A Glance back to Factorization Methods in Heavy-quark Physics

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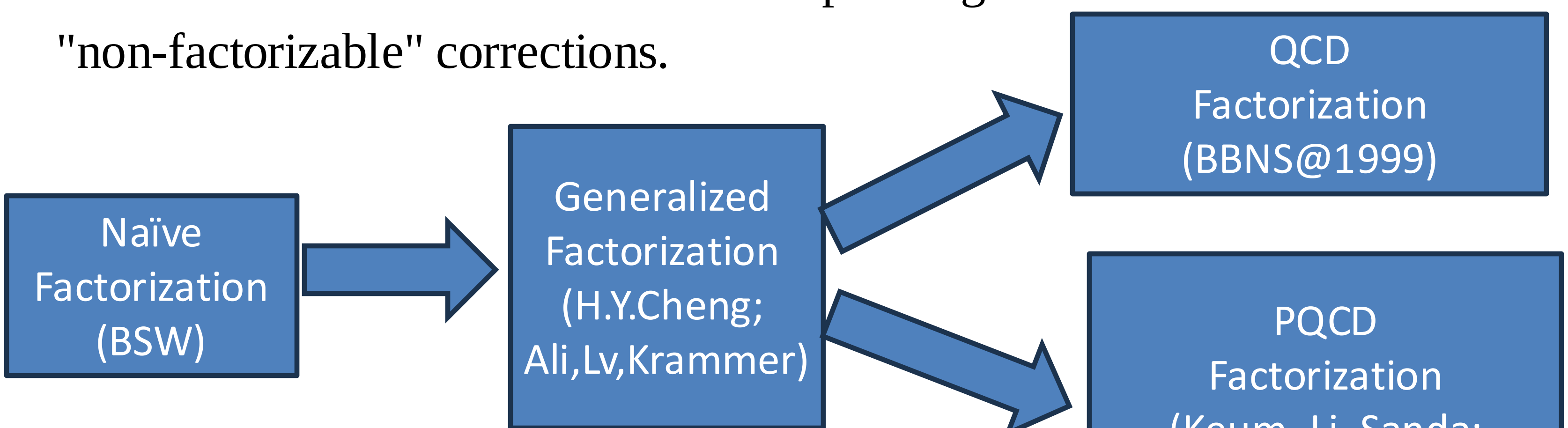
Some old memories

- I was honored to be invited to give a review on BBNS(QCDF) method at the very first meeting of this series - 2002HFCPV@YanTai, but unfortunately, I didn't manage to do it.
- At that time, talking about factorization almost just meant talking about factorizing the matrix element of current-current operators into product of matrix elements of currents and corresponding "non-factorizable" corrections.



Prof. Maozhi Yang present reviews on both BBNS and PQCD method at 1st HPCPV meeting !

Disclaimer

- **Mostly based on my personal impressions. Sorry for many missed progresses made by colleagues!**
- **For a more comprehensive review, one can find the review talk given by Prof. Yuming Wang at 2019HFCPV@Huhhot**
- **Mainly on EFTs based factorization methods**
- **Criticisms and comments are welcome!**

Factorization

因子化不是万能的；
没有因子化是万万不能的。

- The Standard Model is a multiple scale problem (from 100s GeV down to sub eV).
- QCD is a demon which behaves distinctly at high energy (asymptotic freedom) and low energy (color confinement). The mass of first heavy quark, charm quark, just lies on the edge of above two regimes.
- Factorization is to separate dynamics from well-separated scales.
- Factorization is a must-be as we cannot simulate the whole dynamics in particle physics yet.
- Factorization is an approximation (hopefully systematic).

Factorization into EFTs

A Wilsonian matching EFT is an automatic factorization.

Therefore, the factorization formula is written as (Step 0)

$$A_{full} = \sum_i C_i(\mu)\,\langle O_i(\mu)\rangle$$

- **The low energy behaviors of underlying theory are reproduced by EFT interactions/operators sorted by the power counting rules in EFT;**
- **The high energy contributions are encoded into the strength of the couplings, so-called Wilson/short-distance coefficients;**
- **To get precision phenomenology, it requires:**
  - **Perturbative matching to get SD coefficients (+RGE);**
  - **Determinations of LD matrix-elements.**

EFTs in heavy-quark physics

- **NRQCD/pNRQCD: Productions & Decays of quarkonium/Bc/fully heavy baryons/tetraquark**
  - **Exclusive processes: WFs at the origin;**
  - **Inclusive processes: Singlet/Octet mechanisms**
- **HQET: heavy-quark symmetry**

  Decays of B mesons/D mesons/Heavy-light baryons…
- **$\mathrm{SCET_I}$/$\mathrm{SCET_{II}}$:**
  - **EFT version of earlier stage soft/collinear factorization**
  - **Simplification of proof of factorization**

Roadmap of HFP in EFTs

Underlying Theory (String/GUT/SUSY/other new physics models)

Standard Model EFT

Weak Effective Hamiltonian

HQET/SCET/.....

For B meson non-leptonic decays

The scale separation is standard: Standard procedure to write down the operators, do the matchings, apply RGs to resum large logarithms.

QCD Factorization (BBNS@1999)

PQCD Factorization (Keum, Li, Sanda; Ukai,Lv,Yang@2000)

Matching by method of regions

- **Perturbative matching is almost the most important task.**
- **An approach to achieve the asymptotic expansion of the multiple well-separated scales involved loop integrals (especially convenient in DR)** Beneke&Smirnov 1997'

1. **Identify all important momentum regions with homogenous power-counting which give the most important contributions;**
2. **Expand the integrand in each momentum regions w.r.t the power counting and integrate the expanded integrand over whole momentum region in DR;**
3. **Add up contributions from all regions to reproduce the asymptotic expansions of original loop integral;**

Advantages and difficulties

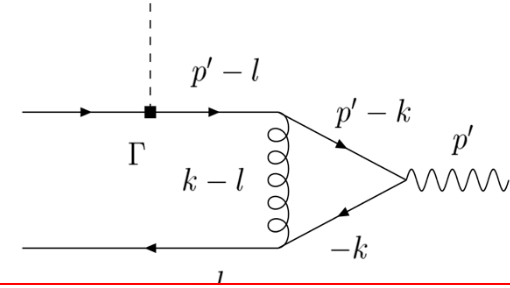
- **Advantages:**

1. **Technical & explicit demonstration of factorization;**
2. **Easy to calculate: each expanded integral involves less scales than original loop integral;**
3. **The low energy regions -> EFT contributions;**
   The large energy regions -> Wilson coefficients
   +(sometimes anomalous dimensions)

- **Difficulties:**

1. **How to identify all the important regions?**
2. **Some expanded integrals may be ill-defined in DR, need extra regularization.**
   e.g. Soft-collinear factorization anomaly/endpoint singularity/rapidity divergence

An example



Photon vertex correction to $\bar{B} \to \gamma l \nu$.

Beneke&Feldman, 2003

$$I = \int [dk] \frac{1}{[(k-l)^2][k^2 - m^2][(p'-k)^2 - m^2]},$$

$$I = \frac{1}{2p' \cdot l} \left(\mathrm{Li}_2 \left(-\frac{2p' \cdot l}{m^2} \right) - \frac{\pi^2}{6} \right) = -\frac{1}{2p' \cdot l} \left\{ \frac{1}{2} \ln^2 \frac{m^2}{2p' \cdot l} + \frac{\pi^2}{3} + \dots \right\},$$

$$= \quad I_c + I_s + I_{hc}$$

The external momenta of the vertex subgraph are: a collinear "photon" momentum $p' = (n_+p', p'_\perp, n_-p') \sim (1, 0, 0)$ with $p'^2 = 0$; a soft "light quark" momentum $l \sim (\lambda^2, \lambda^2, \lambda^2)$ with $l^2 = m^2$; a hard-collinear "light quark" momentum $p' - l \sim (1, \lambda^2, \lambda^2)$ with virtuality λ^2. The two invariants are $2p' \cdot l \sim \lambda^2$ and $m^2 \sim \lambda^4$, so I must be a function of the small dimensionless ratio $m^2/(2p' \cdot l)$. A straightforward calculation gives

$$I_{hc} = \int [dk] \frac{1}{[k^2 - n_+k n_-l][k^2][k^2 - n_+p' n_-k]}$$
$$= -\frac{1}{2p' \cdot l} \left\{ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{2p' \cdot l}{\mu^2} + \frac{1}{2} \ln^2 \frac{2p' \cdot l}{\mu^2} - \frac{\pi^2}{12} \right\}$$

Extra regularization is needed to regularize Ic and Is, but the corresponding singularities in Ic and Is cancel each other

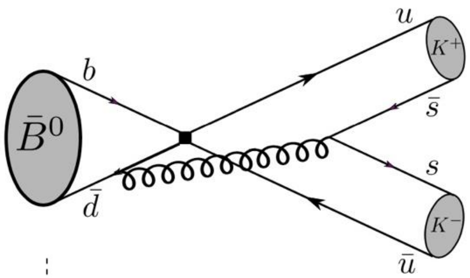
$$I_c = \int [dk] \frac{[-\nu^2]^\delta}{[-n_+k n_-l]^{1+\delta}[k^2 - m^2][k^2 - m^2 - 2p' \cdot k]}$$
$$= -\frac{1}{2p' \cdot l} \left(-\frac{1}{\delta} + \ln \frac{2p' \cdot l}{\nu^2} \right) \left(\frac{1}{\epsilon} - \ln \frac{m^2}{\mu^2} \right)$$

$$I_s = \int [dk] \frac{[-\nu^2]^\delta}{[(k-l)^2]^{1+\delta}[k^2 - m^2][-n_+p' n_-k]}$$
$$= -\frac{1}{2p' \cdot l} \left(\left[\frac{1}{\delta} - \ln \frac{m^2}{\nu^2} \right] \left[\frac{1}{\epsilon} - \ln \frac{m^2}{\mu^2} \right] - \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{m^2}{\mu^2} - \frac{1}{2} \ln^2 \frac{m^2}{\mu^2} + \frac{5\pi^2}{12} \right)$$

Endpoint singularity in B decays

- **Weak annihilation：power suppressed by divergent** **BBNS,2001**

$$A \sim f_B f_K^2 \int_0^1 dx \int_0^1 dy\, \phi_{K^+}(x)\phi_{K^-}(y) \left[\frac{1}{y(1-x\bar{y})} + \frac{1}{\bar{x}^2 y}\right]$$



- **Heavy-to-light form-factors:**

Beneke&Feldmann, 2000

$$f_+^{(\mathrm{HSA})} = \frac{\alpha_s C_F}{4\pi} \frac{\pi^2 f_B f_P M}{N_C E^2} \int_0^1 du \int_0^\infty dl_+ \left\{ \frac{4E-M}{M} \frac{\phi(u)\, \phi_+^B(l_+)}{\bar{u} l_+} \right. ($$

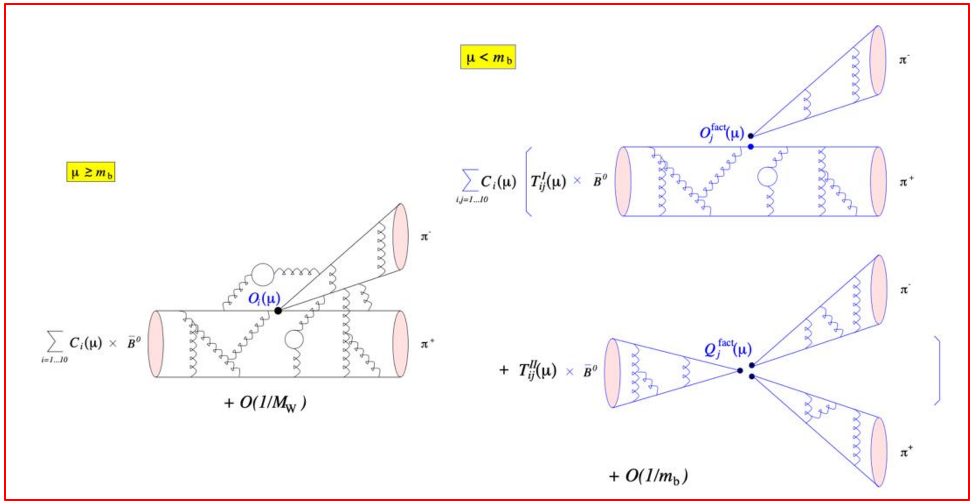
$$\left. + \frac{(1+\bar{u})\,\phi(u)\,\phi_-^B(l_+)}{\bar{u}^2 l_+} + \frac{\mu_P}{2E}\left[\frac{(\phi_p(u) - \phi'_\sigma(u)/6)\,\phi_+^B(l_+)}{\bar{u}^2 l_+} + \frac{4E\phi_p(u)\,\phi_+^B(l_+)}{\bar{u} l_+^2}\right]\right\}$$

- **Parameterizations are needed to do phenomenology!**

$$X_H = \int_0^1 \frac{1}{\bar{v}} = \ln(m_B/\Lambda_{QCD}) + \rho_H e^{i\phi_H} \qquad X_A = \int_0^1 \frac{1}{y} = \ln(m_B/\Lambda_{QCD}) + \rho_A e^{i\phi_A}$$

QCDF for $B \to M_1 M_2$

- **Formfactors cannot be fully factorized into a convolution;**
- **All perturbative corrections at leading power have been calculated up to NNLO in strong coupling, which were mostly done by M.Beneke, X.Q.Li and their collaborators in past 20 years;**



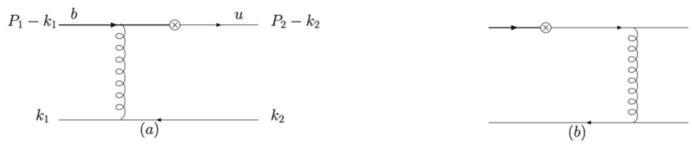
$$\langle M_1 M_2 | C_i O_i | \bar{B} \rangle_{\mathcal{L}_{\text{eff}}} = \sum_{\text{terms}} C(\mu_h) \times \Big\{ F_{B\to M_1} \times \underbrace{T^{\text{I}}(\mu_h, \mu_s)}_{1+\alpha_s+\dots} \star f_{M_2} \Phi_{M_2}(\mu_s)$$

$$+ f_B \Phi_B(\mu_s) \star \Big[\underbrace{T^{\text{II}}(\mu_h, \mu_I)}_{1+\dots} \star \underbrace{J^{\text{II}}(\mu_I, \mu_s)}_{\alpha_s+\dots} \Big] \star f_{M_1} \Phi_{M_1}(\mu_s) \star f_{M_2} \Phi_{M_2}(\mu_s) \Big\}$$

PQCD for $B \to M_1 M_2$

- **Consider transvers mentum to regulate the endpoint singularities;**

Keum, Li, Sanda; Ukai,Lv,Yang,2000



$$H_b^{(0)}(x_1, k_{1T}, x_2, k_{2T}) = -4g^2 C_F \frac{(\eta {P_1}^\mu - {P_2}^\mu)\phi_B^{(+)}(x_1) + {P_2}^\mu \phi_B^{(-)}(x_1)}{\eta(x_1 x_2 \eta m_B^2 + |\mathbf{k}_{1T} - \mathbf{k}_{2T}|^2)} \phi_\pi(x_2).$$

- **Sum the large logarithms by Sudakov factors;**
- **Phenomenologically successful!**

Huge efforts have been made by C.D.Lv, Z.J.Xiao and their students!

For the most recent comprehensive surveys, please see J.Chai, S.Cheng, Y.H. Ju, D.C. Yan, C.D.Lv and Z.J. Xiao, CPC, 2022.

Analogue in $\gamma^* \to J/\psi\,\eta_c$

- **Big success for NRQCD factorization:**
  LO+NLO+NNLO+ +O(v2)+O(v2 NLO)

K.T.Chao, Y.J.Zhang, Z.G.He et al; C.F.Qiao et al;
J.X.Wang, B.Gong et al; F.Feng,J.Yu,W.L.Sang et al;

- **Asymptotic behavior of the amplitude:** **J.X.Wang, J.Yu, D.S.Yang, 2010.**

$$G_{\mathrm{NRQCD}}(Q^2) = C(Q;m_c)\frac{\langle J/\psi(\lambda)|\psi^\dagger \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*(\lambda)\,\chi|0\rangle}{\sqrt{2N_c m_c}}\cdot\frac{\langle\eta_c|\psi^\dagger\chi|0\rangle}{\sqrt{2N_c m_c}} + \mathcal{O}(v^2)\,,$$

$$C^{(0)}(Q;m_c) = 256\pi e_c C_F \alpha_s(\mu_R^2)\,\frac{m_c}{(Q^2)^2}.$$

$$\frac{\mathrm{Re}[C^{(1)}_{\mathrm{asym}}(Q)]}{C^{(0)}_{\mathrm{asym}}(Q)} = \frac{13}{24}\ln^2\frac{Q^2}{m_c^2} - \frac{41}{24}(2\ln 2-1)\ln\frac{Q^2}{m_c^2} + \frac{\beta_0}{4}\ln\frac{\mu_R^2}{Q^2} + \frac{71}{8}\ln 2 + \frac{59}{24}\ln^2 2 - \frac{23}{18} - \frac{\pi^2}{36}.$$

Appearance of double logs at NLO, shocks G.Bodtwin et al (2014). They realized the problem connected with the endpoint singularity!

which is m/Q suppressed comparing with $\gamma^* \to \overline{B_c}B_c$.
Such suppression is caused by a helicity flip!

Tree-level Collinear Factorization

Ma and Si, 2004; Bonder and Chernyak, 2005;

- High twist LCDAs together with endpoint singularity appears!

$$\langle J/\psi(p)\eta_c(k)|J^\mu|0\rangle = iQ_c e\varepsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu(p)p_\alpha k_\beta \mathcal{F}(q^2)$$

$$\sigma(e^+e^- \to J/\psi\eta_c) = 4\pi\alpha^2 Q_c^2 \frac{|\mathcal{F}(s)|^2}{64}\left(1-\frac{4m_h^2}{s}\right)^{\frac{3}{2}}\int_{-1}^{+1} dx(1+x^2)$$

$$\begin{aligned}\mathcal{F}(s) \;=\; & \frac{8\pi\alpha_s(s)}{9} f_{\eta_c} f_{J/\psi}\frac{1}{s^2}\int_0^1 dz_1 dz_2\Big\{\frac{-2m_c^2}{m_{J/\psi}}\psi_\perp^{[2]}(z_1)\phi^{[2]}(\cdots \\ & + m_{J/\psi}\psi_\perp^{[3]}(z_1)\phi^{[2]}(z_2)\left[\frac{1}{z_2^2(1-z_1)} - \frac{1}{z_2(1-z_1)} + \frac{\cdots}{z_1(1-z_2)^2} \cdots \frac{\cdots}{z_1(1-z_2)}\right] \\ & + \frac{2m_{\eta_c}^2}{m_{J/\psi}}\psi_\perp^{[2]}(z_1)\phi_p^{[3]}(z_2)\left[\frac{1}{z_2(1-z_1)^2} + \frac{1}{z_1^2(1-z_2)}\right]\Big\}\cdot\left(1+\mathcal{O}\big(\frac{\Lambda}{\sqrt{s}}\big)\right).\end{aligned}$$

Lead to end-point singularity!

Fit to the expr.!

$$\begin{aligned}\sigma(e^+e^- \to J/\psi\,\eta_c) &\simeq 7.37\text{fb}, \quad \text{for } a = 1, \\ \sigma(e^+e^- \to J/\psi\,\eta_c) &\simeq 20.1\text{fb}, \quad \text{for } a = 1.5, \\ \sigma(e^+e^- \to J/\psi\,\eta_c) &\simeq 31.7\text{fb}, \quad \text{for } a = 1.75.\end{aligned}$$

Phenomenological prediction is severely dependent on parameterization of endpoint singularity!

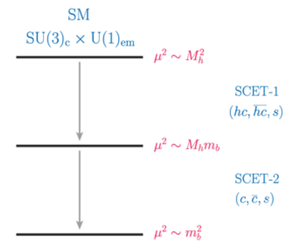
More Examples

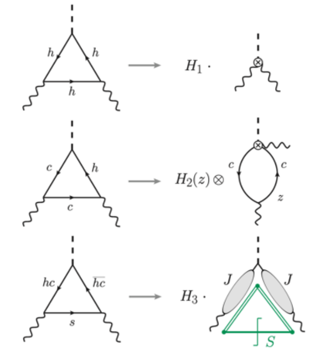
Stolen from Phillip Boeer, seminar@CERN, 2023

- $\rightarrow$ off-diagonal channel in DIS [Beneke et al '20]
- $\rightarrow$ bottom-induced $H \rightarrow \gamma\gamma/gg$ [Neubert et al. '19-'22]
- $\rightarrow$ off-diagonal gluon-thrust [Beneke et al. '22]
- $\rightarrow$ μ-e backscattering [Bell,PB,Felmann '22]
- $\rightarrow$ …
- $\rightarrow$ power corrections in $B \rightarrow h_1 h_2$ (e.g. weak annihilation) [BBNS '00]
- $\rightarrow$ heavy-to-light form factors [Beneke,Feldmann '00, PB'18]
- $\rightarrow$ power corrections in $B \rightarrow \gamma\ell\bar{\nu}_\ell$ [e.g. Beneke,Rohrwild '11]
- $\rightarrow$ QED corrections in $B_s \rightarrow \mu^+\mu^-$ [Beneke, Bobeth, Szafron '18]
- $\rightarrow$ power corrections in $B \rightarrow X_s\gamma$ [Szafron, Hurth '23]
- $\rightarrow$ QED corrections in $B^- \rightarrow \mu^-\bar{\nu}_\mu$ [Cornella et al. '23]
- $\rightarrow$ …

A consistent treatment would be an important breakthrough in controlling the underlying power-expansion!

$H \to \gamma\gamma$ through b quark loop





- **Suppressed by m_b/m_H**
- **Factorization formula**

Z.L.Liu & M.Neubert, JHEP, 2020; J.Wang,2020;

bare factorization contains endpoint singularity

$$\mathcal{M}_b(h \to \gamma\gamma) = H_1(\mu)\,\langle\gamma\gamma|\,O_1(\mu)\,|h\rangle + 2\int_0^1 dz\, H_2(z,\mu)\,\langle\gamma\gamma|\,O_2(z,\mu)\,|h\rangle$$
$$+\, g_\perp^{\mu\nu} H_3(\mu) \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+}\, J(M_h\ell_-,\mu)\, J(-M_h\ell_+,\mu)\, S(\ell_+\ell_-,\mu)\,.$$

- **With the proper subtractions in both jet and soft functions**

$$4[[\bar{H}_2]] \otimes [[\langle O_2\rangle]] = 2H_3 \int_0^\infty \frac{d\omega}{\omega} \int_0^{M_h} \frac{d\ell_-}{\ell_-} S(\omega) J(\omega/\ell_-) \bar{J}(\ell_-)$$

- **Resummation of double logarithms can be done by EFT RGEs:**

$$i\mathcal{M}^{(\mathrm{DL})} \sim \int_0^{m_H} \frac{d\ell_+}{\ell_+} \int_0^{m_H} \frac{d\ell_-}{\ell_-} \theta(\ell_+\ell_- - m_b^2) \exp\left\{-\frac{\alpha_s C_F}{2\pi} \ln\frac{\ell_+}{m_H} \ln\frac{\ell_-}{m_H}\right\}$$

B_c as a playground

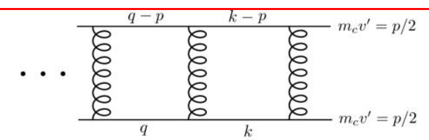
- **The B_c meson as a unique non-relativistic bound state of two flavor heavy quarks, though experimentally hard to measure, can be a good playground for theorists!**
- **$B_c \to \eta_c$ formfactors can be calculated in NRQCD factorization:**

C.F.Qiao, P.Sun, R.L.Zhu, 2012;

- **At large recoil, the NLO corrections contains large double logarithms;**

P.Boeer,G.Bell,T.Feldmann et al, 2023;

The diagrammatic resummation of double logarithms are achieved in a certain limit, and is confirmed by explicit 2-loop results.



Double-logarithmic series governed by implicit integral equations:

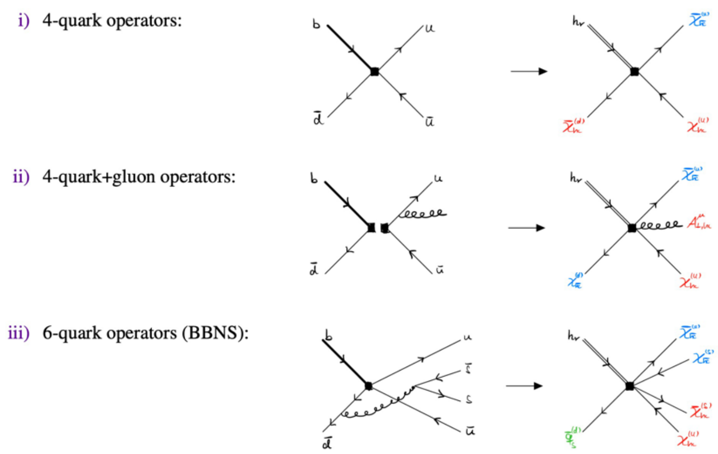
$$f(q_+,q_-) = 1 + \frac{\alpha_s C_F}{2\pi} \int_{q_-}^{p_-} \frac{dk_-}{k_-} \int_{m_c^2/k_-}^{q_+} \frac{dk_+}{k_+} \left(f(k_+,k_-) + \frac{1}{2} f_m(k_+,k_-)\right)$$

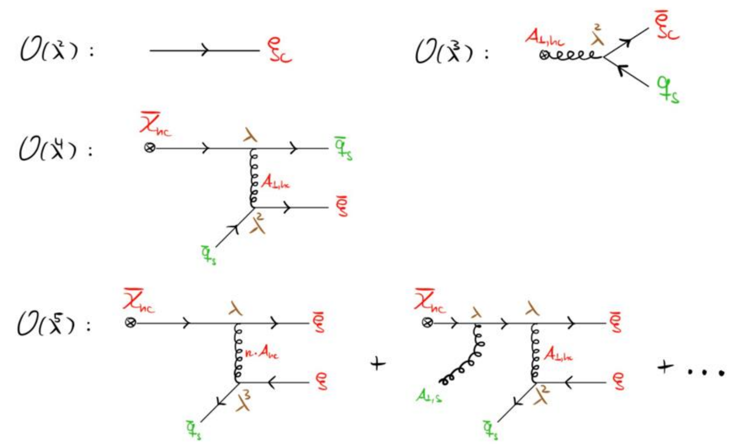
$$f_m(q_+,q_-) = 1 + \frac{\alpha_s C_F}{2\pi} \int_{q_-}^{p_-} \frac{dk_-}{k_-} \int_{m_c^2/k_-}^{q_+} \frac{dk_+}{k_+} f_m(k_+,k_-)$$

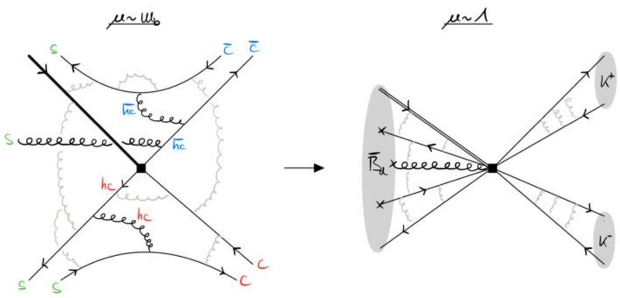
Weak annihilation on the way

Power suppressed soft-collinear interaction is essential for SCET$_{I}$ matching onto SCET$_{II}$

P.Boeer, M.Neubert et al, on going







More soft functions/distributions
will be involved! Much more complicated
than original 6-quark operator involved
factorization formula will appear, more delicated
subtractions are expected!

Lightcone distributions

- Non-perturbative inputs are essential for phenomenological applications;
- The leading twist LCDAs of light mesons are expanded in Gegenbauer polynomials:

$$\phi_\pi(x) = 6\bar{x}x \sum a_n C_n^{\frac{3}{2}}(2x-1),$$

- Lattice calculation of Gegenbauer moments;

RQCD: $a_2(2\ \mathrm{GeV}) = 0.116^{+0.019}_{-0.020}$

LPC: $a_2(2\ \mathrm{GeV}) = 0.258 \pm 0.087,\ a_4(2\ \mathrm{GeV}) = 0.122 \pm 0.056,$

$a_6(2\ \mathrm{GeV}) = 0.068 \pm 0.038$

- The inverse moment: $\int dx \frac{\phi_\pi(x,\mu)}{x} = 3(1 + a_2 + a_4 + a_6 + \cdots);$

Data shows very slow convergence! The inverse moment is very sensitive to the profile of light-meson LCDAs.

Lightcone Distributions-Cont.

Convolutions with hard-kernel at NLO & NNLO, show slow convergence or maybe divergence over Gegenbauer expansion. **L.B.Chen, W.Chen, F.Feng, J.Yu, 2024;**

(m,n)	c_1	c_2	d_1	d_2	d_3
(0,0)	20.25	59.25	91.1250	478.436	696.210
(0,2)	32.75	112.473	170.118	1094.39	2025.84
(0,4)	38.45	147.638	211.902	1541.23	3206.98
(0,6)	42.2571	174.359	241.822	1901.22	4265.06
(2,2)	45.25	192.871	266.472	2178.25	4953.36
(2,4)	50.95	240.181	316.173	2875.57	7237.52
(2,6)	54.7571	274.974	351.380	3415.43	9172.70
(4,4)	56.65	292.970	369.484	3704.29	10222.5
(4,6)	60.4571	331.411	407.102	4337.65	12698.8
(6,6)	64.2643	372.282	446.331	5037.27	15588.4

Table: The numerical values for $\mathcal{T}_{mn}^{(1)} = c_1 L_\mu + c_2$ and $\mathcal{T}_{mn}^{(2)} = d_1 L_\mu^2 + d_2 L_\mu + d_3$, with $0 \leq m, n \leq 6$.

$\pi\gamma^* \to \pi$ formactors at NNLO

Direct Lattice computation of distributions are needed through the Quasi DAs (LAMET); X.D.Ji et al since 2012; see Ji Xu's talk

Un-Scientific Summary

- **No pain no gain!**
- **Keep eyes open!**
- **Salute to all hard-working colleagues!**

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Thanks