Optimizing entanglement and Bell inequality violation in top pair events

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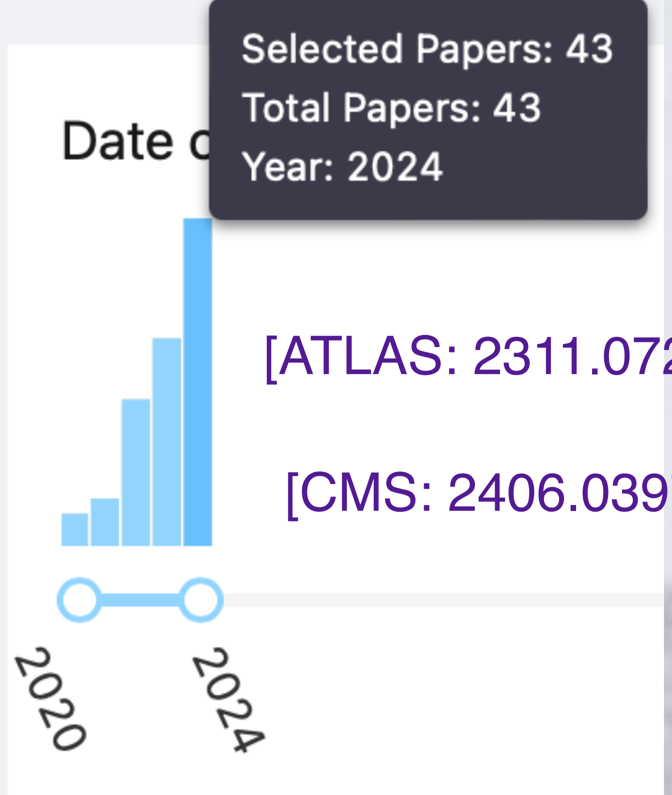
Outline



[ATLAS: 2311.07288]

[CMS: 2406.03976]

Spin correlation

$$A = \frac{N_{\text{like}} - N_{\text{unlike}}}{N_{\text{like}} + N_{\text{unlike}}} = \frac{N(\uparrow\uparrow) + N(\downarrow\downarrow) - N(\uparrow\downarrow) - N(\downarrow\uparrow)}{N(\uparrow\uparrow) + N(\downarrow\downarrow) + N(\uparrow\downarrow) + N(\downarrow\uparrow)}$$

Quantum state
Quantum information

Entanglement
Bell inequality
Discord
…

Spin quantization axis

Fictitious state

Basis dependence

Spin correlation of top pair

- $$A = \frac{N_{\text{like}} - N_{\text{unlike}}}{N_{\text{like}} + N_{\text{unlike}}} = \frac{N(\uparrow\uparrow) + N(\downarrow\downarrow) - N(\uparrow\downarrow) - N(\downarrow\uparrow)}{N(\uparrow\uparrow) + N(\downarrow\downarrow) + N(\uparrow\downarrow) + N(\downarrow\uparrow)} = \langle \sigma_z^t \otimes \sigma_z^{\bar{t}} \rangle$$
  - Need a spin quantization axis z to define $|\uparrow\rangle$ and $|\downarrow\rangle$
  - In practice: **statistically** measured from decay product distribution of top pairs.

- Meausure spin correlation along different directions:
  - $\langle \sigma_x^t \otimes \sigma_x^{\bar{t}} \rangle, \langle \sigma_y^t \otimes \sigma_y^{\bar{t}} \rangle, \langle \sigma_x^t \otimes \sigma_y^{\bar{t}} \rangle, \cdots$
  - reconstructs a "quantum state" $\Longrightarrow$ study quantum information

Quantum tomography at collider
[Y. Afik and J. de Nova, 2203.05582]

$$\hat{\rho} = \frac{1}{4}\left(\hat{I}_2 \otimes \hat{I}_2 + B_i^+ \hat{\sigma}_i \otimes \hat{I}_2 + B_i^- \hat{I}_2 \otimes \hat{\sigma}_i + C_{ij} \hat{\sigma}_i \otimes \hat{\sigma}_j\right)$$

$C_{ij} = \langle \sigma_i^t \otimes \sigma_j^{\bar{t}} \rangle$: spin correlation matrix

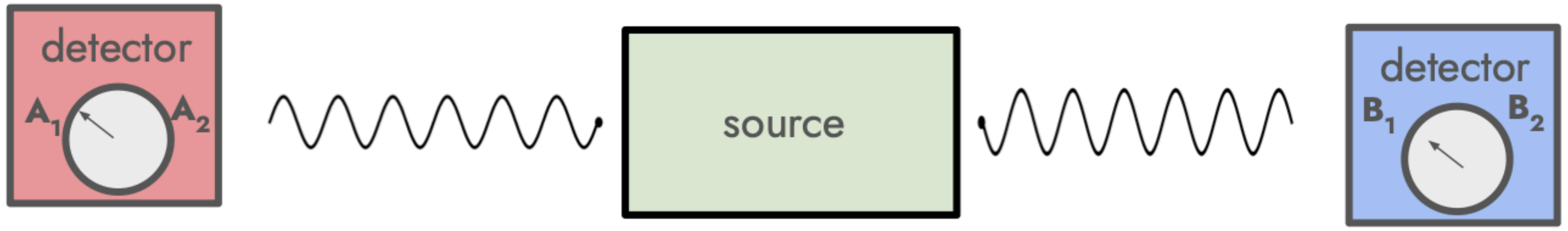
$B_i^{\pm}$: net polarization of t and $\bar{t}$

Quantum Information: Bell (CHSH) inequality [J. Bell, *1964*]

Bell inequality: constructed from four two-outcome measurements $\hat{A}_{1,2}$ and $\hat{B}_{1,2}$

$$\left|\langle \hat{A}_1\hat{B}_1\rangle + \langle \hat{A}_1\hat{B}_2\rangle + \langle \hat{A}_2\hat{B}_1\rangle - \langle \hat{A}_2\hat{B}_2\rangle\right| \leq 2$$

[Clauser et al, *PRL* 23, 880 (1969)]



Depends on the choice of $A_{1,2}$ and $B_{1,2}$. Example:

$$A_1 = \sigma_x \qquad \langle \sigma_x \otimes \sigma_z \rangle = \mathrm{tr}[(\sigma_x \otimes \sigma_z)\rho]$$
$$B_2 = \sigma_z \qquad = C_{13}$$

Bell inequality is violated iff we can find four directions $\vec{a}_{1,2}, \vec{b}_{1,2}$ so that

$$\left|\vec{a}_1 \cdot C \cdot (\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C \cdot (\vec{b}_1 + \vec{b}_2)\right| > 2$$

Scan $\vec{a}_i, \vec{b}_i$ →

$$\mathscr{B}(\rho) = 2\sqrt{\mu_1^2 + \mu_2^2} > 2$$

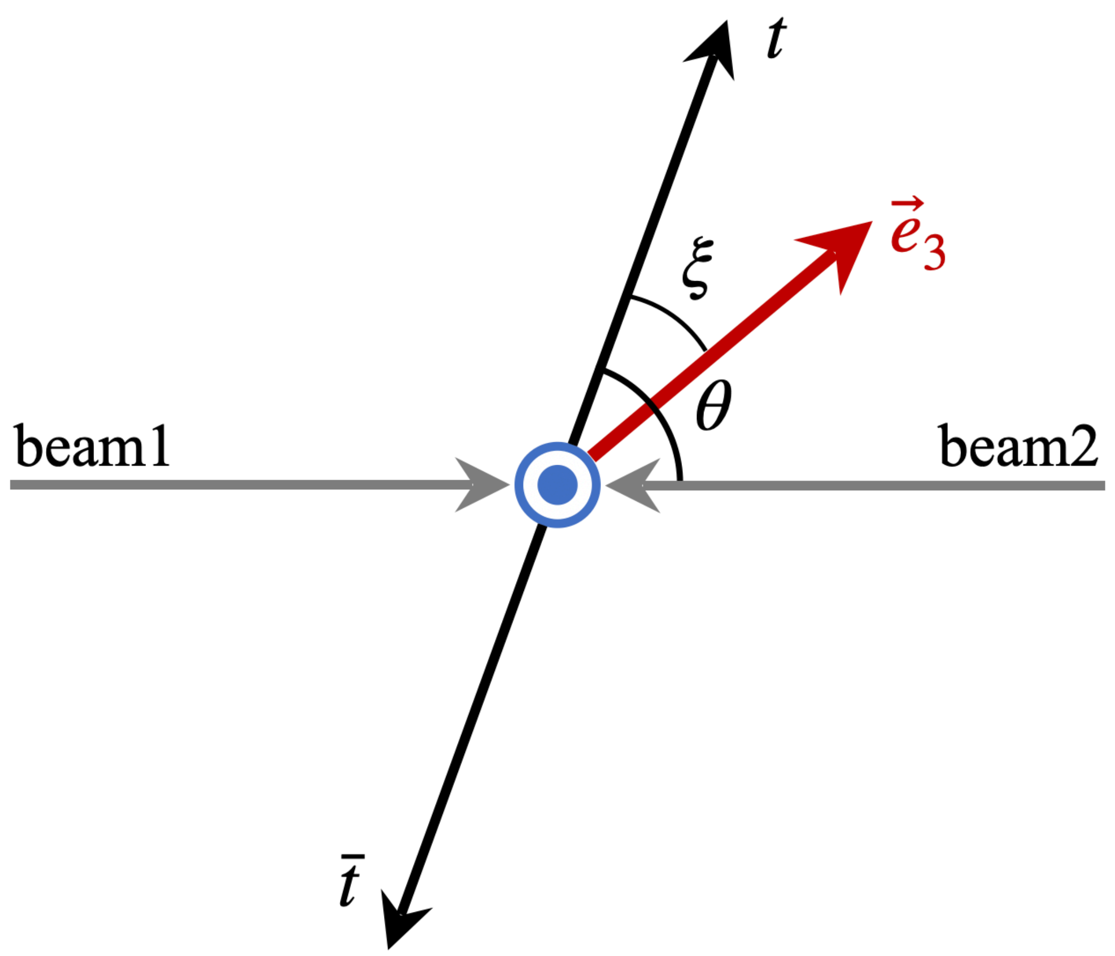
[Horodecki et al, *PLA* 200, 340 (1995)]

μ_1^2, μ_2^2 **are the largest two eigenvalue of $C^T C$.**

Basis dependence of Spin correlation

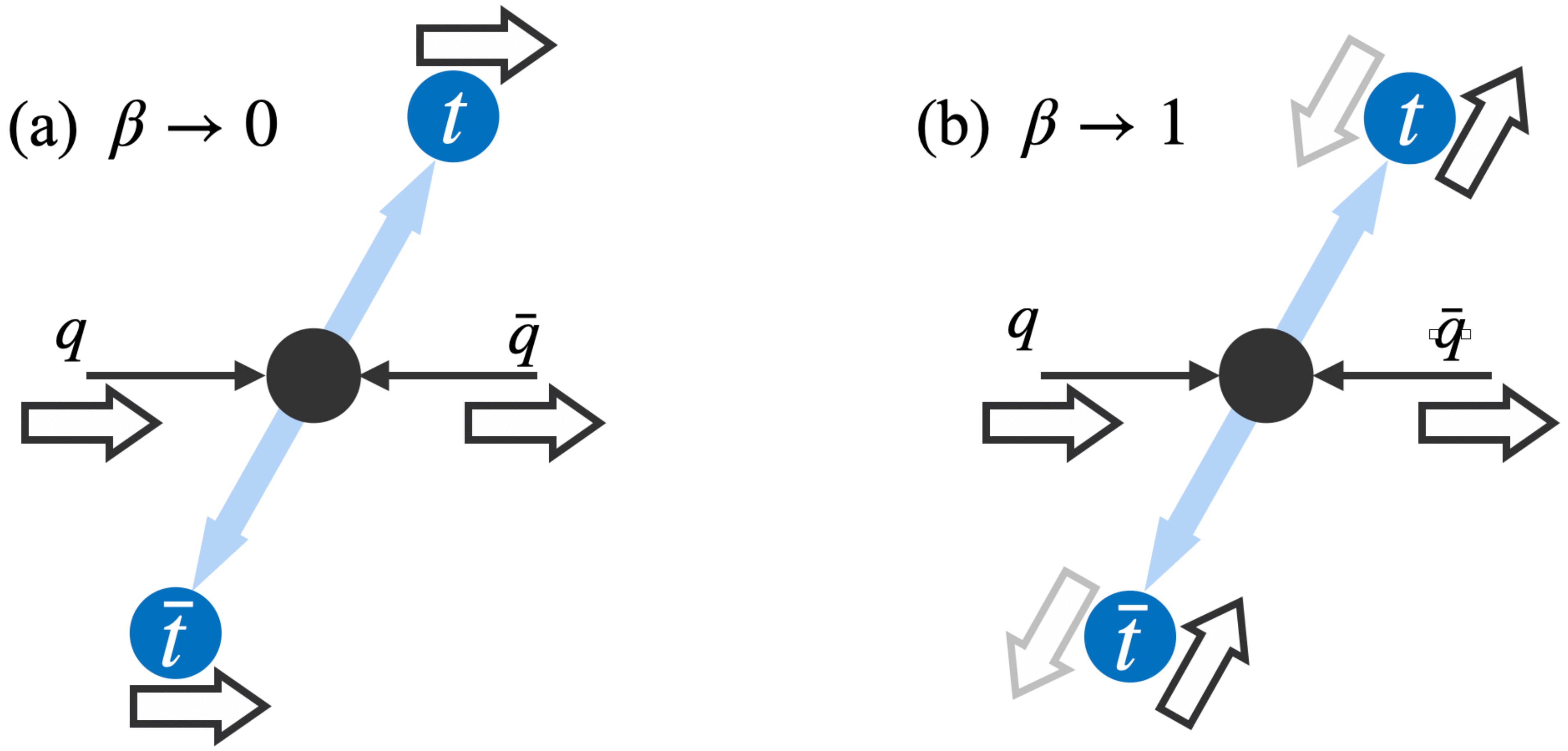
- $A = \dfrac{N_{\text{like}} - N_{\text{unlike}}}{N_{\text{like}} + N_{\text{unlike}}} = \langle \sigma_z^t \otimes \sigma_z^{\bar{t}} \rangle$ clearly depends on the quantization axis

- E.g. 100% spin correlation in $q\bar{q} \to t\bar{t}$, only like-spin states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$



$$\tan \xi = \frac{\tan \theta}{\gamma}$$

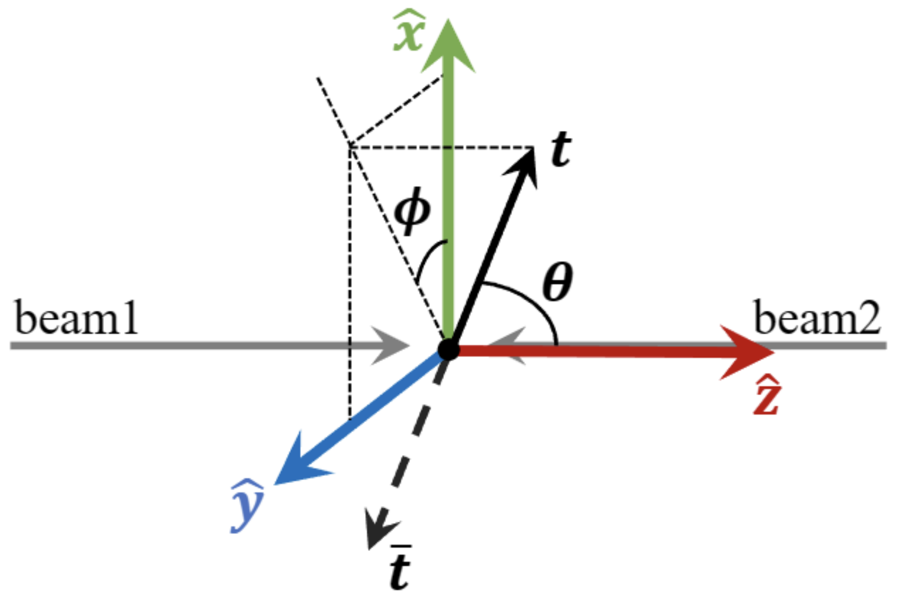
[Parke and Shadmi, *PLB* (1996)]
[Mahlon and Parke, *PLB* (1997)]



Basis dependence of Spin correlation

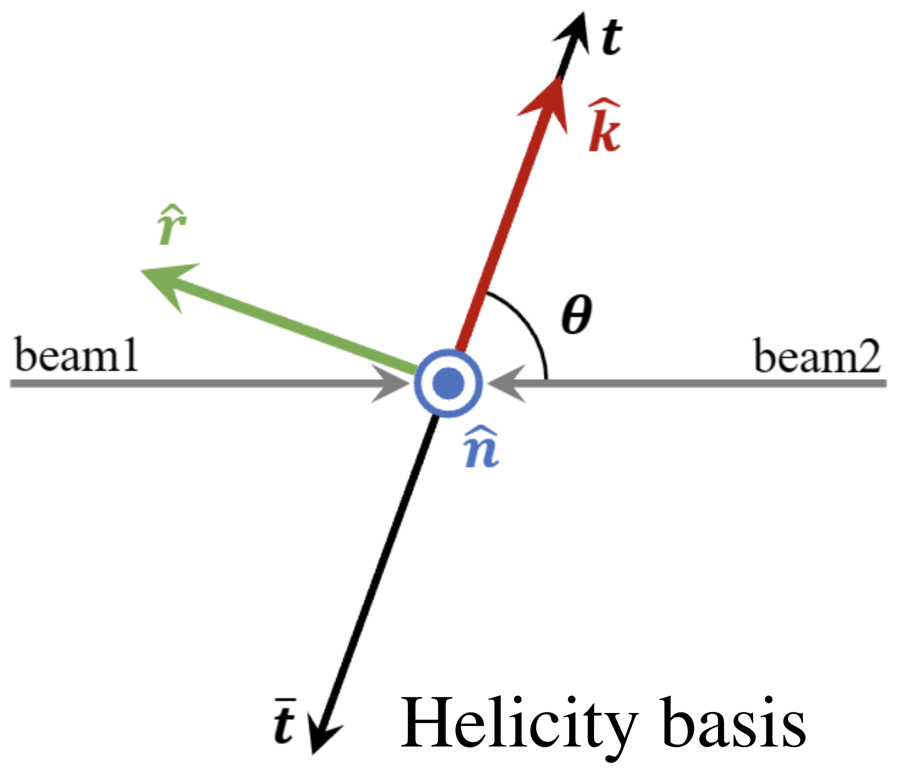
- Easier to undertand from the complete spin correlation matrix

$$\begin{pmatrix} C_{rr} & C_{rn} & C_{rk} \\ C_{nr} & C_{nn} & C_{nk} \\ C_{kr} & C_{kn} & \boxed{C_{kk}} \end{pmatrix} \neq \begin{pmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{yx} & C_{yy} & C_{yz} \\ C_{zx} & C_{zy} & \boxed{C_{zz}} \end{pmatrix}$$



Fixed beam basis

- Two example basis:
  - **Fixed beam basis**: the spin basis $|\uparrow\rangle$ and $|\downarrow\rangle$ are defined as spin eigenstates along $\hat{z}$-direction
  - **Helicity basis**: the spin basis $|\uparrow\rangle$ and $|\downarrow\rangle$ are defined as spin eigenstates along the moving direction of top quark.



Helicity basis

$$C_{\text{hel}} = R^T C_{\text{beam}} R$$

Quantum state: basis independent

$$\hat{\rho} = \frac{1}{4}\left(\hat{I}_2 \otimes \hat{I}_2 + B_i^+ \hat{\sigma}_i \otimes \hat{I}_2 + B_i^- \hat{I}_2 \otimes \hat{\sigma}_i + C_{ij} \hat{\sigma}_i \otimes \hat{\sigma}_j\right)$$

- Choose a different spin quantization basis:

$$\rho \to U^\dagger \rho U, \qquad C_{ij} \to (R^T C R)_{ij}$$

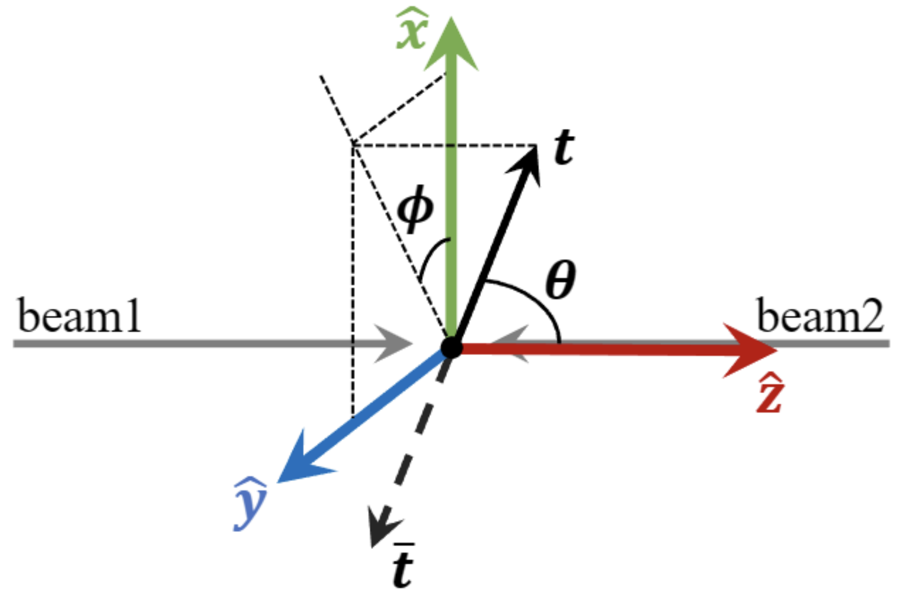
- Intuitively, the properties (e.g. entangled or not) of a **quantum** state should not depends on the basis we used to measure it. **TRUE**
  - **Concurrence:** $\mathscr{C}(\rho) = \max(0,\ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$, $\leftarrow$ Eigenvalues of $M = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$
    $\mathscr{C} > 0$ is entanglement
    $M \to U^\dagger M U$
  - **Bell variable:** $\mathscr{B}(\rho) = 2\sqrt{\mu_1^2 + \mu_2^2}$ $\leftarrow$ Eigenvalues of $C^T C$
    $\mathscr{B} > 2$ is non-local

- ρ_1 and ρ_2 describe the same state if $\rho_1 = U^\dagger \rho_2 U$

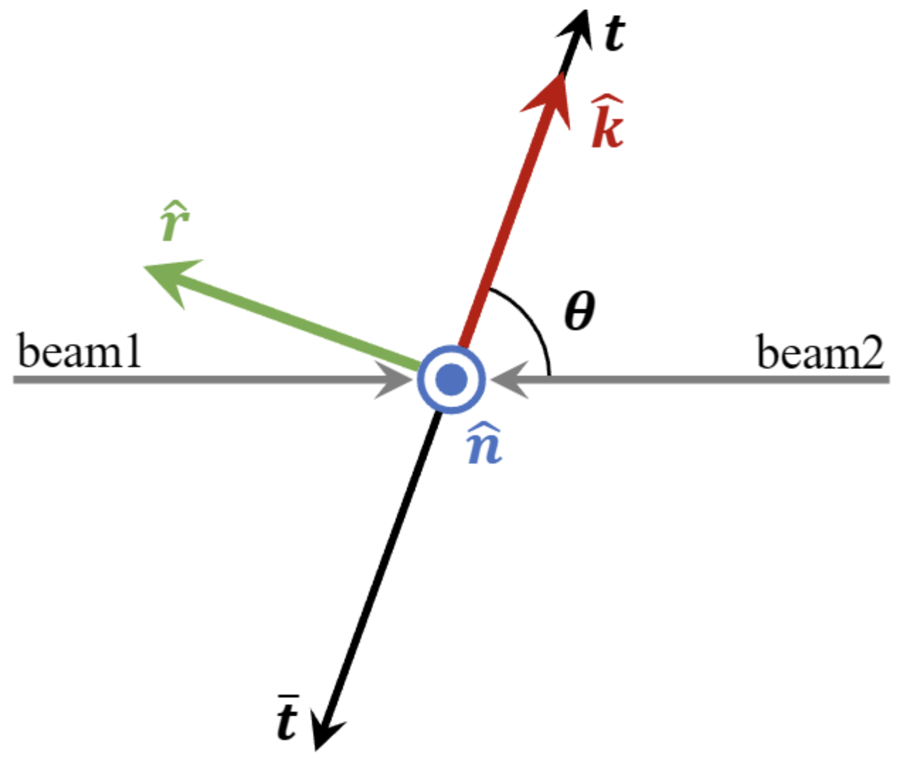
Angular averaged state

In the c.m. frame of $t\bar{t}$

$\mathbf{k} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$



Beam basis: fixed for all $t\bar{t}$ events



Helicity basis different for each $t\bar{t}$ event

- Phase space averaged observable $\Longrightarrow$ averaged state
- **What would happen if the average is done in an event-dependent basis?**

- Example: near threhold, the $e_R^+ e_L^- \to t\bar{t}$ produces a pure state $|\uparrow_z\uparrow_z\rangle$

$$\bar{\rho}^{\text{fixed}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \bar{\rho}^{\text{helicity}} = \begin{pmatrix} \frac{8}{3} & -\frac{\pi}{2} & -\frac{\pi}{2} & \frac{4}{3} \\ -\frac{\pi}{2} & \frac{4}{3} & \frac{4}{3} & -\frac{\pi}{2} \\ -\frac{\pi}{2} & \frac{4}{3} & \frac{4}{3} & -\frac{\pi}{2} \\ \frac{4}{3} & -\frac{\pi}{2} & -\frac{\pi}{2} & \frac{8}{3} \end{pmatrix}$$

$$\mathrm{Tr}[(\bar{\rho}^{\text{fixed}})^2] = 1 \qquad \mathrm{Tr}[(\bar{\rho}^{\text{helicity}})^2] \approx 0.7 < 1$$

$$\rho^{\text{fixed}} \neq U^\dagger \rho^{\text{helicity}} U$$

We are recontructing fictitious states instead of physical quantum states.

Fictitious state: basis dependent

- **What** are fictitious states
  - State recontructed from averaged quantities in event-by-event basis
  - Convex sum of quantum sub-state, but with an event-by-event rotation

$$\text{Quantum state:}\quad \rho_{\text{phy}} = \sum_a \rho_a \qquad C_{\text{phy}} = \sum_a C_a$$

$$\text{Fictitious state:}\quad \rho_{\text{fic}} = \sum_a U_a^\dagger \rho_a U_a \qquad C_{\text{fic}} = \sum_a R_a^T C_a R_a$$

  - $\rho_{\text{fic}} \neq U^\dagger \rho_{\text{phy}} U$
  - Basis dependent: $\rho_{\text{basis1}} \neq U^\dagger \rho_{\text{basis2}} U$

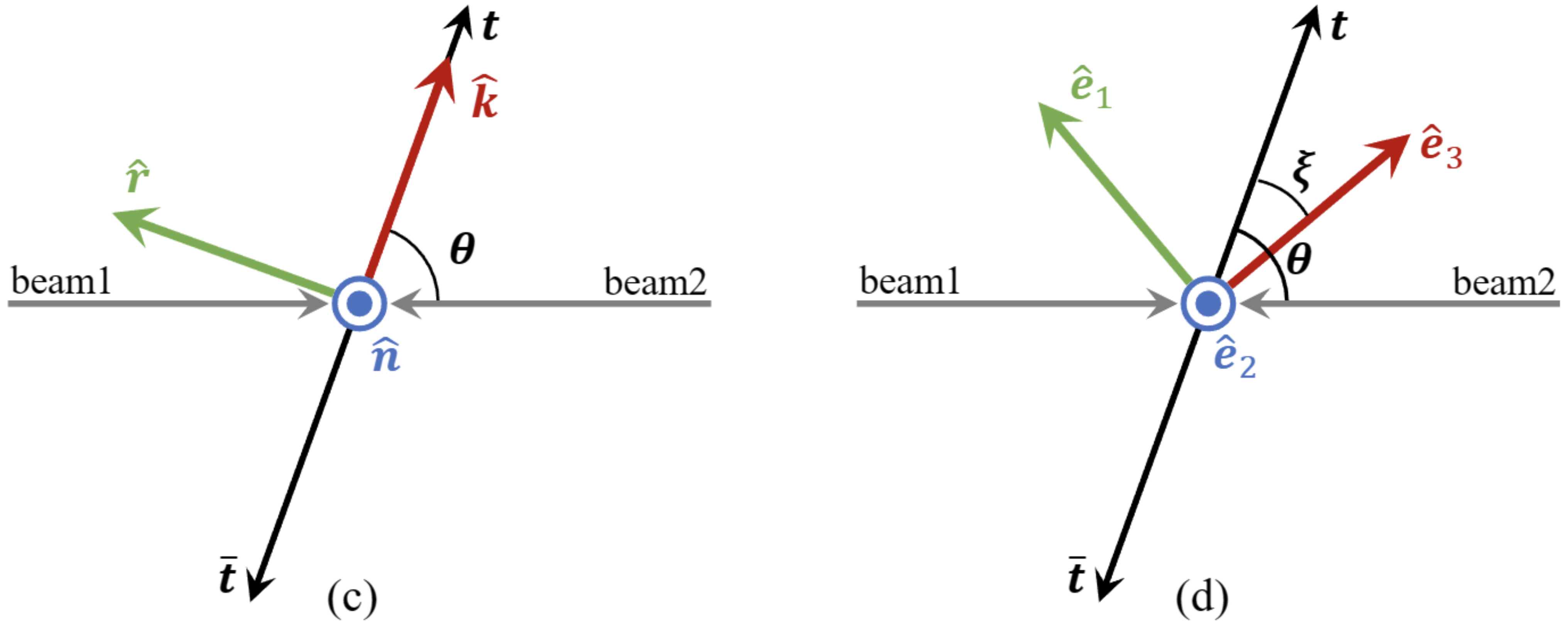
- Fictitious state still **preserves** some quantum properties

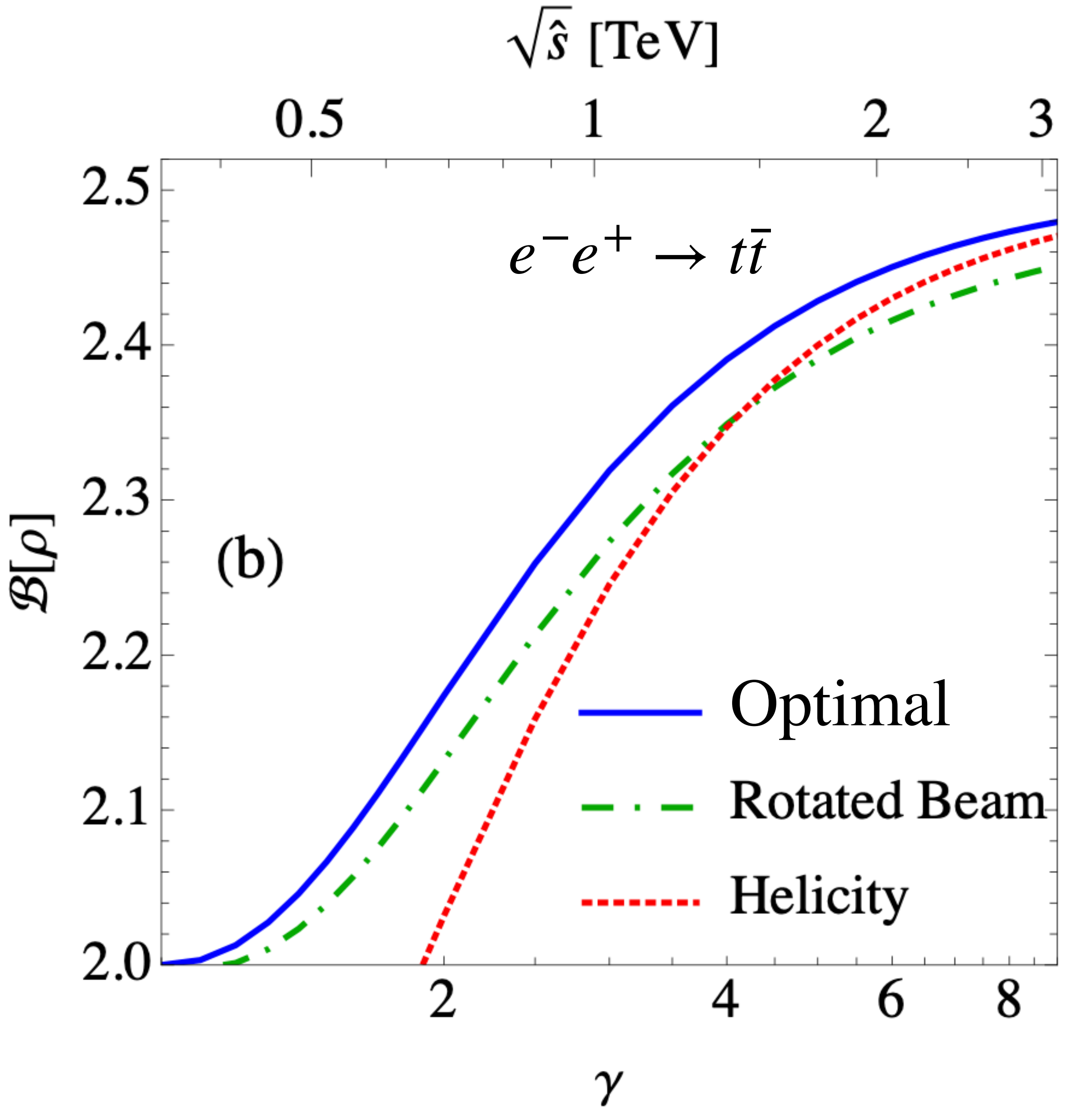
$$\mathscr{C}(\rho_{\text{fic}}) > 0 \implies \mathscr{C}(\rho_a) > 0$$

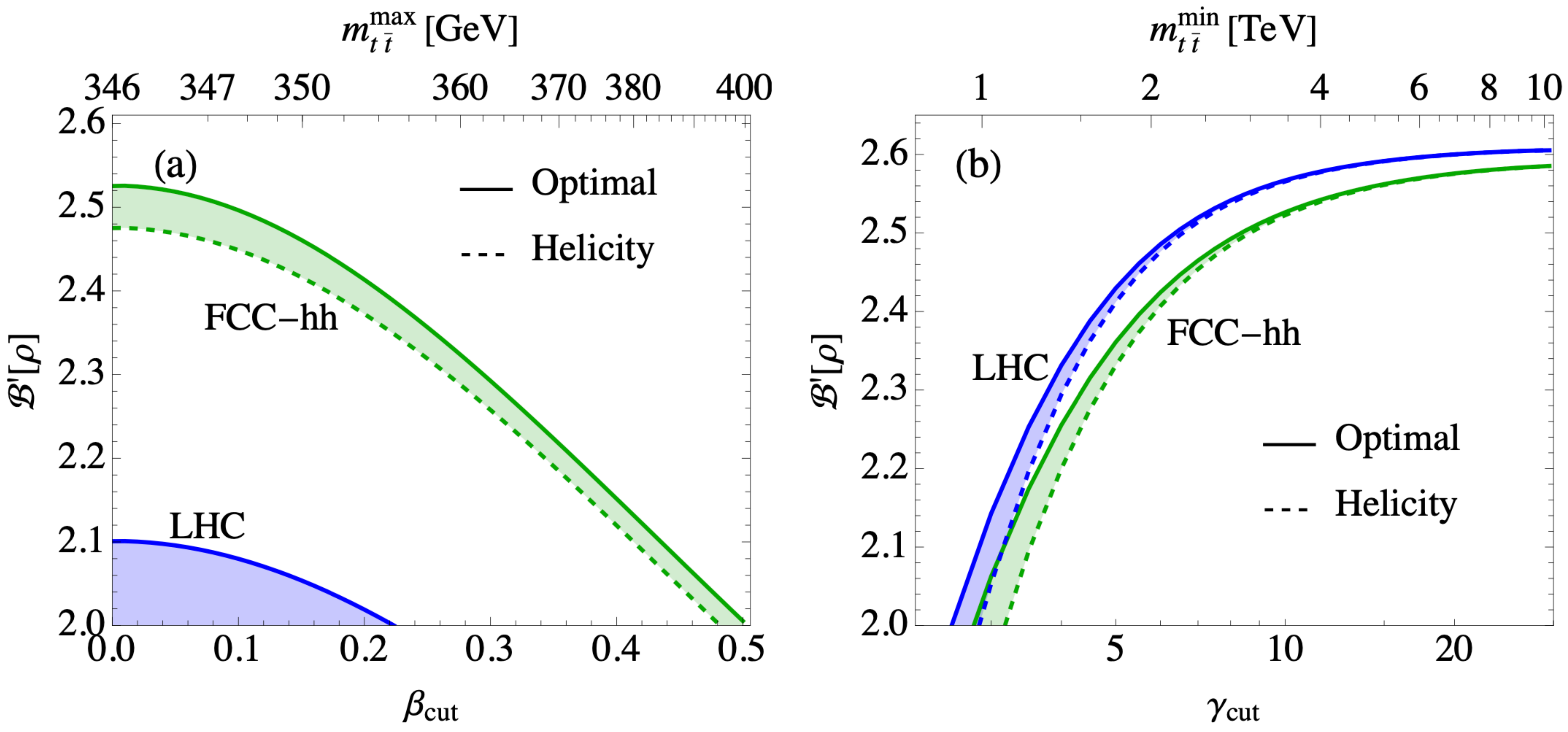
$$\mathscr{B}(\rho_{\text{fic}}) > 2 \implies \mathscr{B}(\rho_a) > 2$$

Optimal Basis Choice

- Event-by-event basis. $\Longrightarrow$ Why helicity basis? Why not others?
- Basis dependent $\Longrightarrow$ optimal basis exists







[**KC**, T. Han and M. Low, 2407.01672]

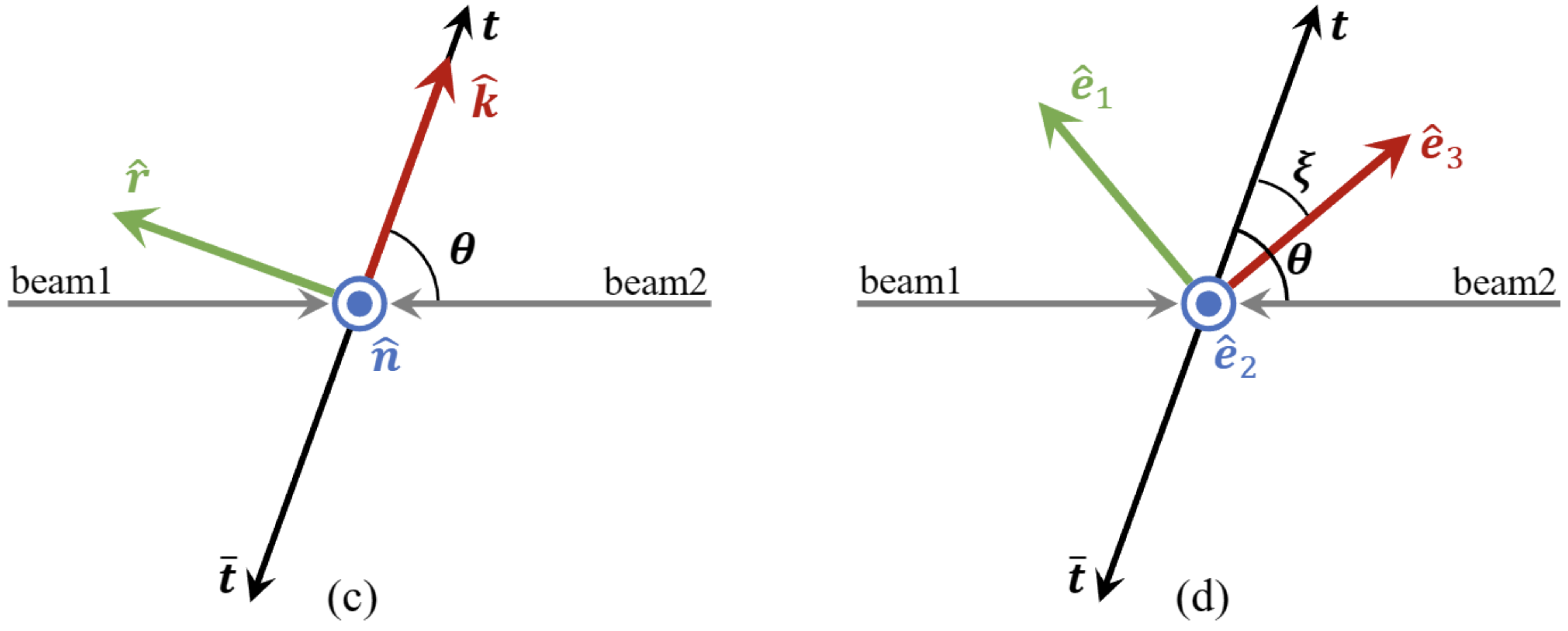
Optimal Basis Choice

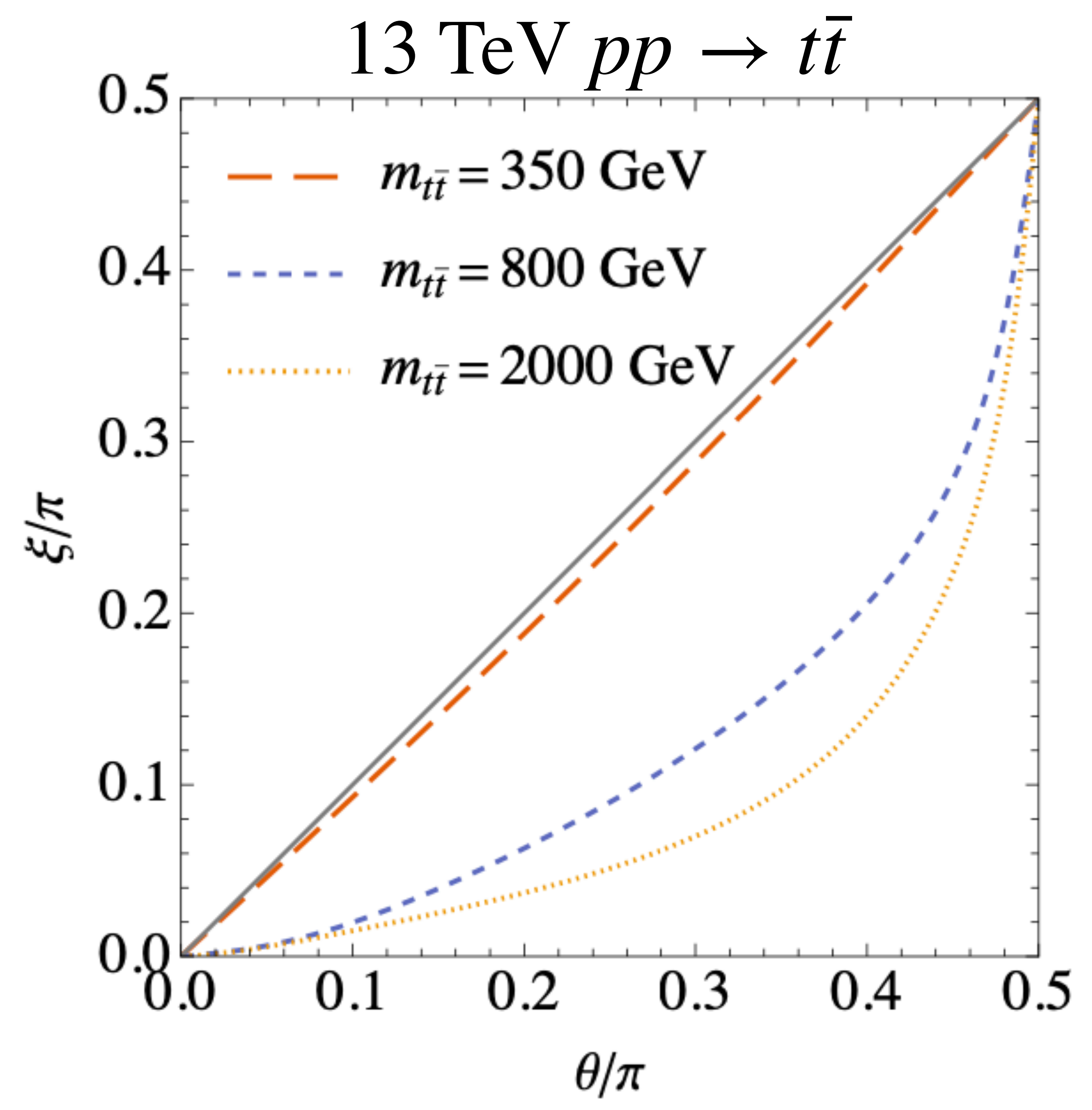
- The optimal basis is the one that diagonalized the correlation matrix [**KC**, T. Han and M. Low, 2311.07288]
- Avoid cancellation during the average

$$\bar{\rho} = \frac{1}{\sigma}\int \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\rho(\Omega)$$

$$\bar{C} = \frac{1}{\sigma}\int \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}C(\Omega)$$

- Same optimal basis for spin correlation, concurrence and Bell variable.





[**KC**, T. Han and M. Low, 2407.01672]

Conclusion:

- The increasingly comprehensive measurement of top pair spin correlation allow us to reconstruct a state and study quantum information.
  - The basis choice in this recontruction introduces arbitrariness.
- Event-by-event basis leads to fictitious state.
  - Fictitious states are basis dependent
  - Can leverage the basis-dependence to optimize the measurement of concurrence and Bell variable.

Phenomenology: Decay Products as Spin Analyzer

[B. Tweedie, 1401.3021]

Top rest frame

- A spin-up top quark $t_\uparrow \to \ell^+ \nu b$: $\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta_\ell} = \frac{1}{2}(1+\cos\theta_\ell)$

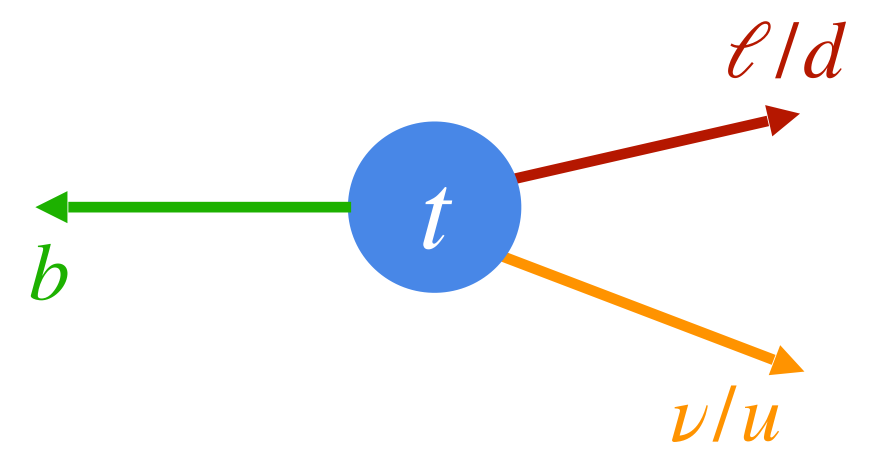
Spin Analyzer	Power
lepton/down-quark	1.00
neutrino/up-quark	-0.34
b-quark or W	$\mp$0.40
soft-quark	0.50

- Or generally, $\rho^t = \frac{1}{2}(I_2 + B_i\sigma_i)$, $\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta_a} = \frac{1}{2}(1 + B_z\kappa_a\cos\theta_a)$

κ_a: spin analysing power of a

$$B_i = 3\langle \ell_i^+ \rangle$$

ℓ_i^+: cosine of the angle between $\overrightarrow{\ell}^+$ and axis $\hat{e}_i$



- For $t\bar{t}$ system,

$$\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^+\sigma_i \otimes I_2 + B_i^- I_2 \otimes \sigma_i + C_{ij}\,\sigma_i\,\sigma_j\right)$$

The density matrix constructed from $t \to \ell^+\nu b$, $\bar{t} \to \ell^-\bar{\nu}\bar{b}$ decay channel is

$$B_i^+ = 3\langle \ell_i^+ \rangle, \quad B_i^- = -3\langle \ell_i^- \rangle, \quad C_{ij} = -9\langle \ell_i^+ \ell_j^- \rangle$$

Backup Top Pait at the LHC

$$\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^{+}\sigma_i \otimes I_2 + B_i^{-} I_2 \otimes \sigma_i + C_{ij}\,\sigma_i \sigma_j\right)$$

$$\bar{\rho} = \frac{1}{\sigma}\int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\rho(\Omega)$$

$$\bar{C} = \frac{1}{\sigma}\int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}C(\Omega)$$

- We are studying phase-space-averaged states

- Example: $D = \mathrm{tr}(C)/3$
  - $D = -0.547$, with $m_{t\bar{t}} < 380$ GeV [ATLAS: 2311.07288]
  - $D = -0.480$, with $m_{t\bar{t}} < 400$ GeV [CMS: 2406.03976]

Article | Open access | Published: 18 September 2024

Observation of quantum entanglement with top quarks at the ATLAS detector

The ATLAS Collaboration

Nature **633**, 542–547 (2024) | Cite this article

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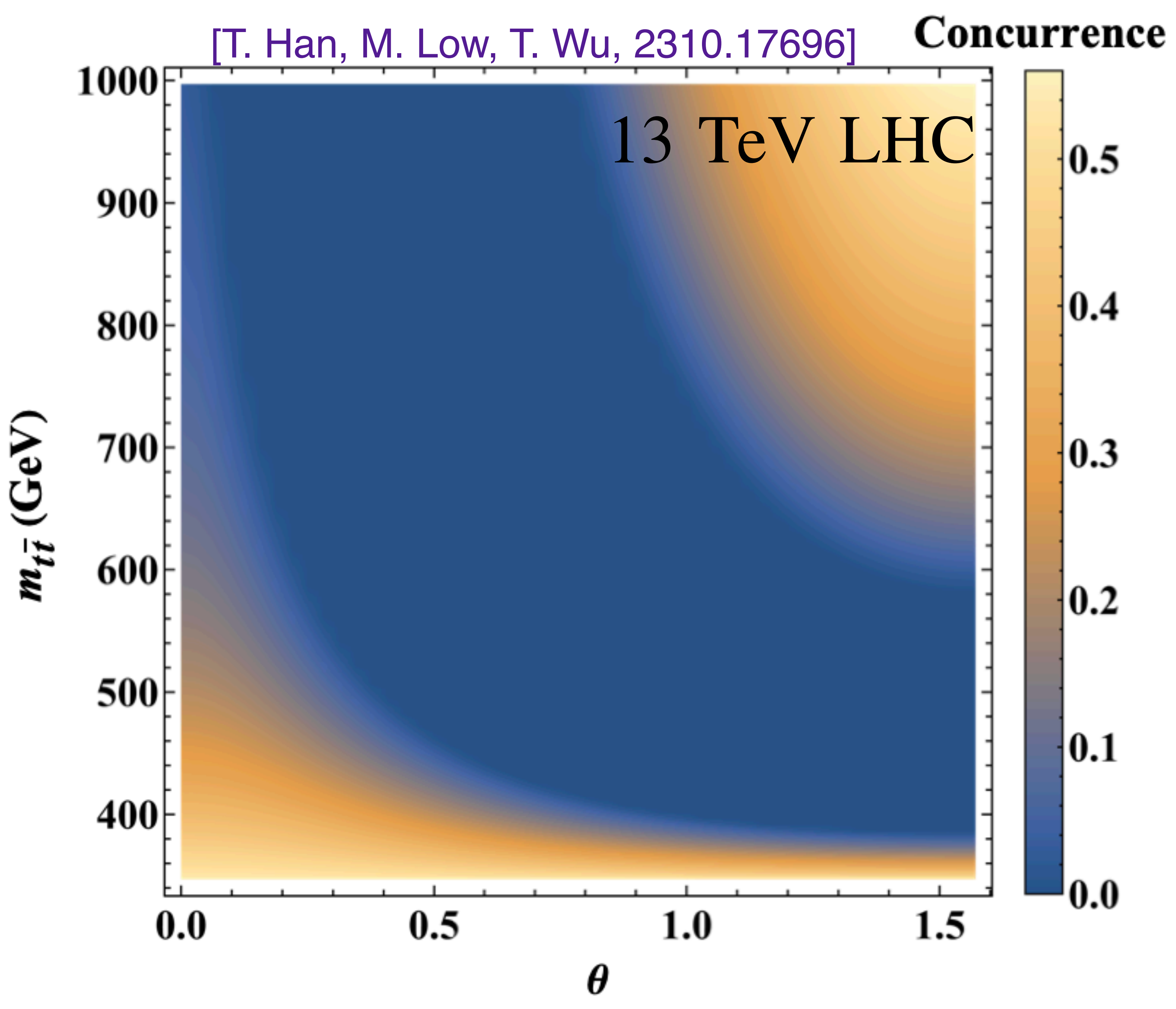
[T. Han, M. Low, T. Wu, 2310.17696]



Backup

- General density matrix (2x2) for 1 qubit, 3 parameters B_i
- General density matrix (4x4) for 2 qubit, 15 parameters

$$\rho = \frac{I_2 + B_i \sigma_i}{2}, \quad \vec{B} = \mathrm{tr}(\vec{\sigma}\rho), \quad |\vec{B}| \leq 1$$

$$\rho = \frac{1}{4}\left(\mathbf{I}_4 + \underline{B_i^+ \sigma_i \otimes \mathbf{I}_2} + \underline{B_i^- \mathbf{I}_2 \otimes \sigma_i} + \underline{C_{ij} \sigma_i \otimes \sigma_j}\right)$$

B_i^+ Polarization vector of qubit A

B_i^- Polarization vector of qubit B

C_{ij} Spin correlation matrix

e.g. Four Bell states:

$$|\Psi_0\rangle = \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$|\Psi_1\rangle = \frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{\sqrt{2}}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$|\Psi_2\rangle = i\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$|\Psi_3\rangle = -\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

e.g. $t\bar{t}$ from QCD production,
Unpolarized: $B^{\pm}$=0
Strongly correlated, $C_{ij} \neq 0$