

Ratios of $c\bar{b}$ meson decay constants in $N^3\text{LO}$ QCD

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Based on

Wei Tao, Zhen-Jun Xiao, JHEP 12 (2023) 152, JHEP 06 (2024) 012

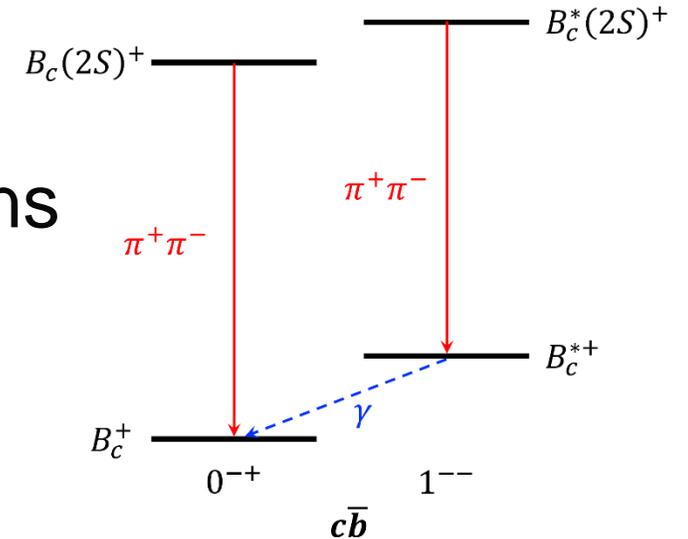
2024年10月25-29日 第21届全国重味物理与CP破坏研讨会@衡阳

Outline

- 1 Introduction
- 2 Calculation Procedure
- 3 Results
- 4 Summary

- To help discover $B_c^*(1S)$ and P-wave $c\bar{b}$ mesons
 - 1998, $B_c(1S)$
 - 2014, $B_c(2S)$ or $B_c^*(2S)$?
 - 2019, $B_c(2S)$ & $B_c^*(2S)$
- To understand similarities and differences in $c\bar{b}$ meson leptonic decays via various currents
- To study new physics by calculating $c\bar{b}$ decay constants involving (axial-)tensor currents
- To test perturbative expansion convergence in Non-Relativistic Quantum Chromodynamics (NRQCD) effective theory

[CDF, PRL(1998)]
 [ATLAS, PRL(2014)]
 [CMS, PRL(2019)]
 [LHCb, PRL(2019)]



Review NRQCD multi-loop calculation for $c\bar{b}$ decay constants

- One-loop for B_c [[E.Braaten,S.Fleming,PRD\(1995\)](#)]
- One-loop for B_c^* [[D.S.Hwang,S.Kim,PRD\(1999\)](#)]
- One-loop with relativistic corrections for B_c & B_c^*
[[J.Lee,W.Sang,S.Kim,JHEP\(2011\)](#)]
- Approximate two-loop for B_c [[A.I.Onishchenko,O.L.Veretin,EPJC\(2007\)](#)]
- Full analytical two-loop for B_c [[L.B.Chen,C.F.Qiao,PLB\(2015\)](#)]
- Three-loop for B_c [[F.Feng,et al.,2208.04302](#)]
- Three-loop for B_c^* [[W.Sang,H.F.Zhang,M.Z.Zhou,PLB\(2023\)](#)]
- Three-loop with relativistic corrections for B_c & B_c^*
[[W.Tao,Z.J.Xiao,R.Zhu, JHEP\(2023\)](#)]

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- 2 **Calculation Procedure**
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Definition

➤ QCD currents

$$j_s = \bar{\psi}_b \psi_c,$$

$$j_p = \bar{\psi}_b \gamma_5 \psi_c,$$

$$j_v^\mu = \bar{\psi}_b \gamma^\mu \psi_c,$$

$$j_a^\mu = \bar{\psi}_b \gamma^\mu \gamma_5 \psi_c,$$

$$j_t^{\mu\nu} = \bar{\psi}_b \sigma^{\mu\nu} \psi_c,$$

$$j_{t5}^{\mu\nu} = \bar{\psi}_b \sigma^{\mu\nu} \gamma_5 \psi_c,$$

$$\{B_c(^1S_0), B_c^*(^3S_1), B_{c0}^*(^3P_0), B_{c1}(^3P_1)\}$$

➤ QCD definition for decay constants

$$\langle 0 | j_p | B_c(q) \rangle \doteq f_{B_c}^p m_{B_c},$$

$$\langle 0 | j_a^\mu | B_c(q) \rangle \doteq f_{B_c}^{a,0} q^\mu,$$

$$\langle 0 | j_v^\mu | B_c^*(q, \varepsilon) \rangle \doteq f_{B_c^*}^{v,i} m_{B_c^*} \varepsilon^\mu,$$

$$\langle 0 | j_t^{\mu\nu} | B_c^*(q, \varepsilon) \rangle \doteq f_{B_c^*}^{t,i0} (q^\mu \varepsilon^\nu - q^\nu \varepsilon^\mu),$$

$$\langle 0 | j_{t5}^{\mu\nu} | B_c^*(q, \varepsilon) \rangle \doteq f_{B_c^*}^{t5,ij} \epsilon^{\mu\nu\alpha\beta} q_\alpha \varepsilon_\beta,$$

$$\langle 0 | j_s | B_{c0}^*(q) \rangle \doteq f_{B_{c0}^*}^s m_{B_{c0}^*},$$

$$\langle 0 | j_v^\mu | B_{c0}^*(q) \rangle \doteq f_{B_{c0}^*}^{v,0} q^\mu,$$

$$\langle 0 | j_a^\mu | B_{c1}(q, \varepsilon) \rangle \doteq f_{B_{c1}}^{a,i} m_{B_{c1}} \varepsilon^\mu,$$

$$\langle 0 | j_t^{\mu\nu} | B_{c1}(q, \varepsilon) \rangle \doteq f_{B_{c1}}^{t,ij} \epsilon^{\mu\nu\alpha\beta} q_\alpha \varepsilon_\beta,$$

$$\langle 0 | j_{t5}^{\mu\nu} | B_{c1}(q, \varepsilon) \rangle \doteq f_{B_{c1}}^{t5,i0} (q^\mu \varepsilon^\nu - q^\nu \varepsilon^\mu),$$

NRQCD factorization

➤ NRQCD expansion

$$\tilde{j}_p = -\varphi_b^\dagger \chi_c,$$

$$\tilde{j}_a^0 = -\tilde{j}_p,$$

$$\tilde{j}_v^i = \varphi_b^\dagger \sigma^i \chi_c,$$

$$\tilde{j}_t^{i0} = i \tilde{j}_v^i,$$

$$\tilde{j}_{t5}^{ij} = -\epsilon^{ijk} \tilde{j}_v^k,$$

$$\tilde{j}_s = -\varphi_b^\dagger \left(\frac{1}{2m_c} + \frac{1}{2m_b} \right) \vec{k} \cdot \vec{\sigma} \chi_c,$$

$$\tilde{j}_v^0 = -\frac{m_b - m_c}{m_b + m_c} \tilde{j}_s,$$

$$\tilde{j}_a^i = \varphi_b^\dagger \left[\left(\frac{1}{2m_c} - \frac{1}{2m_b} \right) \vec{k} + i \left(\frac{1}{2m_c} + \frac{1}{2m_b} \right) \vec{k} \times \vec{\sigma} \right]^i \chi_c,$$

$$\tilde{j}_{t5}^{i0} = i \varphi_b^\dagger \left[\left(\frac{1}{2m_c} + \frac{1}{2m_b} \right) \vec{k} + i \left(\frac{1}{2m_c} - \frac{1}{2m_b} \right) \vec{k} \times \vec{\sigma} \right]^i \chi_c,$$

$$\tilde{j}_t^{ij} = i \epsilon^{ijk} \tilde{j}_{t5}^{k0},$$

➤ NRQCD factorization of decay constants

[G.T.Bodwin,E.Braaten,G.P.Lepage,PRD(1995)]

$$f_{B_c}^p = \sqrt{\frac{2}{m_{B_c}}} \underbrace{C_p(m_b, m_c, \mu_f, \mu)}_{\text{mathing coefficients}} \underbrace{\langle 0 | \chi_b^\dagger \psi_c | B_c(\mathbf{P}) \rangle}_{\text{NRQCD LDMEs}} (\mu_f) + \mathcal{O}(v^2)$$

$$f_{B_c^*}^{v,i} = \sqrt{\frac{2}{m_{B_c^*}}} \underbrace{C_{v,i}(m_b, m_c, \mu_f, \mu)}_{\text{mathing coefficients}} \underbrace{\langle 0 | \chi_b^\dagger \sigma \cdot \epsilon \psi_c | B_c^*(\mathbf{P}) \rangle}_{\text{NRQCD LDMEs}} (\mu_f) + \mathcal{O}(v^2)$$

Matching formula

➤ QCD-NRQCD matching formula

$$\sqrt{Z_{2,b}^{\text{OS}} Z_{2,c}^{\text{OS}} Z_J^{\text{OS}}} \left(Z_J^{\overline{\text{MS}}} \right) \Gamma_J = C_J (\bar{C}_J) \sqrt{\tilde{Z}_{2,b}^{\text{OS}} \tilde{Z}_{2,c}^{\text{OS}} \tilde{Z}_J^{-1}} \tilde{\Gamma}_J$$

- $Z_J(\tilde{Z}_J)$: QCD(NRQCD) current renormalization constant
- $Z_2(\tilde{Z}_2)$: QCD(NRQCD) heavy quark field renormalization constant
- $\Gamma_J(\tilde{\Gamma}_J)$: QCD(NRQCD) current vertex function

➤ Relations between matching coefficients

$$\frac{f_{B_c}^p}{f_{B_c}^{a,0}} = \frac{C_p}{C_{a,0}} = \frac{\tilde{Z}_p}{\tilde{Z}_{a,0}} = 1$$

$$\frac{f_{B_c^*}^{v,i}}{f_{B_c^*}^{t,i0}} = \frac{C_{v,i}}{C_{t,i0}} = \frac{\tilde{Z}_{v,i}}{\tilde{Z}_{t,i0}} = 1 \quad \tilde{Z}_s = \tilde{Z}_{v,0}$$

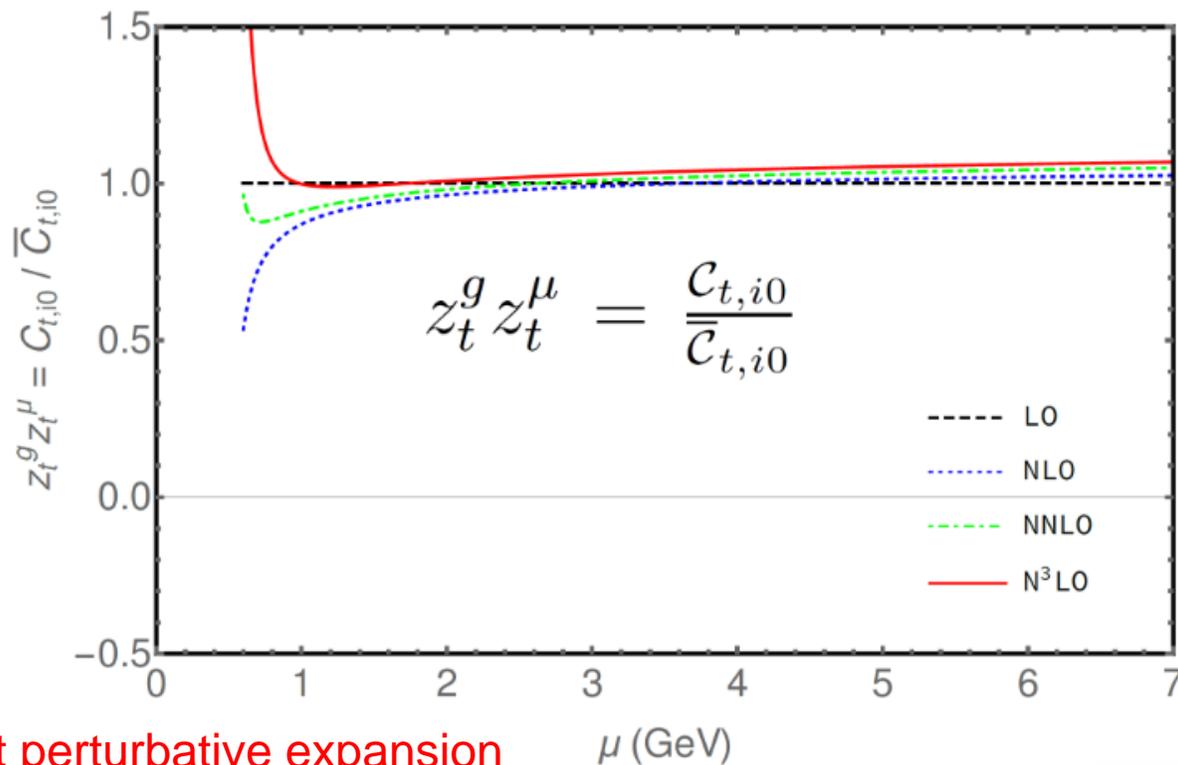
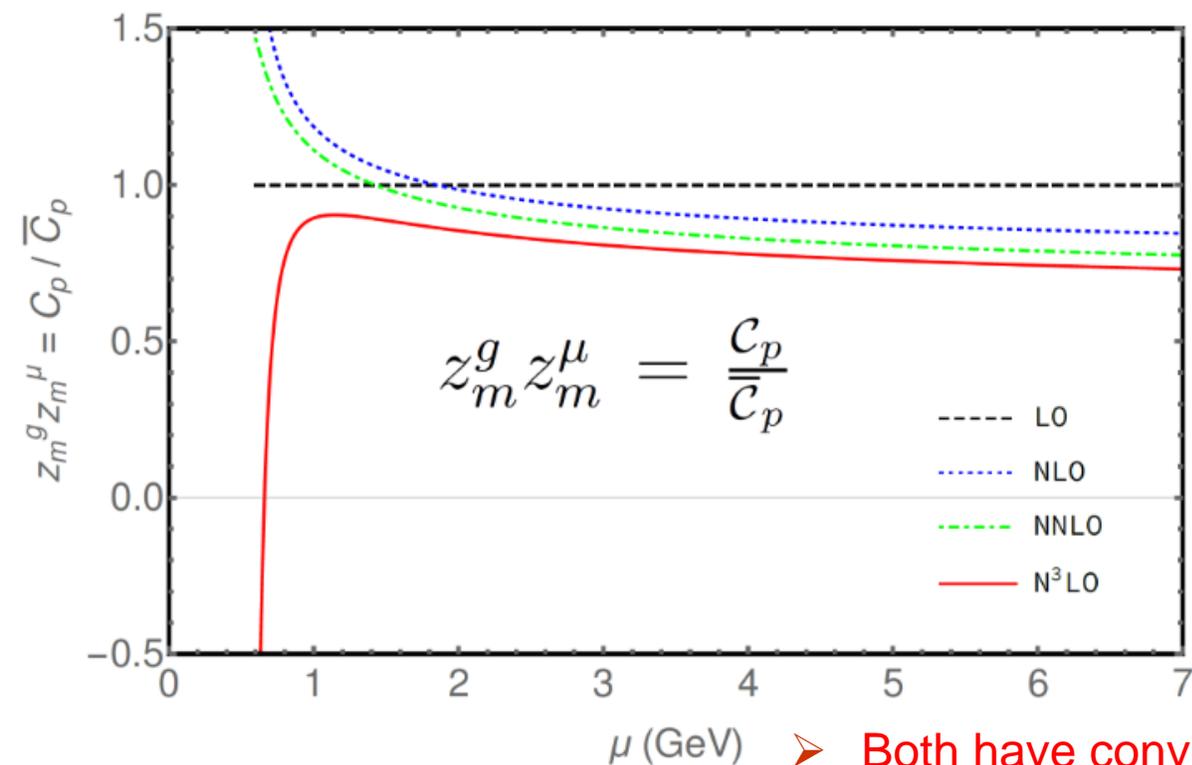
$$\frac{f_{B_{c0}^*}^s}{f_{B_{c0}^*}^{v,0}} = \frac{C_s}{C_{v,0}} = \frac{m_b Z_{m,b}^{\text{OS}} + m_c Z_{m,c}^{\text{OS}}}{m_b Z_{m,b}^{\text{OS}} - m_c Z_{m,c}^{\text{OS}}} \frac{m_b - m_c}{m_b + m_c}$$

Solve for Z_J^{OS}

- Ratio of QCD current OS to $\overline{\text{MS}}$ renormalization constants

$$\frac{C_J}{\overline{C}_J} = \frac{Z_J^{\text{OS}}}{Z_J^{\overline{\text{MS}}}} = z_J^g z_J^\mu + \mathcal{O}(\epsilon),$$

- Verify the correctness of $z_t^g z_t^\mu = \frac{C_{v,i}}{\overline{C}_{t,i0}}$

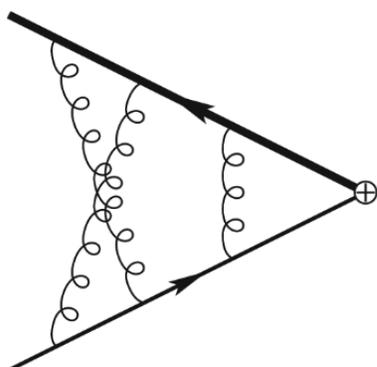


$\mu_f \in [0.4, 1.2, 7] \text{ GeV}$

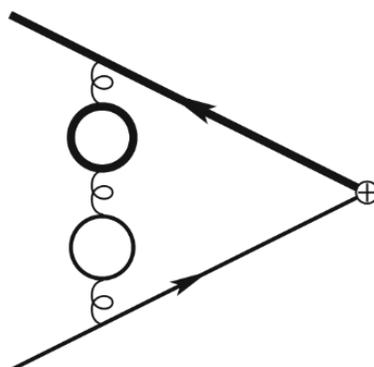
- Both have convergent perturbative expansion
- Both have weak dependence on renormalization scale
- Both are independent of factorization scale

Feynman diagrammatic renormalization at $\mathcal{O}(\alpha_s^3)$

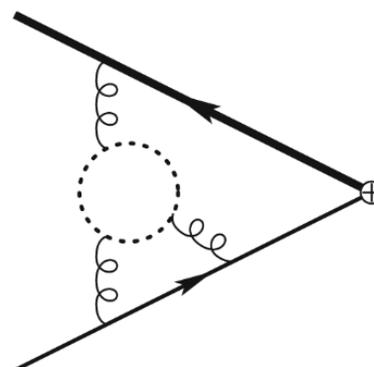
- Tree diagram inserted with one $\mathcal{O}(\alpha_s^3)$ counter-term vertex
- One-loop diagram inserted with one $\mathcal{O}(\alpha_s^2)$ counter-term vertex
- One-loop diagram inserted with two $\mathcal{O}(\alpha_s)$ counter-term vertexes
- Two-loop diagrams inserted with one $\mathcal{O}(\alpha_s)$ counter-term vertex
- Three-loop diagrams



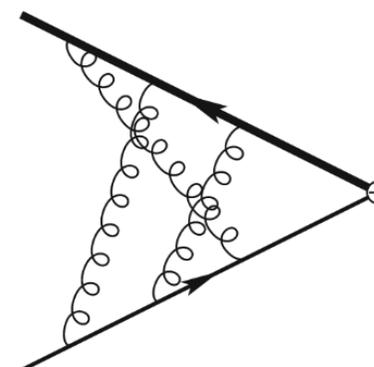
(a) (CA - 2CF) CF CF



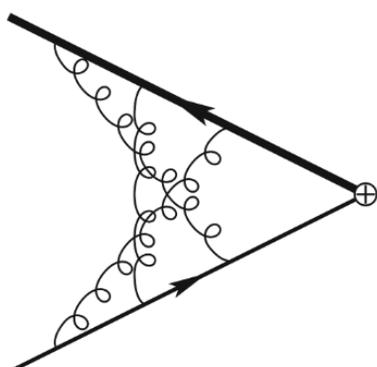
(b) TF nb TF nc CF



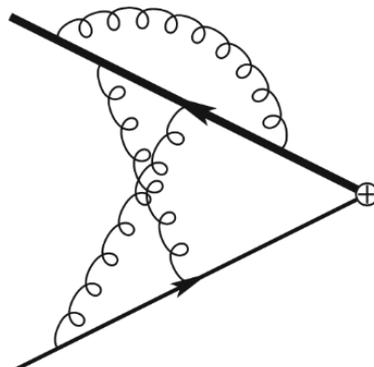
(c) CA CA CF



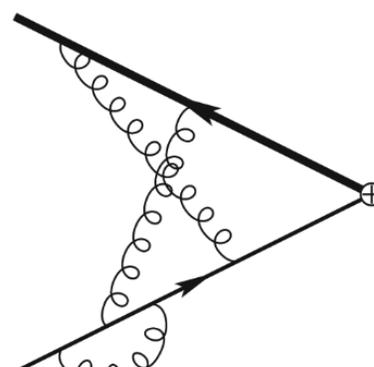
(d) (CA - 2CF) (CA - 2CF) CF



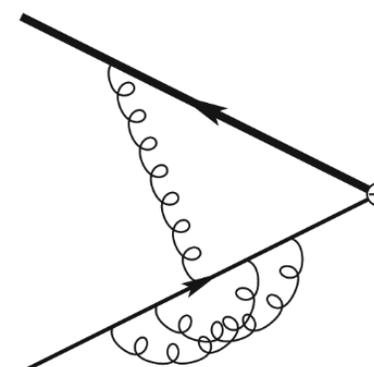
(e) (CA - 2CF) (CA - CF) CF



(f) (CA - 2CF) (CA - CF) CF



(g) (CA - 2CF) (CA - 2CF) CF



(h) (CA - 2CF) (CA - CF) CF

Multi-loop calculation process

- FeynCalc: [V.Shtabovenko, R.Mertig, F.Orellana, CPC(2025)]
 - Generate Feynman diagrams & amplitudes;
 - Express amplitudes in terms of Feynman integrals;
 - Minimize the total number of families of Feynman integrals.
- FIRE / Kira / FiniteFlow: [A.V.Smirnov, F.S.Chukharev, CPC(2020)]
[J.Klappert, et al., CPC(2021)]
[T.Peraro, JHEP(2019)]
 - Reduce Feynman integrals to master integrals.
- Kira + FIRE + Mathematica code:
 - Minimize the total number of families of master integrals.
- AMFlow + FiniteFlow / Kira: [X.Liu, Y.Q.Ma, CPC(2023)]
 - Calculate master integrals.
- Renormalization.

268 three-loop diagrams
→ $\mathcal{O}(100)$ families of Feynman integrals
→ 26 families of master integrals

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Expression of C_J

$$x \equiv \frac{m_c}{m_b},$$

$$L_\mu \equiv \ln \frac{\mu^2}{m_b m_c},$$

$$L_{\mu_f} \equiv \ln \frac{\mu_f^2}{m_b m_c}.$$

$$\begin{aligned} C_J(\mu_f, \mu, m_b, m_c) = & 1 + \frac{\alpha_s^{(n_l)}(\mu)}{\pi} C_J^{(1)}(x) \\ & + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi} \right)^2 \left[\frac{C_J^{(1)}(x)}{4} \beta_0^{(n_l)} L_\mu + \frac{\tilde{\gamma}_J^{(2)}(x)}{2} L_{\mu_f} + C_J^{(2)}(x) \right] \\ & + \left(\frac{\alpha_s^{(n_l)}(\mu)}{\pi} \right)^3 \left\{ \frac{C_J^{(1)}(x)}{16} \beta_0^{(n_l)^2} L_\mu^2 + \left[\frac{C_J^{(1)}(x)}{16} \beta_1^{(n_l)} + \frac{C_J^{(2)}(x)}{2} \beta_0^{(n_l)} \right] L_\mu \right. \\ & + \frac{\tilde{\gamma}_J^{(2)}(x)}{4} \beta_0^{(n_l)} L_\mu L_{\mu_f} + \left[\frac{\partial \tilde{\gamma}_J^{(3)}(L_{\mu_f}; x)}{4 \partial L_{\mu_f}} - \frac{\tilde{\gamma}_J^{(2)}(x)}{8} \beta_0^{(n_l)} \right] L_{\mu_f}^2 \\ & \left. + \frac{1}{2} \left[C_J^{(1)}(x) \tilde{\gamma}_J^{(2)}(x) + \tilde{\gamma}_J^{(3)}(L_{\mu_f} = 0; x) \right] L_{\mu_f} + C_J^{(3)}(x) \right\} + \mathcal{O}(\alpha_s^4), \end{aligned}$$

➤ One-loop

$$C_p^{(1)}(x) = C_{a,0}^{(1)}(x) = \frac{3}{4} C_F \left(\frac{x-1}{x+1} \ln x - 2 \right);$$

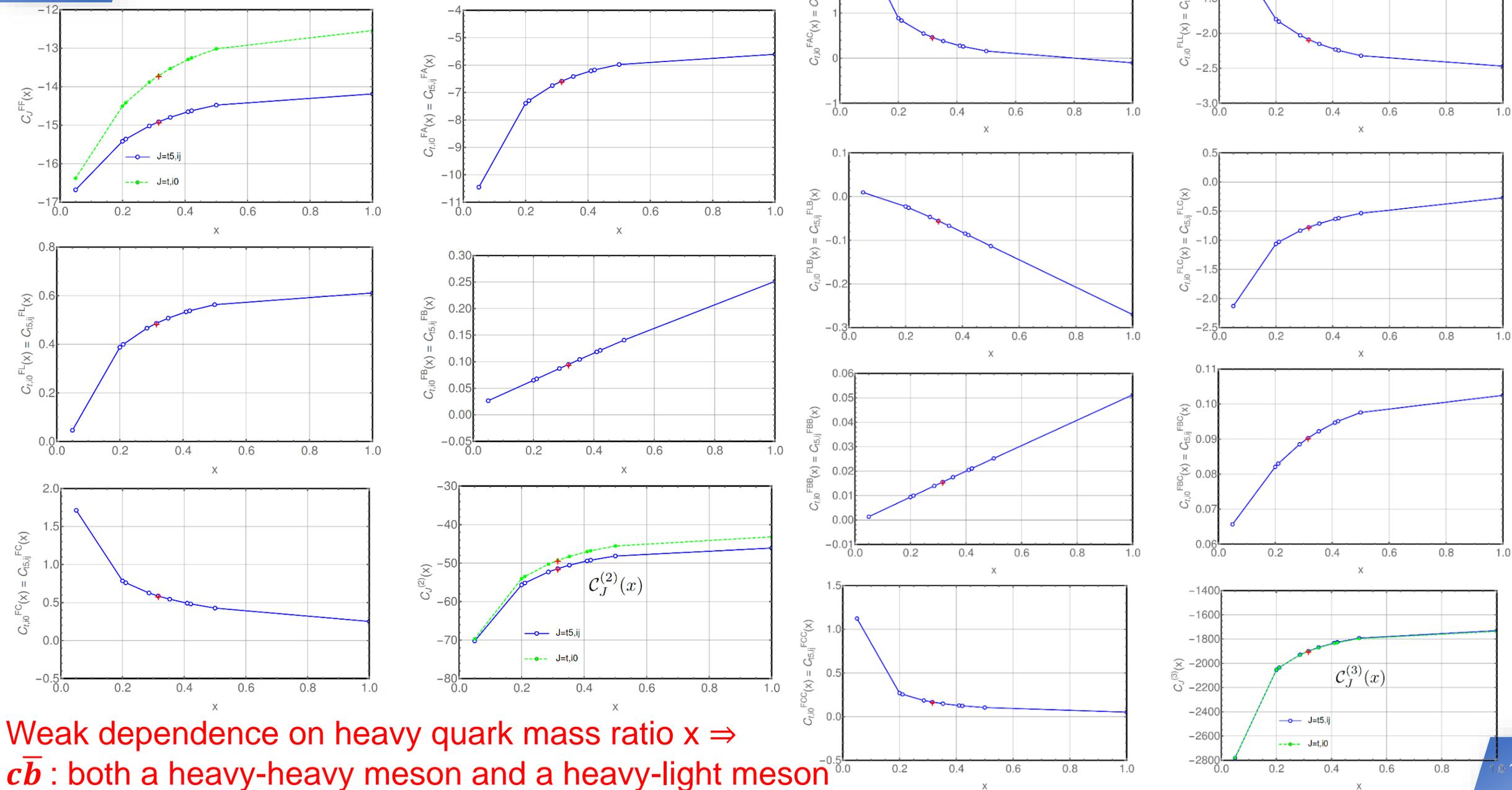
$$C_{v,i}^{(1)}(x) = C_{t,i0}^{(1)}(x) = C_{t5,ij}^{(1)}(x) = \frac{3}{4} C_F \left(\frac{x-1}{x+1} \ln x - \frac{8}{3} \right);$$

$$C_s^{(1)}(x) = \frac{3}{4} C_F \left(\frac{x-1}{x+1} \ln x - \frac{2}{3} \right), \quad C_{v,0}^{(1)}(x) = \frac{3}{4} C_F \left(\frac{x+1}{x-1} \ln x - \frac{2}{3} \right);$$

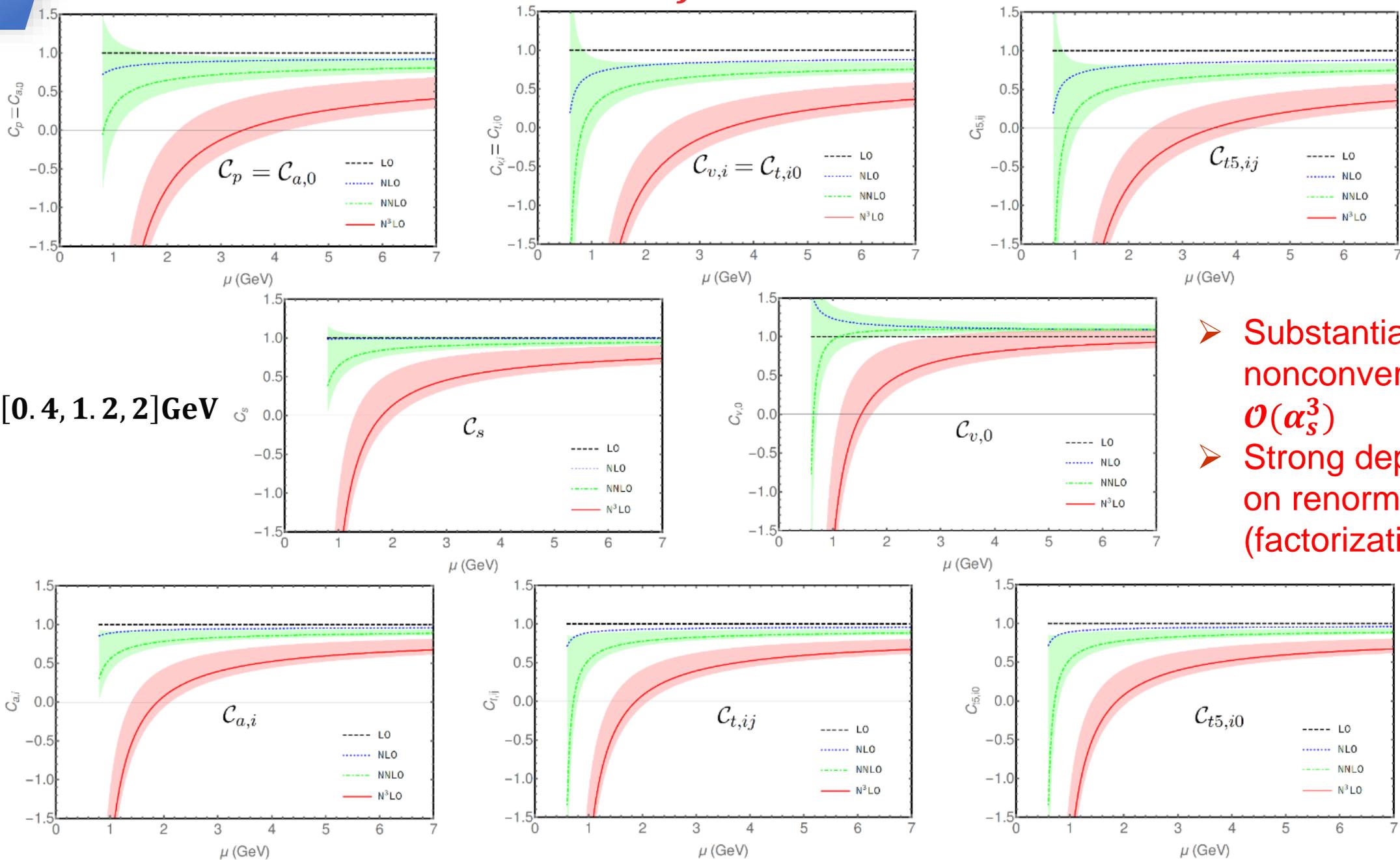
$$C_{a,i}^{(1)}(x) = C_{t,ij}^{(1)}(x) = C_{t5,i0}^{(1)}(x) = \frac{3}{4} C_F \left(\frac{x-1}{x+1} \ln x - \frac{4}{3} \right).$$

3.2

Mass dependence of C_J



Weak dependence on heavy quark mass ratio $x \Rightarrow$
 $c\bar{b}$: both a heavy-heavy meson and a heavy-light meson

Scale dependence of C_J 

- Substantial nonconvergence at $\mathcal{O}(\alpha_s^3)$
- Strong dependence on renormalization (factorization) scale

3.4

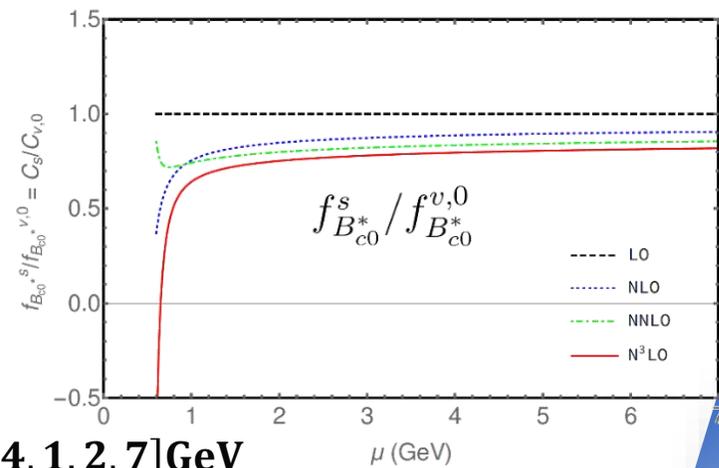
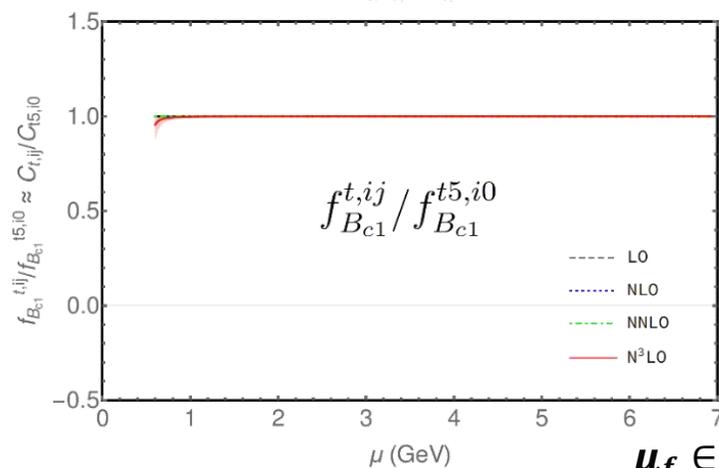
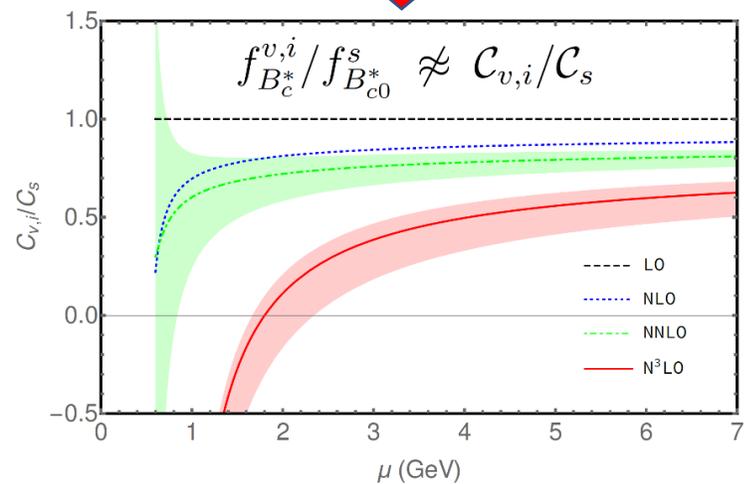
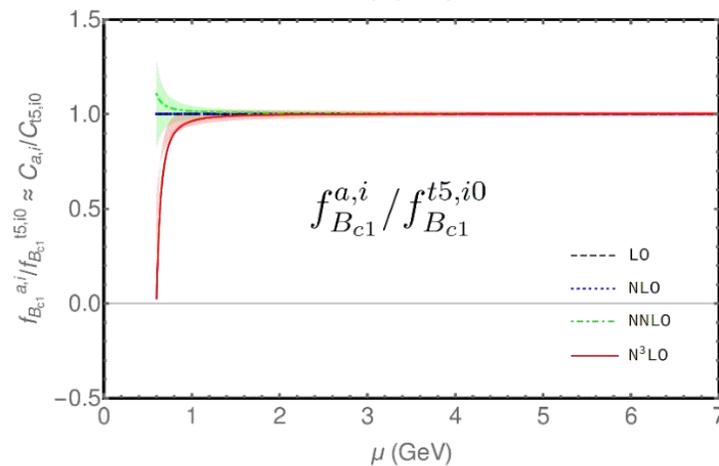
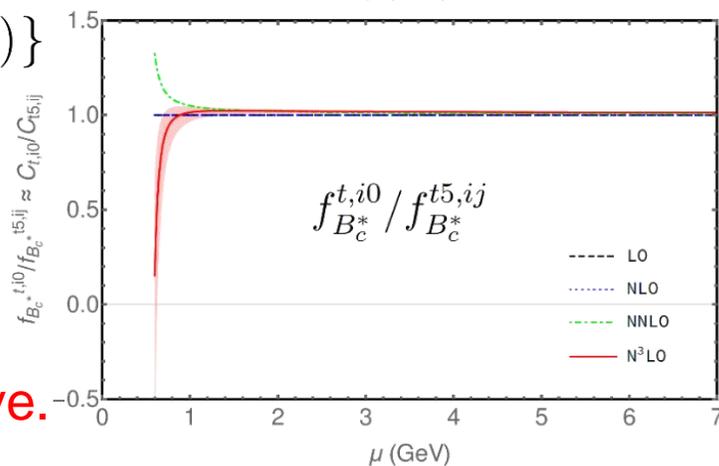
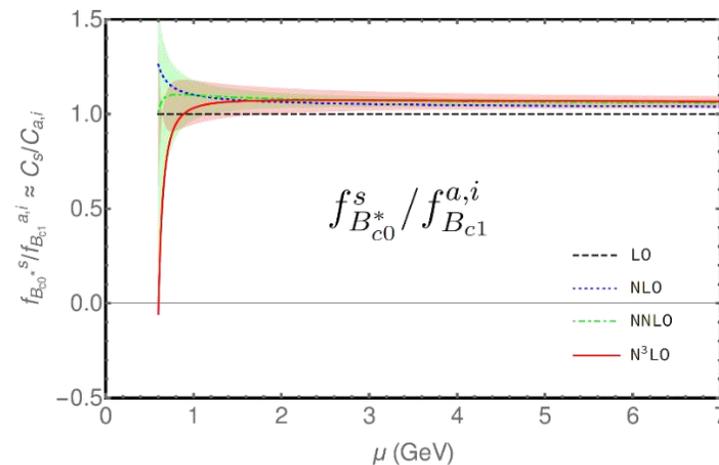
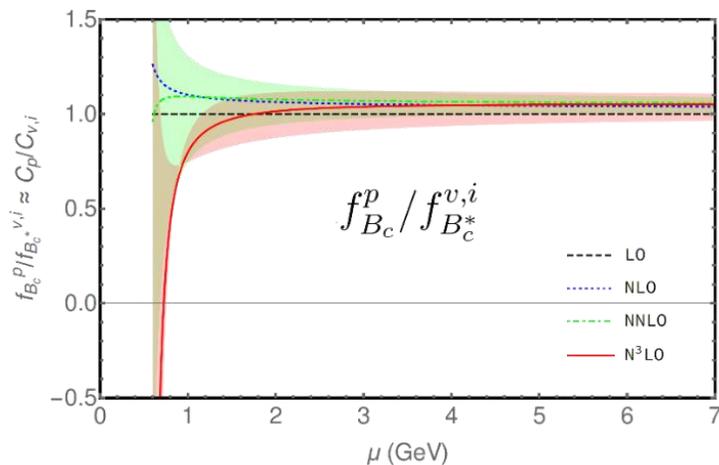
Ratios of decay constants

$$\frac{f_{X_1}^{J_1}}{f_{X_2}^{J_2}} \approx \frac{C_{J_1} \tilde{f}_{X_1}^{J_1}}{C_{J_2} \tilde{f}_{X_2}^{J_2}} \approx \frac{C_{J_1}}{C_{J_2}}$$

$$X \in \{B_c(1S_0), B_c^*(3S_1), B_{c0}^*(3P_0), B_{c1}(3P_1)\}$$

Convergent N³LO correction with weak scale-dependence.

Cannot apply heavy-quark spin symmetry between S-wave and P-wave.



$\mu_f \in [0.4, 1.2, 7] \text{ GeV}$

	LO	NLO	NNLO	N ³ LO
$\frac{C_{v,i}}{C_s}$	1	$0.84400^{+0+0.03953+0+0}_{-0-0.06519-0-0}$	$0.75905^{+0.05944-0.07471-0.00114-0.04304}_{-0.09541+0.05127+0.00091+0.02692}$	$0.38591^{+0.08251-0.65866-0.00062-0.09918}_{-0.20176+0.23967+0.00030+0.05279}$
$\frac{f_{B_c}^p}{f_{B_c^*}^v,i}$	1	$1.05200^{+0+0.02173-0-0}_{-0-0.01318+0+0}$	$1.06946^{+0.05704+0.01393-0.00203-0.00473}_{-0.09157-0.01228+0.00153+0.00593} \pm 0.01746$	$1.03816^{+0.08313-0.06443-0.00519+0.00294}_{-0.12195+0.01312+0.00417-0.00050} \pm 0.04966$
$\frac{f_{B_{c0}}^s}{f_{B_{c1}}^{a,i}}$	1	$1.05200^{+0+0.02173-0-0}_{-0-0.01318+0+0}$	$1.07143^{+0.02139+0.01592-0.00037+0.00122}_{-0.03434-0.01315+0.00021-0.00007} \pm 0.01943$	$1.07225^{+0.04668-0.00573-0.00135+0.00411}_{-0.05482-0.00602+0.00094-0.00131} \pm 0.02126$
$\frac{f_{B_c^*}^{t,i0}}{f_{B_c^*}^{t5,ij}}$	1	1	$1.01298^{+0+0.01312+0.00074-0.00276}_{-0-0.00575-0.00068+0.00186} \pm 0.01298$	$1.01822^{+0.00417+0.00481+0.00124-0.00267}_{-0.00670-0.00559-0.00111+0.00143} \pm 0.01822$
$\frac{f_{B_{c1}}^{a,i}}{f_{B_{c1}}^{t5,i0}}$	1	1	$1.00430^{+0.00713+0.00435-0.00010-0.00207}_{-0.01145-0.00190+0.00005+0.00178} \pm 0.00430$	$1.00045^{+0.01638-0.00875-0.00041-0.00229}_{-0.01767+0.00141+0.00030+0.00208} \pm 0.00816$
$\frac{f_{B_{c1}}^{t,ij}}{f_{B_{c1}}^{t5,i0}}$	1	1	1	$0.99964^{+0.00042-0.00067-0.00001-0.00009}_{-0.00067+0.00021+0.00001+0.00011} \pm 0.00036$
$\frac{f_{B_{c0}}^s}{f_{B_{c0}}^{v,0}}$	1	$0.87385^{+0+0.03197+0.00455-0.01197}_{-0-0.05271-0.00485+0.01906}$	$0.82199^{+0+0.03401+0.00886-0.01078}_{-0-0.04341-0.00974+0.02050}$	$0.78047^{+0+0.03860+0.01240-0.00839}_{-0-0.05744-0.01390+0.01920}$

$\mu_f = 1.2_{-0.8}^{+5.8}$ GeV, $\mu = 3_{-1.5}^{+4.0}$ GeV, $m_b = 4.75 \pm 0.5$ GeV, $m_c = 1.5 \pm 0.5$ GeV, **► Uncertainty ordering:**

$$\tilde{f}_{X_1}^{J_1} / \tilde{f}_{X_2}^{J_2} = 1 \pm \left(\left| (C_{J_1}/C_{J_2})_{\alpha_s^2} \right| + \left| (C_{J_1}/C_{J_2})_{\alpha_s^3} \right| \right) \quad \Delta_{\mu_f} \gtrsim \Delta_{\text{LDME}} \gtrsim \Delta_{\mu} \gg \Delta_{m_c} \gtrsim \Delta_{m_b}$$

► Hierarchical relationship among $c\bar{b}$ meson decay constants

$$f_{B_c}^p = f_{B_c}^{a,0} > f_{B_c^*}^{v,i} = f_{B_c^*}^{t,i0} > f_{B_c^*}^{t5,ij} > f_{B_{c0}}^{v,0} > f_{B_{c0}}^s > f_{B_{c1}}^{a,i} \gtrsim f_{B_{c1}}^{t5,i0} \gtrsim f_{B_{c1}}^{t,ij}$$

- Using $C_{v,i} \equiv C_{t,i0}$, obtain QCD tensor current on-shell renormalization constant, and then calculate matching coefficients of tensor currents.
- Invoking heavy-quark spin symmetry, and approximating decay constant ratio as matching coefficient ratio, obtain convergent **N³LO** corrections to ratios of $c\bar{b}$ meson decay constants.

Thank you!