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Gravitational Waves Sourced from QCD phase transition at finite density

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第 17 届粒子物理、核物理和宇宙学交叉学科前沿问题研讨会

Based on: Hui-wen Zheng, Yi Lu, Fei Gao, Si-xue Qin, and Yu-xin Liu, (2023), Phys. Rev. D 109, 114013 (2024), arXiv: 2312.00382

Hui-wen Zheng, Fei Gao, Ligong Bian, Si-xue Qin, and Yu-xin Liu, arXiv: 2407.03795

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1. Introduction of the Effective Potential

Definition of the effective potential

Introducing the local external source $J(x)$:

$$W[J] = -\mathrm{Ln} \int D\varphi \exp\left\{ -I(\varphi) - \int dx \; J(x)\varphi(x) \right\},$$

$$\frac{\delta W[J]}{\delta J(x)} = \bar{\varphi}(x),$$

which defines $\bar{\varphi}(x)$ as the dynamic variable. Based on Legendre transformation:

$$\Gamma[\bar{\varphi}] = W[J] - \int dx \; J(x)\bar{\varphi}(x).$$

Cornwall-Jackiw-Tomboulis (CJT) effective potential

Introducing the bilocal external source $J(x,y)$:

$$W_{\mathrm{CJT}}[J] = -\mathrm{Ln}\int D\varphi \exp\left\{-I(\varphi) - \int dxdy\ \varphi(x)J(x,y)\varphi(y)\right\},$$
$$\frac{\delta W_{\mathrm{CJT}}[J]}{\delta J(x,y)} = S(x,y),$$

with $S(x,y) = \langle\varphi(x)\varphi(y)\rangle$. Above equation defines $S(x,y)$ as the dynamic variable. After applying the Legendre transformation:

$$\Gamma_{\mathrm{CJT}}[S] = W_{\mathrm{CJT}}[J] - \int dxdy\ J(x,y)S(x,y),$$
$$= \mathrm{TrLn}[S] - \mathrm{Tr}[SS_0^{-1}] - \Gamma_2[S],$$

where S is the dressed propagator, S_0 is the free propagator, and $\Gamma_2[S]$ is the sum of all two-particle irreducible diagrams expressed in terms of S.

J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D **10**, 2428-2445 (1974)

The problem of the CJT effective potential

$\Gamma_{\mathrm{CJT}}[S]$ **does not admit a ground state, its vacuum energy is negative infinite, and its all extrema are unstable**.

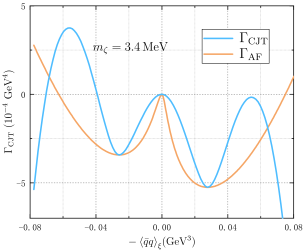
R. W. Haymaker, T. Matsuki, and F. Cooper, Phys. Rev. D 35, 2567 (1987).

In bare vertex approximation, an alternative choice is the auxiliary field (AF) potential:

$$\Gamma_{\mathrm{AF}}[\Sigma] = \mathrm{TrLn}[S] + \frac{1}{2}\mathrm{Tr}[\Sigma S],$$

H. Kleinert, Phys. Lett. B **62**, 429 (1976)

Roberts, Williams, Prog. Part. Nucl. Phys. **33**, 477-575 (1994)



which is obtain via the Hubbard-Stratonovich transformation from the CJT generating functional:

$$\int D(\Phi)\exp\left[-\frac{1}{2}\left(\Phi-\psi\bar{\psi}\right)\frac{\delta^2\Gamma_2[S]}{\delta S^2}\left(\Phi-\psi\bar{\psi}\right)\right] = \text{const},$$

Beyond the bare vertex approximation

The new effective potential should satisfy 3 conditions:

- The dynamic variable is strictly the self energy:

$$\frac{\delta W_\Sigma [L]}{\delta L} = -\Sigma [S + L] \, ,$$

- Generating functional satisfies in the bare vertex:

R. W. Haymaker, T. Matsuki and F. Cooper, Phys. Rev. D **35**, 2567 (1987)

$$W_\Sigma = W_{\rm CJT} - \frac{1}{2} K \left(\frac{\delta^2 \Gamma_2 [S]}{\delta S^2} \right)^{-1} K,$$

- Equation of motion strictly satisfies:

$$\frac{\delta \Gamma_\Sigma [\Sigma]}{\delta \Sigma[S]} = -\left(S_0^{-1} + \Sigma[S] \right)^{-1} + S.$$

which guarantees the lower bound of the energy.

The new effective potential

Applying the generalized Legendre transformation to CJT generating functional:

H. W. Zheng, Y. Lu, F. Gao, S. X. Qin, and Y. X. Liu, (2023), Phys. Rev. D 109, 114013 (2024), arXiv: 2312.00382

$$W_\Sigma[L] = W_{\rm CJT}[K] + \Gamma_2[S] - \Gamma_2[S+L] - {\rm Tr}[SK],$$
$$K = \Sigma[S] - \Sigma[S+L],$$

Its first-order derivative is:

$$\frac{\delta W_\Sigma[L]}{\delta L} = -\Sigma[S+L],$$

Legendre transformation:

$$\Gamma_\Sigma[\Sigma] = {\rm TrLn}\left[S_0^{-1} + \Sigma\right]^{-1} - {\rm Tr}[1] - \Gamma_2[(S+L)] + {\rm Tr}[(S+L)\Sigma],$$
$$\frac{\delta\Gamma_\Sigma[\Sigma]}{\delta\Sigma} = -\left(S_0^{-1} + \Sigma[S+L]\right)^{-1} + (S+L)_\Sigma,$$

Technical problem of the new effective potential

Equation of motion --(Functional integration)--> Effective potential

For a scalar field $\varphi(x)$, assuming its equation of motion:

$$\delta\Gamma(\varphi)/\delta\varphi(x) = F(\varphi) = 0.$$

Then its effective potential is:

$$\Gamma(\varphi) = \int dx \delta\varphi(x) F(\varphi).$$

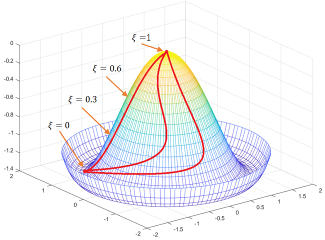
However, functional integration is difficult.

Homotopy method

We can choose a homotopy trajectory to get the potential:

$$\Gamma(\varphi(\xi))-\Gamma(\varphi(\xi=0)) = \int_0^{\xi} d\xi' \int dx \frac{d\varphi(x)}{d\xi'} F(\varphi),$$

which reduces the functional integration into 1-dim integration.



A schematic view of homotopy trajectories.

Homotopy method for the new effective potential:

$$\Gamma_\Sigma\left[\Sigma_\xi\right]-\Gamma_\Sigma\left[\Sigma_1\right] = \int_0^{\xi} d\xi' \mathrm{Tr}\left\{\frac{d\Sigma_{\xi'}}{d\xi'}\left[-\left(S_0^{-1}+\Sigma_{\xi'}\right)^{-1}+S_{\xi'}\right]\right\}.$$

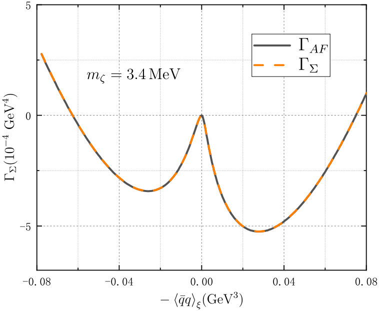
H. W. Zheng, Y. Lu, F. Gao, S. X. Qin, and Y. X. Liu, (2023), Phys. Rev. D 109, 114013 (2024), arXiv: 2312.00382

Homotopy method

In the bare vertex approximation, the comparison between the AF potential and the newly defined potential:

$$\Gamma_{\mathrm{AF}}[S] = \mathrm{TrLn}[S] + \frac{1}{2}\mathrm{Tr}[\Sigma S],$$

$$\Gamma_\Sigma\left[\Sigma_\xi\right] - \Gamma_\Sigma\left[\Sigma_1\right] = \int_0^\xi d\xi' \mathsf{Tr}\left\{\frac{d\Sigma_{\xi'}}{d\xi'}\left[-\left(S_0^{-1} + \Sigma_{\xi'}\right)^{-1} + S_{\xi'}\right]\right\},$$



2. Introduction of Dyson-Schwinger equation

Introduction of Dyson-Schwinger equation

Euclidean QCD action:

Roberts, Williams, Prog. Part. Nucl. Phys. **33**, 477-575

$$S_{QCD} = \int d^4x[\bar{q}(\not{D} + m_0)q(x) + \frac{1}{4}(F^a_{\mu\nu})^2],$$

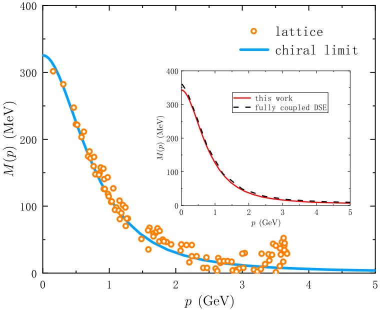
$$S_{GF} = \int d^4x[\partial_\mu \bar{\omega}_a(\delta^{ab}\partial_\mu - g_0 f^{abc}A^c_\mu)\omega_b - \frac{1}{2\xi_0}(\partial_\mu A^a_\mu)^2],$$

Taking the variation of the field:

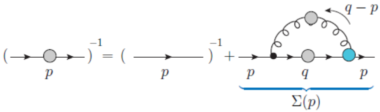
$$\int \mu[\bar{q}, q, A_\mu]\frac{\delta}{\delta\bar{q}(x)} exp[-S_{QCD} - S_{GF} + \int d^4x(\bar{\eta}q + \bar{q}\eta + J_\mu A_\mu)] = 0.$$

Then we obtain the Dyson-Schwinger equation.

Introduction of Dyson-Schwinger equation



H. W. Zheng, Y. Lu, F. Gao, S. X. Qin, and Y. X. Liu, (2023), Phys. Rev. D 109, 114013 (2024), arXiv: 2312.00382

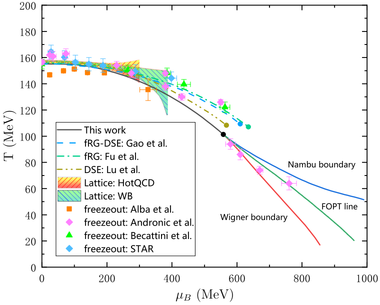


Longitudinal and transverse Slavnov-Taylor identity:

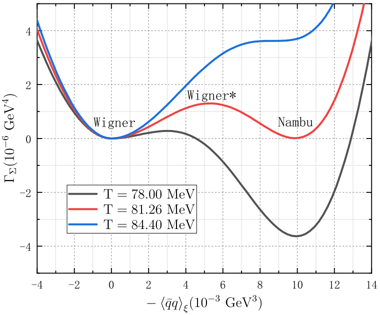
$$\Gamma_\mu(\tilde{\omega}_n, \vec{p}, \tilde{\omega}_m, \vec{q}) = F(k^2)\,\frac{A(\tilde{\omega}_n, \vec{p}) + A(\tilde{\omega}_m, \vec{q})}{2}\gamma_\mu + Z_{A,L}^{-1/2}(k^2)\Delta_B P^L_{\mu\nu} k_\nu \sigma_{\mu\nu} + Z_{A,T}^{-1/2}(k^2)\Delta_B P^T_{\mu\nu} k_\nu \sigma_{\mu\nu}$$

L. Chang, Y. X. Liu, and C. D. Roberts, Phys. Rev. Lett. 106, 072001 (2011), F. Gao, J. Papavassiliou, and J. M. Pawlowski, Phys. Rev. D 103, 094013 (2021)

QCD phase diagram



QCD phase diagram



Effective potential in first-order phase transition

The effective potential intuitively shows the coexistence of two phases in the first-order phase transition (FOPT) region.

Thermal quantities

number density:

$$n_q(T,\mu_q) = 2N_c \int \frac{d^3k}{(2\pi)^3}[f_q^+(k;T,\mu_q) - f_q^-(k;T,\mu_q)],$$

$$f_q^{\pm}(k;T,\mu_q) = \frac{\Phi(T,\mu_q)x_\pm^2 + 2\Phi(T,\mu_q)x_\pm + 1}{x_\pm^3 + 3\Phi(T,\mu_q)x_\pm^2 + 3\Phi(T,\mu_q)x_\pm + 1},$$

$$x_\pm(k;T,\mu_q) = \exp[(E_q(k;T,\mu_q) \mp \mu_q)/T],$$

$$E_q(k;T,\mu_q) = \sqrt{k^2 + M_q^2},$$

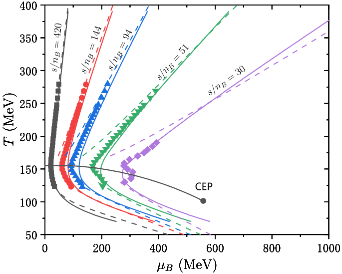
entropy density:

$$s_{QCD}(T,\mu) = s_{latt}(T,\mu=0) + \delta s(T,\mu),$$

$$\delta s(T,\mu) = s(T,\mu) - s(T,0) = \int_0^{\mu_q} d\mu_q{}' \frac{\partial n_q(T,\mu')}{\partial T}.$$

W. J. Fu et al. Phys. Rev. D 92, 116006 (2015), arXiv:1508.06504 [hep-ph].

Y. Lu et al. (2023), arXiv:2310.16345 [hep-ph].



Isentropic trajectories. Solid line: DSE from this work; scatter points: the lattice; dashed line (above): the improved ideal quark gas; dashed line (below): HRG.

J. N. Guenther et al. Nucl. Phys. A 967, 720 (2017), arXiv:1607.02493 [hep-lat].

3. Cosmic trajectories and gravitational waves

Stochastic gravitational waves background (SGWB)

Recently, Pulsar Timing Array (PTA) collaborations have found SGWB:

Z. Arzoumanian et al. (NANOGrav), Astrophys. J. Lett. 905, L34 (2020)

S. Chen et al. (EPTA), Mon. Not. Roy. Astron. Soc. 508, 4970 (2021)

Supported by two evidences:

1. Common-spectrum process;
2. Hellings-Downs correlation.

Meanwhile, many studies estimate the strong FOPT at $T \sim 10\,\mathrm{MeV}$, which is in the QCD FOPT region.

G. Agazie et al. (NANOGrav), Astrophys. J. Lett. 951, L8 (2023)

L. Bian, S. Ge, J. Shu, B. Wang, X. Y. Yang and J. Zong, Phys. Rev. D **109**, no.10, L101301 (2024)

Mechanism for QCD FOPT

Large lepton asymmetries ($Y_{L_f} = n_f/s$) can induce the QCD FOPT in early Universe.

F. Gao and I. M. Oldengott, Phys. Rev. Lett. 128, 131301 (2022)

BBN and CMB constrain the total lepton asymmetry $\left|Y_{L_e} + Y_{L_\mu} + Y_{L_\tau}\right| < 1.2 \times 10^{-2}$, instead of the individual.

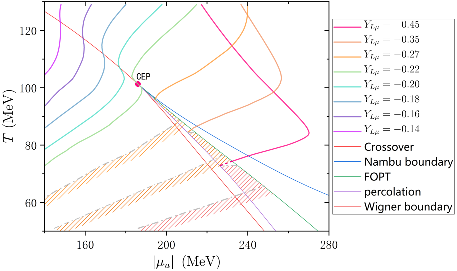
I. M. Oldengott and D. J. Schwarz, EPL 119, 29001 (2017)

Consider the conservation laws:

$$Y_{L_\alpha} = \frac{n_\alpha + n_{\nu_\alpha}}{s},$$

$$Y_B = \sum_i \frac{B_i n_i}{s},$$

$$Q = \sum_i \frac{Q_i n_i}{s}.$$



Cosmic trajectories for the scenario $Y_{L_e} = 0$, $Y_{L_\mu} = -Y_{L_\tau}$.

H. W. Zheng, F. Gao, L. Bian, S. X. Qin, and Y. X. Liu, arXiv: 2407.03795

Gravitational waves

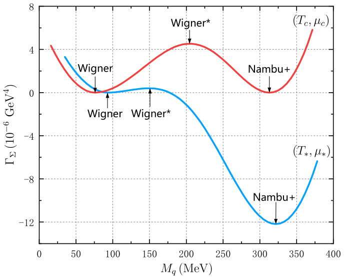
$$\Omega_s(f) = \mathcal{D}\,\widetilde{\Omega}_s \Upsilon(\tau_{sw}) \left(\frac{\kappa_s \alpha_*}{1+\alpha_*}\right)^2 (H_* R_*)\, \mathcal{S}(f/f_s),$$

with α_* the phase transition (PT) strength, β/H_* the inverse PT duration.

$$\Gamma(T,\mu) \simeq T\left(\frac{S_3}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{S_3}{T}\right) \times \max(T^3, \mu^3, r^{-3}, (V''_{\rm eff})^{3/2}, \phi^3),$$

with

$$S_3 = 4\pi \int_0^\infty dr r^2 \left[\frac{1}{2}\left(\frac{d\phi}{dr}\right)^2 + V_{\rm eff}(\phi)\right].$$



Effective potential at PT temperature and percolation temperature along the cosmic trajectory.

PT parameter

$$\dot{\rho_r} + 3\frac{\dot{a}}{a}(\rho_r + p) = 0,$$

$$\alpha_* = \left.\frac{\Delta\theta}{\rho_r(T,\ \mu)}\right|_{(T_*,\mu_*)},$$

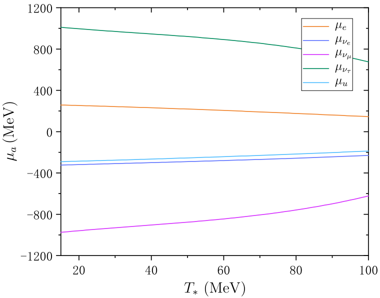
with

$$\Delta\theta = \frac{1}{4}T\Delta s + \frac{1}{4}\mu_q \Delta n_q - \Delta P,$$

ρ_r collects electron, muon, all neutrinos, photon and u/d quarks in the false vacuum.

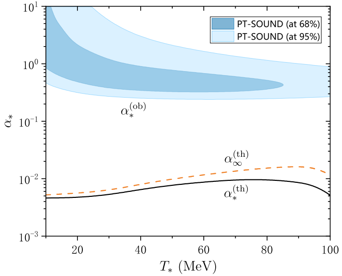
$$\beta = -\frac{d}{dt}\left(\frac{S_3}{T}\right)\Big|_{t_*} = 4H_*\rho_r\frac{d}{d\rho_r}\left(\frac{S_3}{T}\right)\Big|_{(T_*,\mu_*)},$$

where the derivative lies along the cosmic trajectory.

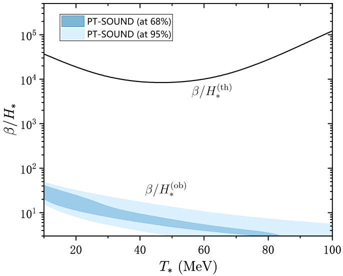


The lepton and quark chemical potentials along the percolation line.

Numerical results



PT strength.



Inverse PT duration.

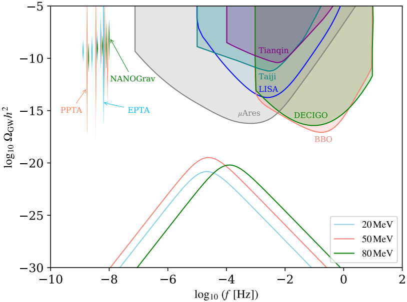
$$\alpha_\infty = \frac{\Delta P_{LO}}{\rho_r},$$

with

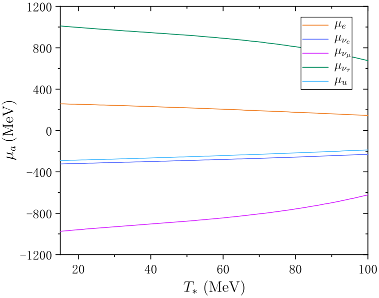
$$\Delta P_{LO} = \sum_{a'} \int_{m_f}^{m_t} dm^2 \int \frac{d^3p}{(2\pi)^3} \frac{f_{F/B}(E, T, \mu_{a'})}{2E},$$

with $\sum_{a'}$ summing over all particles taking part in the phase transition. α_∞ determine whether the bubble walls can runaway.

Gravitational waves spectrum



The gravitational waves spectra of the QCD FOPT.



The lepton and quark chemical potentials along the percolation line.

the GW signals of NANOGrav are very unlikely sourced from the chiral phase transition of QCD.

The large chemical potential from the conservation laws significantly suppresses the GW signal.

Summary

- Defining a new effective potential, providing a rigorous solution to the problem of the CJT formalism lacking an energy lower bound.
- Homotopy method, simplifying the functional integration.
- Calculating the QCD phase diagram and thermal quantities.
- Computing the gravitational waves from QCD theory, with the conservation laws, we suggest that the GW signals of NANOGrav are very unlikely sourced from the chiral phase transition of QCD.

Thank you!

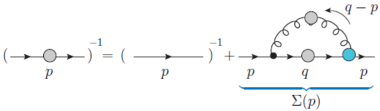
Appendix

Introduction of Dyson-Schwinger equation

The gluon propagator using lattice input:

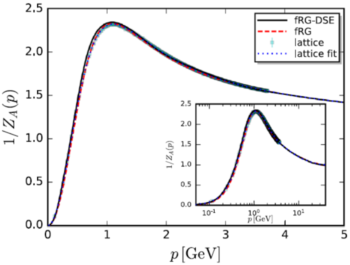
F. Gao, Papavassiliou, Pawlowski, Phys. Rev. D **103**, no.9, 094013

$$D_{\mu\nu}(k^2) = P^T_{\mu\nu} D_T(k^2 + m_g^2) + P^L_{\mu\nu} D_T(k^2),$$



Thermal gluon mass:

$$m_g^2 = g^2 [\frac{N_c}{3} T^2 + N_f (\frac{T^2}{6} + \frac{\mu^2}{2\pi^2})],$$



Longitudinal and transverse projection operators:

$$P^T_{\mu\nu} = \begin{cases} 0, & \mu \ or \ \nu = 4 \\ \delta_{ij} - \frac{\vec{k}_i \vec{k}_j}{\vec{k}^2}, & \mu, \ \nu = 1, 2, 3 \end{cases},$$

$$P^L_{\mu\nu} = \delta_{\mu\nu} - k_\mu k_\nu / k^2 - P^T_{\mu\nu}$$

S. X. Qin et al., Phys. Lett. B **722**, 384-388;

L. Chang, Y. X. Liu, Roberts, Phys. Rev. Lett. **106**, 072001

Introduction of Dyson-Schwinger equation

Longitudinal and transverse Ward-Green-Takahashi identity:

L. Chang, Y. X. Liu, and C. D. Roberts, Phys. Rev. Lett. 106, 072001 (2011),

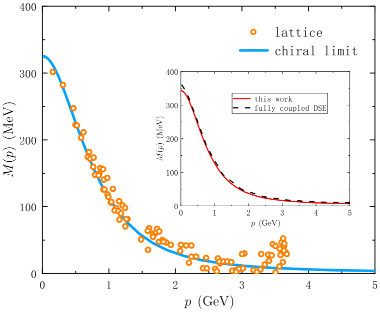
F. Gao, J. Papavassiliou, and J. M. Pawlowski, Phys. Rev. D 103, 094013 (2021)

$$\Gamma_\mu(\tilde{\omega}_n,\vec{p},\tilde{\omega}_m,\vec{q}) = F(k^2)\frac{A(\tilde{\omega}_n,\vec{p})+A(\tilde{\omega}_m,\vec{q})}{2}\gamma_\mu + Z_{A,L}^{-1/2}(k^2)\Delta_B P^L_{\mu\nu}k_\nu\sigma_{\mu\nu} + Z_{A,T}^{-1/2}(k^2)\Delta_B P^T_{\mu\nu}k_\nu\sigma_{\mu\nu}$$

with

$$\Delta_B = \frac{\tilde{B}(\tilde{\omega}_n,\vec{p})-\tilde{B}(\tilde{\omega}_m,\vec{q})}{\tilde{\omega}_n^2+\vec{p}^2-\tilde{\omega}_m^2-\vec{q}^2},$$

$$\tilde{B}(\tilde{\omega}_n,\vec{p}) = B(\omega_0\ \mathrm{sgn}(\omega_n)+i\mu_q,l_p),$$



L. Chang, Y. X. Liu, and C. D. Roberts, Phys. Rev. Lett. 106, 072001 (2011),

F. Gao, J. Papavassiliou, and J. M. Pawlowski, Phys. Rev. D 103, 094013 (2021)

with $k=(\Omega_{nm},\vec{q}-\vec{p})$, $l_p=(\vec{p}^2+{\omega_n}^2-{\omega_0}^2)^{1/2}$, $Z_{A,L}=D_L(k^2)k^2$ and $Z_{A,T}=D_T(k^2)k^2$.

Comparison with lattice/experiment

	$\langle\bar{q}q\rangle^{\frac{1}{3}}\ (\zeta = 2\,\mathrm{GeV})$	f_π	$m^\zeta(\zeta = 2\,\mathrm{GeV})$
Our work	$273.1\,\mathrm{MeV}$	$91.5\,\mathrm{MeV}$	$3.67\,\mathrm{MeV}$
lattice/experiment	$272(5)\,\mathrm{MeV}$	$92.1\,\mathrm{MeV}$	$3.42\,\mathrm{MeV}$

A. Bazavov, *et al.* PoS **LATTICE2010**, 083 (2010),

P. A. Boyle, *et al.* Phys. Rev. D **93**, no.5, 054502 (2016)

where f_π is from Pagels-Stokar formula:

H. Pagels and S. Stokar, Phys. Rev. D 20, 2947 (1979).

$$f_\pi^2 = \frac{3}{4\pi^2}\int_0^\Lambda dp^2 \frac{p^2 M\left(p^2\right)}{A\left(p^2\right)\left[p^2 + M^2\left(p^2\right)\right]^2}\left[M\left(p^2\right) - \frac{p^2}{2}\frac{dM\left(p^2\right)}{dp^2}\right].$$

m^ζ is determined by Gell-Mann-Oakes-Renner (GOR) relation:

H. Pagels and S. Stokar, Phys. Rev. D 20, 2947 (1979).

$$f_\pi^2 m_\pi^2 = -2m^\zeta\left\langle\bar{q}q\right\rangle.$$