sin\theta)$$

$$\varepsilon_z^l \equiv \cos\theta J_l(l\beta\sin\theta)$$



Epp V., Guselnikova U., Angular momentum of radiation from a charge in circular and spiral motion, Physics Letters A, 2019

Relativistic Landau states

"Spin-up" state

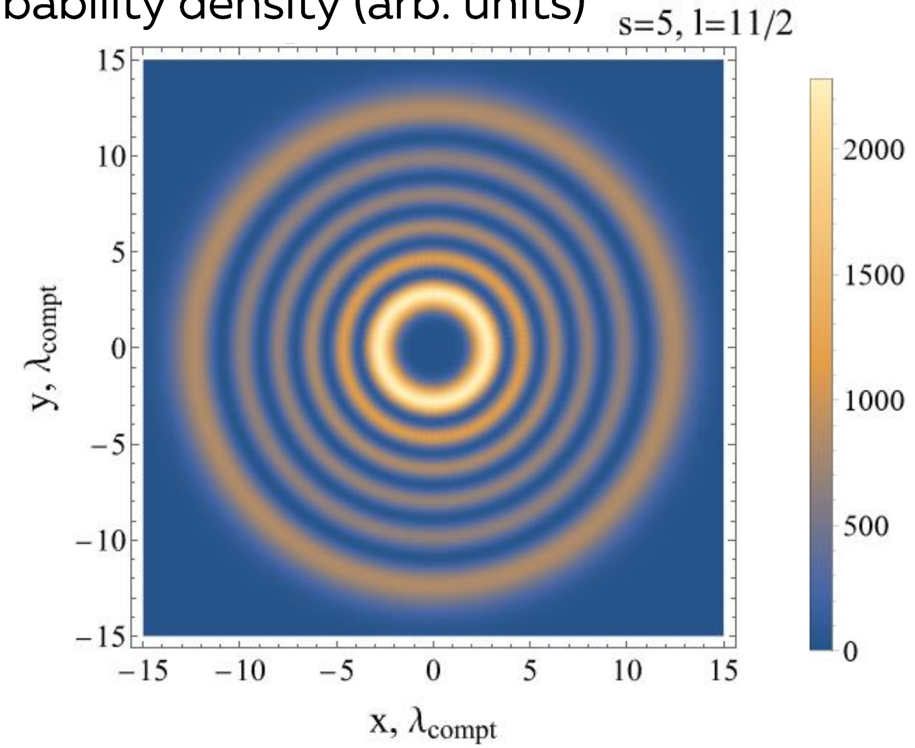
some radial function

$$\Psi^{\uparrow}_{s,\ell}(x) = N^{\uparrow} \begin{pmatrix} (m+\varepsilon)\Phi_{s,\ell-1/2}(\rho)e^{-i\varphi/2} \\ 0 \\ p_z\Phi_{s,\ell-1/2}(\rho)e^{-i\varphi/2} \\ -ieH\Phi_{s,\ell+1/2}(\rho)e^{i\varphi/2} \end{pmatrix} e^{-it\varepsilon_{s,\ell}+i\ell\varphi+ip_z z}$$

"Spin-down" state

$$\Psi^{\downarrow}_{s,\ell}(x) = N^{\downarrow} \begin{pmatrix} 0 \\ (m+\varepsilon)\Phi_{s,\ell+1/2}(\rho)e^{i\varphi/2} \\ -2i(\ell+s+1/2)\Phi_{s,\ell-1/2}(\rho)e^{-i\varphi/2} \\ -p_z\Phi_{s,\ell+1/2}(\rho)e^{i\varphi/2} \end{pmatrix} e^{-it\varepsilon_{s,\ell}+i\ell\varphi+ip_z z}$$

Probability density (arb. units)



Energy

$$\varepsilon_{s,\ell} = \sqrt{m^2 + p_z^2 + 2|e|H(s+\ell+1/2)}$$

radial quantum number

TAM projection (half-integer)

$$\hat{j}_z \Psi^{\updownarrow}_{s,\ell} = \left(\hat{\ell}_z + \frac{1}{2}\widehat{\Sigma}_z\right)\Psi^{\updownarrow}_{s,\ell} = \ell\Psi^{\updownarrow}_{s,\ell}$$

Photon transverse momentum

faculty of PHYSICS

Transverse momentum of the photon

$$\kappa = \sqrt{(\varepsilon - \varepsilon')^2 - (p_z - p'_z)^2} \geq 0$$

Radiation in the critical field

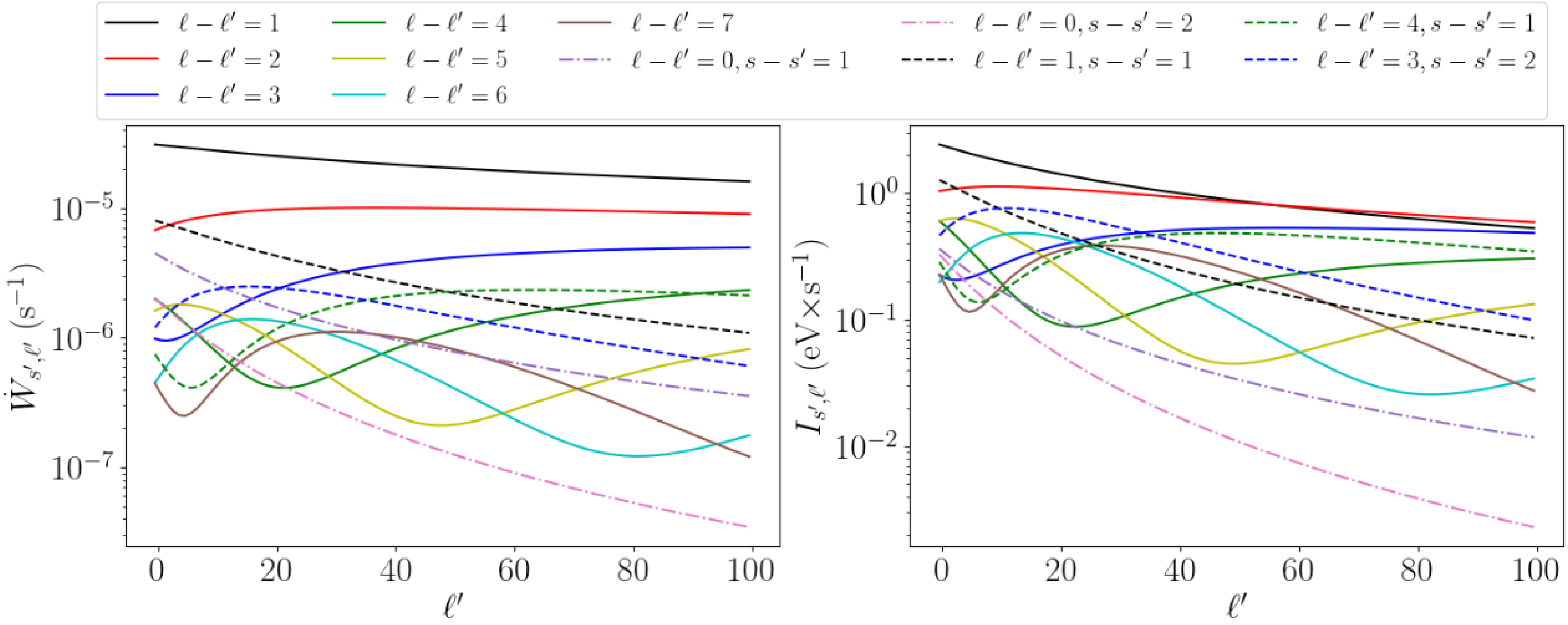


Figure 1: The emission probability (32) (left) and the corresponding intensity (33) (right) for $H = H_c$, $p_z = 10^{-3}mc$, and no spin-flip. For the solid lines $s = s' = 20$, the dashed lines correspond to the twisted photons with a simultaneous change of the radial quantum number $s \to s' \neq s$, the dash dotted lines correspond to the untwisted photons with the TAM $j_z = \ell - \ell' = 0$.

More radiation in the critical field

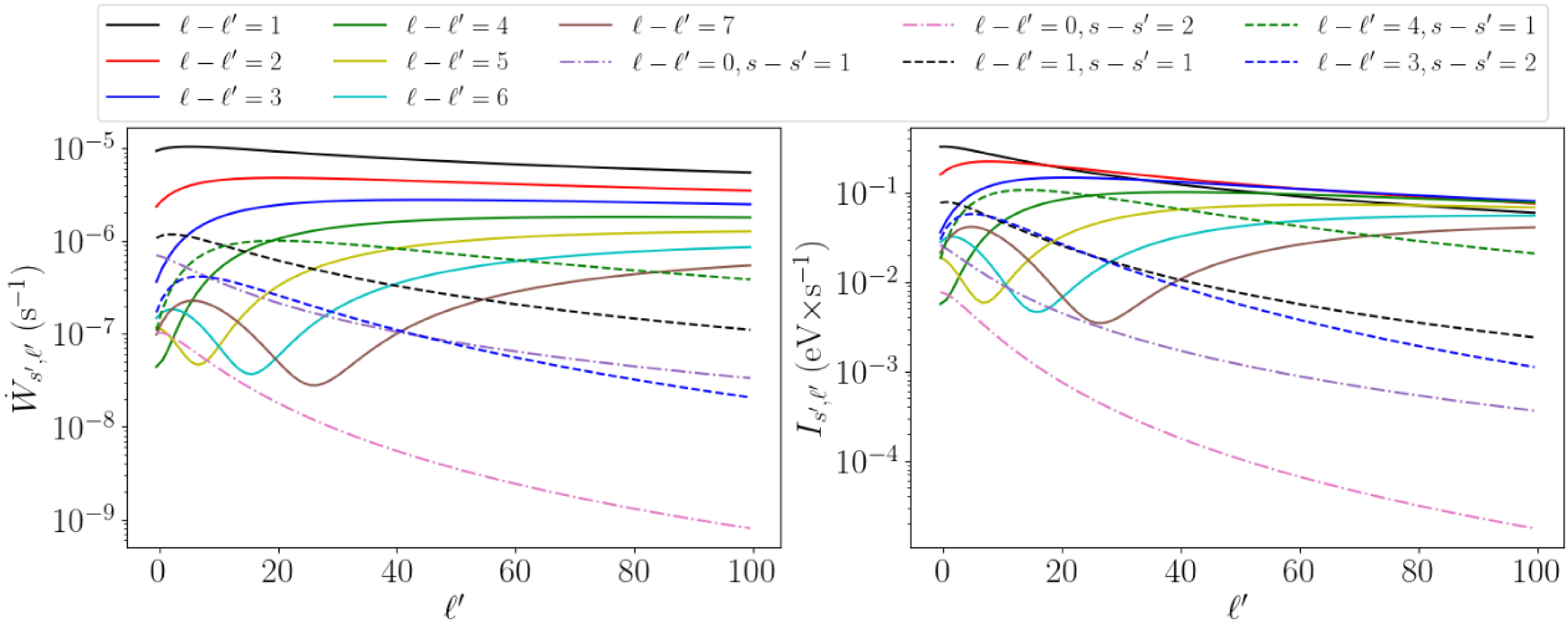


Figure 2: The same as in Fig 1, but for $H = 0.1H_c$ and $s = 5$.

Sokolov-Ternov effect

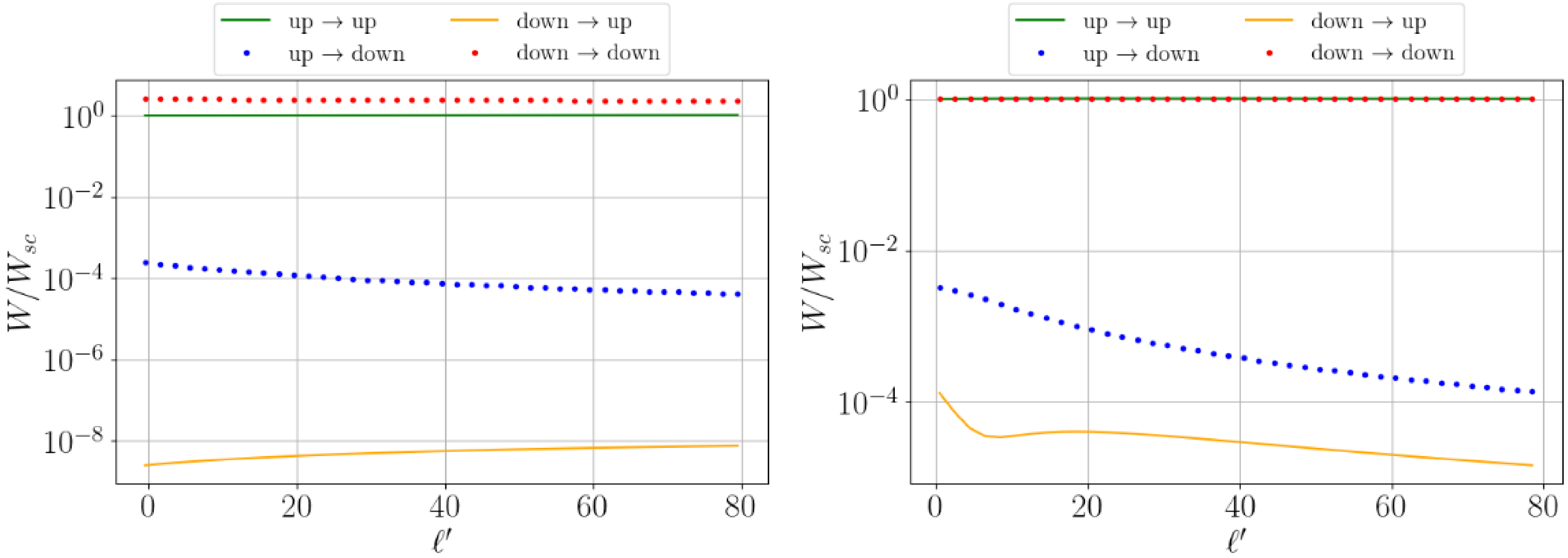


Figure 4: The ratio of four possible types of transition probabilities to the probability of emission by a scalar charge derived in [15]. $H = 10^{-3}H_c$ (left) and $H = H_c$ (right); $\ell - \ell' = 3$, $s = s' = 20$ for all transitions.

Dependence on p_z

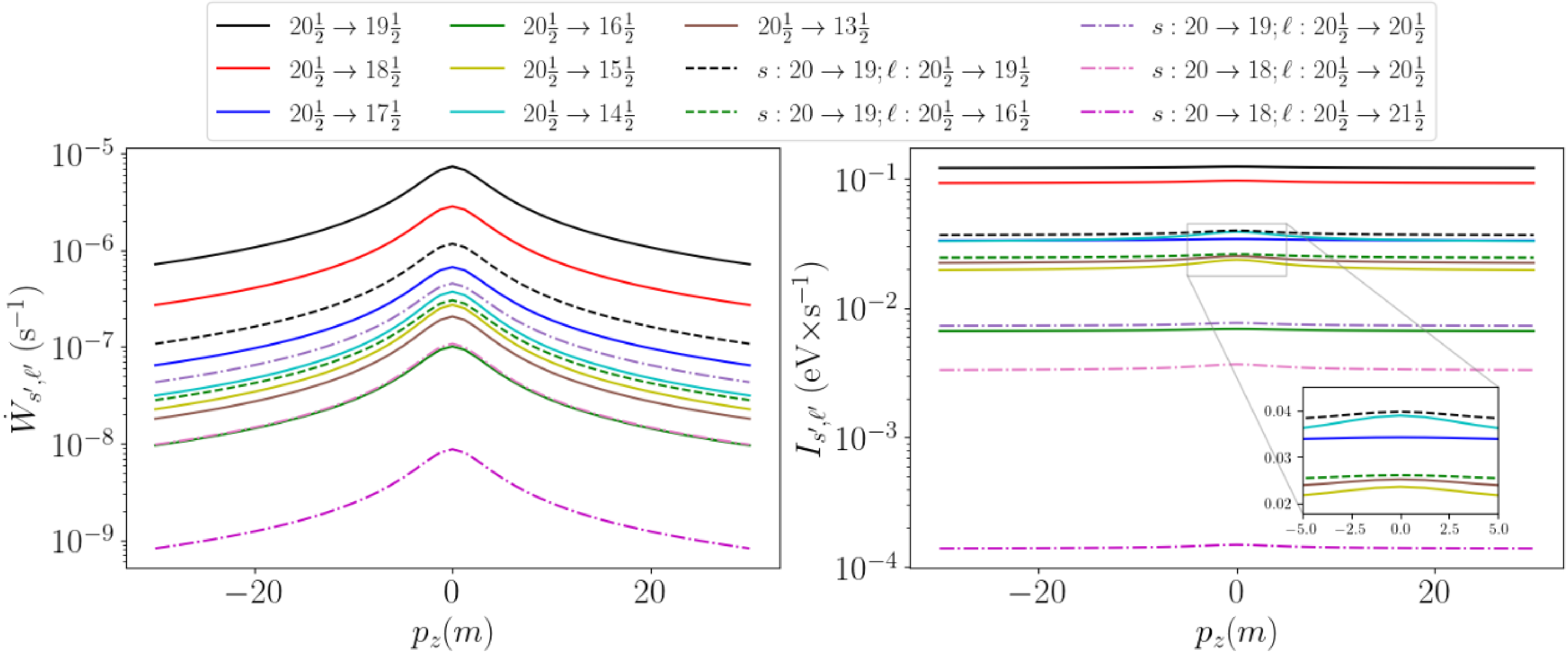


Figure 5: The dependence of the emission probability (left) and the intensity (right) on the electron momentum p_z for $H = 0.1H_c$, $s = s' = 20$. The transition $20\frac{1}{2} \to 19\frac{1}{2}$ means $\ell = 20\frac{1}{2}$, $\ell' = 19\frac{1}{2}$, $s = s' = 20$; those with $\ell : 20\frac{1}{2} \to 20\frac{1}{2}$ correspond to the untwisted photons with $j_z = 0$. The green line overlaps with the pink dashed one on the left; the cyan line on the left overlaps with the blue one on the right. The magenta dash-dotted line corresponds to increase of the electron OAM during the emission (so that the photon TAM is $\ell - \ell' = -1$).