A progress in the inverse matrix method in QCD sum rules

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In collaboration with Yi-Peng Xing, Run-Hui Li

Inspired by
H. n. Li, Phys. Rev. D 104, no.11, 114017 (2021)
H. n. Li and H. Umeeda, Phys. Rev. D 102, 114014 (2020)

Outline

- Introduction
- Inverse matrix method
  - -- m_ρ
  - -- f_ρ
- Discussions
- Summary and outlook

Introduction

QCD sum rules

M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 147, 385-447 (1979)
M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B 147, 448-518 (1979)

- Determination of quark mass
- Calculation of form factors for mesons and baryons
- Calculation of hadron spectra and decay constants
- Calculation of parameters in some effective theories
- Study of properties of exotic states
- …

"You can get anything you want from QCD Sum Rules."

D. B. Leinweber, Annals Phys. 254, 328-396 (1997)

Of course not!

Applied to the ρ meson – Traditional method

$$\Pi_{\mu\nu}(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T\{J_\mu(x)J_\nu(0)\}|0\rangle = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)$$

$$J_\mu = (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)/\sqrt{2}$$

$$\int_0^\infty ds\frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_0^\infty ds\frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

$$\rho^h(s) = f_V^2\delta(s-m_V^2) + \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s)\theta(s-s_0)$$

$$\frac{f_V^2}{m_V^2-q^2} = \frac{1}{\pi}\int_0^{s_0} ds\frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

$$f_V^2 e^{-m_V^2/M^2} = \frac{1}{\pi}\int_0^{s_0} ds\,\mathrm{Im}\Pi^{\mathrm{pert}}(s)e^{-s/M^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{M^2} + 2\frac{\langle m_q\bar{q}q\rangle}{M^2} - \frac{112\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{M^4}$$

H. n. Li and H. Umeeda, Phys. Rev. D 102, 114014 (2020)

Applied to the ρ meson – Traditional method

$$f_V^2 e^{-m_V^2/M^2} = \frac{1}{\pi}\int_0^{s_0} ds \mathrm{Im}\Pi^{\mathrm{pert}}(s) e^{-s/M^2} + \frac{1}{12\pi}\frac{\langle \alpha_s G^2\rangle}{M^2} + 2\frac{\langle m_q \bar{q}q\rangle}{M^2} - \frac{112\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{M^4}$$

$$\Rightarrow \boxed{\frac{\partial f_V^2}{\partial s_0} > 0} \qquad \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s) = \frac{1}{4\pi^2}\left(1+\frac{\alpha_s}{\pi}\right) > 0$$

There is no perfect parameter selection!

$$\rho^h(s) = f_V^2\delta(s-m_V^2) + \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s)\theta(s-s_0)$$

The model is too simple!

Inverse matrix method

-- m_ρ

Inverse problem

- Inverse matrix method can be viewed as a method to solve inverse problems
- Examples:

  --- CT imaging

  --- Electrodynamics: $\int \frac{\rho(\vec{x}')dV'}{4\pi\varepsilon_0 r} = \varphi(\vec{x})$ --- integral equation

$$\int_0^\infty ds \frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_0^\infty ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q \bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

Hadron level

--- unknown spectral density

QCD level --- input

- Inverse problem has become a very useful and mature method in the field of applied mathematics --- See 熊傲昇's talk
- Well-posed (and ill-posed):

  --- Existence

  --- Uniqueness

  --- Stability

Traditional method revisited and its improvements

$$\int_0^\infty ds \frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_0^\infty ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle \alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q \bar{q} q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa \alpha_s \langle \bar{q} q\rangle^2}{(q^2)^3}$$

Traditional method:

- OPE at $q^2 = -\infty$

$$\hat{B}_M \equiv \lim_{\substack{Q^2, n \to \infty \\ Q^2/n = M^2}} \frac{1}{(n-1)!}(Q^2)^n \left(-\frac{d}{dQ^2}\right)^n,$$

- $\rho^h(s) \to \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s)$ when $s \to \infty$

Improvements:

…

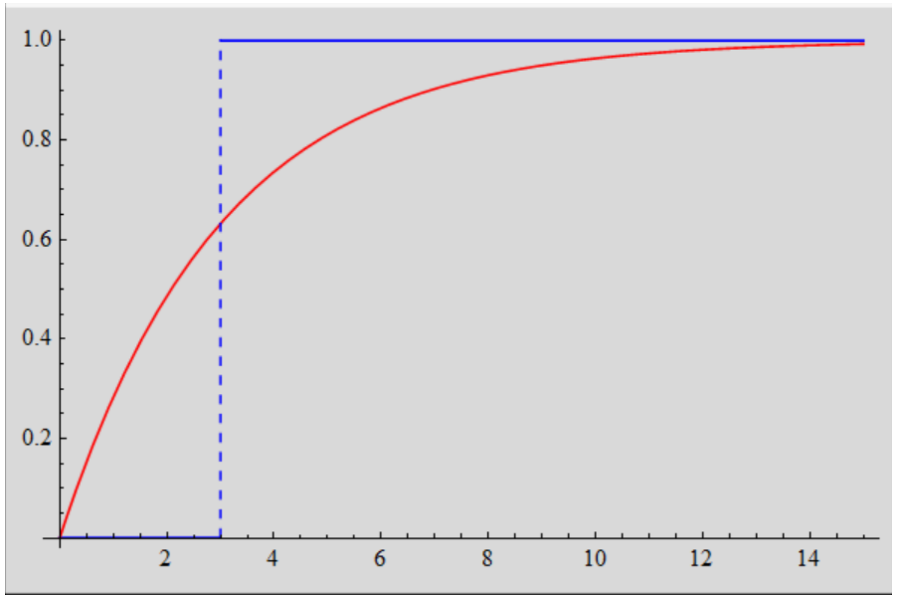
- Fit in a selected (q^2 or M^2) interval

Li-21

$$\int_0^\infty ds \frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_0^\infty ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle \alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q \bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

- Expand ρ^h using Laguerre polynomials
- Expand the two sides at $q^2 = -\infty$ --- OPE at $q^2 = -\infty$

integral equation --> matrix equation ==> ρ^h



"Soft wall" & "hard wall"

$$\rho(s) = \Delta\rho(s,\Lambda) + \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s)(1-\exp(-s/\Lambda))$$

Two parameters:

- N
- Λ

Li-21

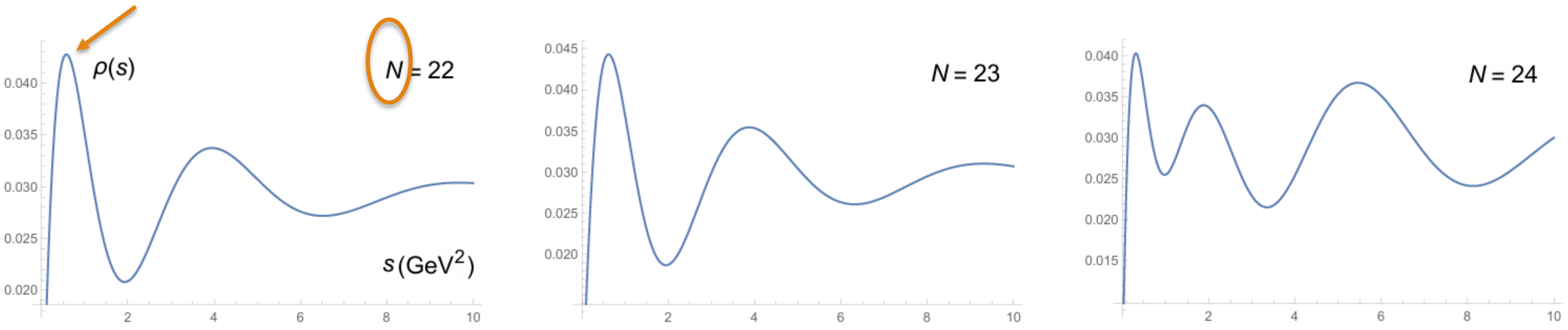
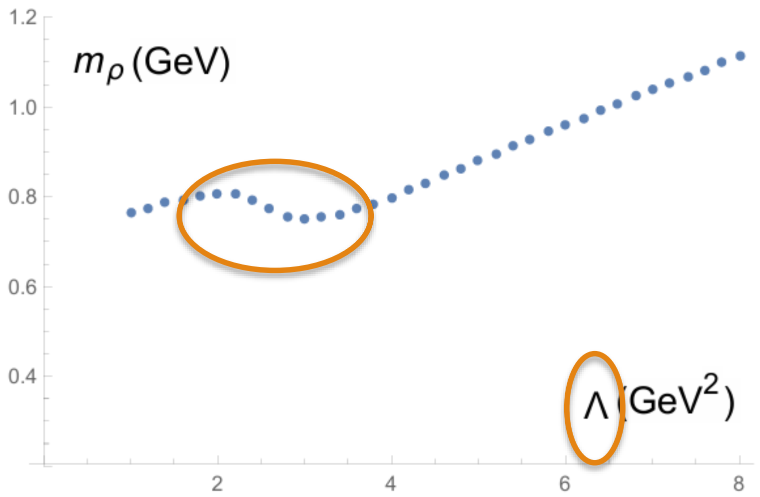


FIG. 4: Solutions to $\rho(s)$ for $\Lambda = 2.5$ GeV^2 with the expansions up to $N = 22$, 23 and 24 generalized Laguerre polynomials $L_n^{(1)}(y)$.



$$m_\rho = (0.78 \pm 0.07) \text{ GeV}.$$

$$f_\rho^2 \approx \int_0^\infty ds \Delta\rho(s, \Lambda),$$

$$f_\rho \approx 0.20 \text{ GeV},$$

This work -- m_ρ

$$\int_0^{\Lambda} ds \frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_0^{\Lambda} ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle \alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q \bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

$\rho^h(s) \to \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s)$ when $s \to \infty$

$$\int_0^1 dy \frac{\rho^h(y)}{x-y} = c\int_0^1 dy \frac{1}{x-y} - \frac{c_2}{x^2\Lambda^2} - \frac{c_3}{x^3\Lambda^3}.$$

$$\rho^h(y) = \sum_{n=0}^{N+1} a_n P_n(2y-1),$$

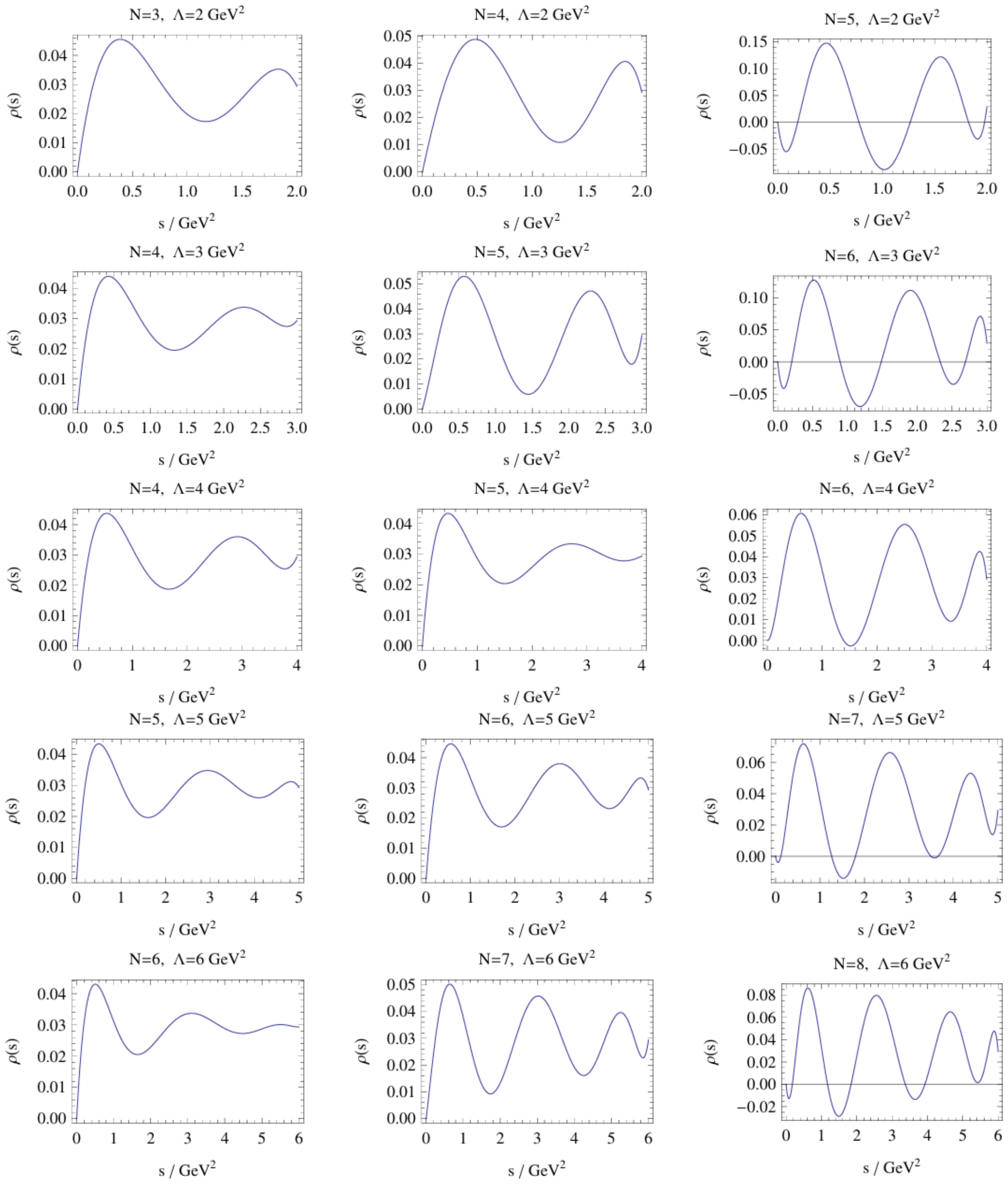
$$\frac{1}{x-y} = \sum_{m=0}^{N-1}\frac{y^m}{x^{m+1}}$$

$$\sum_{n=0}^{N+1}\left(\int_0^1 dy\, y^m P_n(2y-1)\right) a_n = c\frac{1}{m+1} + \cdots, \quad m = 0, \cdots, N-1,$$

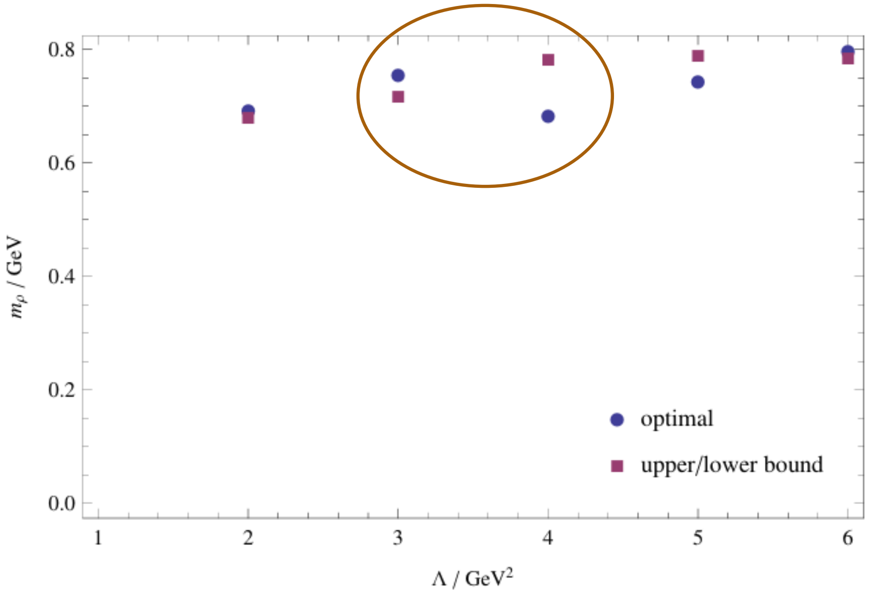
$$\sum_{n=0}^{N+1} a_n P_n(-1) = 0, \quad \sum_{n=0}^{N+1} a_n P_n(1) = c.$$

- Λ and N are parameters
- Two differences:
  - --- Hard wall
  - --- Legendre

Numerical results for m_ρ



Stability criterion $\rho^h > 0$



$$m_\rho = 0.68\sim0.78 \text{ GeV}$$

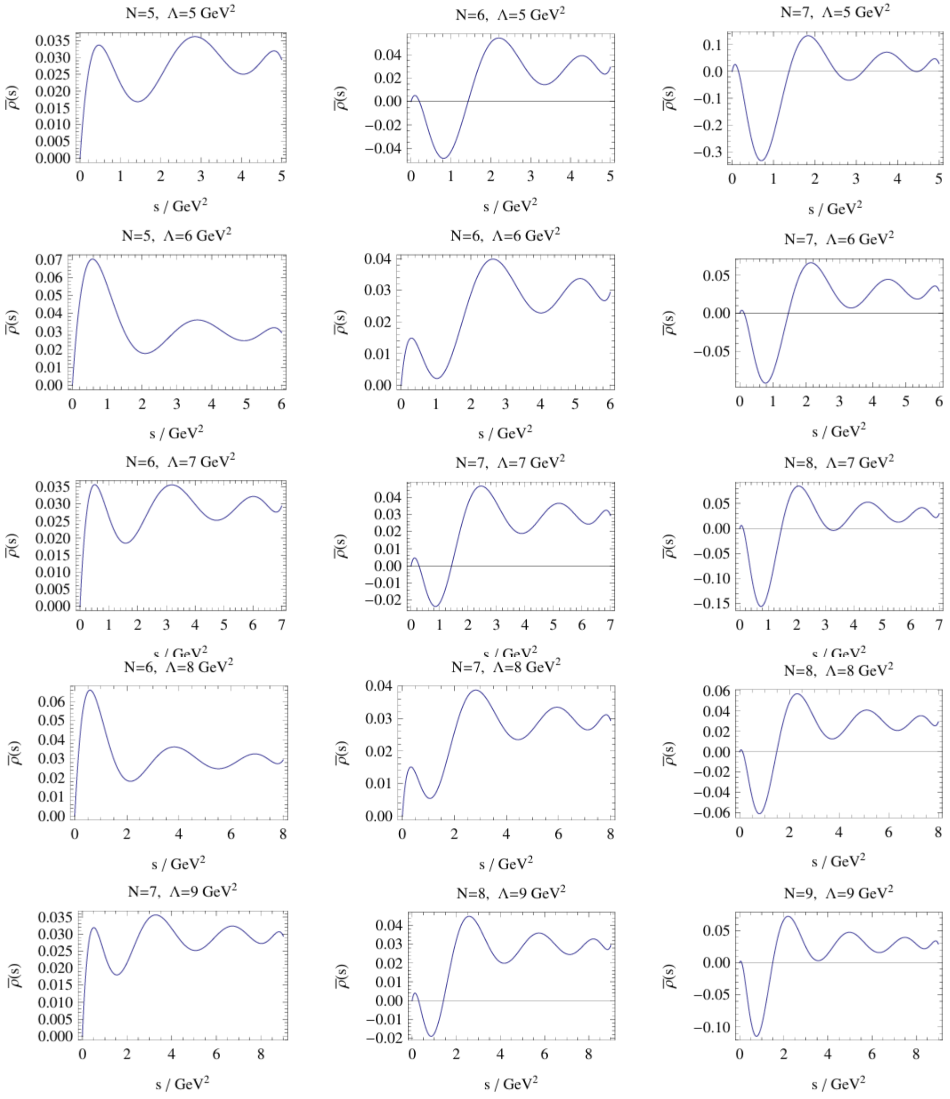
Inverse matrix method

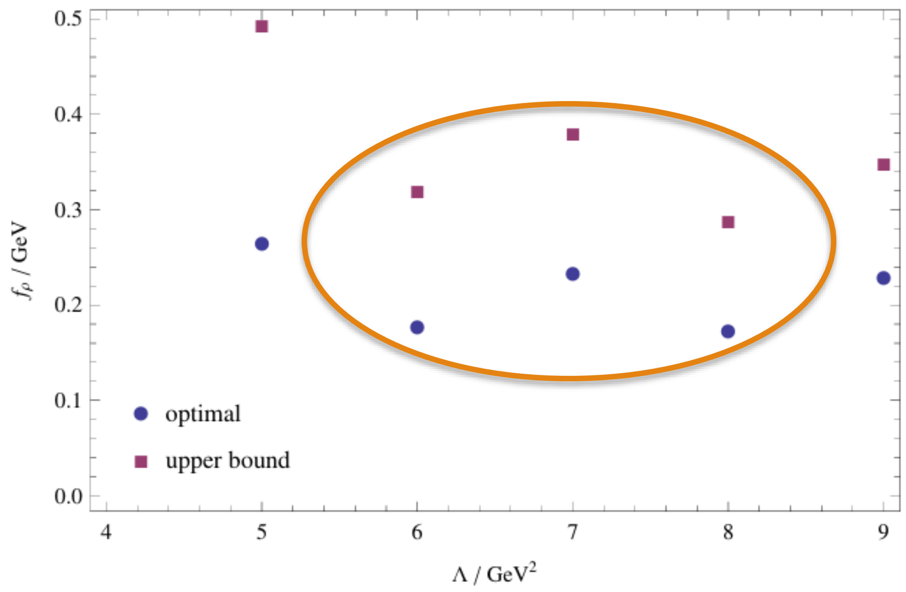
-- f_ρ

This work -- f_ρ -- version 1

$$\frac{f_\rho^2}{m_\rho^2 - q^2} + \int_0^\Lambda ds \frac{\bar{\rho}^h(s)}{s - q^2} = c \int_0^\Lambda ds \frac{1}{s - q^2} + \frac{c_2}{(q^2)^2} + \frac{c_3}{(q^2)^3},$$

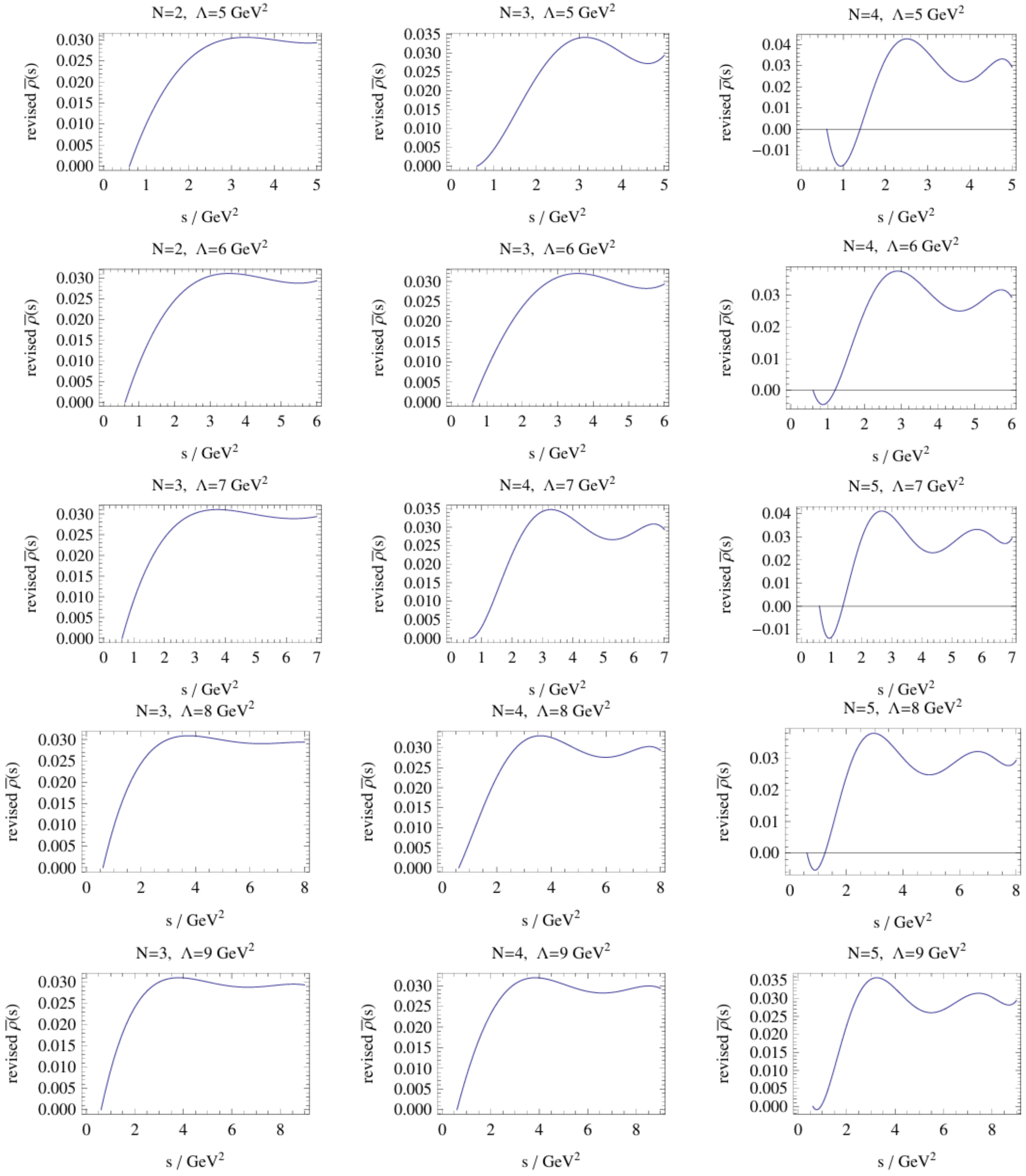
$$a_{N+2} = f_\rho^2/\Lambda.$$





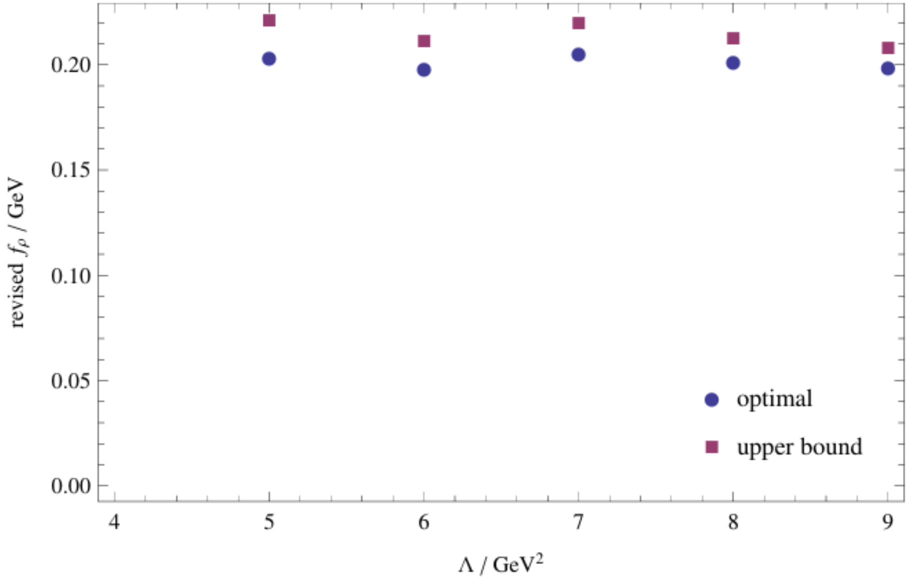
$$f_\rho = (0.17\text{-}0.38)\ \text{GeV}.$$

This work -- f_ρ -- version 2



$$\frac{f_\rho^2}{m_\rho^2 - q^2} + \int_{m_\rho^2}^{\Lambda} ds \frac{\bar{\rho}^h(s)}{s - q^2} = c \int_0^{\Lambda} ds \frac{1}{s - q^2} + \frac{c_2}{(q^2)^2} + \frac{c_3}{(q^2)^3}.$$

ground state + low excited states + ~~continuous spectrum~~



$$f_\rho = (0.20\text{-}0.22)\ \text{GeV}.$$

Discussions

Discussions

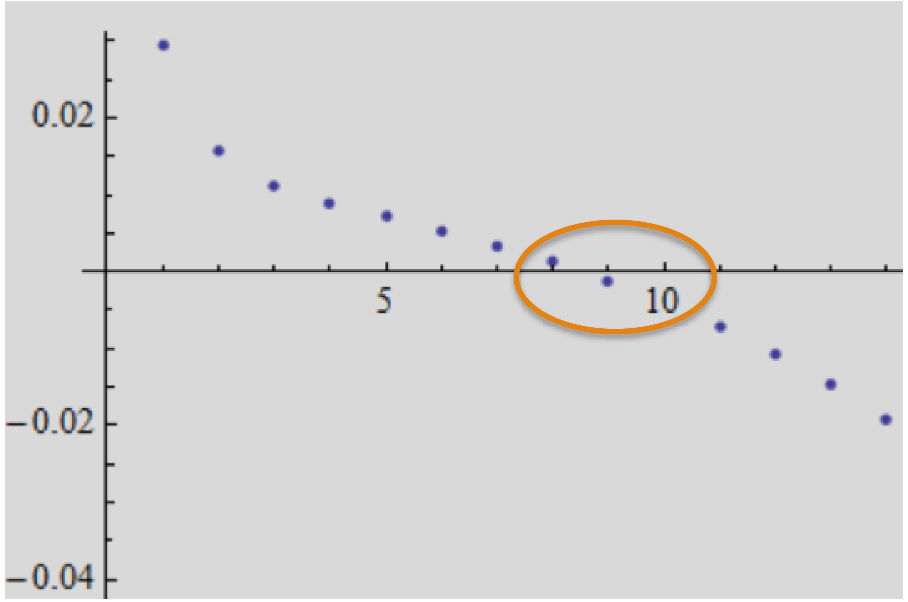
- We nearly have no assumptions
- The stability criterion plays an important role

--- $\rho^h > 0$ --- not bad --- Talking about pictures

--- $\frac{\partial f_\rho}{\partial N} = 0$ (--- $\frac{\partial f_\rho}{\partial \Lambda} = 0$)

--- $|a_N|$ is the smallest

$$\rho^h(y) = \sum_{n=0}^{N+1} a_n P_n(2y-1),$$



Summary and outlook

Summary and outlook

- Inverse matrix method is introduced to extract m_ρ and f_ρ

$$\int_0^\Lambda ds \frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_0^\Lambda ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

$$\frac{f_\rho^2}{m_\rho^2-q^2} + \int_0^\Lambda ds \frac{\bar{\rho}^h(s)}{s-q^2} = c\int_0^\Lambda ds \frac{1}{s-q^2} + \frac{c_2}{(q^2)^2} + \frac{c_3}{(q^2)^3},$$

$$\frac{f_\rho^2}{m_\rho^2-q^2} + \int_{m_\rho^2}^\Lambda ds \frac{\bar{\rho}^h(s)}{s-q^2} = c\int_0^\Lambda ds \frac{1}{s-q^2} + \frac{c_2}{(q^2)^2} + \frac{c_3}{(q^2)^3}.$$

- Used to
  --- extract the masses of hadrons (ground and excited states),
  --- form factors, …

Thank you for your attention!

Backup

Soft wall and hard wall

$$\rho(s) = \Delta\rho(s,\Lambda) + \frac{1}{\pi}\mathrm{Im}\Pi^{\mathrm{pert}}(s)(1-\exp(-s/\Lambda))$$ "Soft wall"

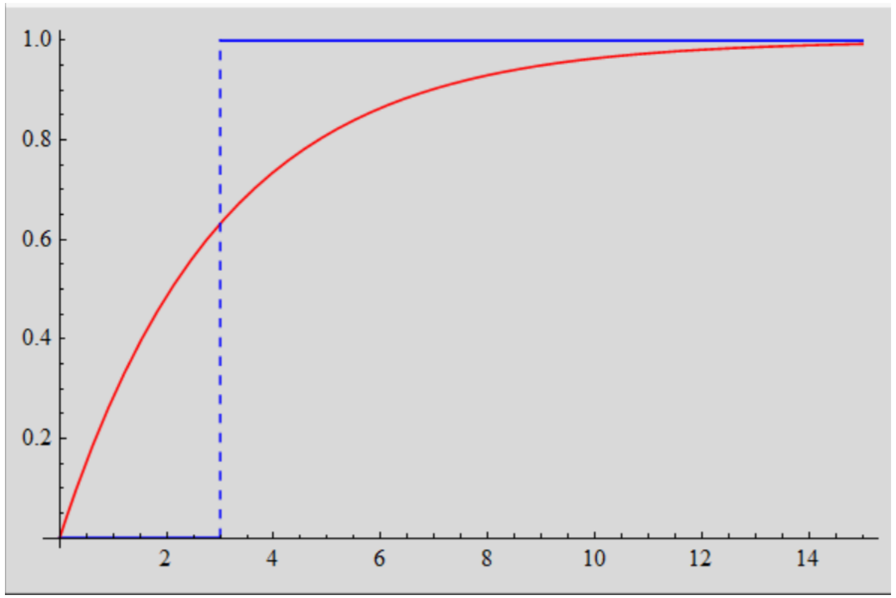
$$\int_0^\infty ds \frac{\Delta\rho(s,\Lambda)}{s-q^2} = \int_0^\infty ds \frac{ce^{-s/\Lambda}}{s-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

$$\rho(s) = \rho(s)\theta(\Lambda - s) + \rho(s)\theta(s-\Lambda)$$ "Hard wall"

$$\int_0^\Lambda ds \frac{\rho(s)}{s-q^2} = \frac{1}{\pi}\int_0^\Lambda ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{12\pi}\frac{\langle\alpha_s G^2\rangle}{(q^2)^2} + 2\frac{\langle m_q\bar{q}q\rangle}{(q^2)^2} + \frac{224\pi}{81}\frac{\kappa\alpha_s\langle\bar{q}q\rangle^2}{(q^2)^3}$$

$\Delta\rho$: $[0,+\infty]$ --- Laguerre
ρ: $[0,\Lambda]$ --- Legendre

Li-20: fit in $[0,\Lambda]$ using Legendre



"Soft wall" &
"hard wall"