

Parity Violation on Bell Tests

Yong Du (杜勇)

The 2024 International Workshop on the High Energy CEPC,
Hangzhou, Oct 24, 2024

Based on

2409.15418, with Xiao-Gang He, Chia-Wei Liu, Jian-Ping Ma



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Editorial | Published: 08 September 2022

Survey the foundations

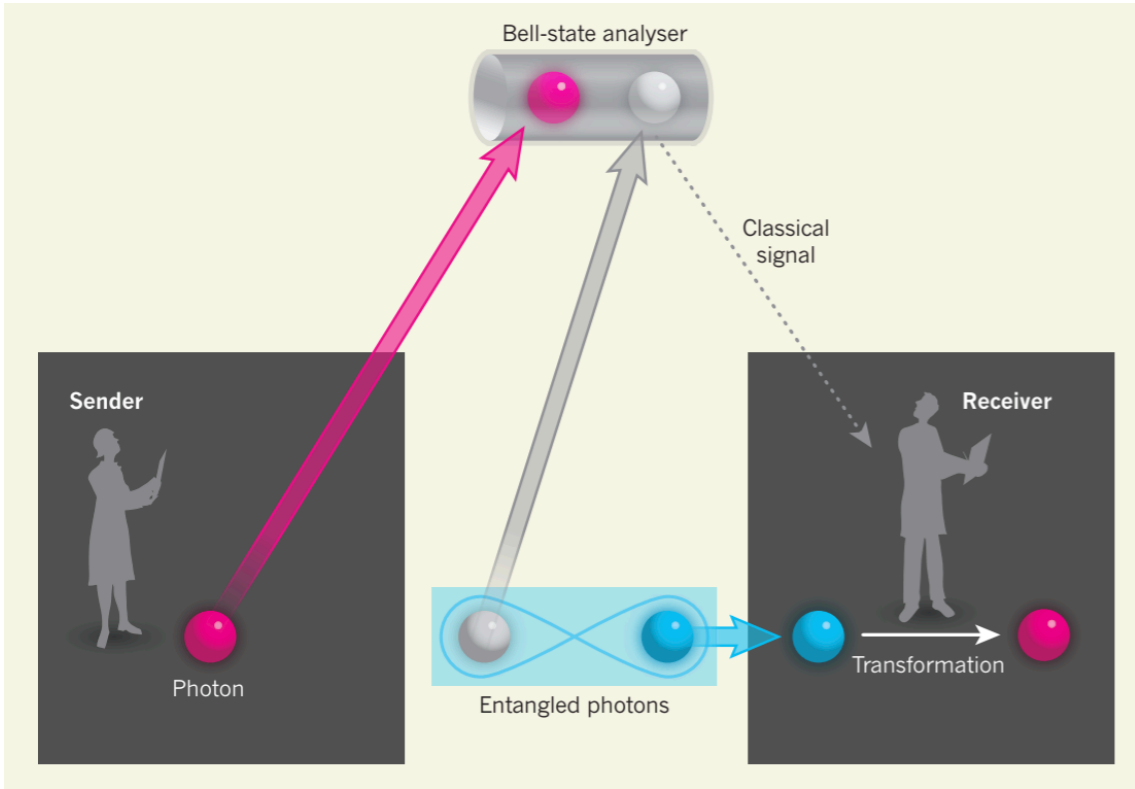
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It is easy to dismiss research into the foundations of quantum mechanics as irrelevant to physicists in other areas. Adopting this attitude misses opportunities to appreciate the richness of quantum mechanics.

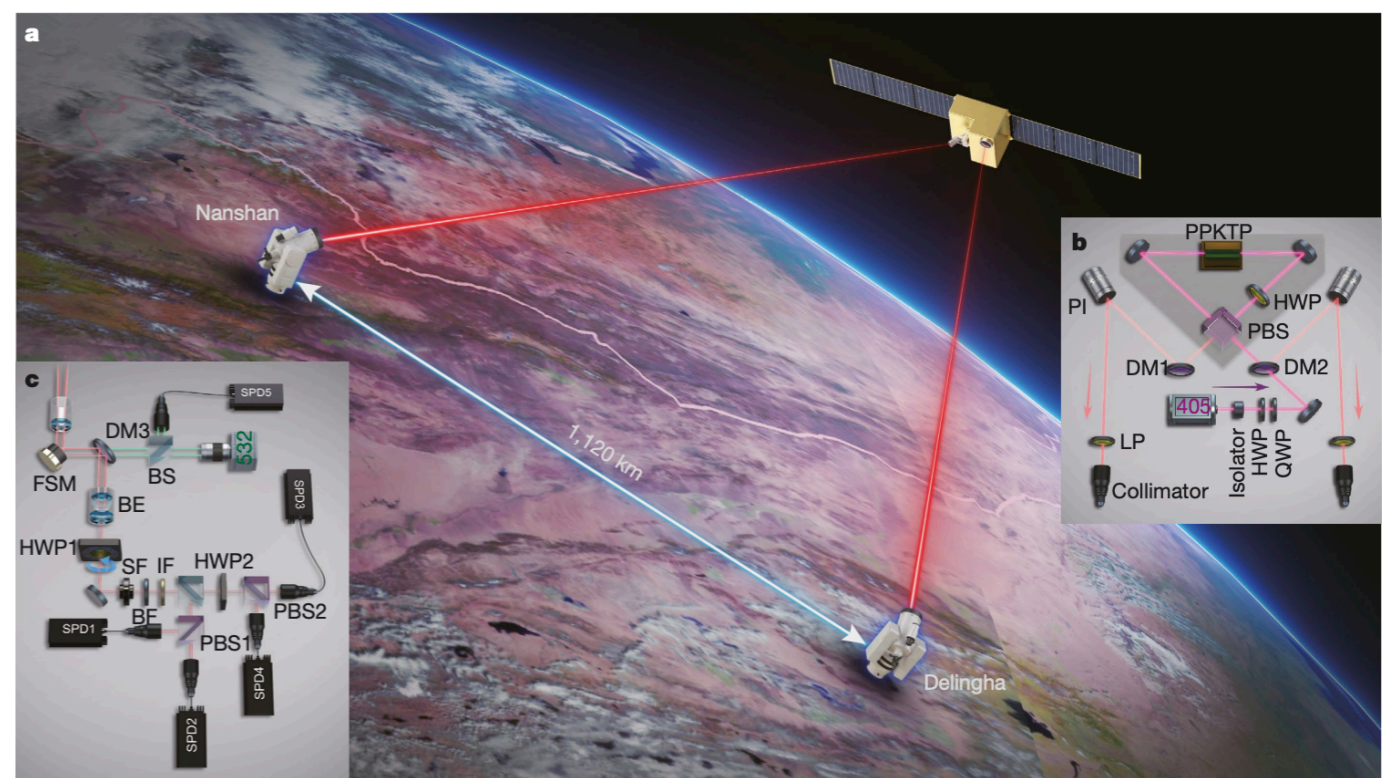
Introduction

Quantum teleportation with entangled photons



Gisin, Nature, 2017

Quantum Cryptography and new protocols: Quantum Key Distribution using entangled photons

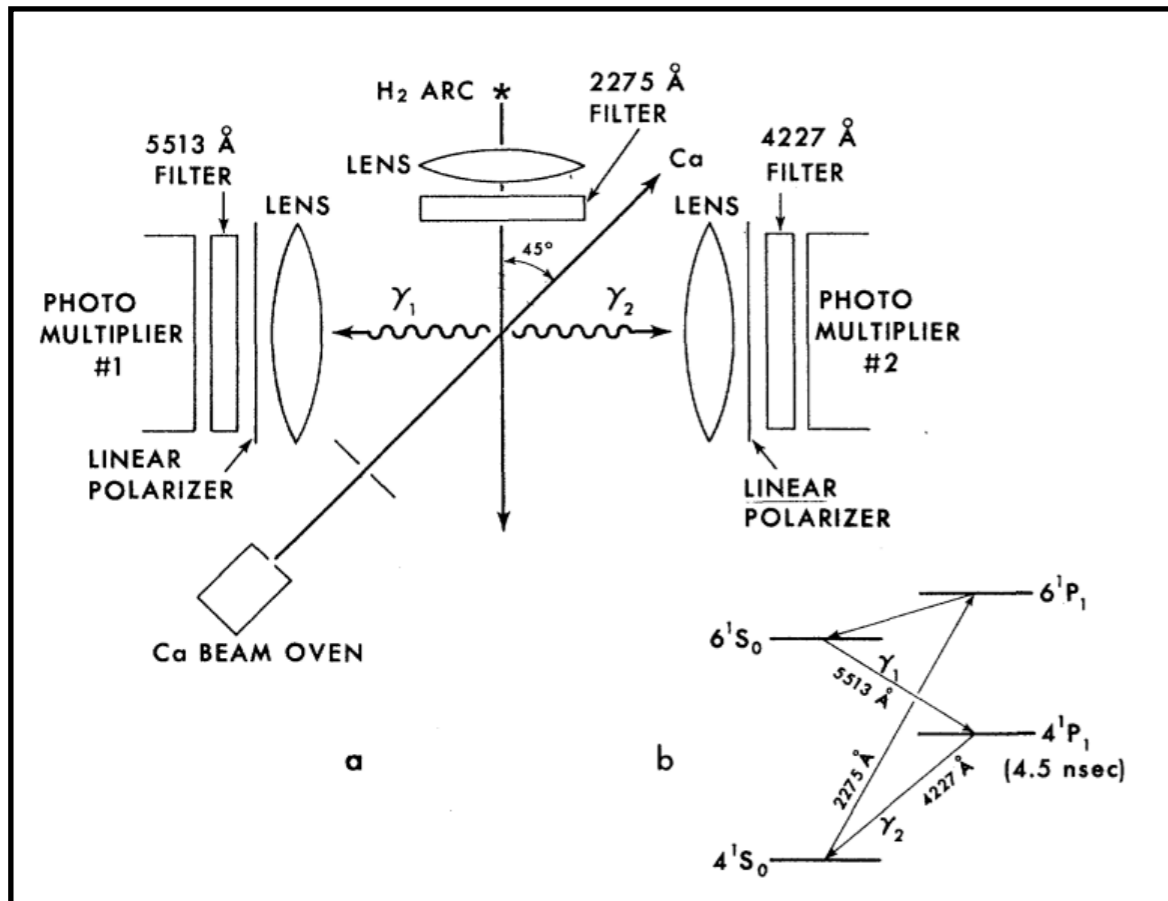


Yin et al, Nature 2020

Introduction

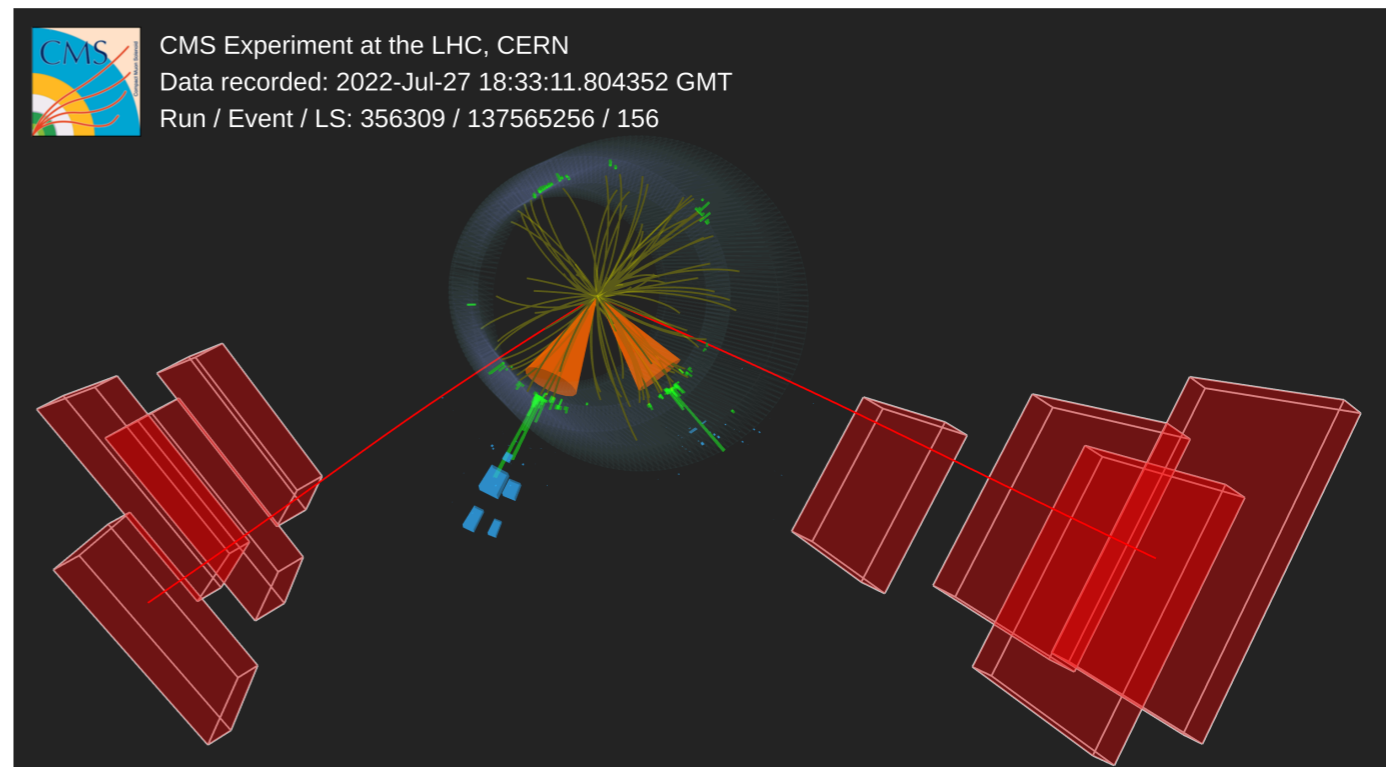
Practically, it is not that challenging to prepare entangled pairs

Cascade photons



Kocher & Commins, PRL, 1967

$t\bar{t}$ at the LHC

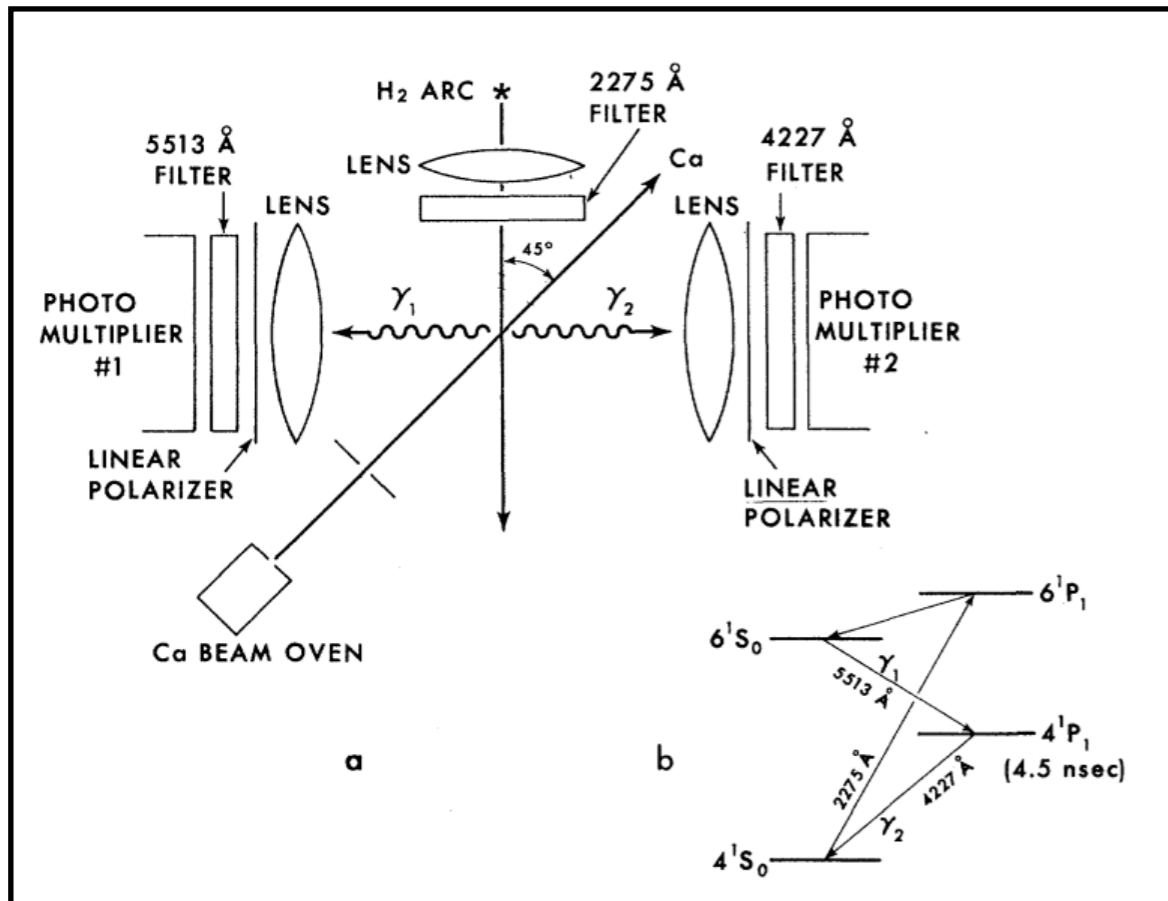


CMS-PHO-EVENTS-2022-033

Introduction

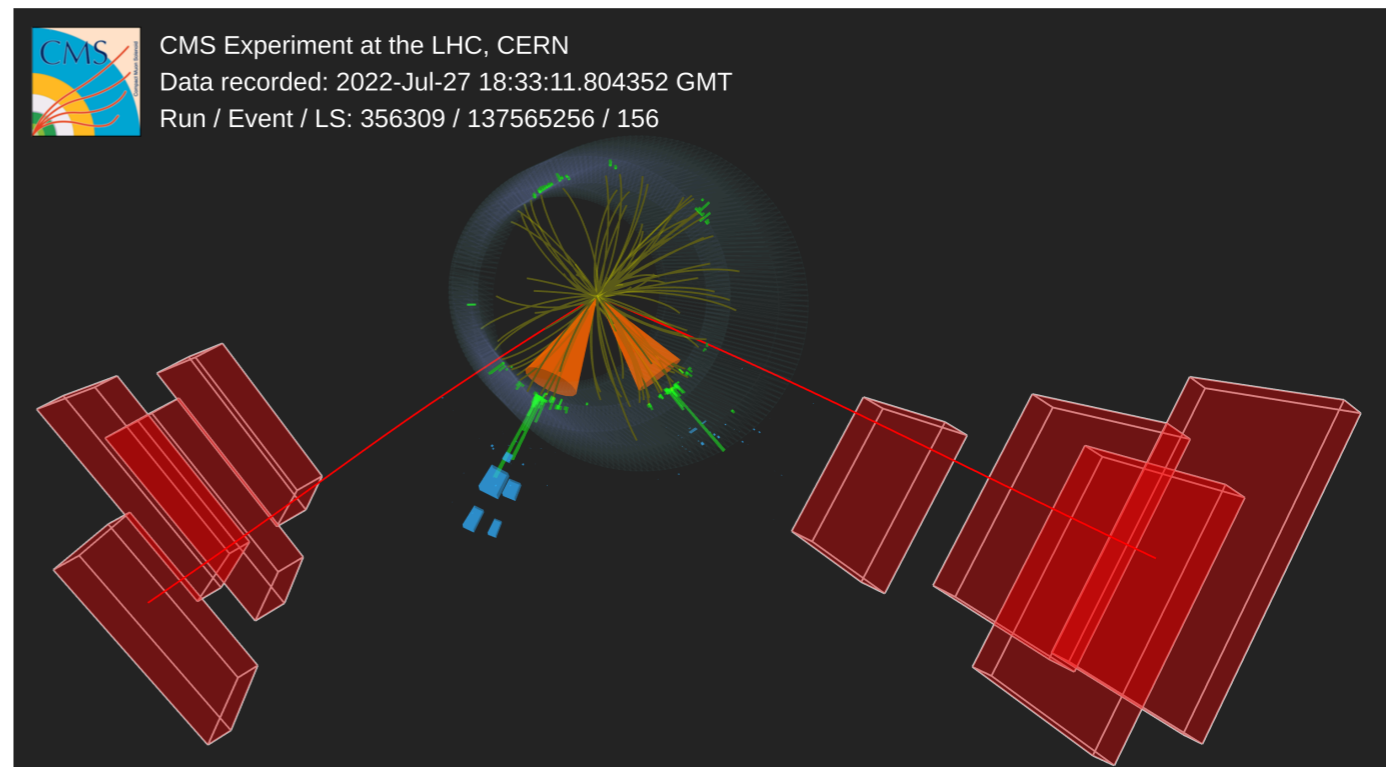
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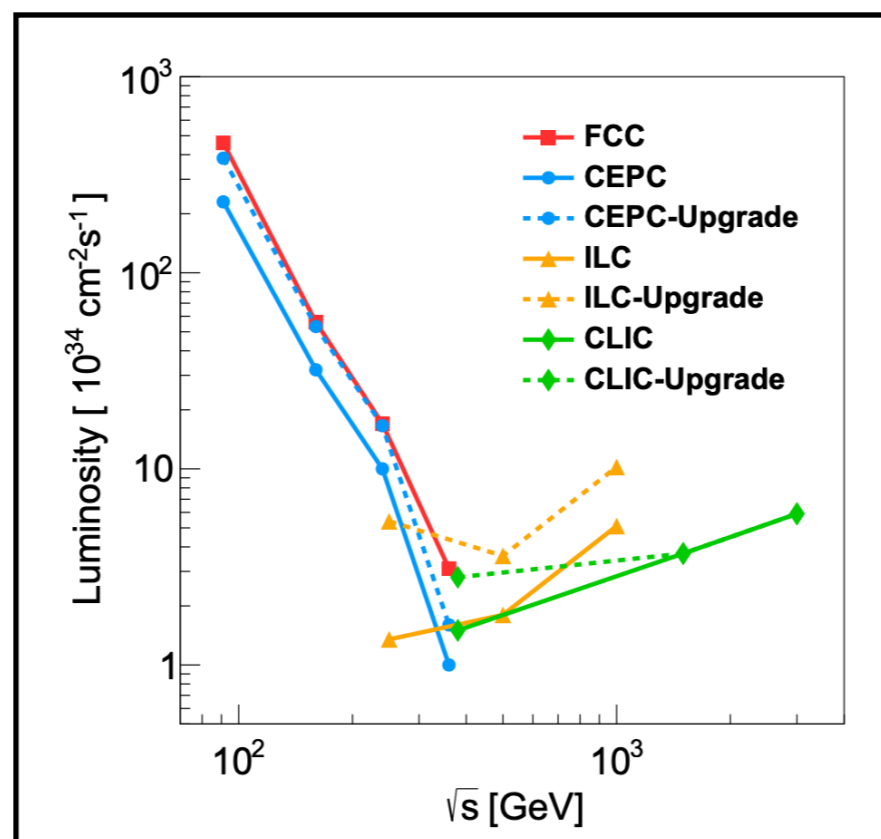


CMS-PHO-EVENTS-2022-033

The challenges lie in performing a loophole free test, and now there are well-established methods for that. *Luminosity will probably be the key ingredient as we'll see later.*

Introduction

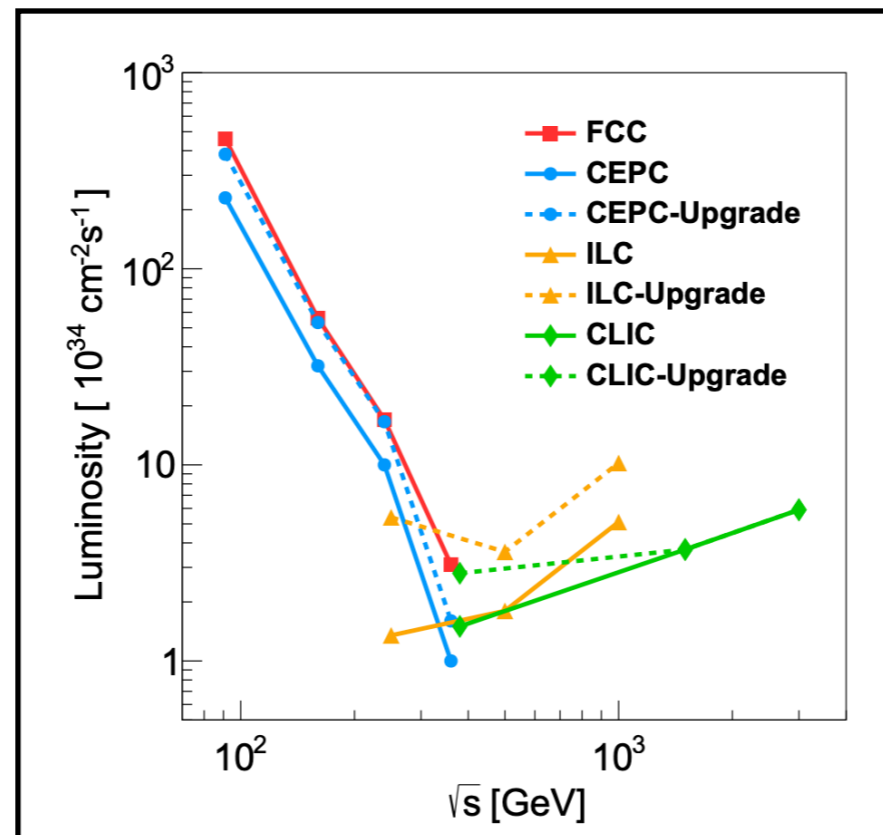
The high-luminosity of CEPC/FCC-ee, especially the TeraZ option, makes it ideal for this kind of studies with, for example, $f\bar{f}$ pairs.



CEPC Snowmass 2021, 2205.08553

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CEPC Snowmass 2021, 2205.08553

Fermions are also special to quantum mechanics, thus promising candidates for excluding some local hidden variable theories.

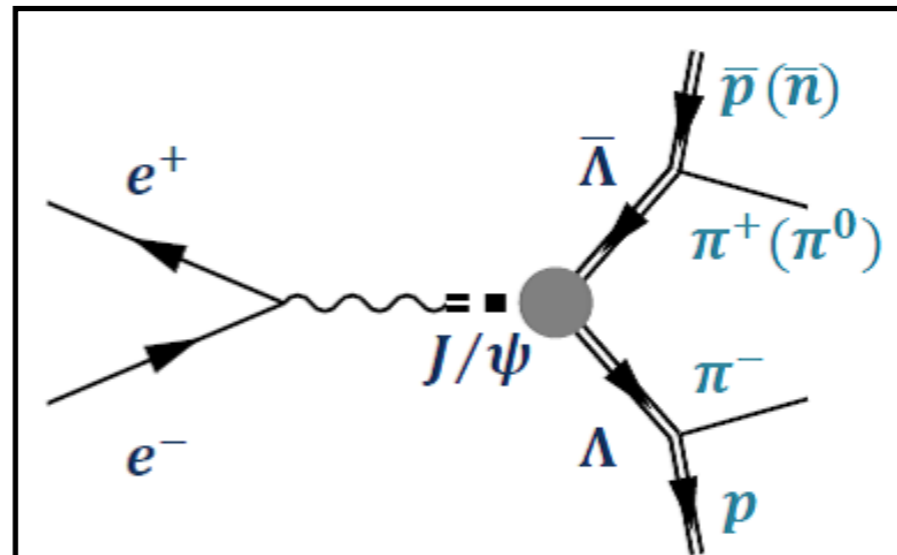
Formalism and Exp Extraction

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The entangled spin-half bipartite system can be properly described by the spin-density matrix expanded in the $SU(2) \otimes SU(2)$ Hilbert space:

$$\rho = \frac{1}{4} \left(I_4 + \sum_i B_i^+ s^i \otimes I_2 + \sum_j B_j^- I_2 \otimes s^j + \sum_{i,j} C_{ij} s^i \otimes s^j \right)$$

Polarization of the subsystem

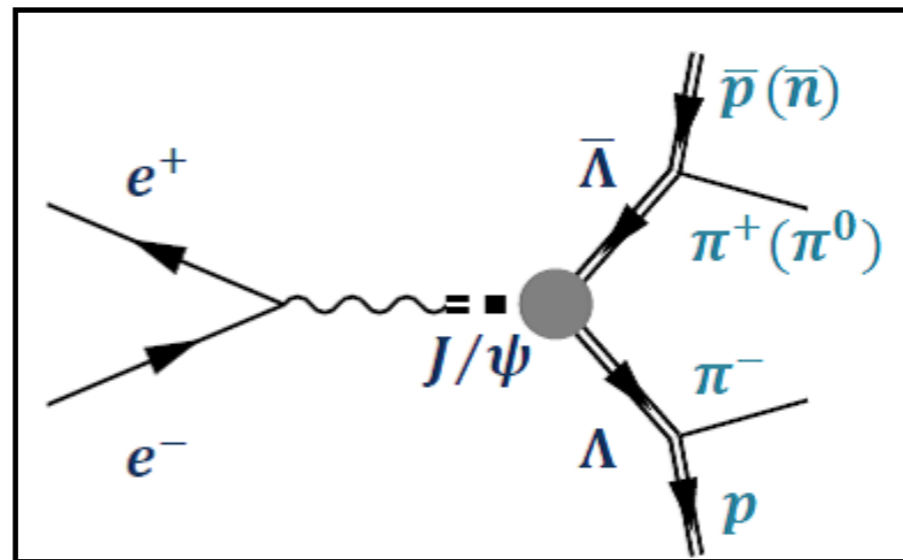


$$\frac{d\Gamma_{\Lambda}}{d\Omega_p}(\vec{s}_1, \hat{l}_p) \propto 1 + \alpha \vec{s}_1 \cdot \hat{l}_p$$

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Spin correlation

$$\frac{d\sigma}{d\Omega_k d\Omega_p d\Omega_{\bar{p}}} \propto \text{Tr} \left[\rho \left(1 + \alpha s_1 \cdot \hat{l}_p \right) \left(1 - \bar{\alpha} s_2 \cdot \hat{l}_{\bar{p}} \right) \right]$$

Equally applies to LHC, CEPC, FCC-ee ($e^+e^-/pp \rightarrow \tau^+\tau^-/t\bar{t}$ for instance!)

QT = Entanglement = Bell Nonlocality?

The underlying assumption is that the fermion pair is entangled, which is not guaranteed to be present!

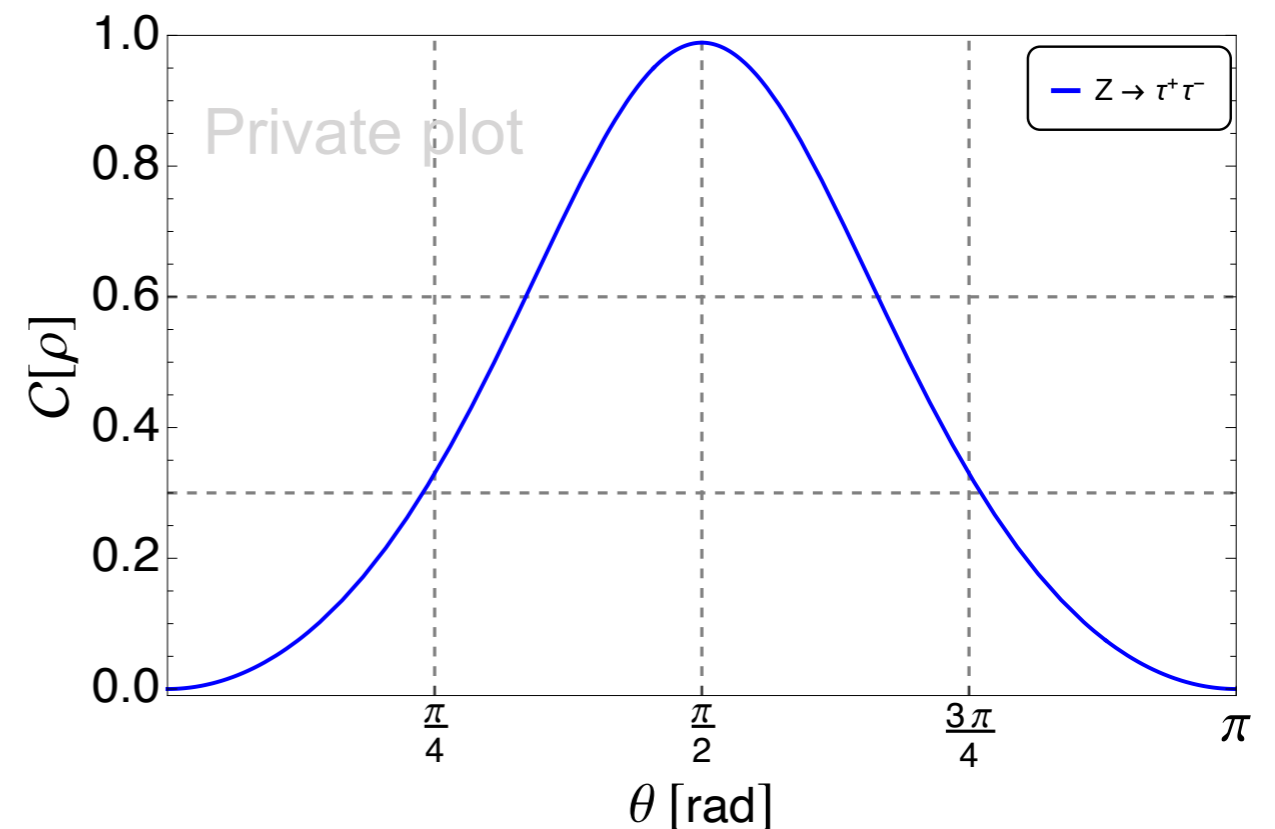
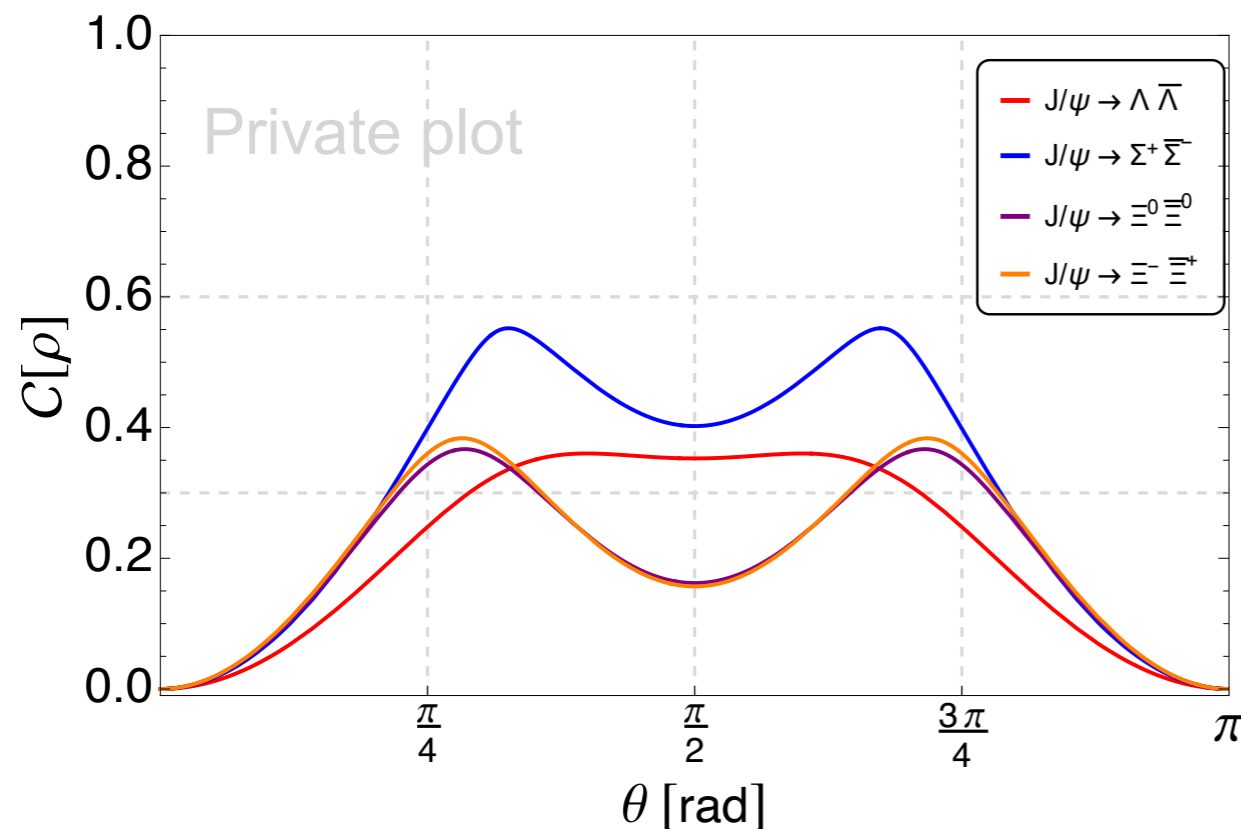
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Concurrence quantifies the entanglement of the fermion pair

$$C(\rho) = \max[0, 2\lambda_{\max} - \text{Tr } R]$$

$$R = \sqrt{\sqrt{\rho}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\sqrt{\rho}}$$



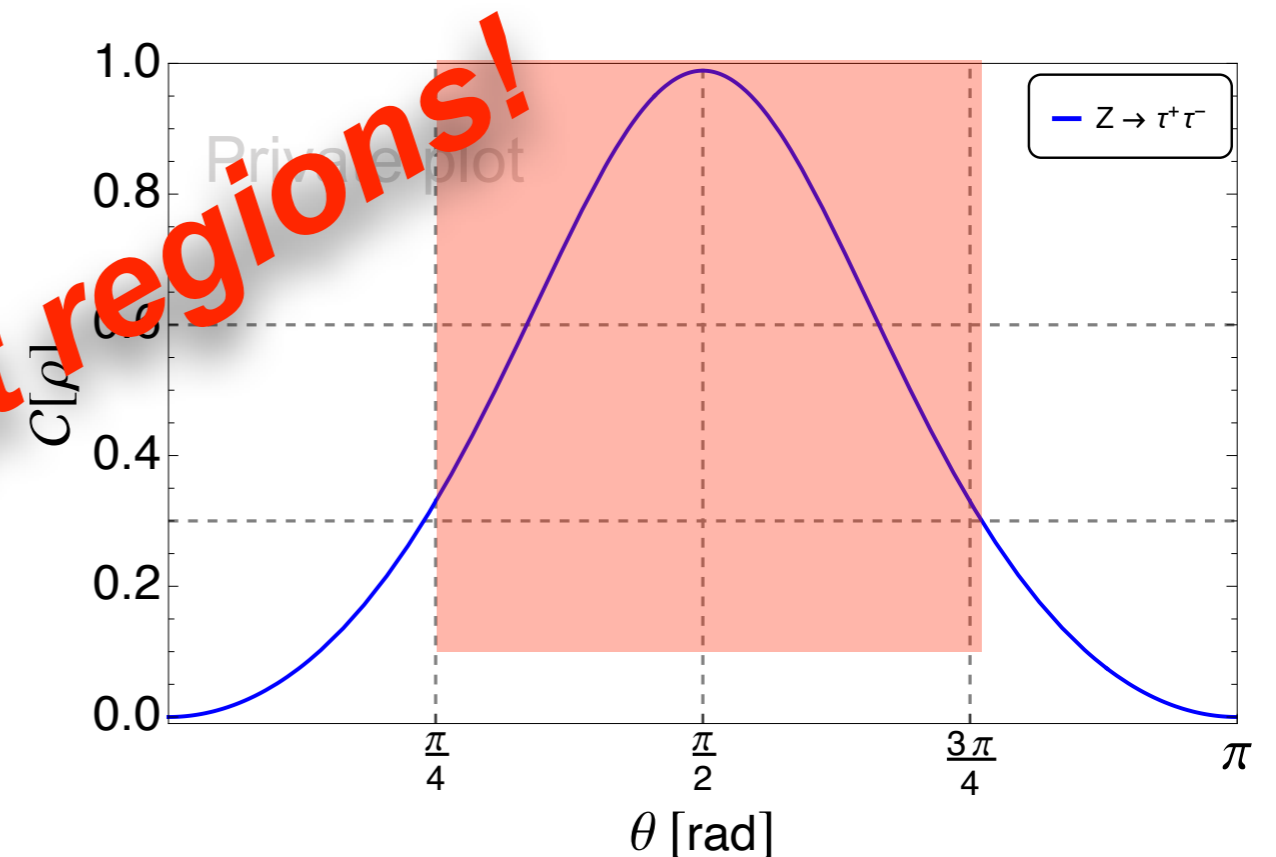
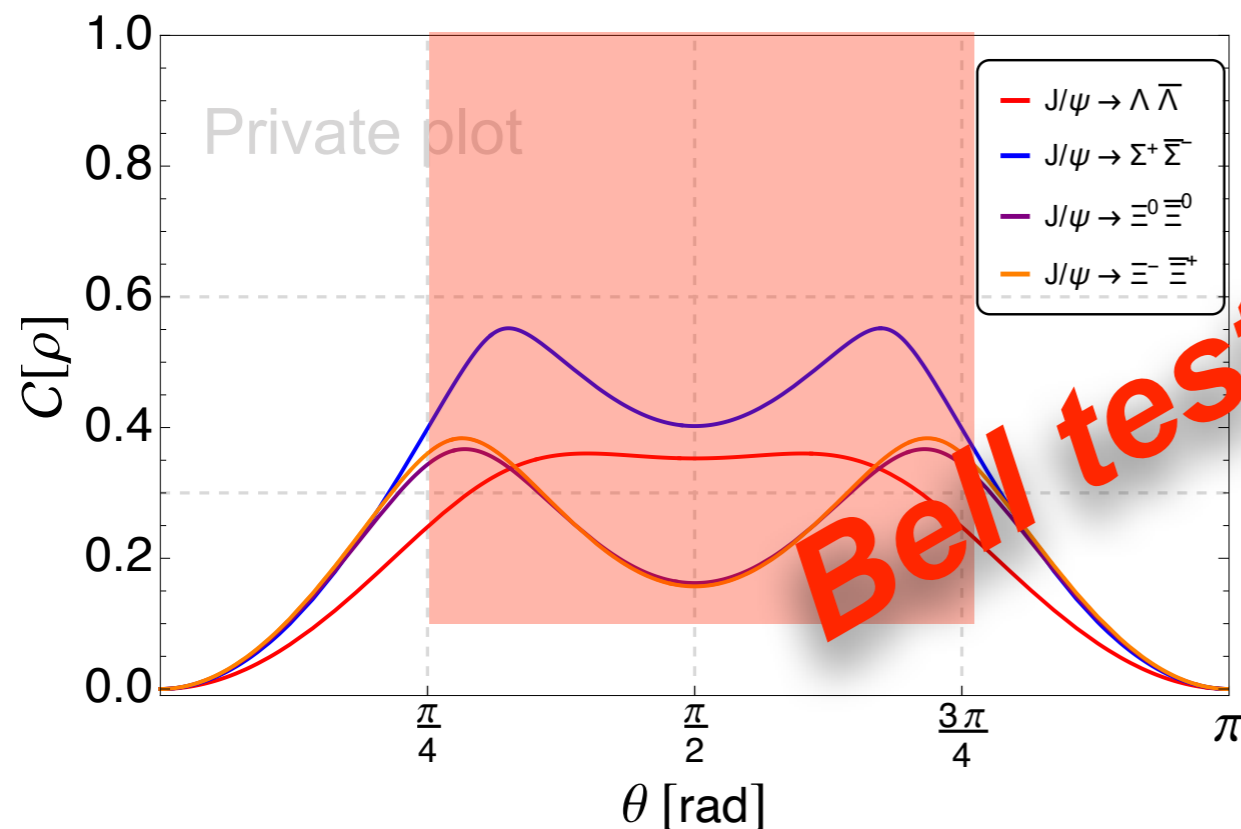
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Bell test regions!

QT = Entanglement = Bell Nonlocality?

The original Bell inequality requires simultaneously adjusting two directions with a spacelike separation randomly, thus practically very challenging.

The CHSH inequality instead avoids this simultaneity:

$$\mathcal{B}(\rho) = 2\sqrt{\mu_1 + \mu_2} \leq 2\sqrt{2} \quad \mu_{1,2} \text{ the largest two eigenvalues of } C^T C$$

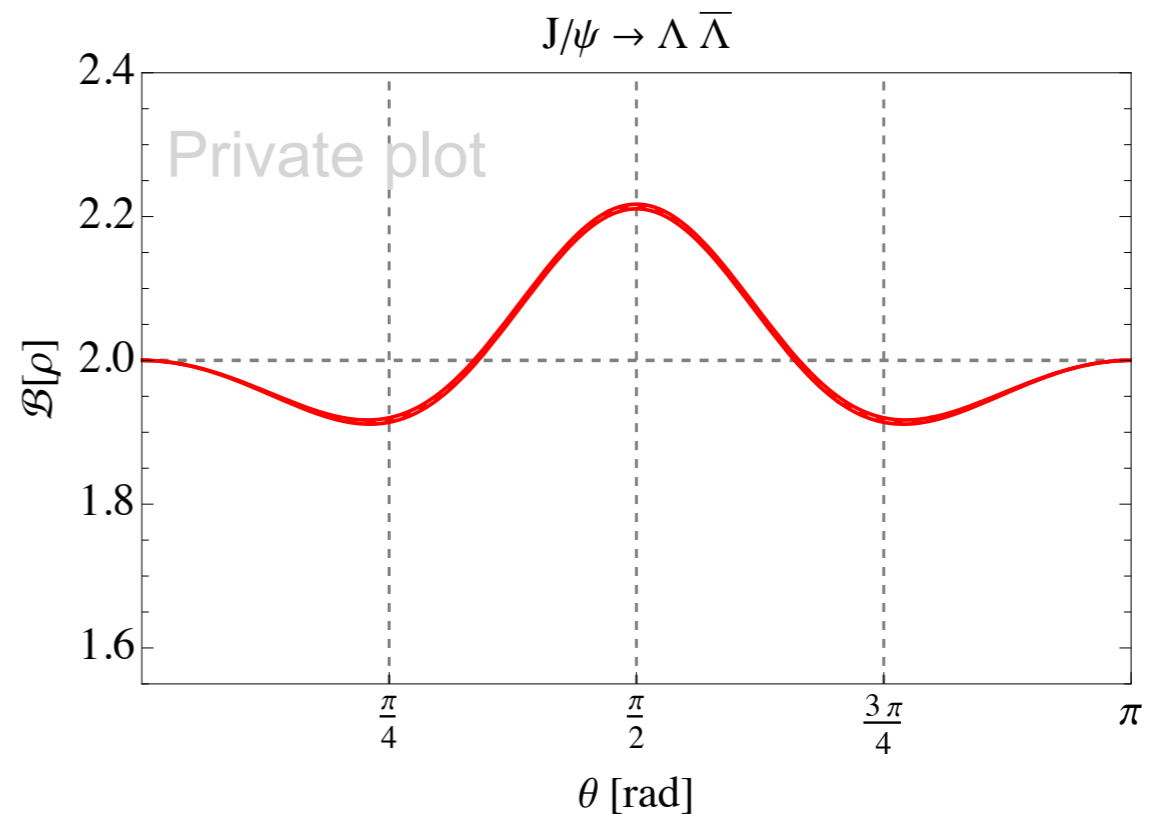
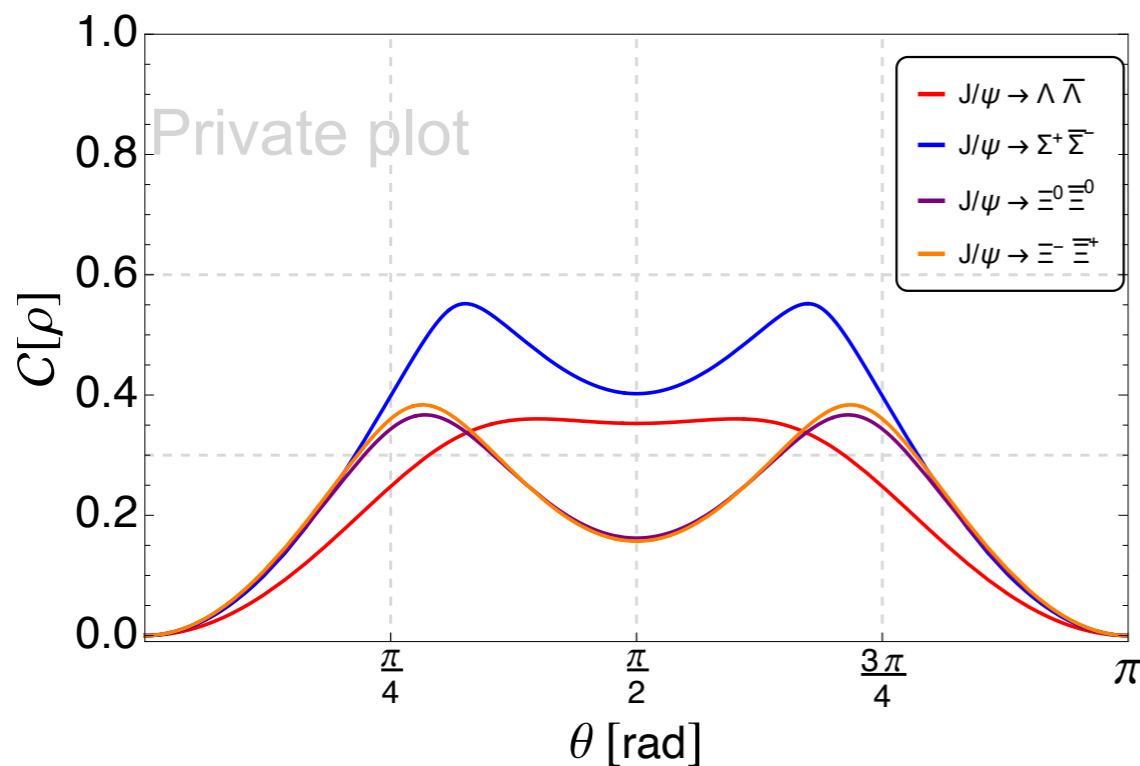
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See also Wu et al, [2406.16298](#)

Quantum entanglement \supset Bell inequality violation

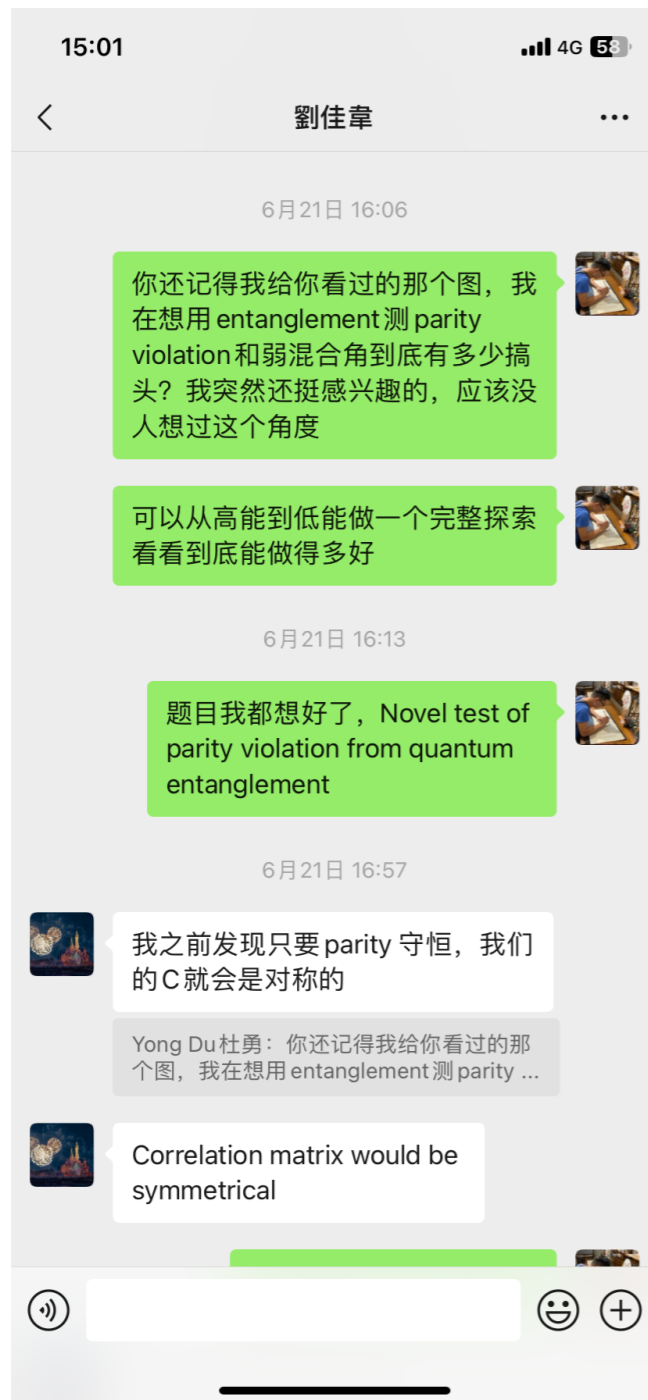
Parity on Bell Tests

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Q1: Isolating $P V$, impact on Bell tests?

Q2: What are our targets?

Parity on Bell Tests

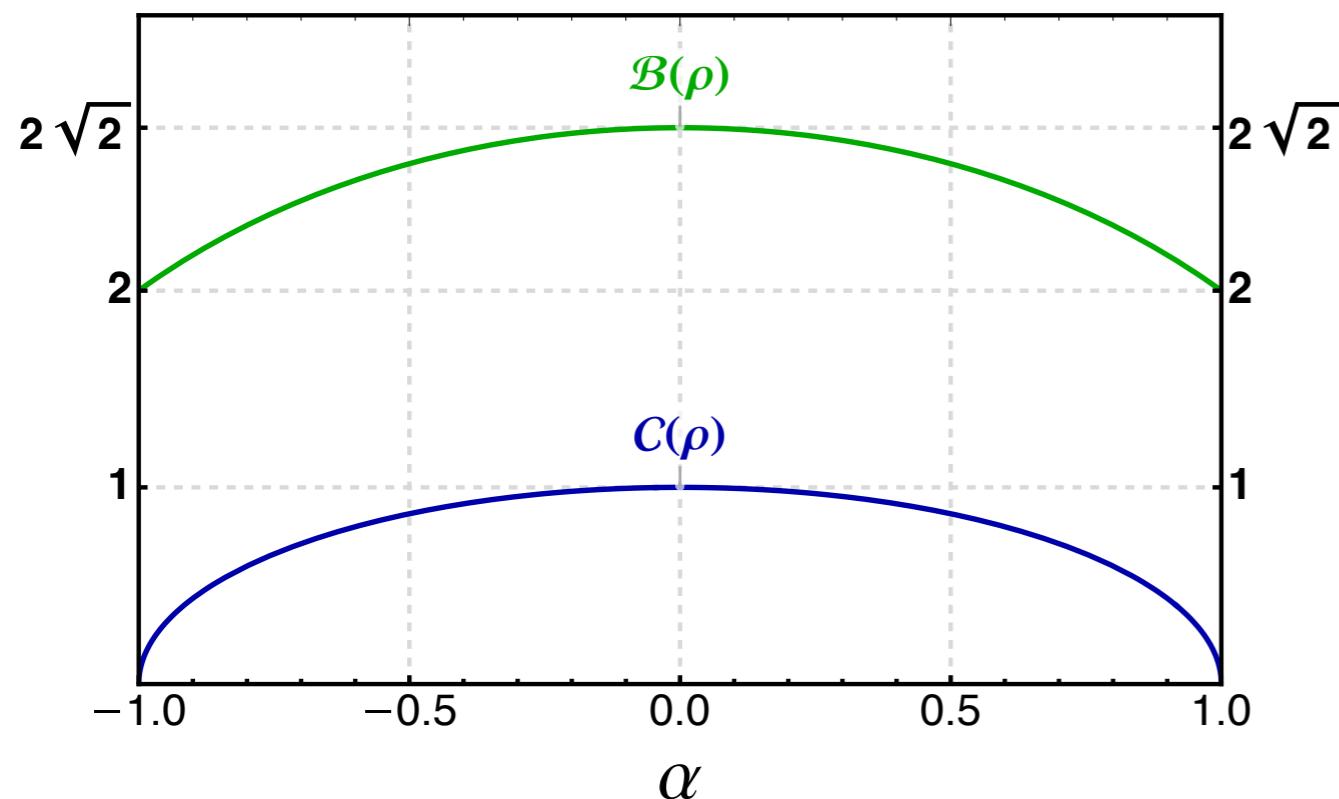
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The silly simple *spin-0 case* ($h \rightarrow f_1 \bar{f}_2$): $\mathcal{L} = h \bar{f}_1 (g_S - g_P \gamma_5) f_2$

$$|\Psi\rangle = \frac{S + P}{\sqrt{2(|S|^2 + |P|^2)}} |\uparrow\downarrow\rangle + \frac{S - P}{\sqrt{2(|S|^2 + |P|^2)}} |\downarrow\uparrow\rangle$$



$$\alpha = \frac{2\text{Re}(S^*P)}{|S|^2 + |P|^2}$$

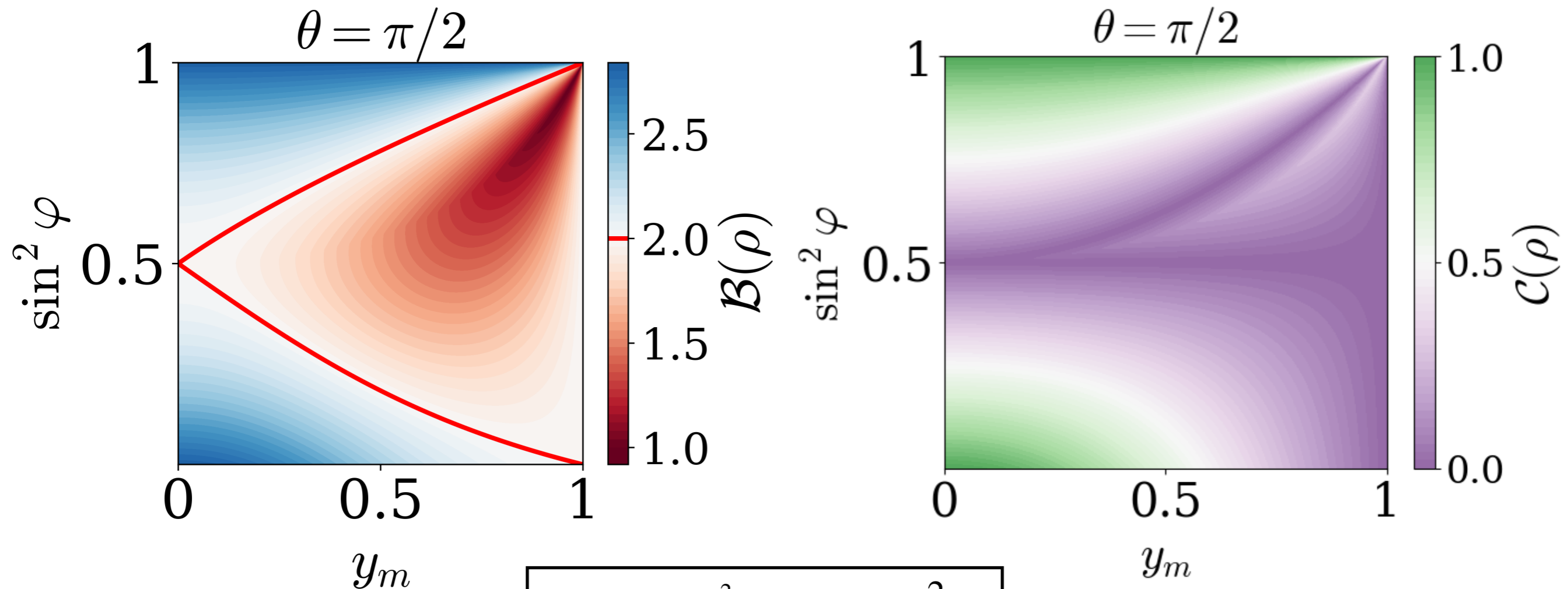
$$S = \sqrt{m_i^2 - (m_1 + m_2)^2} g_S$$

$$P = \sqrt{m_i^2 - (m_1 - m_2)^2} g_P$$

Parity on Bell Tests

The simplest case is the spin-1/2 bipartite system resulting from spin-0 and spin-1 particle decays

The not that simple *spin-1 case* ($V \rightarrow f\bar{f}$): $\mathcal{L} = V^\mu \bar{f}\gamma_\mu (F_V + F_A\gamma_5) f$



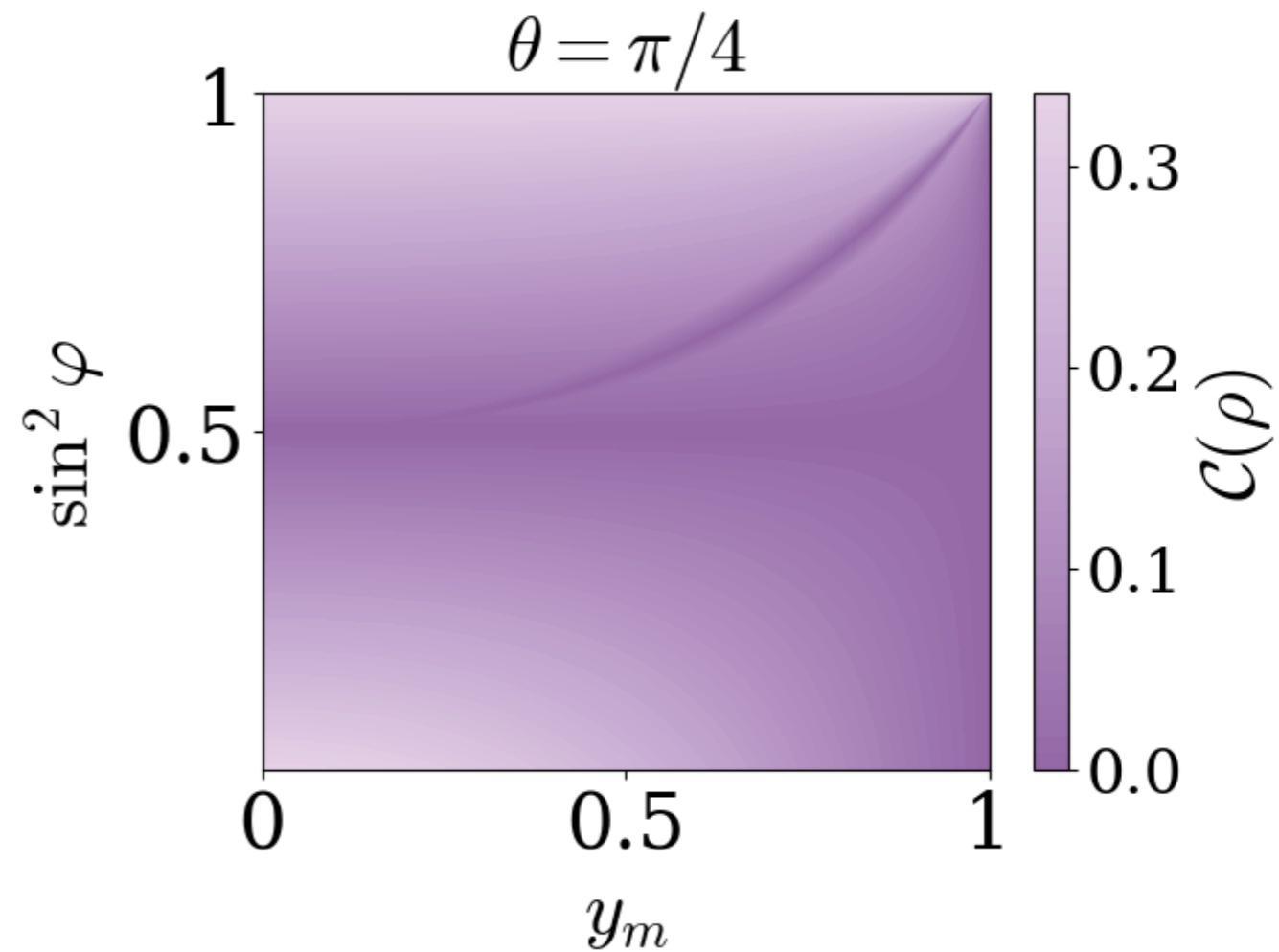
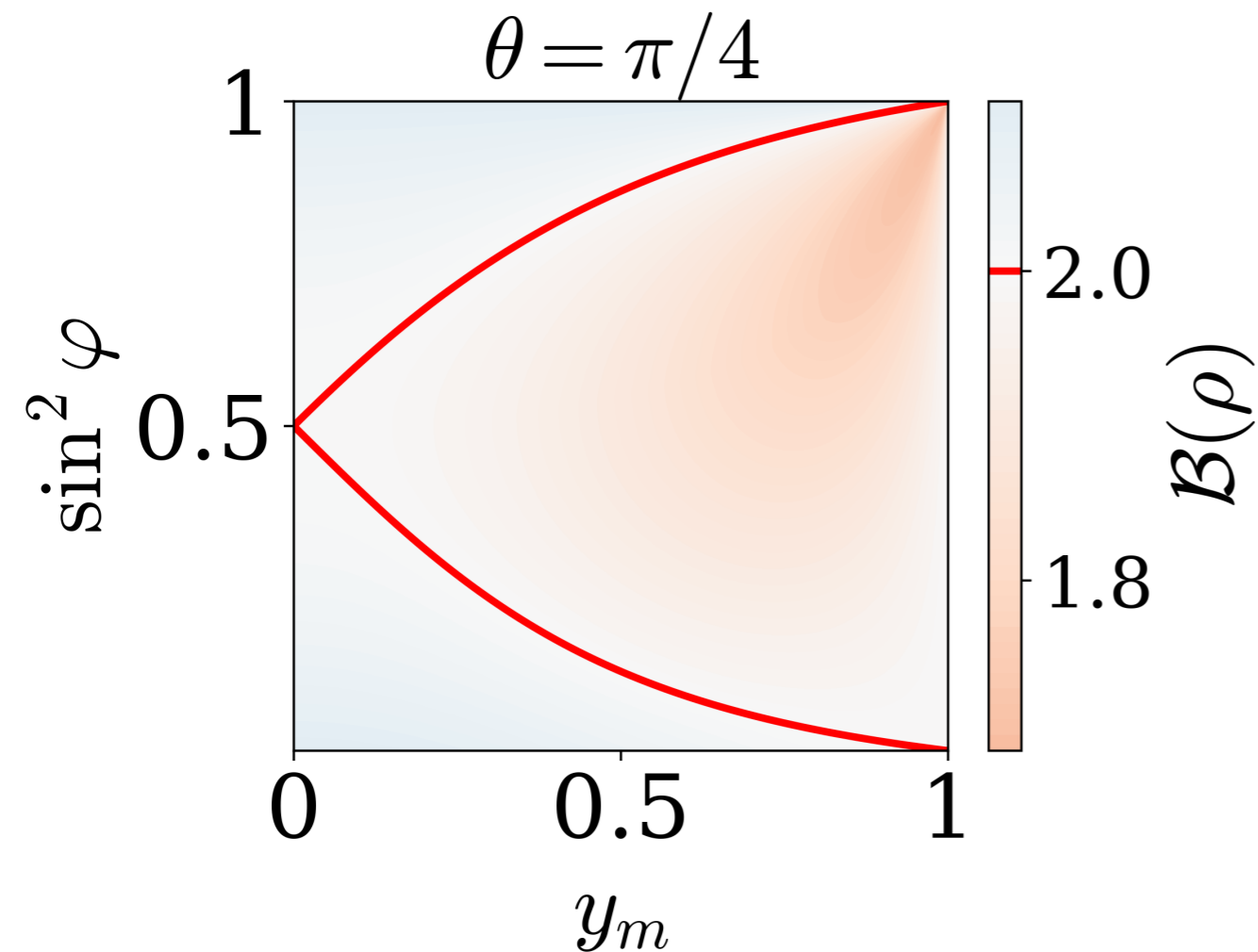
$$\sin^2 \varphi = \frac{F_A^2}{F_A^2 + F_V^2} \quad y_m = \frac{2m_f}{m_V}$$

YD, He, Liu, Ma, [2409.15418](#)

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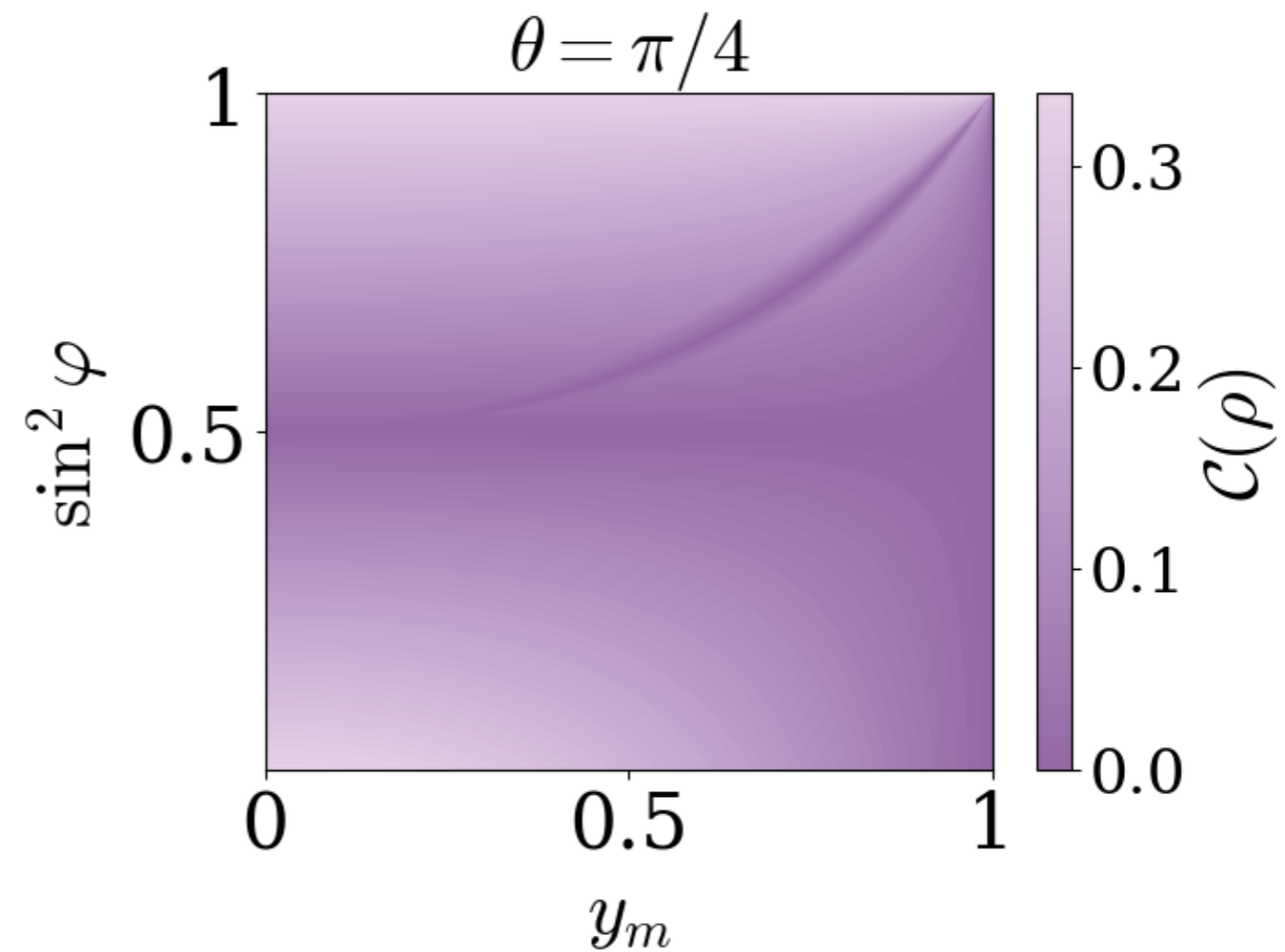
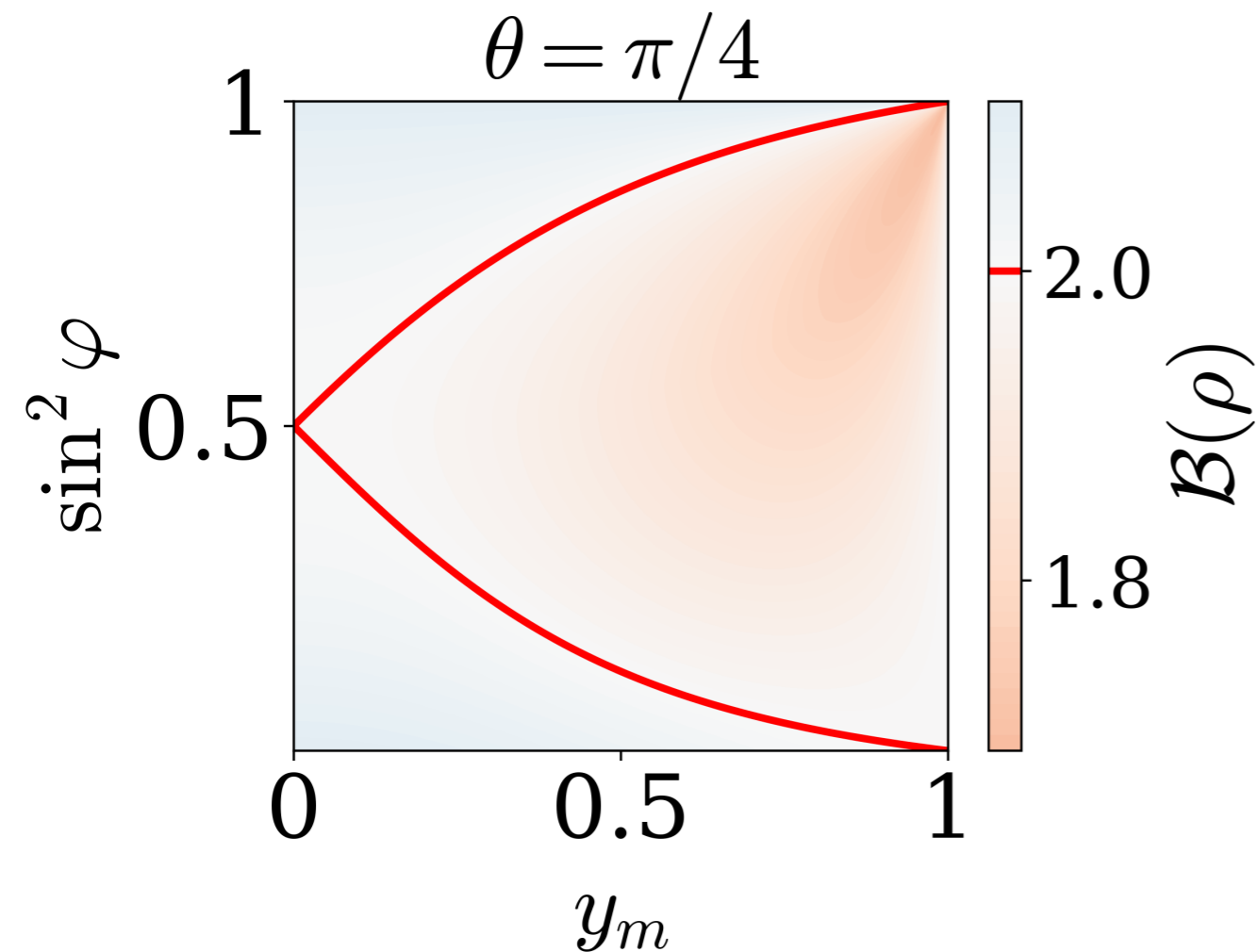


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Stat. improvement!

YD, He, Liu, Ma, [2409.15418](#)

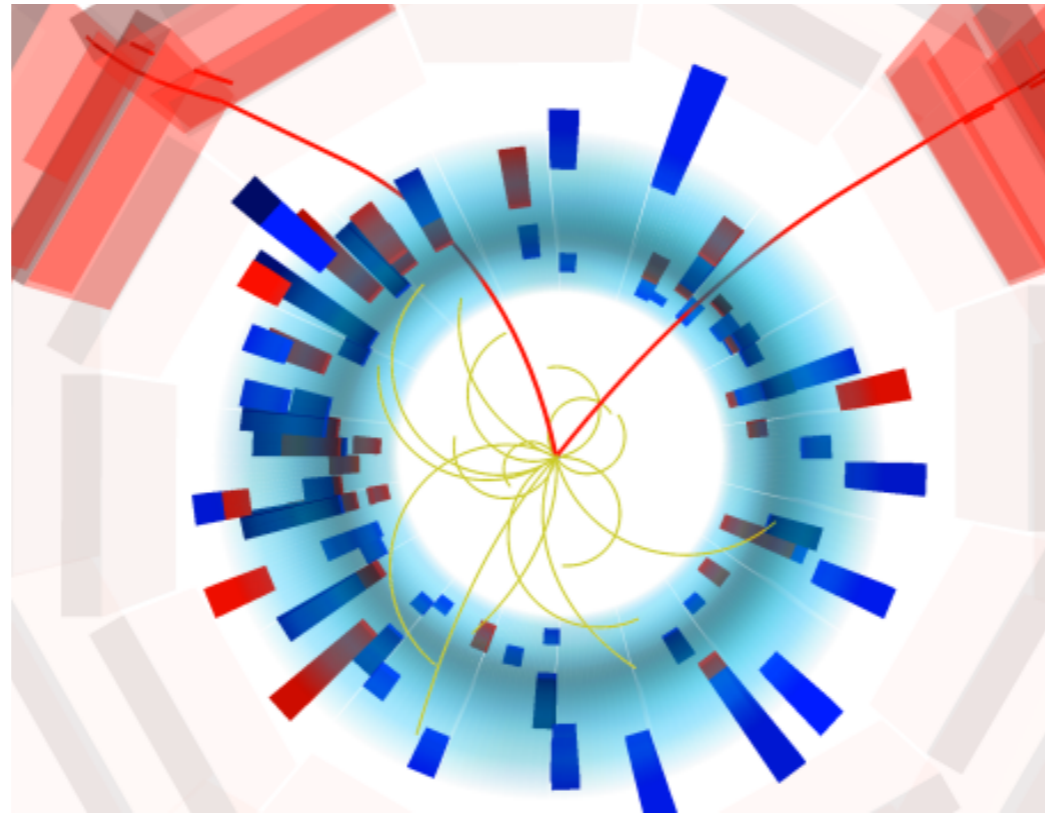
Environmental effects

What is missing? Interactions with the surroundings (as has always been).

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Figure credit: CMS collaboration

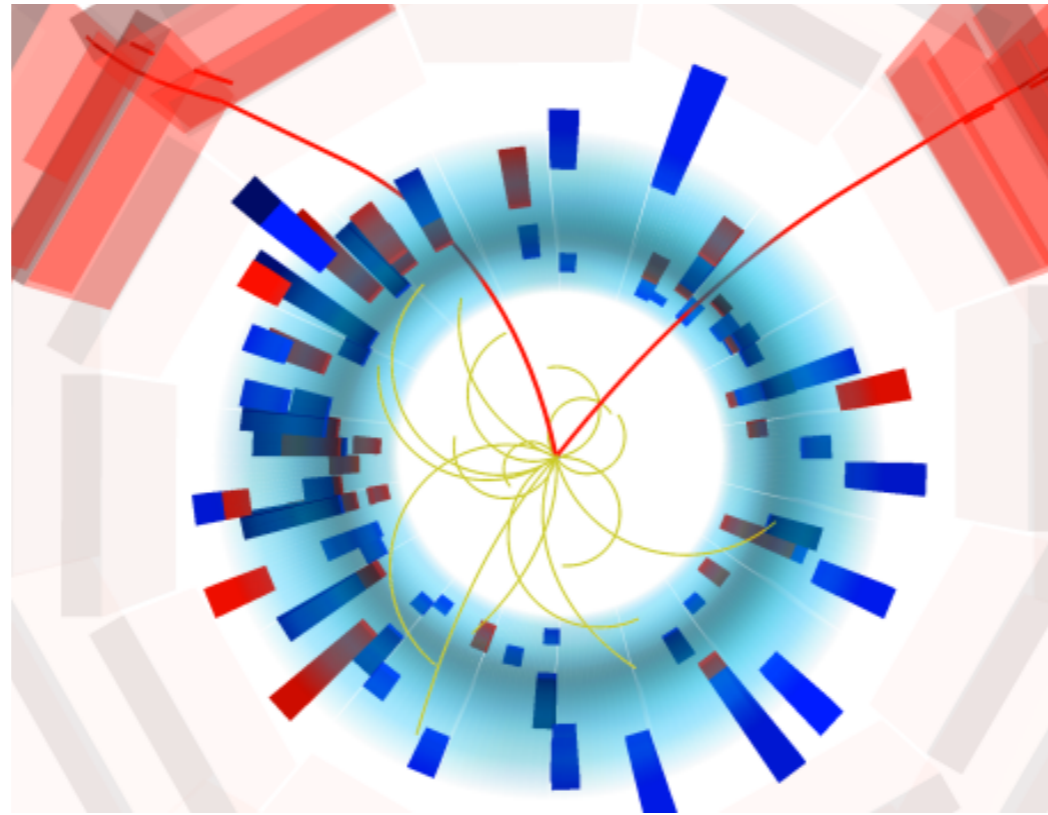


This environmental effect is largely overlooked in literature. As I will show soon, [this](#) ignorance may lead to misunderstandings in interpreting the physical results.

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This environmental effect is largely overlooked in literature. As I will show soon, [this ignorance may lead to misunderstandings in interpreting the physical results.](#)

To isolate the effects from the magnetic field and to make our point, we focus on particles decay before hitting the detector. Good examples are τ , Λ_c^+ and Ξ^- for LEP/CEPC/FCC-ee and BESIII, respectively.

Environmental effects

For the momenta, the magnetic field simply induces a rotation along the \hat{z} direction due to the Lorentz force:

$$R_p = e^{-i\mathbf{J} \cdot \left(\frac{q\mathbf{H}}{m\gamma}\right)t}$$

Environmental effects

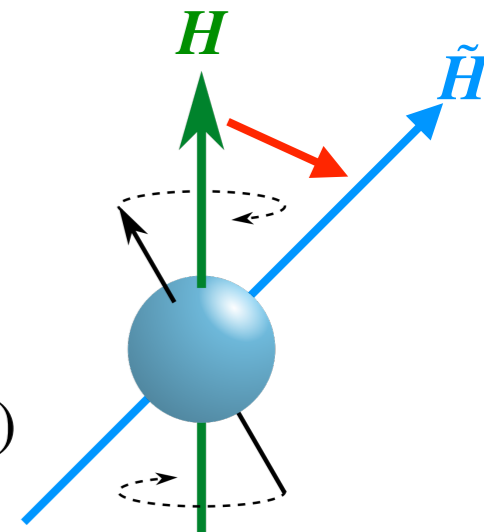
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For the spins, induction instead of spin precession as described by the Bargmann-Michel-Telegdi equation: Bargmann, Michel, Telegdi, PRL 1959

$$\frac{d\mathbf{S}(t)}{dt} = \frac{ge}{2m}\mathbf{S}(t) \times \left[\gamma\mathbf{H} + (1 - \gamma)\frac{\mathbf{H} \cdot \mathbf{v}}{v^2}\mathbf{v} \right] \equiv \mathbf{S}(t) \times \tilde{\mathbf{H}}$$

First-order Magnus expansion is sufficient $|\tilde{\mathbf{H}}| \approx \frac{ge\tau_f |\mathbf{H}| \gamma}{m} \sim \mathcal{O}(10^{-2} \sim 10^{-4})$




$$R_s = e^{-i\mathbf{J} \cdot \Omega_1(t)} \quad \Omega_1(t) = \int_0^t dt' \frac{ge}{2m} \left[\gamma\mathbf{H} + (1 - \gamma)\frac{\mathbf{H} \cdot \mathbf{v}(t')}{v^2}\mathbf{v}(t') \right]$$

Environmental effects

Due to the magnetic effect


YD, He, Liu, Ma, 2409.15418

$$\rho(0,0) = \frac{1}{4} \left(I_4 + \sum_i B_i^+ s^i \otimes I_2 + \sum_j B_j^- I_2 \otimes s^j + \sum_{i,j} C_{ij} s^i \otimes s^j \right)$$

$$\rho(t_1, t_2) = \frac{1}{4} \left(I_4 + \sum_i B_i^+ s^i \otimes I_2 + \sum_j B_j^- I_2 \otimes s^j + \sum_{i,j} C_{ij} s^i \otimes s^j \right)$$

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Statistical average is taken over the decay time through a Gaussian PDF:

$$p(t_1, t_2) = \frac{1}{2\pi\sigma_{\text{TOF}}^2} e^{-\frac{(t_1 - \tau)^2 + (t_2 - \tau)^2}{2\sigma_{\text{TOF}}^2}}$$

$$\sigma_{\text{TOF}}^{\text{BESIII}} = 33 \text{ ps}$$

$$\sigma_{\text{TOF}}^{\text{LEP}} = 150 \text{ ps}$$

* a Poisson PDF instead barely affects our conclusion

Decay time correlation is ignored as we lack this info (*also crucial for a loophole free test!*), thus high luminosity would possibly be urgently needed for a loophole free test at colliders.

Environmental effects

Q: How large is this environmental effect?

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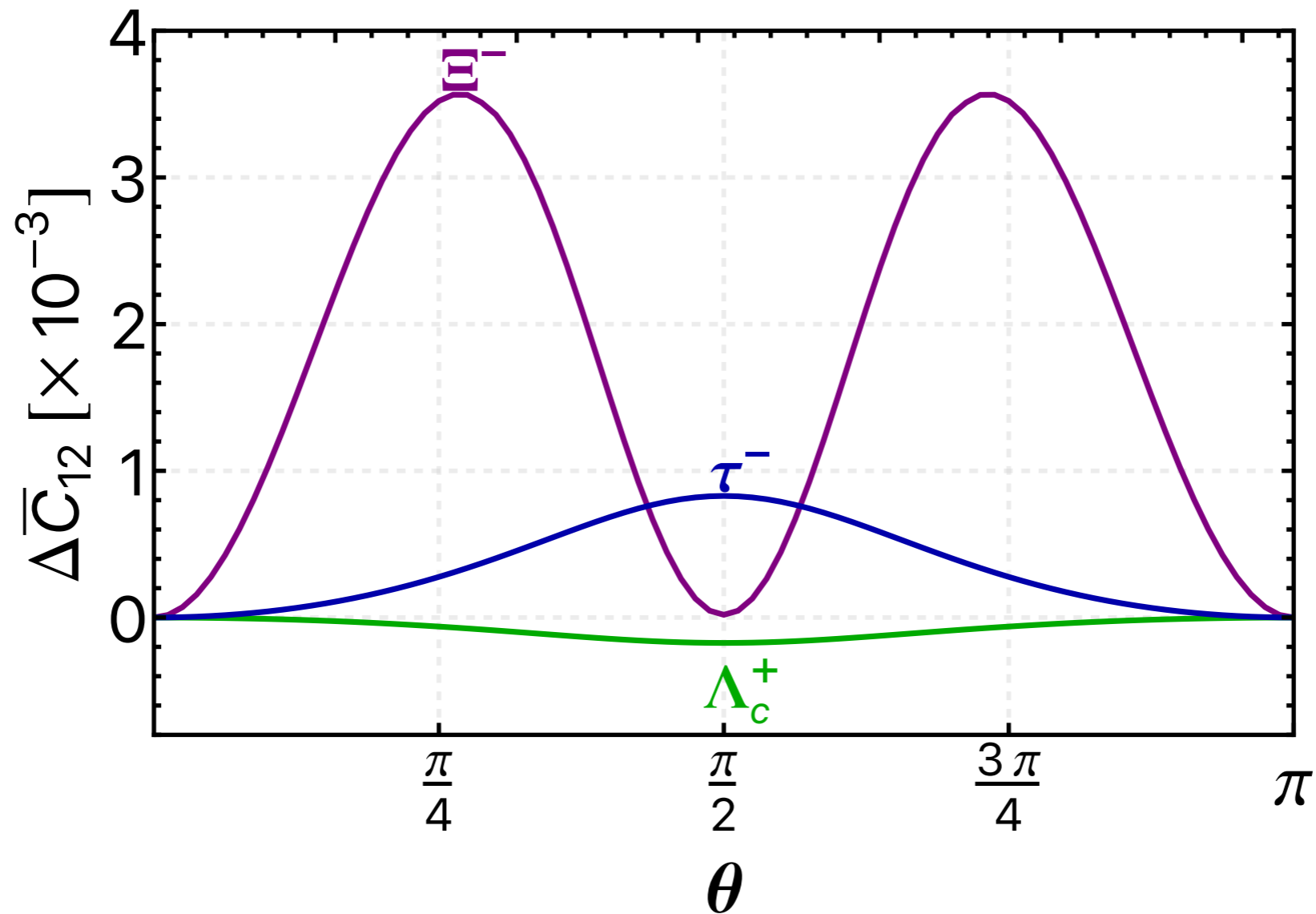
Rotational invariance puts constraints on the generic form of R (backup slide), and P and CP invariance will lead to, for instance, $C_{12} = C_{21}$ under P or CP invariance

$$\Delta \bar{C}_{12} \equiv \bar{C}_{12} - \bar{C}_{21}$$

$\Delta \bar{C}_{12} \neq 0$ would correspond to a spurious P and/or CP violation due to interaction with the environmental magnetic field.

Environmental effects

Spurious P and/or CP violation can be of $\mathcal{O}(10^{-3})$ for $|\mathbf{H}| = 1$ tesla.

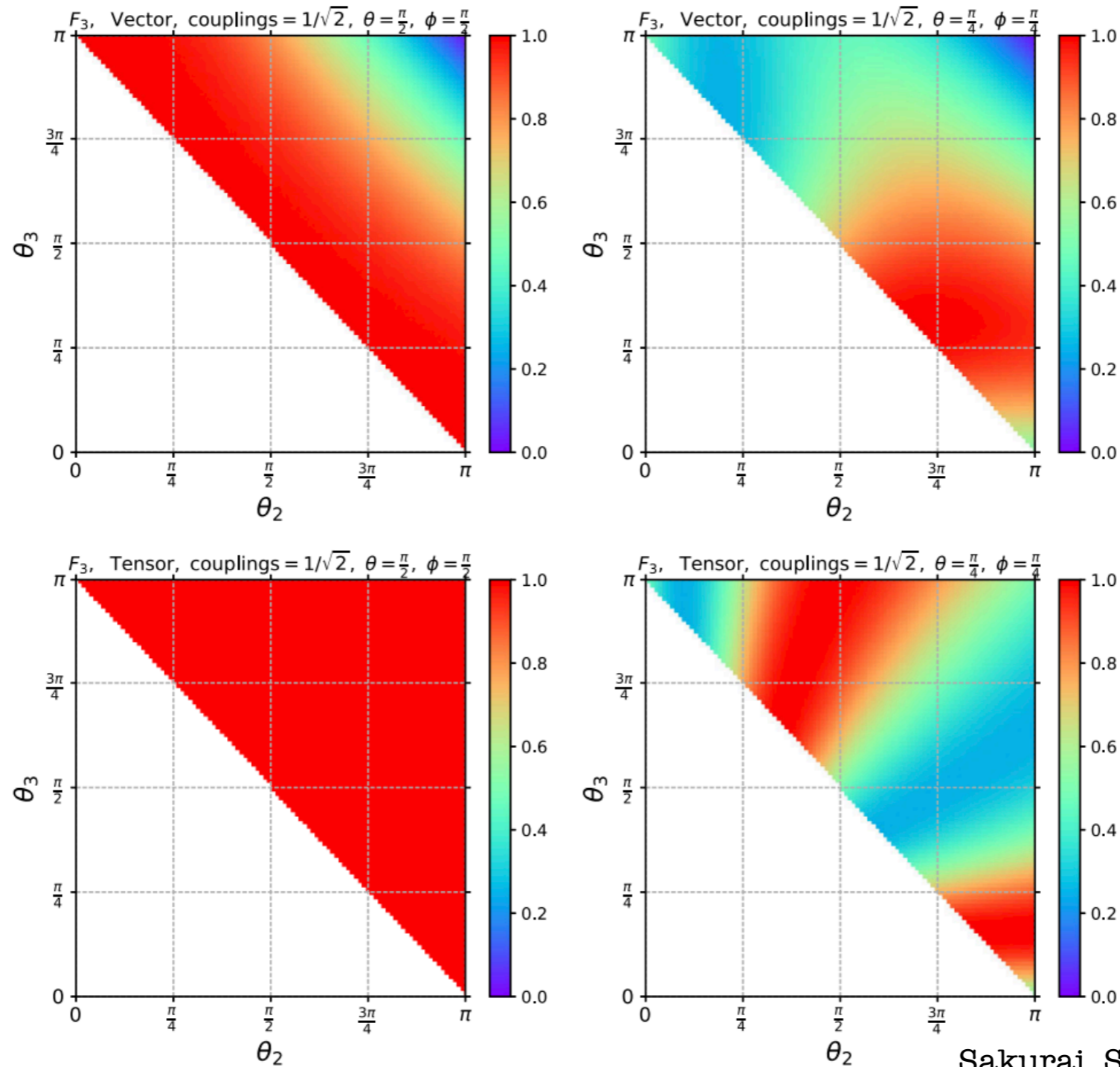


Non-negligible and may become observable at a future high-lumi electron collider!

New Physics/Paradigm

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The spin correlation can be easily modified by the presence of new physics, *e.g.*, U(1) gauge boson or 4-fermion operators: New physics in the heatmap



Sakurai, Spannowsky, PRL 2024

New Physics/Paradigm

Up to now, the discussion, though free of referring to any specific local hidden variable theory, however does rely on the knowledge of a quantum one.

Q: Do we have to?

New Physics/Paradigm

Challenging the validity of general QFT can be achieved from the simple spin-0 h decay:

$$\rho = \frac{1}{4} \left(\mathbf{I}_4 + \sum_i B_i^+ s^i \otimes \mathbf{I}_2 + \sum_j B_j^- \mathbf{I}_2 \otimes s^j + \sum_{i,j} C_{ij} s^i \otimes s^j \right)$$

Assuming no special direction for nature, *i.e.*, SO(3) invariance alone (in the rest frame of h)

$$\vec{B}^+ = b_{1k} \hat{k}, \quad \vec{B}^- = b_{2k} \hat{k} \quad C_{ij} = c_0 \delta_{ij} + c_2 \epsilon_{ijl} \hat{k}_l + c_5 \left(\hat{k}_i \hat{k}_j - \frac{\delta_{ij}}{3} \right)$$

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Angular momentum conservation immediately leads to (3 independent parameters):

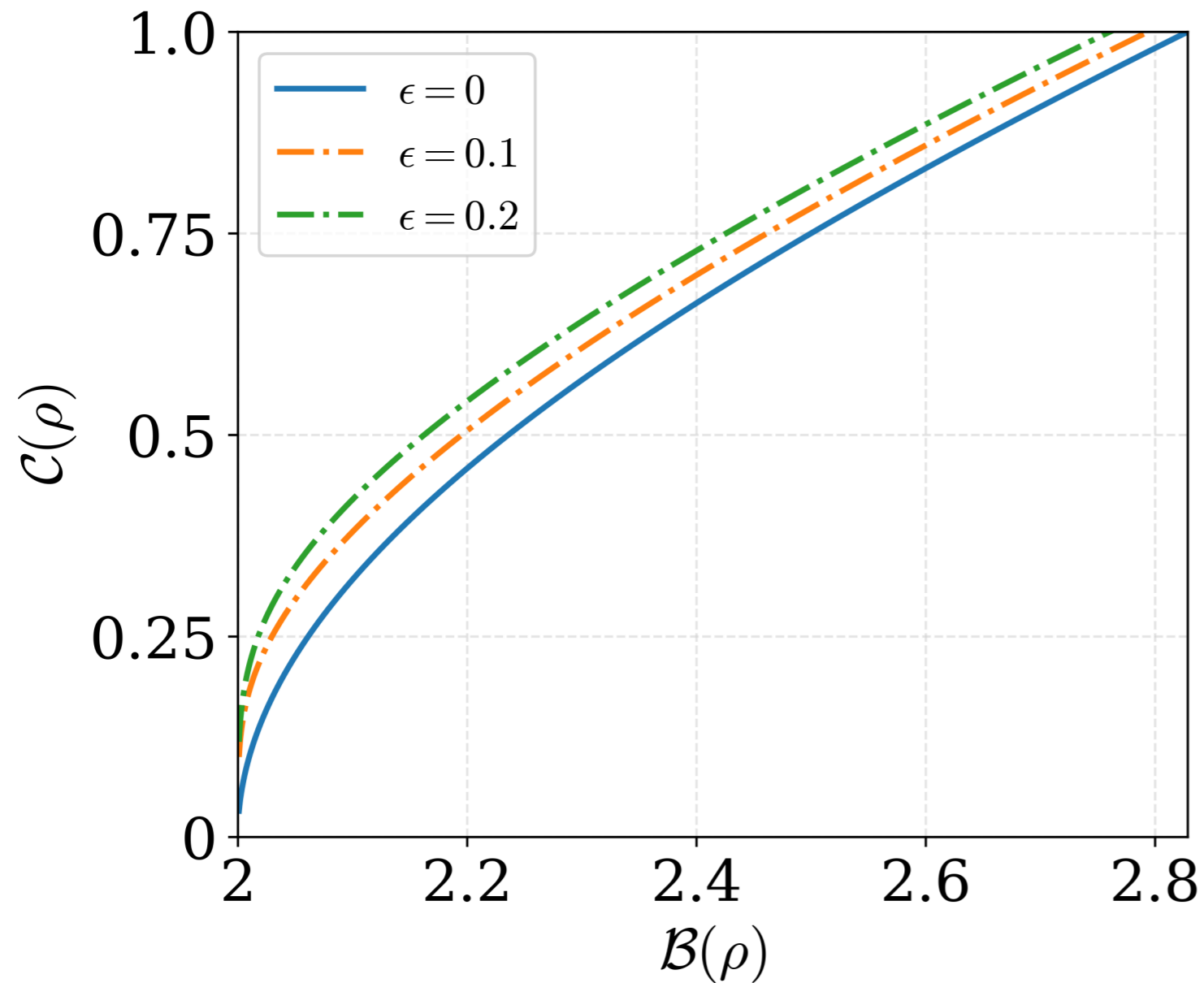
$$b_{1k} = -b_{2k} \quad c_0 = -1 - \frac{2}{3}c_5$$

$$\mathcal{B}(\rho) = 2\sqrt{2 - b_{1k}^2 - \epsilon} \quad \mathcal{C}(\rho) = \frac{1}{2} \left[(\mathcal{B}(\rho)^2 - 4)(\mathcal{B}(\rho)^2 - 4 + 4\epsilon) \right]^{\frac{1}{4}}$$

$\epsilon = 1 - b_{1k}^2 - c_2^2 - (1 + c_5)^2$. In *any* QFT, $\epsilon = 0$ is guaranteed and *unprovenly* utilized for fitting.

New Physics/Paradigm

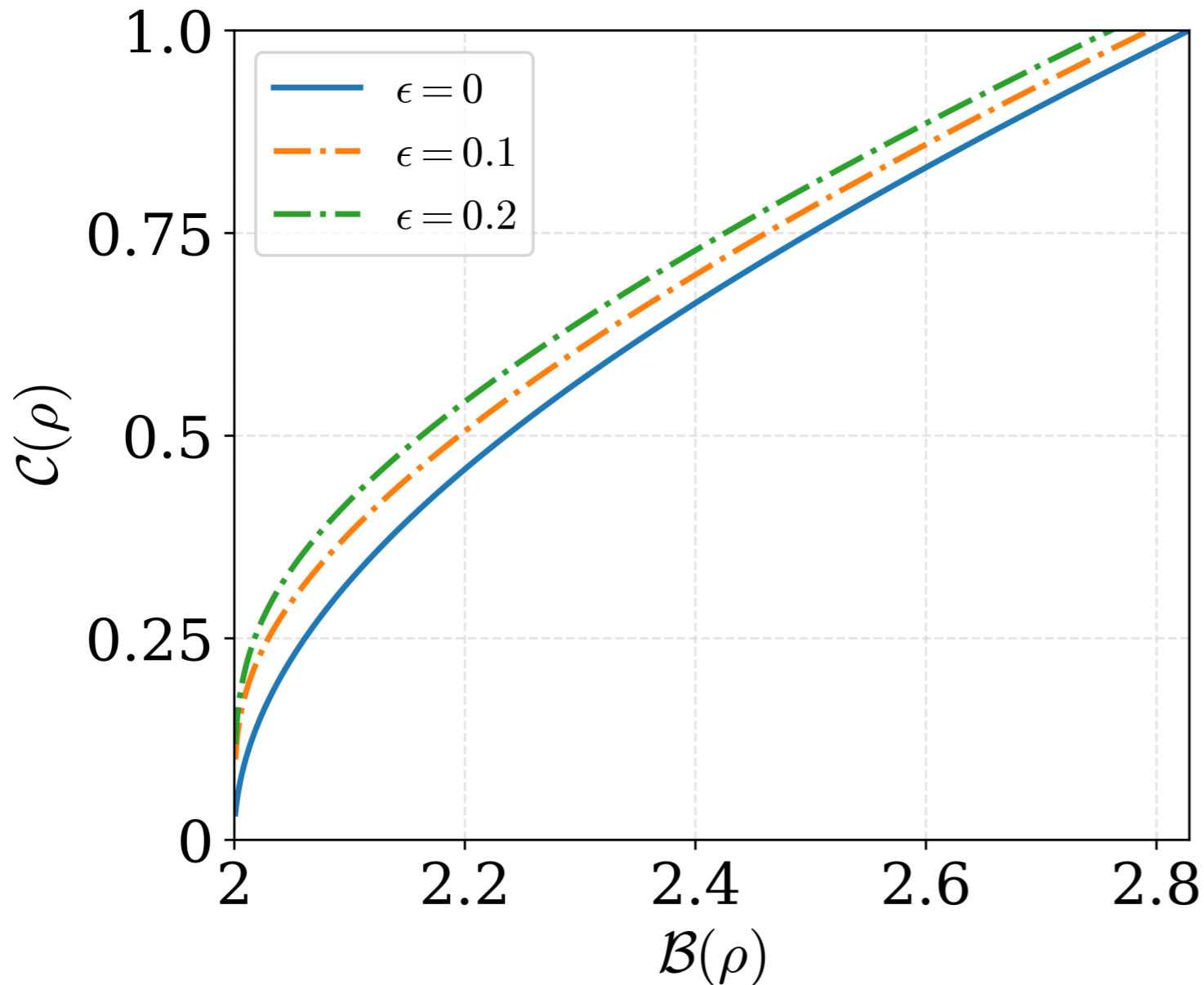
Free test of QFT along with the Bell tests



If $\epsilon \neq 0$ were observed, new paradigm beyond the QFT will be needed!

New Physics/Paradigm

Free test of QFT along with the Bell tests



Measurement of the Decay Parameters of the Λ^0 Particle*

JAMES W. CRONIN AND OLIVER E. OVERSETH†

Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received 26 September 1962)

The decay parameters of $\Lambda^0 \rightarrow \pi^- + p$ have been measured by observing the polarization of the decay protons by scattering in a carbon-plate spark chamber. The experimental procedure is discussed in some detail. A total of 1156 decays with useful proton scatters was obtained. The results are expressed in terms of polarization parameters, α , β , and γ given below:

$$\alpha = 2 \operatorname{Re} s p^* / (|s|^2 + |p|^2) = +0.62 \pm 0.07,$$

$$\beta = 2 \operatorname{Im} s p^* / (|s|^2 + |p|^2) = +0.18 \pm 0.24,$$

$$\gamma = |s|^2 - |p|^2 / (|s|^2 + |p|^2) = +0.78 \pm 0.06,$$

where s and p are the s - and p -wave decay amplitudes in an effective Hamiltonian $s + p \boldsymbol{\sigma} \cdot \mathbf{p} / |\mathbf{p}|$, where \mathbf{p} is the momentum of the decay proton in the center-of-mass system of the Λ^0 , and $\boldsymbol{\sigma}$ is the Pauli spin operator. The helicity of the decay proton is positive. The ratio $|p|/|s|$ is $0.36_{-0.06}^{+0.05}$ which supports the conclusion that the $K\Lambda N$ parity is odd. The result $\beta = 0.18 \pm 0.24$ is consistent with the value $\beta = 0.08$ expected on the basis of time-reversal invariance.

$$\epsilon = -0.025 \pm 0.154$$

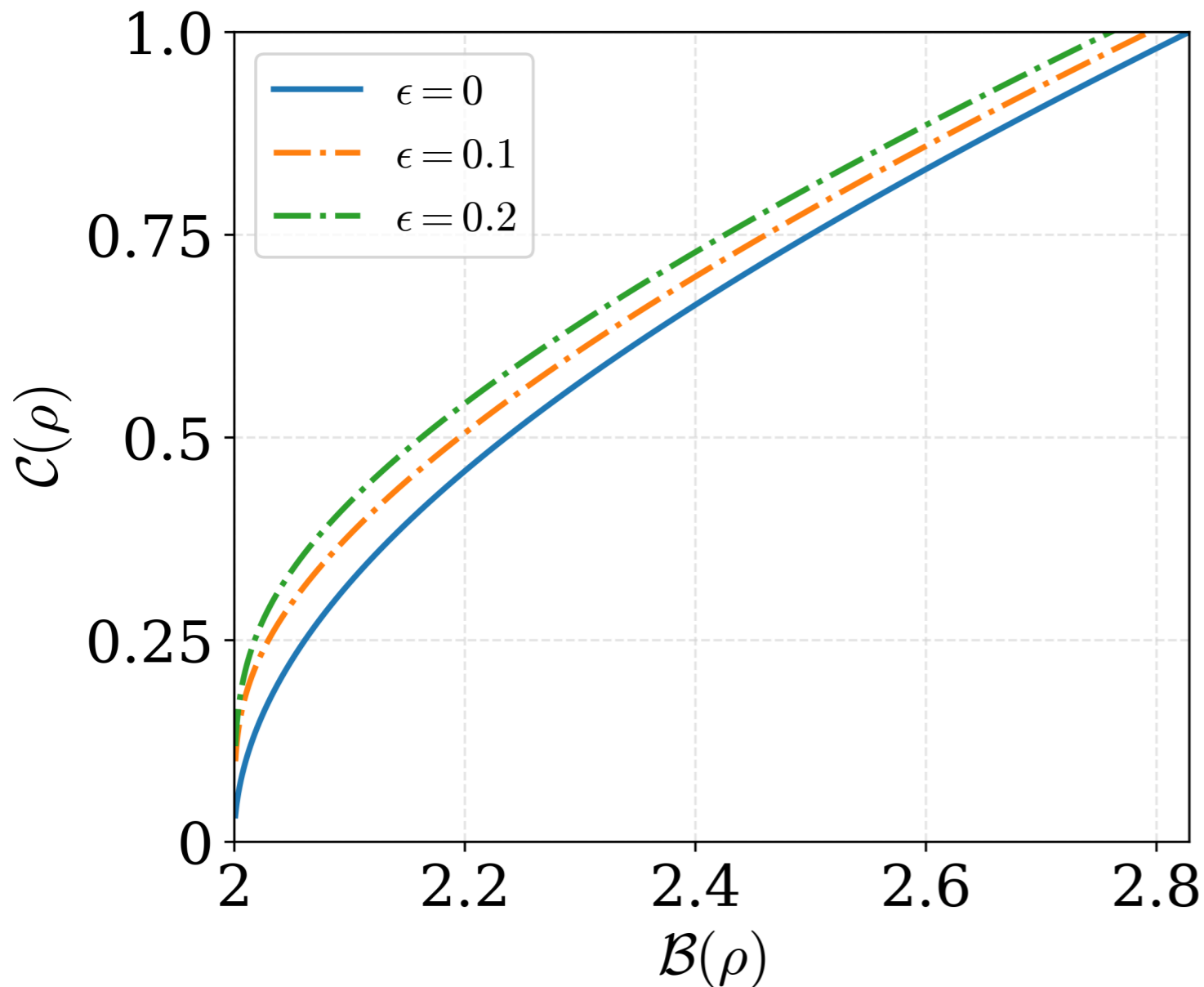
(Hint for probability non-conservation)

* Not mentioning BESIII here as $\epsilon = 0$ is taken as the starting point.

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$$\begin{aligned}\alpha &= 2 \operatorname{Re} s p^* / (|s|^2 + |p|^2) = +0.62 \pm 0.07, \\ \beta &= 2 \operatorname{Im} s p^* / (|s|^2 + |p|^2) = +0.18 \pm 0.24, \\ \gamma &= |s|^2 - |p|^2 / (|s|^2 + |p|^2) = +0.78 \pm 0.06,\end{aligned}$$

where s and p are the s - and p -wave decay amplitudes in an effective Hamiltonian $s + p \boldsymbol{\sigma} \cdot \mathbf{p} / |\mathbf{p}|$, where \mathbf{p} is the momentum of the decay proton in the center-of-mass system of the Λ^0 , and $\boldsymbol{\sigma}$ is the Pauli spin operator. The helicity of the decay proton is positive. The ratio $|p|/|s|$ is $0.36_{-0.06}^{+0.05}$ which supports the conclusion that the $K\Lambda N$ parity is odd. The result $\beta = 0.18 \pm 0.24$ is consistent with the value $\beta = 0.08$ expected on the basis of time-reversal invariance.

$$\epsilon = -0.025 \pm 0.154$$

(Hint for probability non-conservation)

* Not mentioning BESIII here as $\epsilon = 0$ is taken as the starting point.

If $\epsilon \neq 0$ were observed, new paradigm beyond the QFT will be needed!

Time for data reanalysis is NOW!

Summary

- ❖ The entangled fermion pair can be utilized for testing QE and BN. We found parity violation could significantly modify the spin correlations of the bipartite system for both spin-0 and spin-1 particles.
- ❖ The largely overlooked environmental effect was examined and we found a spurious P and/or CP violation of $\mathcal{O}(10^{-4} \sim 10^{-3})$ can be induced. This has to be subtracted for a genuine determination of P and CP violation at a future high-lumi lepton collider.
- ❖ We also propose a free test of the QFT framework using the simplest spin-0 decay and encourage our experimental colleagues to do such tests NOW.

Backup

ρ constraints

Rotational invariance puts constraints on the generic form of R :

$$B_1(\hat{\mathbf{p}}, \hat{\mathbf{k}}) = \hat{\mathbf{p}}b_{1p}(\omega) + \hat{\mathbf{k}}b_{1k}(\omega) + \hat{\mathbf{n}}b_{1n}(\omega),$$

$$B_2(\hat{\mathbf{p}}, \hat{\mathbf{k}}) = \hat{\mathbf{p}}b_{2p}(\omega) + \hat{\mathbf{k}}b_{2k}(\omega) + \hat{\mathbf{n}}b_{2n}(\omega),$$

$$C^{ij}(\hat{\mathbf{p}}, \hat{\mathbf{k}}) = \delta^{ij}c_0(\omega) + \epsilon^{ijk} \left(\hat{\mathbf{p}}^k c_1(\omega) + \hat{\mathbf{k}}^k c_2(\omega) + \hat{\mathbf{n}}^k c_3(\omega) \right) + \left(\hat{\mathbf{p}}^i \hat{\mathbf{p}}^j - \frac{1}{3} \delta^{ij} \right) c_4(\omega) + \left(\hat{\mathbf{k}}^i \hat{\mathbf{k}}^j - \frac{1}{3} \delta^{ij} \right) c_5(\omega) \\ + \left(\hat{\mathbf{p}}^i \hat{\mathbf{k}}^j + \hat{\mathbf{k}}^i \hat{\mathbf{p}}^j - \frac{2}{3} \omega \delta^{ij} \right) c_6(\omega) + (\hat{\mathbf{p}}^i \hat{\mathbf{n}}^j + \hat{\mathbf{n}}^i \hat{\mathbf{p}}^j) c_7(\omega) + (\hat{\mathbf{k}}^i \hat{\mathbf{n}}^j + \hat{\mathbf{n}}^i \hat{\mathbf{k}}^j) c_8(\omega),$$

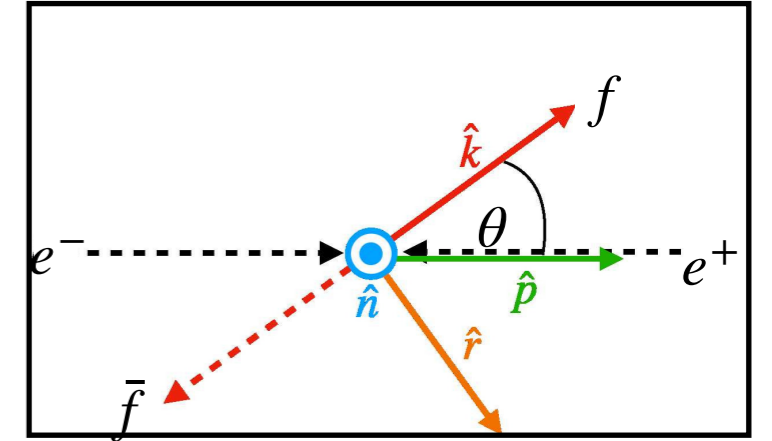
Further constraints from discrete symmetries such as P and CP:

$$b_{1p}(\omega) = b_{2p}(\omega) = b_{1k}(\omega) = b_{2k}(\omega) = c_1(\omega) = c_2(\omega) = c_7(\omega) = c_8(\omega) = 0, \quad (\text{from P invariance})$$

$$b_{1m}(\omega) = b_{2m}(\omega), \quad m = p, k, n \quad \text{and} \quad c_i(\omega) = 0, \quad i = 1, 2, 3. \quad (\text{from CP invariance})$$

ρ for the spin-1 case

The spin density matrix for a spin-1 decay process:



$$\vec{B}^{\pm} = \frac{1}{\bar{N}} \sqrt{1 - y_m^2} \left(y_m c_{\theta} \hat{p} + (1 + (1 - y_m) c_{\theta}^2) \hat{k} \right) \text{Re} \left(\frac{F_A}{F_V} \right),$$

$$C_{ij} = \frac{1}{\bar{N}} \left[\frac{1}{3} \bar{N} \delta_{ij} + (1 - (1 - y_m^2) \left| \frac{F_A}{F_V} \right|^2) (\hat{p}_i \hat{p}_j - \frac{1}{3} \delta_{ij}) - ((1 - y_m) c_{\theta} (1 - (1 + y_m) \left| \frac{F_A}{F_V} \right|^2)) (\hat{p}_i \hat{k}_j + \hat{k}_i \hat{p}_j - \frac{2}{3} c_{\theta} \delta_{ij}) \right. \\ \left. + (1 - y_m) (1 + c_{\theta}^2 (1 - y_m)) (\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij}) + \sqrt{1 - y_m^2} s_{\theta} \left((\hat{p}_i \hat{n}_j + \hat{n}_i \hat{p}_j) - (1 - y_m) c_{\theta} (\hat{k}_i \hat{n}_j + \hat{n}_i \hat{k}_j) \right) \text{Im} \left(\frac{F_A}{F_V} \right) \right],$$

$$\bar{N} = \frac{1}{2} \left[1 + c_{\theta}^2 + y_m^2 s_{\theta}^2 + (1 - y_m^2) (1 + c_{\theta}^2) \left| \frac{F_A}{F_V} \right|^2 \right],$$

$$y_m = \frac{2m_f}{m_V}$$