Study of channel radius in beam driven plasma wakfield

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Outline

1 Background

2 Adiabatic sheath model

3 Conclusion

Background

Beam-driven plasma wakefield accelerators (PWFAs)

Review

- In 1985, Pisin Chen *et al.* proposed a new acceleration mechanism driven by charged particles;
- In 2007, people used 85 cm plasma to double the energy of a 42 GeV electron beam, with the maximum acceleration gradient of 52 GeV/m;

Nonlinear wakefield

- In the blowout regime: high current drive electron beam, beam density $n_b \gg$ plasma density n_e;
- $\sim$100 GeV/m acceleration gradient; $\sim$MT/m transverse focusing field; fs short-period pulses.

VOLUME 54, NUMBER 7 PHYSICAL REVIEW LETTERS 18 FEBRUARY 1985

Acceleration of Electrons by the Interaction of a Bunched Electron Beam with a Plasma

Pisin Chen[a]
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
and
J. M. Dawson, Robert W. Huff, and T. Katsouleas
Department of Physics, University of California, Los Angeles, California 90024

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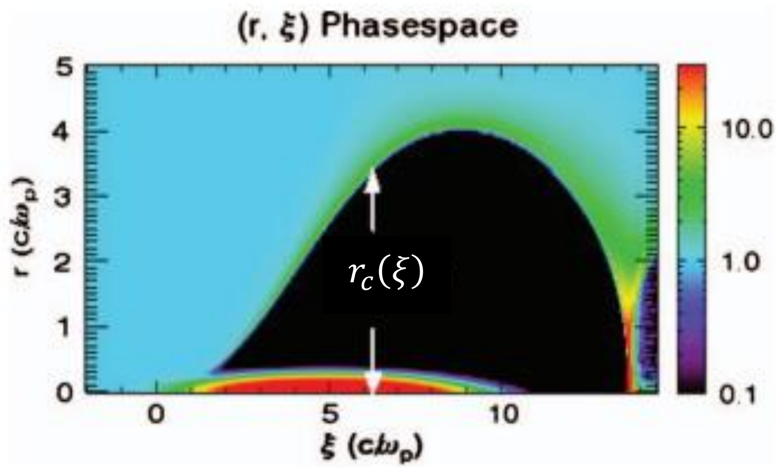
LETTERS

Energy doubling of 42 GeV electrons in a metre-scale plasma wakefield accelerator

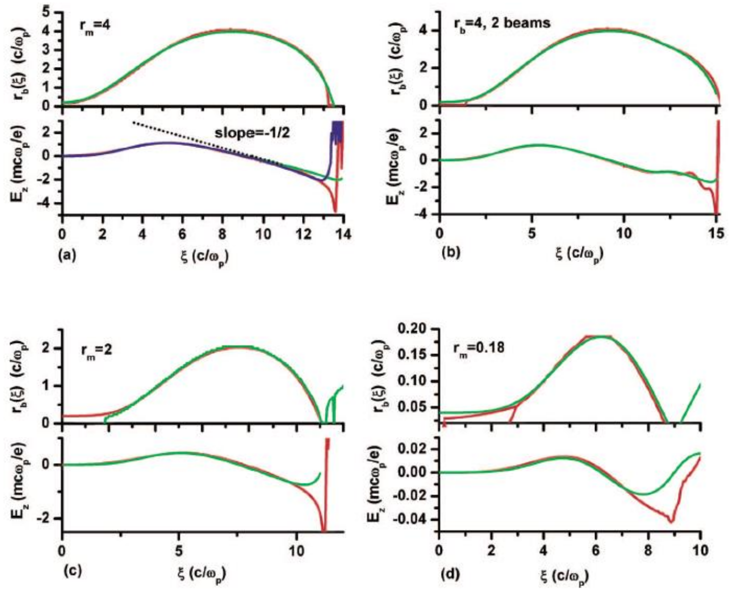
Ian Blumenfeld[1], Christopher E. Clayton[2], Franz-Josef Decker[1], Mark J. Hogan[1], Chengkun Huang[2], Rasmus Ischebeck[1], Richard Iverson[1], Chandrashekhar Joshi[2], Thomas Katsouleas[3], Neil Kirby[1], Wei Lu[2], Kenneth A. Marsh[2], Warren B. Mori[2], Patric Muggli[3], Erdem Oz[3], Robert H. Siemann[1], Dieter Walz[1] & Miaomiao Zhou[2]



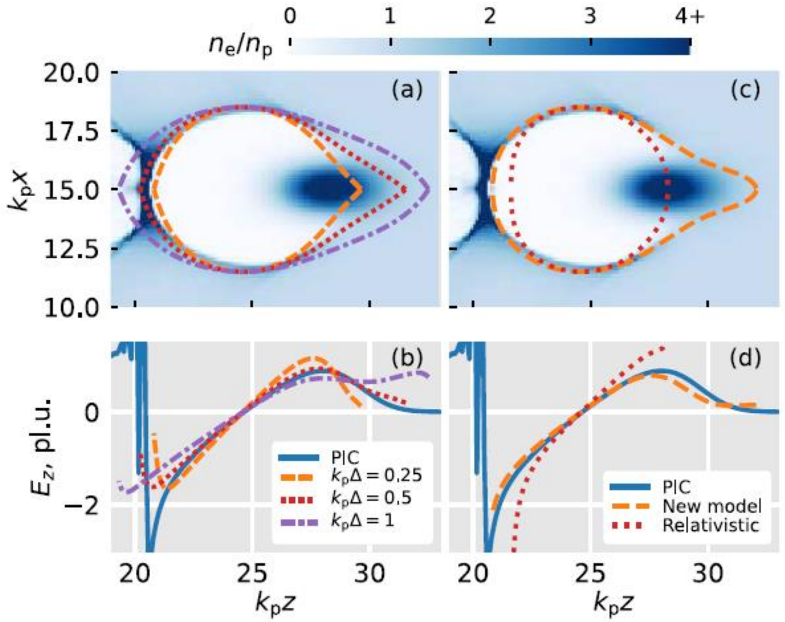
[1] W. Lu *et al.*, *Phys. Plasmas*, 13, 056709 (2006)

Why is the channel radius important

- The wakefield potential scale with the channel radius $\psi_0 \sim r_c^2/4$
- The acceleration and deceleration field depends on the wakefield potential $E_z = \frac{d\psi_0}{d\xi}$
- The E_z distribution is very important for the PWFA design
- For high transformer ratio PWFA, detailed study of r_c and ψ_0 is critical



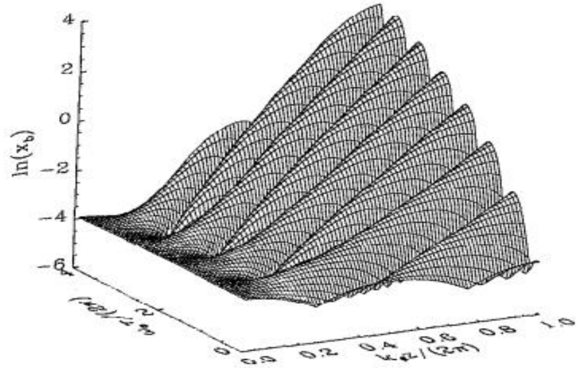
W. Lu et al., Phys. Plasmas 13, 056709 (2006);



A. Golovanov et al., PhysRevLett.130.105001 (2023)

Hosing instability in the long ion channel

the amplitude increases exponentially with time and distance



- Hosing instability : a major problem in PWFA.

  for the channel centroid [1]:

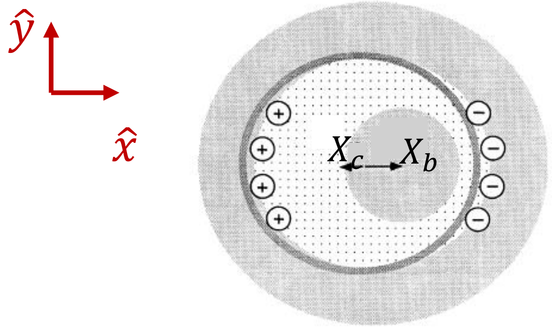
$$\frac{\partial^2 X_c(\xi,t)}{\partial \xi^2} + C_r(\xi)C_\psi(\xi)\omega_0^2 X_c(\xi,t) = C_r(\xi)C_\psi(\xi)\omega_0^2 X_b(\xi,t),$$

  where X_c is the channel centroid, X_b is the beam centroid,

  $\xi = t - z$ is the longitudinal co-moving coordinate,

  $\omega_0 = 1/\sqrt{2}$ (normalized), $\omega_\beta = 1/\sqrt{2\gamma_b}$ (γ_b is the beam Lorentz factor)

- In the adiabatic nonrelativistic limit in ref. [2], $C_r(\xi)C_\psi(\xi) = 1$.



However, it can be found that,

the oscillation frequency of X_c is not ω_0.

[1] D. H. Whittum et al., Phys. Rev. Lett. 67. 991 (1991) [2] C. Huang et al., PhysRevLett. 99. 255001 (2007)

The balancing radius of long ion channel

- Consider narrow electron beam, the linear density $\Lambda(\xi) = \int_0^\infty n_b(\xi, r')\, r' dr'$

In theory, there are two forms of balancing radius:

a) If the electron sheath follows the δ function[1]:

$$\left(\frac{r_\delta^3}{4} + r_\delta\right)\frac{d^2 r_\delta}{d\xi^2} + \left(1 + \frac{r_\delta^2}{2}\right)\left(\frac{dr_\delta}{d\xi}\right)^2 + \frac{r_\delta^2}{4} = \Lambda(\xi),$$

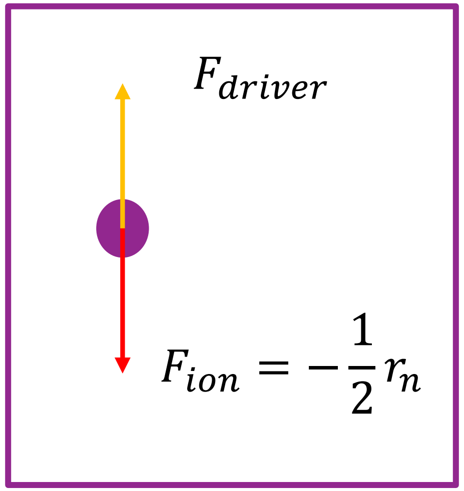
when $dr_\delta/d\xi \approx 0$,the balancing radius is $r_{\delta 0} = 2\sqrt{\Lambda}$;

b) For the boundary electrons,

the attractive force from the uniform ion background: $F_{ion} = -\frac{1}{2} r_n$,

the repulsive force from the electron driver: $F_{driver} = \frac{\Lambda}{r_n}$,

the balancing radius is the neutralization radius: $r_n = \sqrt{2\Lambda}$;

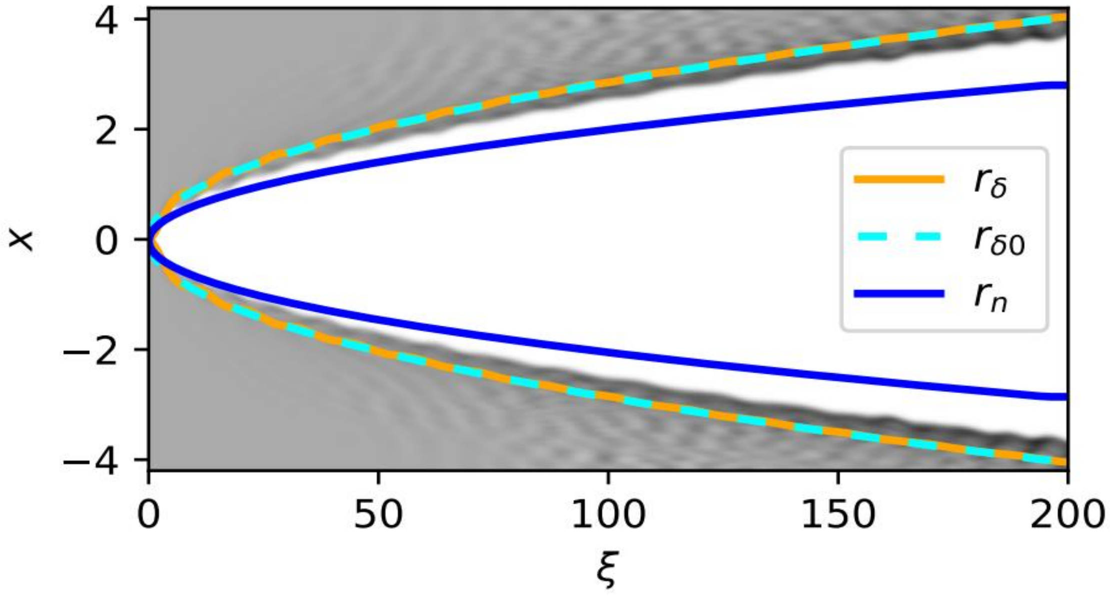


[1] A. Golovanov et al., PhysRevLett.130.105001 (2023)

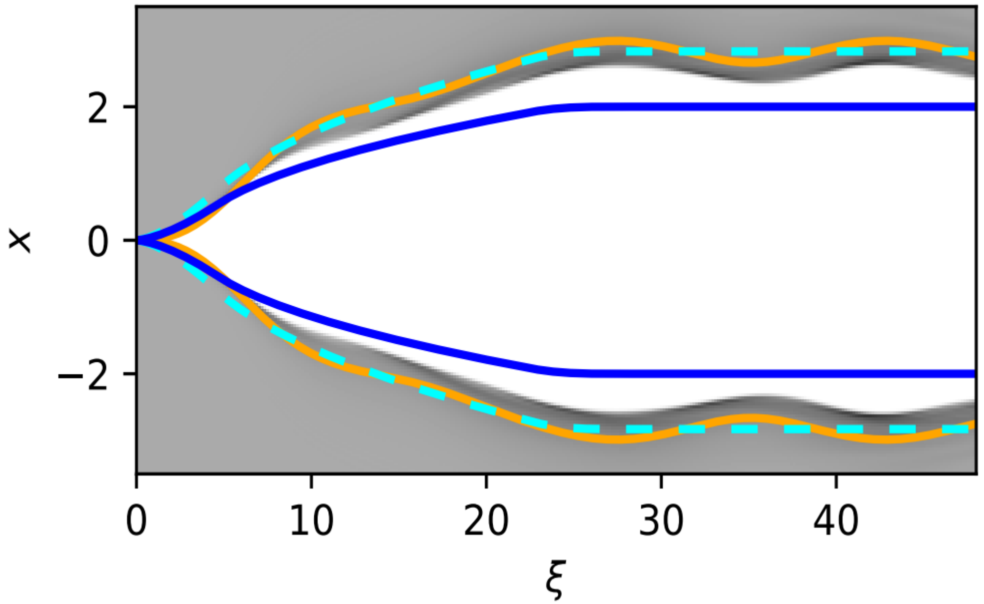
Through the simulations by QuickPIC, we find the fact that

the actual balanced channel radius is between r_n and r_δ.

Near-adiabatic blowout:



Near-stationary blowout:



Adiabatic assumption: $\frac{d\Lambda}{\Lambda d\xi} \ll 1$, $\frac{dr_c}{r_c d\xi} \ll 1$, $\partial_r \ll \partial_\xi$.

Adiabatic sheath model

Maxwell's equations for pseudo-potential

- Quasistatic approximation： $\partial_s \ll \partial_\xi$
- Poisson-like equation: $-\nabla_\perp^2 \begin{bmatrix} A \\ \phi \end{bmatrix} = \begin{bmatrix} j \\ \rho \end{bmatrix}$,
- Pseudo-potential: $\psi = \phi - A_z$,
- The source: $S(\xi, r) = \rho - j_z = \rho_b + \rho_i + \rho_e - j_{bz} - j_{ez} = 1 - n_e(1 - v_z)$,where $j_{bz} \approx \rho_b$

$$n_e = \begin{cases} 0, & r < r_c(\xi) \\ n_e, & r \geq r_c(\xi) \end{cases}$$

When $r \geq r_c(\xi), n_e$ is to be determined

"b" represents electron beam, "e" represents plasma electrons, v_z is the velocity of plasma electron

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = S(\xi, r) = \begin{cases} 1, & r < r_c(\xi) \\ 1 - n_e(1 - v_z), & r \geq r_c(\xi) \end{cases}$$

Boundary conditions

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = 1 - n_e(1 - v_z)$$

$$\psi|_{r=r_c(\xi)} = \psi_c,$$

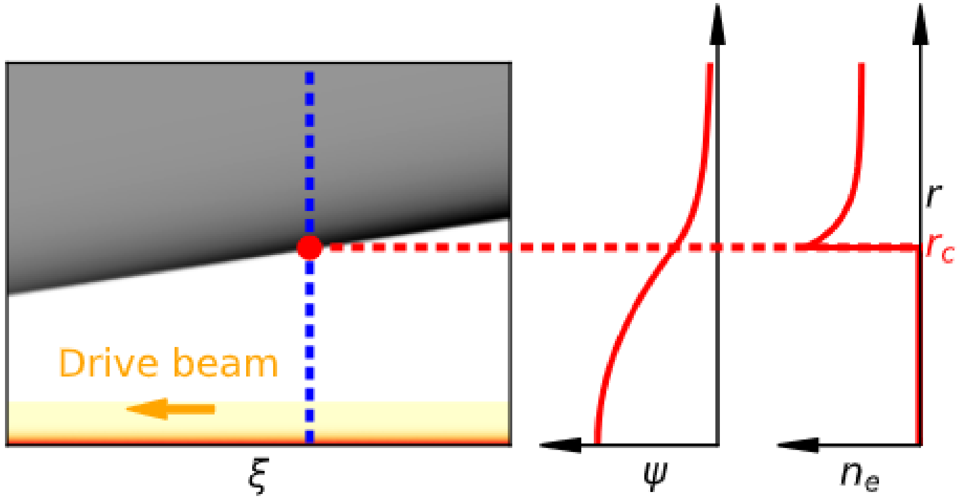
ψ_c is to be determined

$$\psi|_{r<r_c(\xi)} = \psi_c + \frac{r_c^2}{4} - \frac{r^2}{4},$$

$$\frac{\partial}{\partial r}\psi|_{r=r_c(\xi)} = -\frac{r_c}{2},$$

$$\lim_{r\to\infty}\psi = 0$$

Try to solve $\psi|_{r>r_c(\xi)}$



The velocity of plasma electron

A constant of the motion for any plasma electron : $\gamma(1 - v_z) = 1 + \psi$

- Lorentz factor of the plasma electron: $\gamma = \frac{1}{\sqrt{1+v_r^2+v_z^2}} \approx \frac{1}{\sqrt{1+v_z^2}}$

For $r > r_c$, the transverse motion of plasma electrons is negelected (the adiabatic assumption)

Therefore,

$$v_z(r) = \frac{2}{1 + [1 + \psi(r)]^2} - 1$$

Forces outside the bubble

$$F_r|_{r>r_c} = -E_r + v_z B_\theta = 0$$

wakefields | driver beam | sheath

Gauss's Law: $E_r(r) = \frac{r}{2} - \frac{\Lambda}{r} - \frac{1}{r}\int_0^r n_e(r')r'dr'$

Stokes' Theorem: $B_\theta(r) = -\frac{\Lambda}{r} - \frac{1}{r}\int_0^r n_e(r')v_z(r')r'dr'$

$n_e = 0$ when $r < r_c(\xi)$

Therefore, $F_r|_{r>r_c} = \frac{1}{r}\left[-\frac{r^2}{2} + (1-v_z)\Lambda + \int_0^r n_e(r')r'dr' - v_z\int_0^r n_e(r')v_z(r')r'dr'\right] = 0,$

The balancing radius $r_c = \sqrt{2[1-v_z(r=r_c)]\Lambda} = 2\sqrt{\frac{\Lambda}{1+\frac{1}{(1+\psi_c)^2}}}.$

The sheath equations

For $r > r_c$, the coupled equations are:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = 1 - n_e(1 - v_z)$$

$$v_z = \frac{2}{1+(1+\psi)^2} - 1$$

$$-\frac{r^2}{2} + (1 - v_z)\Lambda + \int_0^r n_e(r')r'dr' - v_z\int_0^r n_e(r')v_z(r')r'dr' = 0$$

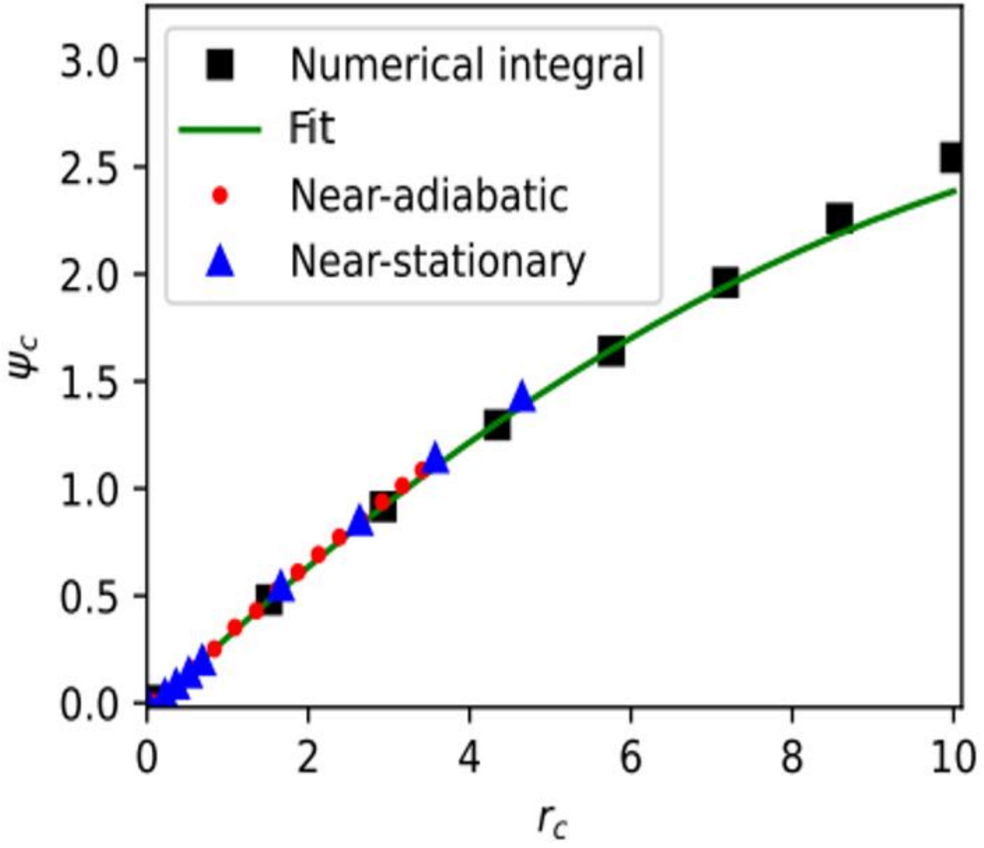
Boundary conditions:

$$\frac{\partial}{\partial r}\psi|_{r=r_c(\xi)} = -\frac{r_c}{2},$$

$$\lim_{r\to\infty}\psi = 0$$

Numerical results of the shooting method

large scale blowout radius



polynomial fit：

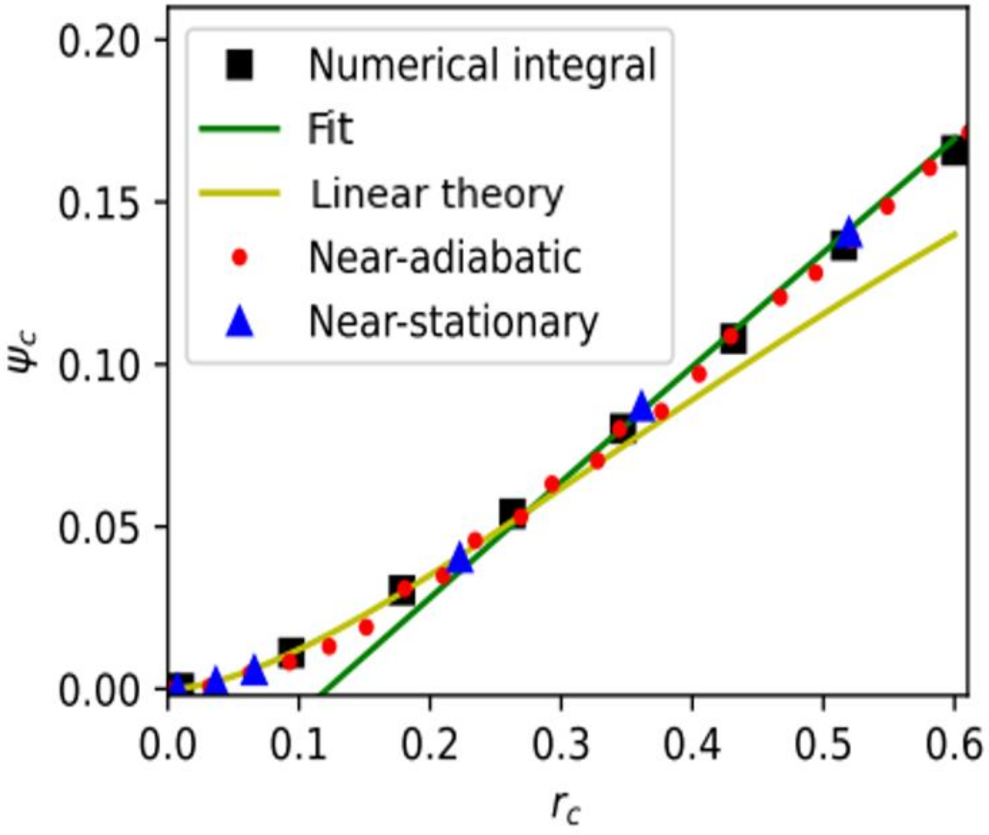
$$\psi_c \approx -0.012 r_c^2 + 0.363 r_c - 0.044$$

"Near-adiabatic" and "Near-stationary" are the simulation results in the near-adiabatic bubble and near-stationary bubble respectively.

This fit is a good estimation for $r_c \lesssim 8$.

Numerical results in small scale

small scale blowout radius



For small radius ($r_c \ll 1, \psi_c \ll 1, \Lambda \ll 1, \psi \ll 1$):

$$v_z \approx -\psi, n_e \approx 1 - (\Lambda/r)(\partial\psi/\partial r)$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = 1 - n_e(1 - v_z) \xrightarrow{\text{simplified to}} \frac{\partial^2}{\partial r^2}\psi + \frac{1}{r}\frac{\partial}{\partial r}\psi - \psi = 0$$

Modified Bessel function of the second kind $K_\nu(x)$ is solution of the differential equation[1]:

$$x^2 y'' + xy' - (x^2 + \nu^2)y = 0,$$

and $K_0(x) \approx -\ln\frac{x}{2} - 0.577$

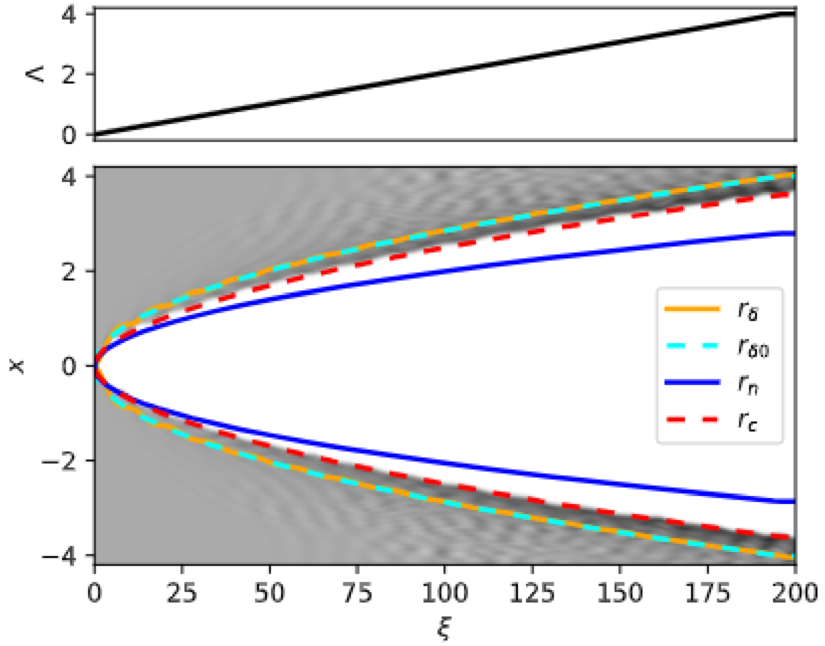
Thus, linear theory: $\boldsymbol{\psi = \frac{r_c^2}{2} K_0(r)}$, $\boldsymbol{\psi_c = \frac{r_c^2}{2} K_0(r_c)}$

Sum up: $\psi_c = \begin{cases} \frac{r_c^2}{2} K_0(r_c), & r_c \lesssim 0.3 \\ -0.012 r_c^2 + 0.363 r_c - 0.044, & 0.3 \lesssim r_c \lesssim 8 \end{cases}$

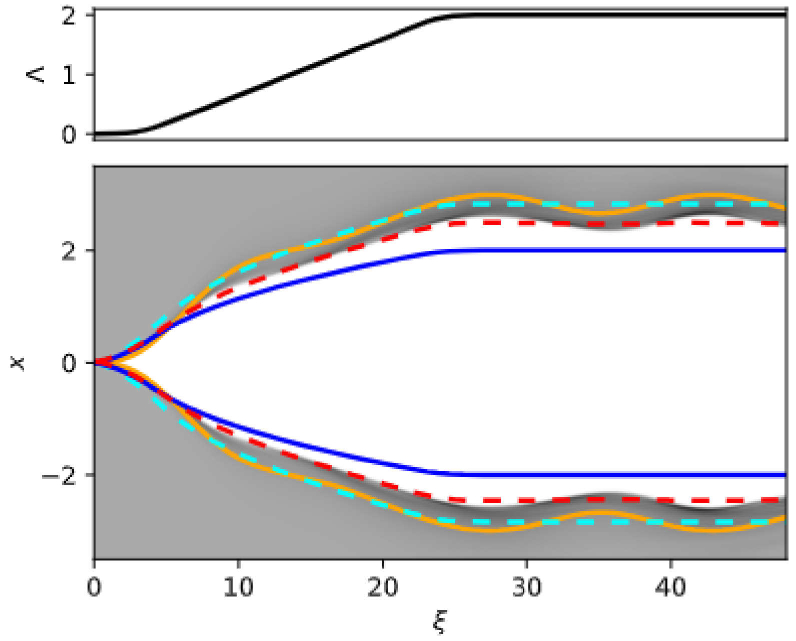
[1] A. Jeffrey and H.-H. Dai, Handbook of Mathematical Formulas and integrals (Academic Press, Elsevier, 2008) 4th edition.

the balancing radius in adiabatic sheath model

the balancing radius : $r_c = \sqrt{2[1 - v_z(r = r_c)]\Lambda} = 2\sqrt{\frac{\Lambda}{1+\frac{1}{(1+\psi_c)^2}}}$



Near-adiabatic blowout



Near-stationary blowout

The balancing radius in our model r_c best matches the simulation results.

Conclusion

- For an adiabatic sheath, we have obtained the coupled equations of ψ, v_e and n_e.
- Balancing radius of channel $r_c = 2\sqrt{\frac{\Lambda}{1+\frac{1}{(1+\psi_c)^2}}}$ is between r_n and r_δ.
- We have found ψ and ψ_c for all r:
  - $\psi(r) = \begin{cases} \psi_c + \frac{r_c^2}{4} - \frac{r^2}{4}, & r < r_c \\ \frac{r_c^2}{2} K_0(r), & r \geq r_c, \text{if } r_c \lesssim 0.3 \end{cases}$
  - If $r_c > 0.3$, $\psi(r)$ can be obtained by numerically solving the coupled equations.
  - $\psi_c = \begin{cases} \frac{r_c^2}{2} K_0(r_c), & r_c \lesssim 0.3 \\ -0.012 r_c^2 + 0.363 r_c - 0.044, & 0.3 \lesssim r_c \lesssim 8 \end{cases}$

Thank you!

The blowout radius

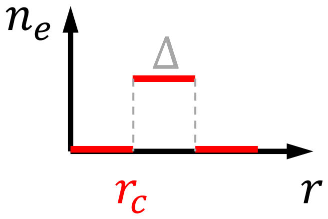
- The blowout radius $r_c(\xi)$ is determined by

① the current distribution of the drive beam； ② the density distribution of sheath electrons.[1];

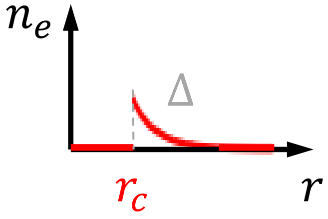
- the drive beam current $\Lambda(\xi) = \int_0^\infty n_b(\xi, r')\, r' dr' = \frac{2I}{I_A}$, also represents the normalized linear density,

n_b — the density, I — the instantaneous current, $I_A \approx 17$ kA;

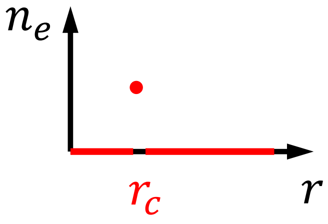
- Existing theories have simplified sheath to square or exponential distributions [2-4].



Square distribution



Exponential distribution



δ function distribution

[1] W. Lu *et al.*, *Phys. Plasmas*, 13, 056709 (2006) [2] S. A. Yi *et al.*, *Phys. Plasma*, 20, 013108 (2013)
[3] J. Thomas *et al.*, *Phys. Plasma*, 23, 053108 (2016) [4] A. A. Golovanov *et al.*, *Quantum Electron.*, 46, 295 (2016)

The sheath equations

- Poisson-like equation: : $-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = \rho - j_z = \begin{cases} 1, & r < r_c(\xi) \\ 1 - n_e(1 - v_z), & r \geq r_c(\xi) \end{cases}$,

  n_e – density of plasma electrons， v_z – longitudinal velocity of plasma electrons。

  ① when $r < r_c(\xi)$, $\psi|_{r<r_c(\xi)} = \psi_c + \frac{r_c^2}{4} - \frac{r^2}{4}$, where $\psi|_{r=r_c(\xi)} = \psi_c$;

  ② **Boundary conditions:** $\frac{\partial}{\partial r}\psi|_{r=r_c(\xi)} = -\frac{r_c}{2}$, $\lim_{r\to\infty}\psi = 0$。

- **a constant of plasma electrons motion：** $\gamma - \gamma v_z = 1 + \psi$。

- Gauss'law : $E_r(r) = \frac{r}{2} - \frac{\Lambda}{r} - \frac{1}{r}\int_0^r n_e(r')r'dr'$， Stokes' Theorem: $B_\theta(r) = -\frac{\Lambda}{r} - \frac{1}{r}\int_0^r n_e(r')v_z(r')r'dr'$,

  Forces in equilibrium for electrons outside the ion channel :

$$F_r|_{r>r_c} = -E_r + v_z B_\theta = \frac{1}{r}\left[-\frac{r^2}{2} + (1 - v_z)\Lambda + \int_0^r n_e(r')r'dr' - v_z\int_0^r n_e(r')v_z(r')r'dr'\right] = 0.$$