GRIFFIN: A C++ library for higher-order electroweak corrections

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L. Chen and A. Freitas,
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`github.com/lisongc/GRIFFIN/releases`

Motivation

Tools for EW precision calculation pre-2022:

- **ZFITTER/DIZET, TOPAZØ, ...:** Bardin et al. '99 Montagna et al. '98
  - rad. corr. packages developed for LEP era
  - SM prediction for EWPOs and (diff.) cross-sections ($e^+e^- \to f\bar{f}$)
  - Full NLO corrections + partial higher orders
  - QED ISR/FSR corrections through analyt. formulae
  - Can be linked with MC codes (**KoralZ**, ...) Jadach et al. '80s–99
  - Difficult to expand and maintain
    (Fortran77, not fully gauge-invariant framework, ...)

- Modern fitting tools (**Gfitter, HEPfit, GAPP**): Baak et al. '14 de Blas et al. '19; Erler '00
  - own implementations of rad. corr.
    [only EWPOs (pseudo-obs.), not full observables]
  - extensions to higher orders, different schemes and models require custom work

Motivation

Goal of the GRIFFIN* project:

- New EW library that is modular / object-oriented (C++)
- Based on manifestly gauge-invariant setup
- Repository of existing calculations
- Can be extended to include ...
  - ... higher orders
  - ... different input parameter schemes
  - ... BSM physics (also SMEFT/HEFT)
  - ... new processes or new EWPOs
- Can be linked to MC generators and global fitting packages (QED more effectively handled with MC generators)

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* Gauge-invariant Resonance In Four-Fermion INteractions

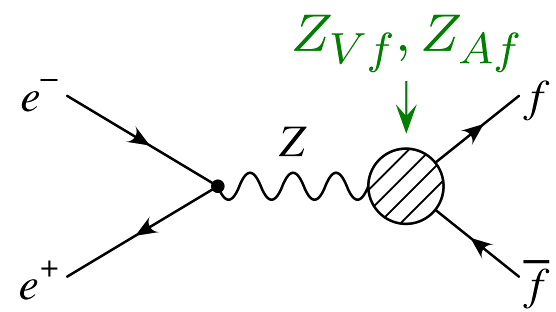
Standard EWPOs

Fermi constant:

electroweak corrections

$$\frac{G_F}{\sqrt{2}} = \frac{e^2}{8 s_\mathrm{W}^2 M_\mathrm{W}^2}(1 + \Delta r)$$

Z pole asymmetries:



$$A_\mathrm{FB} \equiv \frac{\sigma_\mathrm{F} - \sigma_\mathrm{B}}{\sigma_\mathrm{F} + \sigma_\mathrm{B}} = \frac{3}{4}\mathcal{A}_e \mathcal{A}_f$$

Final state τ pol. $\langle \mathcal{P}_\tau \rangle = -\mathcal{A}_\tau$

$$\mathcal{A}_f = \frac{2(1 - 4\sin^2\theta^f_\mathrm{eff})}{1 + (1 - 4\sin^2\theta^f_\mathrm{eff})^2} \qquad \sin^2\theta^f_\mathrm{eff} = \frac{1}{4|Q_f|}\left[1 - \mathrm{Re}\,\frac{Z_{Vf}}{Z_{Af}}\right]$$

Z$\to f\bar{f}$ partial widths:

$$\Gamma_{ff} = C\left[(Z_{Vf})^2 + (Z_{Af})^2\right] = C'\, F^f_A \left[(1 - 4|Q_f|\sin^2\theta^f_\mathrm{eff})^2 + \left(\mathrm{Im}\,\frac{Z_{Vf}}{Z_{Af}}\right)^2\right]$$

Theory calculations for EWPOs: Status

Available results for Δr, $\sin^2\theta^f_{\rm eff}$, Γ_{ff}:

- Many seminal works on 1-loop and leading 2-loop corrections

  Veltman, Passarino, Sirlin, Marciano, Bardin, Hollik, Riemann, Degrassi, Kniehl, ...

- Full 2-loop results

  Freitas, Hollik, Walter, Weiglein '00
  Awramik, Czakon '02
  Onishchenko, Veretin '02
  Awramik, Czakon, Freitas, Weiglein '04
  Awramik, Czakon, Freitas '06
  Hollik, Meier, Uccirati '05,07
  Awramik, Czakon, Freitas, Kniehl '08
  Freitas '14
  Dubovyk, Freitas, Gluza, Riemann, Usovitsch '16,18

- Partial higher orders: $\mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^2)$, $\mathcal{O}(\alpha_{\rm t}^2\alpha_{\rm s})$, $\mathcal{O}(\alpha_{\rm t}^3)$, $\mathcal{O}(\alpha_{\rm t}\alpha_{\rm s}^3)$, $\mathcal{O}(N_f^3\alpha^3)$, $\mathcal{O}(N_f^2\alpha^2\alpha_{\rm s})$

  Chetyrkin, Kühn, Steinhauser '95
  Faisst, Kühn, Seidensticker, Veretin '03
  Boughezal, Tausk, v. d. Bij '05
  Schröder, Steinhauser '05
  Chetyrkin et al. '06
  Boughezal, Czakon '06
  Chen, Freitas '20

$\alpha_{\rm t} = \frac{y_{\rm t}^2}{4\pi}$

$N_f =$ closed fermion loop

Full amplitude $e^+e^- \to f\bar{f}$

EWPOs like $\sin^2\theta^f_{\text{eff}}$, Γ_{ff} only contain **leading** EW corrections **on Z-pole**

For complete prediction need:

- Sub-leading corrections
- Off-peak matrix element
- QED/QCD ISR/FSR contributions

Pole expansion

Expand amplitude for $e^+e^- \to f\bar{f}$ about **complex pole** $s_0 \equiv \overline{M}_Z^2 + i\overline{M}_Z\overline{\Gamma}_Z$:

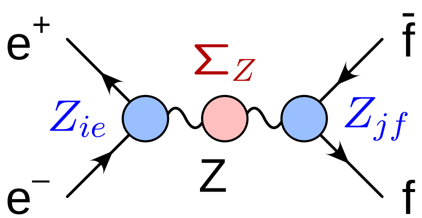
$\longrightarrow$ All terms are individually gauge-invariant

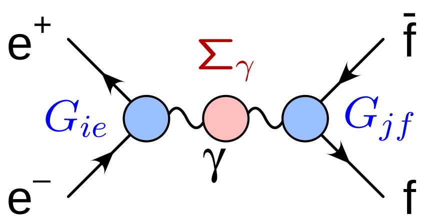
$$\mathcal{M}_{ij} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0)S'_{ij} + \ldots \qquad (i,j = V,A)$$

$$R_{ij} = \left.\frac{Z_{ie}Z_{jf}}{1 + \Sigma'_Z}\right|_{s=s_0}$$

$$S_{ij} = \left[\frac{Z_{ie}Z'_{jf} + Z'_{ie}Z_{jf}}{1 + \Sigma'_Z} - \frac{Z_{ie}Z_{jf}\Sigma''_Z}{2(1 + \Sigma'_Z)^2} + \frac{G_{ie}G_{jf}}{s + \Sigma_\gamma} + B_{ij}\right]_{s=s_0}$$

$$S'_{ij} = \ldots$$





Pole expansion

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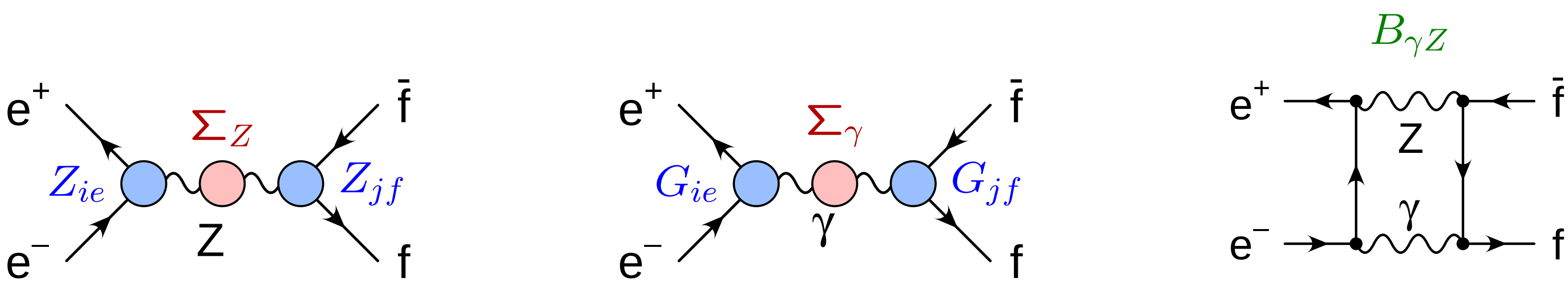
$$\mathcal{M}_{ij} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0)S'_{ij} + \ldots \qquad (i,j = V, A)$$

$$R_{ij} = \left.\frac{Z_{ie}Z_{jf}}{1 + \Sigma'_Z}\right|_{s=s_0} + B^R_{\gamma Z,ij} + B^{RL}_{\gamma Z,ij} \ln(1 - \tfrac{s}{s_0})$$

$$S_{ij} = \left[\frac{Z_{ie}Z'_{jf} + Z'_{ie}Z_{jf}}{1 + \Sigma'_Z} - \frac{Z_{ie}Z_{jf}\Sigma''_Z}{2(1 + \Sigma'_Z)^2} + \frac{G_{ie}G_{jf}}{s + \Sigma_\gamma} + B_{ij}\right]_{s=s_0}$$
$$+ B^S_{\gamma Z,ij} + B^{SL}_{\gamma Z,ij} \ln(1 - \tfrac{s}{s_0})$$

$$S'_{ij} = \ldots$$



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$$R_{ij} = \left.\frac{Z_{ie}Z_{jf}}{1+\Sigma'_Z}\right|_{s=s_0} + B^{R}_{\gamma Z,ij} + B^{RL}_{\gamma Z,ij}\ln(1-\tfrac{s}{s_0})$$

$$= 4 I^3_e I^3_f \sqrt{F^e_A F^f_A}\left[Q^e_i Q^f_j + \ldots\right], \qquad Q^f_V = 1 - 4|Q_f|\sin^2\theta^f_{\rm eff}, \quad Q^f_A = 1$$

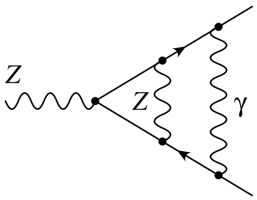
terms related to
Im Z_{ie}, Im Z_{jf}, Im Σ_Z
and the γZ box

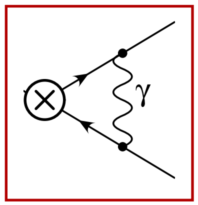
Express R_{ij} in terms of $\sin^2\theta^f_{\rm eff}$ and F^f_A (accurate up to NNLO)

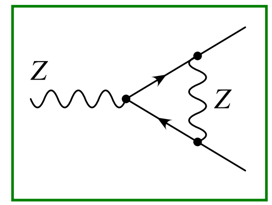
Initial-/final-state radiation

Factorization of massive EW corrections and QED/QCD ISR/FSR:

$$Z_{if}^{\text{tot}} = R_f^i \times Z_{if}, \qquad R_f^{\mathrm{V}}(s) \equiv \frac{\mathcal{M}^{\text{QED/QCD}}_{V^*\to f\bar{f}}}{\mathcal{M}^{\text{Born}}_{V^*\to f\bar{f}}}, \qquad R_f^{\mathrm{A}}(s) \equiv \frac{\mathcal{M}^{\text{QED/QCD}}_{A^*\to f\bar{f}}}{\mathcal{M}^{\text{Born}}_{A^*\to f\bar{f}}},$$



$=$


$+$ finite, with $\otimes =$


$\mathcal{R}_V^f$, $\mathcal{R}_A^f$: QED/QCD radiation factors;

FSR known inclusively to $\mathcal{O}(\alpha_s^4)$, $\mathcal{O}(\alpha^2)$, $\mathcal{O}(\alpha\alpha_s)$ Chetyrkin, Kühn, Kwiatkowski '96
Kataev '92; Baikov, Chetyrkin, Kühn, Rittinger '12

ISR via structure functions with LL resummation Kureav, Fadin '85
Montagna, Nicrosini, Piccinini '97
Ablinger, Blümlein, De Freitas, Schönwald '20

or compute exclusively using MC methods,

e.g. **KKMC**, Arbuzov, Jadach, Wąs, Ward, Yost '20

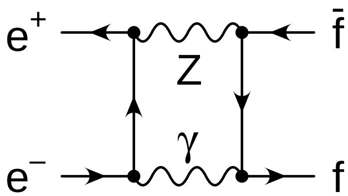
SHERPA_YFS, Krauss, Price, Schönherr '22

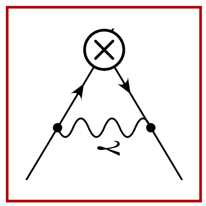
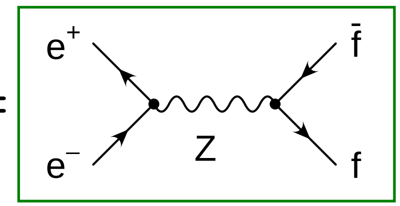
POWHEG_EW Barzè, Montagna, Nason, Nicrosini, Piccinini '12,13

ISR-FSR inteference (IFI)

QED soft IR singular pieces for IFI also factorize:

$$B_{ij(1)} = B^{\text{tot}}_{ij(1)} - \mathcal{M}_{ij(0)}\, 2Q_eQ_f\left[R_{e(1)}(t) - R_{e(1)}(u)\right]$$



$=$  $+$ finite, with $\otimes =$ 

Off-peak contribution

Pole expansion works well in window of few GeV about Z pole, but not beyond

$$\mathcal{M}_{ij}^{\text{exp},s_0} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0)S'_{ij} + \dots$$

Outside this window, use $\mathcal{M}_{ij}^{\text{noexp}}$ without expansion in s and Dyson summation:

$$\mathcal{M}_{ij} = \frac{R_{ij}}{s - s_0} + \mathcal{M}_{ij}^{\text{noexp}} - \mathcal{M}_{ij}^{\text{exp},\overline{M}_Z}$$

To avoid double counting and cancel unphys. pole at $s=\overline{M}_Z^2$ in $\mathcal{M}_{ij}^{\text{noexp}}$:

$$M_{ij}^{\text{exp},M_Z^2} = \mathcal{T}_\alpha \left\{ \left[\frac{R_{ij}}{s - s_0} \right]_{s_0 = \overline{M}_Z^2 - i\overline{M}_Z \alpha \overline{\Gamma}_{Z(1)}} \right\}$$

$\mathcal{T}_x$ = Taylor operator in x

- See also Dittmaier, Huber '09
- Could also use complex-mass scheme to compute $\mathcal{M}_{ij}^{\text{noexp}}$

Denner, Dittmaier, Roth, Wieders '05; Denner, Dittmaier '06

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$$\mathcal{M}_{ij} = \frac{R_{ij}}{s - s_0} + \mathcal{M}_{ij}^{\mathrm{noexp}} - \mathcal{M}_{ij}^{\mathrm{exp},\overline{M}_Z}$$

Implementation in GRIFFIN v1.0/1.1: ↑ @NNLO (for $\frac{R_{ij}}{s - s_0}$); ↖ ↗ @NLO (for $\mathcal{M}_{ij}^{\mathrm{noexp}}$ and $\mathcal{M}_{ij}^{\mathrm{exp},\overline{M}_Z}$)

Off-peak contribution

Pole expansion works well in window of few GeV about Z pole, but not beyond

$$\mathcal{M}_{ij}^{\text{exp},s_0} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0) S'_{ij} + \ldots$$

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Implementation in GRIFFIN v1.0/1.1:

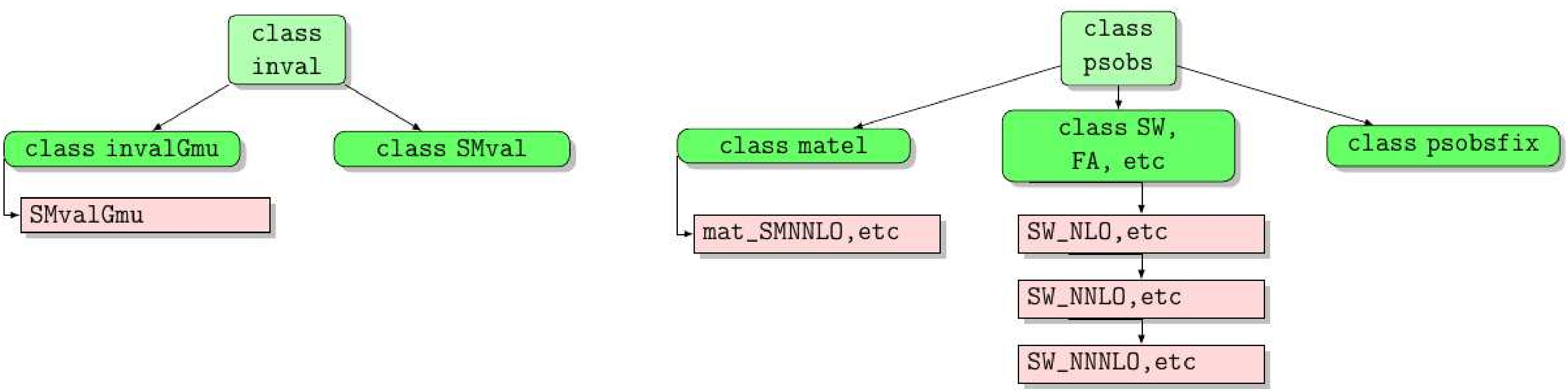
$\uparrow$ @NNLO (for $\frac{R_{ij}}{s-s_0}$)

$\nwarrow$ $\nearrow$ @NLO (for $\mathcal{M}_{ij}^{\text{noexp}}$ and $\mathcal{M}_{ij}^{\text{exp},\overline{M}_Z}$)

SM predictions for EWPOs (Δr, $\sin^2\theta^f_{\text{eff}}$, F^f_A) at NNLO+

Structure of the library

`class inval`		`class psobs`	
input parameters (in the SM)		output observables	
Boson masses and widths	$M_{\mathrm{W,Z,H}}$ $\Gamma_{\mathrm{W,Z}}$	pesudo-observables defined at Z-peak	$F_{V,A}$, $\sin^2\theta^f_{eff}$ $\Gamma_{Z\to f\bar{f}}$, Δr, etc.
Fermion masses	$m^{\mathrm{OS}}_{\mathrm{e},\mu,\tau}$ $m^{\overline{\mathrm{MS}}}_{\mathrm{d,u,s,c}}(M_{\mathrm{Z}})$ $m^{\mathrm{OS}}_{\mathrm{t}}$	amplitude coefficients under pole scheme	R , S , and S'
Couplings	$\alpha(0)$ $\Delta\alpha \equiv 1-\alpha(0)/\alpha(M^2_{\mathrm{Z}})$ $\alpha^{\overline{\mathrm{MS}}}_s(M^2_{\mathrm{Z}})$, G_μ	(polarized) matrix element square near Z-peak	$\mathrm{Re}\, M_{ij}M^*_{kl}$



Sample program


```
#include <iostream>
using namespace std;

#include "EWPOZ2.h"
#include "xscnnlo.h"
#include "SMval.h"

int main()
{
  SMval myinput;  // convert masses from PDG values to complex pole scheme
  myinput.set(al, 1/137.03599976);
  myinput.set(MZ, 91.1876);
  myinput.set(MW, 80.377);
  myinput.set(GamZ, 2.4952);
  myinput.set(GamW, 2.085);
  myinput.set(MH, 125.1);
  myinput.set(MT, 172.5);
  myinput.set(MB, 2.87);
  myinput.set(Delal, 0.059);
  myinput.set(als, 0.1179);

  cout << endl << "Complex-pole masses: MW=" << myinput.get(MWc) << ", MZ="
    << myinput.get(MZc) << endl << endl;
```

Sample program (2)


```
// compute matrix element for ee->dd with vector coupling in initial
// state and vector coupling in final state
int ini = ELE, fin = DQU, iff = VEC, off = VEC;

cout << "=== Matrix element for ee->dd (i=e, f=d) ===" << endl << endl;

// compute vertex form factors:
FA_SMNNLO FAi(ini, myinput), FAf(fin, myinput);
SW_SMNNLO SWi(ini, myinput), SWf(fin, myinput);
cout << "F_A^i (NNLO+) = " << FAi.result() << endl;
cout << "F_A^f (NNLO+) = " << FAf.result() << endl;
cout << "sineff^i (NNLO+) = " << SWi.result() << endl;
cout << "sineff^f (NNLO+) = " << SWf.result() << endl;
cout << endl;
```

Sample program (3)


```
  double cme,          // center-of-mass energy
         cost = 0.5; // scattering angle
  Cplx res1, res2;

  cout << "SM matrix element M_VV for cos(theta)=" << cost << ": " << endl;
  // compute matrix element for ee->dd using SM form factors:
  mat_SMNNLO M(ini, fin, iff, off, FAi, FAf, SWi, SWf, cme*cme, cost,
   myinput);
  cout << "sqrt(s)\t\ttot. result\t\toff-resonance contrib." << endl;
  for(cme = 10.; cme <= 190.; cme += 20.)
  {
    M.setkinvar(cme*cme, cost);
    res1 = M.result();
    res2 = M.resoffZ();
    cout << cme << " \t" << res1 << " \t" << res2 << endl;
  }
  cout << endl;

  return 0;
}
```

Sample program (output)


```
Complex-pole masses: MW=80.35, MZ=91.1535

=== Matrix element for ee->dd (i=e, f=d) ===

F_A^i (NNLO+) = (0.034499,0)
F_A^f (NNLO+) = (0.0345443,0)
sineff^i (NNLO+) = (0.231172,0)
sineff^f (NNLO+) = (0.230985,0)

SM matrix element M_VV for cos(theta)=0.5:
sqrt(s)        tot. result                   off-resonance contrib.
10       (0.000316739,-5.58082e-06)      (0.000309429,-5.53734e-06)
30       (3.53793e-05,-5.99317e-07)      (2.84458e-05,-5.59139e-07)
50       (1.25851e-05,-1.90789e-07)      (6.4247e-06,-1.59184e-07)
70       (6.07798e-06,-5.97311e-08)      (1.19433e-06,-4.81728e-08)
90       (-7.31188e-07,-3.55673e-06)     (8.7104e-09,-1.80673e-09)
110      (3.14635e-06,-1.62001e-07)      (4.59289e-07,1.10821e-08)
130      (2.12596e-06,-7.90095e-08)      (1.82894e-06,1.92144e-08)
150      (1.5668e-06,-5.34561e-08)       (3.83515e-06,2.49419e-08)
170      (1.20884e-06,-3.97403e-08)      (6.35319e-06,2.97998e-08)
190      (9.60973e-07,-3.33532e-08)      (9.31833e-06,3.12732e-08)
```

Comparison GRIFFIN 1.0 vs. DIZET 6.45

❑ Numerical Results:

$$|\rho_Z^f| = \frac{2\sqrt{2}F_A^f}{G_\mu M_Z^2}$$

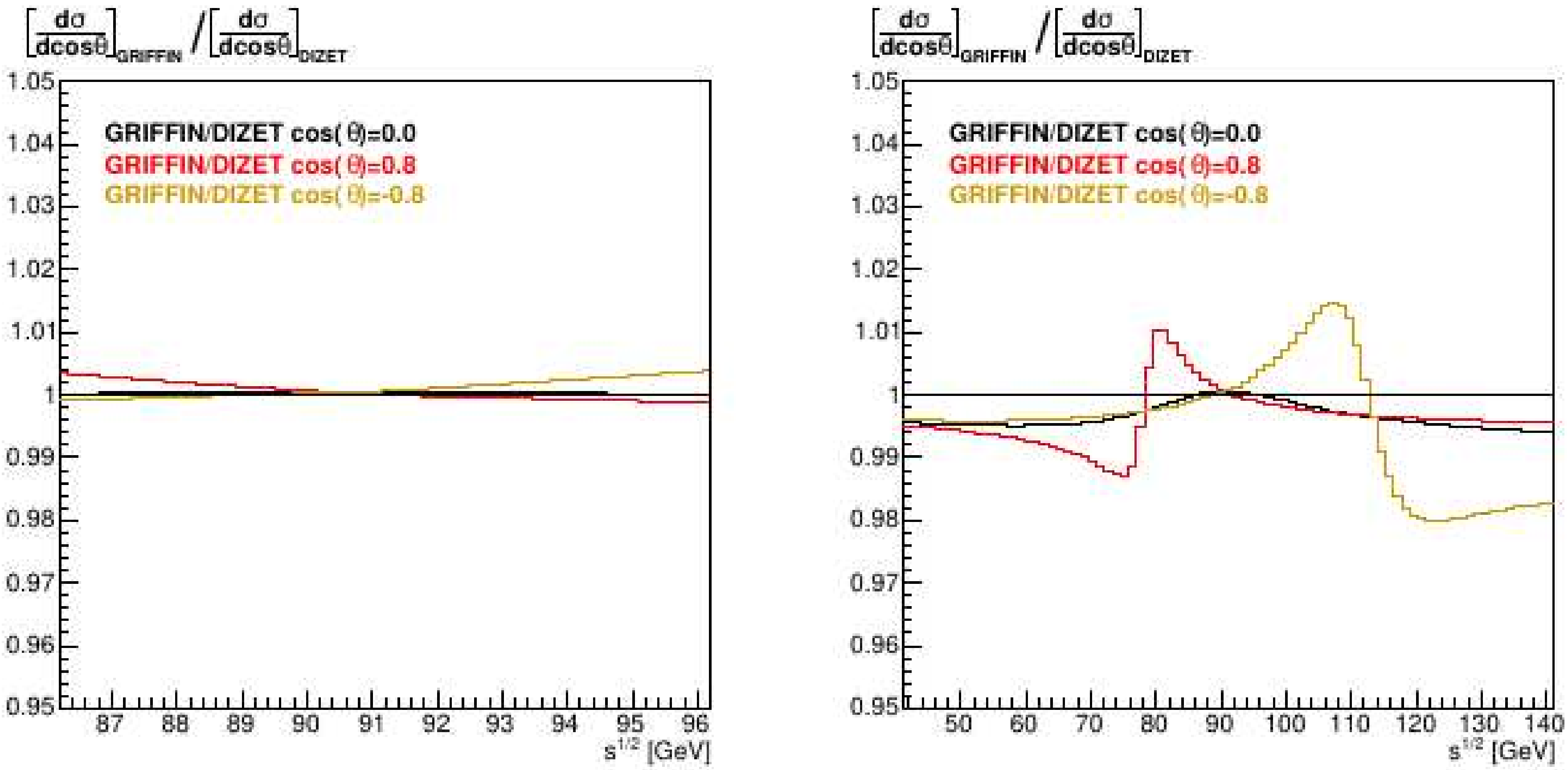
	$\lvert\rho_Z^f\rvert$		$\sin^2\theta_{\rm eff}^f$		$\Gamma_{Z\to f\bar{f}}$	
	DIZET 6.45	GRIFFIN	DIZET 6.45	GRIFFIN	DIZET 6.45	GRIFFIN
$\nu\bar{\nu}$	1.00800	1.00814	0.231119	NAN	0.167206	0.167197
$\ell\bar{\ell}$	1.00510	1.00519	0.231500	0.231534	0.083986	0.083975
$u\bar{u}$	1.00578	1.00573	0.231393	0.231420	0.299938	0.299958
$d\bar{d}$	1.00675	1.00651	0.231266	0.231309	0.382877	0.382846
$b\bar{b}$	0.99692	0.99420	0.232737	0.23292	0.376853	0.377432

	DIZET 6.45	GRIFFIN all orders	GRIFFIN $\mathcal{O}(\alpha, \alpha^2, \alpha_t\alpha_s, \alpha_t\alpha_s^2)$
Δr	3.63947×10^{-2}	3.68836×10^{-2}	3.63987×10^{-2}

- ❑ Not a **one-one-one match.** (no leading N3LO implemented in dizet v.6.45)
- ❑ most numbers are in agreement up to at least **4-digit**. The actual discrepancy is in the realm of missing N3(4)LO.
- ❑ fictitious discrepancies stem from the input scheme/definition of the form factors/EWPOs.

Comparison GRIFFIN 1.0 vs. DIZET 6.45

Ratios of differential cross-sections for $e^+e^- \to \mu^+\mu^-$ for different θ:



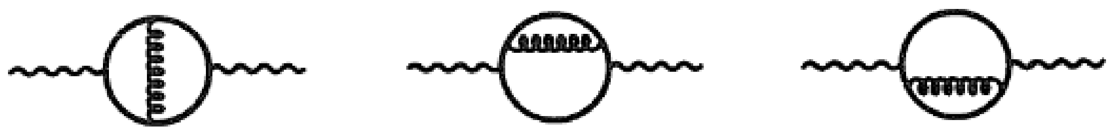
- $\lesssim \mathcal{O}(10^{-3})$ agreement near Z-pole ($\sim$NNLO precision)
- %-level agreement away from Z pole (NLO prec., different implementations)

[Note: enhanced corrections when tree-level matrix element is small]

Changes in GRIFFIN 1.1

Released Sept. 24

- Implementation of $f\bar{f} \to f\bar{f}$ (e.g. Bhabha scattering)
- Implementation of $\mathcal{O}(\alpha_f \alpha_s)$ corrections off-Z-resonance
  ($\alpha_f =$ EW corr. with closed fermion loops)
  $\to$ new calculation



- Some technical improvements
- Adoption of `cmake` build system; compilation to link library

Outlook

Future upgrades:

- Higher-order off-resonance corrections, e.g.
  $\mathcal{O}(\alpha\alpha_s)$, Heller, v.Manteuffel, Schabinger, Spiesberger '20
  Bonciani et al. '21
  $\mathcal{O}(N_f\alpha^2)$
- SMEFT $d{=}6$ operator effects
- W production and decay (a.k.a. charged-current DY)

Try out the code: **`github.com/lisongc/GRIFFIN/releases`**

Feedback welcome!

Backup slides

Implementation of higher-order corrections

Corrections entering through $\delta\rho$:			
	drho2aas	$\mathcal{O}(\alpha_t\alpha_s)$	[3, 4]
	drho2a2	$\mathcal{O}(\alpha_t^2)$	[5–9]
*	drho3aas2	$\mathcal{O}(\alpha_t\alpha_s^2)$	[10, 11]
*	drho3a2as	$\mathcal{O}(\alpha_t^2\alpha_s)$	[12, 13]
*	drho3a3	$\mathcal{O}(\alpha_t^3)$	[12, 13]
*	drho4aas3	$\mathcal{O}(\alpha_t\alpha_s^3)$	[14–16]
Full corrections to F_A^f, $\sin^2\theta_{\rm eff}^f$:			
*	res2ff	$\mathcal{O}(\alpha_f^2)$	[17–19]
*	res2fb	$\mathcal{O}(\alpha_f\alpha_b)$	[17–20]
*	res2bb	$\mathcal{O}(\alpha_b^2)$	[21–25]
*	res2aas	$\mathcal{O}(\alpha\alpha_s)$	[26–28] (correction to internal gauge-boson self-energies)
*	res2aasnf	$\mathcal{O}(\alpha\alpha_s)$	[29–34] (non-factorizable final-state corrections for $f = q$)
*	res3fff	$\mathcal{O}(\alpha_f^3)$	[35]
*	res3ffa2as	$\mathcal{O}(\alpha_f^2\alpha_s)$	[36]

Z lineshape

- Deconvolution of initial-state QED radiation:

$$\sigma[e^+e^- \to f\bar{f}] = \mathcal{R}_{\text{ini}}(s,s') \otimes \sigma_{\text{hard}}(s')$$

Kureav, Fadin '85
Berends, Burgers, v. Neerven '88
Kniehl, Krawczyk, Kühn, Stuart '88
Beenakker, Berends, v. Neerven '89
Bardin et al. '91; Skrzypek '92
Montagna, Nicrosini, Piccinini '97

Soft photons (resummed) + collinear photons

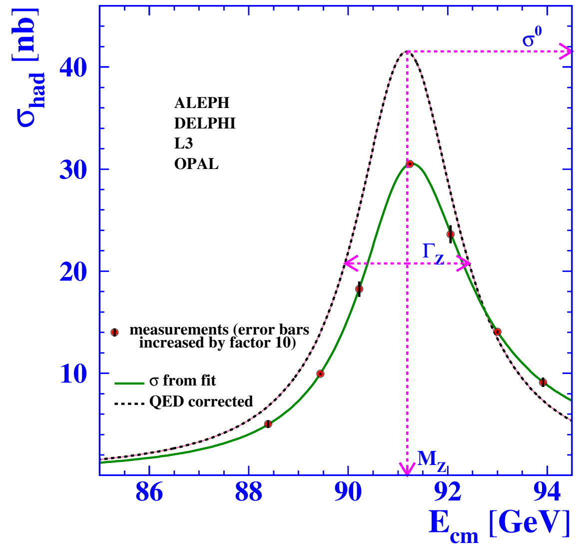
$$\mathcal{R}_{\text{ini}} = \sum_n \left(\frac{\alpha}{\pi}\right)^n \sum_{m=0}^{n} h_{nm} \ln^m\left(\frac{s}{m_e^2}\right)$$

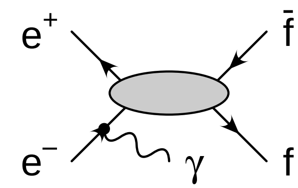
Universal (m=n) logs known to $n = 6$,
also some sub-leading terms

Ablinger, Blümlein, De Freitas, Schönwald '20

Exclusive description: MC tools

LEP EWWG '05

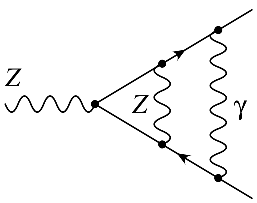




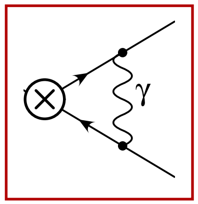
Initial-/final-state radiation

Factorization of massive EW corrections and QED/QCD ISR/FSR:

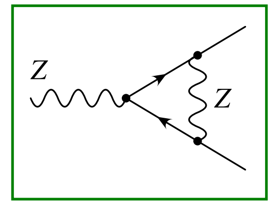
$$Z_{if}^{\text{tot}} = R_f^i \times Z_{if}, \qquad R_f^{\mathsf{V}}(s) \equiv \frac{\mathcal{M}^{\text{QED/QCD}}_{V^* \to f\bar{f}}}{\mathcal{M}^{\text{Born}}_{V^* \to f\bar{f}}}, \qquad R_f^{\mathsf{A}}(s) \equiv \frac{\mathcal{M}^{\text{QED/QCD}}_{A^* \to f\bar{f}}}{\mathcal{M}^{\text{Born}}_{A^* \to f\bar{f}}},$$



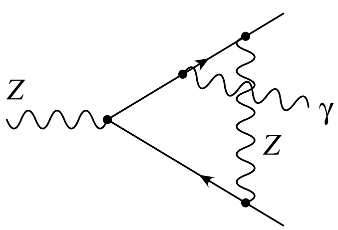
$=$



$+$ finite, with $\otimes =$



Additional non-factorizable contributions, e.g.



$\rightarrow$ Incorporated in F_A^f, $\sin^2 \theta^f_{\text{eff}}$ form factors

$\rightarrow$ Known at $\mathcal{O}(\alpha\alpha_s)$ Czarnecki, Kühn ’96
Harlander, Seidensticker, Steinhauser ’98

$\rightarrow$ Currently not known at $\mathcal{O}(\alpha^2)$ and beyond

Pole expansion

Expand amplitude for $e^+e^- \to f\bar{f}$ about **complex pole** $s_0 \equiv \overline{M}_Z^2 + i\overline{M}_Z\overline{\Gamma}_Z$:

$$\mathcal{M}_{ij} = \frac{R_{ij}}{s - s_0} + S_{ij} + (s - s_0)S'_{ij} + \ldots \qquad (i,j = V,A)$$

Current state of art: R @ NNLO + leading higher orders
S @ NLO
S' @ (N)LO

For future ee colliders: (at least) one order more!

Cross-section: $\sigma_Z = \dfrac{R}{(s - \overline{M}_Z^2)^2 + \overline{M}_Z^2\overline{\Gamma}_Z^2} + \sigma_{\text{non-res}}$

In exp. studies: $\sigma \sim \dfrac{1}{(s - M_Z^2)^2 + s^2\Gamma_Z^2/M_Z^2}$

$$\overline{M}_Z = M_Z/\sqrt{1 + \Gamma_Z^2/M_Z^2} \approx M_Z - 34\text{ MeV}$$

$$\overline{\Gamma}_Z = \Gamma_Z/\sqrt{1 + \Gamma_Z^2/M_Z^2} \approx \Gamma_Z - 0.9\text{ MeV}$$