A unified formulation for systematical calculation of finite-volume effects at one-loop level

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Based on [Ze-Rui Liang and De-Liang Yao, JHEP **12**, 029 (2022)]

Outline

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1. Introduction

Finite Volume Corrections

- The **finite volume correction (FVC)** for a given quantity Q is given by

$$\delta Q = Q(L) - Q(\infty)$$

  - $Q(L)$ and $Q(\infty)$ are calculated in the finite volume and infinite volume, respectively.
- FVC are not only the theoretical interest, but also the need in the precise extraction of physical results in lattice QCD simulation.
- The **Lüscher formula** provides an approach to calculate FVC to masses.
  - Lüscher formula relate the finite size mass shift to an integral of a special amplitude, evaluated in the infinite volume. [M. Lüscher, Commun. Math. Phys. 104, 177-20(61986)]
  - Its application to the study of the FVC to the masses, pions, nucleon and heavy mesons, has been made.

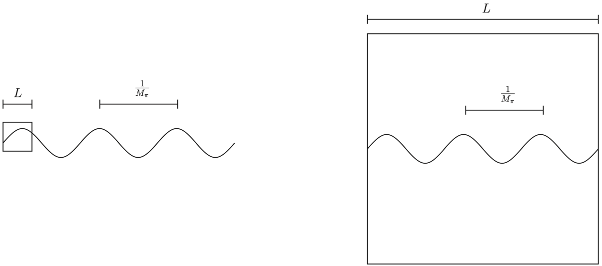
    [G. Colangelo, *et al*, EPJC 33, (2004)], [G. Colangelo, *et al*, NPB 721, (2005)], [G. Colangelo, *et al*, PRD 82, 034506 (2010)]
  - This approach fails in generating exponential terms beyond leading order.
  - A resummed version [G. Colangelo, *et al*, NPB 721, (2005)] or a Lüscher-formula-like asymptotic [G. Colangelo, *et al*, PLB 590 (2004), 258-264] expression was proposed. But the feasibility of the Lüscher formula approach is rather limited.

Chiral perturbation theory

- At finite volume, another systematical and popular tool to evaluate FVC is **ChPT**.

  [J. Gasser, H. Leutwyler, PLB 184 (1987) 83], [J. Gasser, H. Leutwyler, PLB 188 (1987) 477], [J. Gasser, H. Leutwyler, NPB 307 (1988) 763]

  - The Lagrangian is the same as the infinite case.
  - In a cubic box, momentum is discretized where the boundary conditions are imposed.



(a) ϵ-regime. (b) p-regime.

Fig. from [Alessio Giovanni Willy Vaghi, PHD thesis, (2015)]

  - We are interested in ChPT for p-regime:

$$M_\pi L \gg 1$$

Chiral perturbation theory

- A multitude of works concerning FVC based on ChPT have been done :
  - Masses:
    [S. R. Beane, PRD 70, 034507 (2004)], [L. S. Geng, *et al*, PRD 84, 074024 (2011)], [L. Alvarez-Ruso, *et al*, PRD 88, 054507 (2013)], [D. L. Yao, PRD 97, 034012 (2018)], [D. Severt, *et al*, CTP. 72, 075201 (2020)]
  - Decay constants: [D. Becirevic, *et al*, PRD 69, 054010 (2004)], [L. S. Geng, *et al*, PRD 89, 113007 (2014)]
  - Nucleon electric dipole moments: [T. Akan, *et al*, PLB 736, 163-168 (2014)]
  - Scalar form factors in $K_{\ell 3}$ semi-leptonic decay: [K. Ghorbani, *et al*, EPJC 71, 1671 (2011)]
  - FVC to forward Compton scattering off the nucleon: [J. L. de la Parra, *et al*, PRD 103, 034507 (2021)]
  - . . .
- Calculations of FVC in ChPT are tedious :
  - Complexity occurs in the one-loop analyses.
  - Automation of the one-loop calculations of FVC is still unavailable.
  - Expressions of the results for a given quantity might be different in form.
- **Our work**
  - Intend to give a unified description of the one-loop tensor integrals in a finite volume.
  - Generalize tensor decomposition of the one-loop tensor integrals to the FVC case, and derive a compact formula for the tensor coefficients.
  - Investigate the feasibility of the PV reduction of the tensor integrals.

2. One-loop tensor integrals at finite volume

Definition of loop integrals for FVC

- General form of one-loop N-point rank-P tensor integrals

$$T^{N,\mu_1,\cdots,\mu_P} = \frac{1}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{\mu_1}\cdots k^{\mu_P}}{D_1 D_2 \cdots D_N}\ , \quad D_j = [(k+p_{j-1})^2 - m_j^2 + i0^+]$$

with $p_0 = 0$, $j = 1, 2, \cdots, N$, and an infinitesimal imaginary part $i0^+$.

- The **finite-volume tensor integrals**
  - In a cubic box of volume $V = L^3$, the periodic boundary conditions $\mathbf{k}_n = \frac{2\pi\mathbf{n}}{L}$

$$\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} F(\mathbf{k}) \to \frac{1}{L^3}\sum_{\mathbf{n}} F(\mathbf{k}_n)\ , \quad \mathbf{n} \equiv (n_1, n_2, n_3) \text{ with } n_i \in \mathbb{Z}$$

  - The tensor integrals at finite volume are

$$T_V^{N,\mu_1,\cdots,\mu_P} = \frac{1}{i}\left(\int \frac{1}{L^3}\sum_{\mathbf{n}}\int \frac{\mathrm{d}k^0}{2\pi}\right) \frac{k^{\mu_1}\cdots k^{\mu_P}}{D_1 D_2 \cdots D_N} \equiv \frac{1}{i}\int_V \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k^{\mu_1}\cdots k^{\mu_P}}{D_1 D_2 \cdots D_N}\ .$$

Definition of loop integrals for FVC

- ☞ Poisson summation formula

$$\frac{1}{L^3}\sum_{\mathbf{n}} F(\mathbf{k}_n) = \sum_{\mathbf{n}} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} e^{i\mathbf{l}_k\cdot\mathbf{k}} F(\mathbf{k}).$$

- ☞ Then the finite-volume tensor integrals are

$$T_V^{N,\mu_1,\cdots,\mu_P} = \sum_{\mathbf{n}} \frac{1}{i}\int_V \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-i\,l_k\cdot k} \frac{k^{\mu_1}\cdots k^{\mu_P}}{D_1 D_2\cdots D_N}\ ,$$

  with a four vector $l_k^\mu = (0, \mathbf{n}L) = n^\mu L$. $|\mathbf{n}| = 0$ represents the infinite-volume contribution.

- ❑ The difference between the infinite and finite cases defines the FVC, and the **tensor integrals for FVC** are

$$\widetilde{T}^{N,\mu_1,\cdots,\mu_P} = \sum_{\mathbf{n}\neq 0} \frac{1}{i}\int_V \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-i\,l_k\cdot k} \frac{k^{\mu_1}\cdots k^{\mu_P}}{D_1 D_2\cdots D_N}\ .$$

Decomposition of the FVC tensor

- Considering the discretization effects at finite volume, a unit space-like vector $n^\mu = (0, \mathbf{n})$ is introduced.

$$\widetilde{L}^{\mu_1 \cdots \mu_P} = \{\underbrace{g \cdots g}_{s} p \cdots p \underbrace{n \cdots n}_{r}\}^{\mu_1 \cdots \mu_P}_{i_{2s+1} \cdots i_{P-2s-r}} ,$$

  - ☞ $2s$ out of P Lorentz indices are distributed over the metric tensors and any pair of them are symmetrical.
  - ☞ the n-vectors occupy r Lorentz indices from the remaining ones.
  - ☞ the rest Lorentz indices are assigned to the momenta
  - ☞ the number of terms

$$\frac{P!}{2^s s!r!(P-2s-r)!}$$

Examples

- Some instructive examples

$$
\begin{aligned}
\{pp\cdots p\}^{\mu_1\mu_2\cdots\mu_P}_{i_1 i_2\cdots i_P} &= p^{\mu_1}_{i_1}\, p^{\mu_2}_{i_2} \cdots p^{\mu_P}_{i_P} \ , \\
\{pn\}^{\mu_1\mu_2}_{i_1} &= p^{\mu_1}_{i_1}\, n^{\mu_2} + n^{\mu_1}\, p^{\mu_2}_{i_1} \ , \\
\{ppn\}^{\mu_1\mu_2\mu_3}_{i_1 i_2} &= p^{\mu_1}_{i_1}\, p^{\mu_2}_{i_2}\, n^{\mu_3} + p^{\mu_1}_{i_1}\, n^{\mu_2}\, p^{\mu_3}_{i_2} + n^{\mu_1}\, p^{\mu_2}_{i_1}\, p^{\mu_3}_{i_2} \ , \\
\{pnn\}^{\mu_1\mu_2\mu_3}_{i_1} &= p^{\mu_1}_{i_1}\, n^{\mu_2}\, n^{\mu_3} + n^{\mu_1}\, p^{\mu_2}_{i_1}\, n^{\mu_3} + n^{\mu_1}\, n^{\mu_2}\, p^{\mu_3}_{i_1} \ , \\
\{gn\}^{\mu_1\mu_2\mu_3} &= g^{\mu_1\mu_2}\, n^{\mu_3} + g^{\mu_1\mu_3}\, n^{\mu_2} + g^{\mu_2\mu_3}\, n^{\mu_1} \ , \\
\{gpn\}^{\mu_1\mu_2\mu_3\mu_4}_{i_1} &= g^{\mu_1\mu_2}\,(p^{\mu_3}_{i_1}\, n^{\mu_4} + n^{\mu_3}\, p^{\mu_4}_{i_1}) + g^{\mu_1\mu_3}\,(p^{\mu_2}_{i_1}\, n^{\mu_4} + n^{\mu_2}\, p^{\mu_4}_{i_1}) \\
&\quad + g^{\mu_1\mu_4}\,(p^{\mu_2}_{i_1}\, n^{\mu_3} + n^{\mu_2}\, p^{\mu_3}_{i_1}) + g^{\mu_2\mu_3}\,(p^{\mu_1}_{i_1}\, n^{\mu_4} + n^{\mu_1}\, p^{\mu_4}_{i_1}) \\
&\quad + g^{\mu_2\mu_4}\,(p^{\mu_1}_{i_1}\, n^{\mu_3} + n^{\mu_1}\, p^{\mu_3}_{i_1}) + g^{\mu_3\mu_4}\,(p^{\mu_1}_{i_1}\, n^{\mu_2} + n^{\mu_1}\, p^{\mu_2}_{i_1}) \ , \\
\{gg\}^{\mu_1\mu_2\mu_3\mu_4} &= g^{\mu_1\mu_2}\, g^{\mu_3\mu_4} + g^{\mu_1\mu_3}\, g^{\mu_2\mu_4} + g^{\mu_1\mu_4}\, g^{\mu_2\mu_3} \ .
\end{aligned}
$$

Decomposition of the FVC tensor integrals

- The one-loop tensor integrals can be decomposed into the form as

$$\widetilde{T}^{N,\mu_1\cdots\mu_P} = \sum_{\mathbf{n}\neq 0} \widehat{T}^{N,\mu_1\cdots\mu_P}$$

with

$$\widehat{T}^{N,\mu_1\cdots\mu_P} = \sum_{s=0}^{[P/2]} \sum_{r=0}^{P-2s} \sum_{\substack{i_{2s+1}=1,\\ \cdots\\ i_{P-2s-r}=1}}^{N-1} \{\underbrace{g\cdots g}_{s}\, p\cdots p\, \underbrace{n\cdots n}_{r}\}^{\mu_1\cdots\mu_P}_{i_{2s+1}\cdots i_{P-2s-r}} \widehat{T}^{N}_{\underbrace{0\cdots 0}_{2s}\, i_{2s+1}\cdots i_{P-2s-r}\, \underbrace{N\cdots N}_{r}} \,.$$

- $[P/2]$ is the floor function.
- Tensor coefficient $\widehat{T}^{N}_{0\cdots 0 i_{2s+1}\cdots i_{P-2s-r} N\cdots N}$ is invariant with respect to permutation of the subscripts i_j', i.e. $\widehat{C}_{001233} = \widehat{C}_{002133}$.
- the subscripts "N" are unique in the finite volume.

Examples

- Decomposition of the FVC tensor integrals up to rank 3

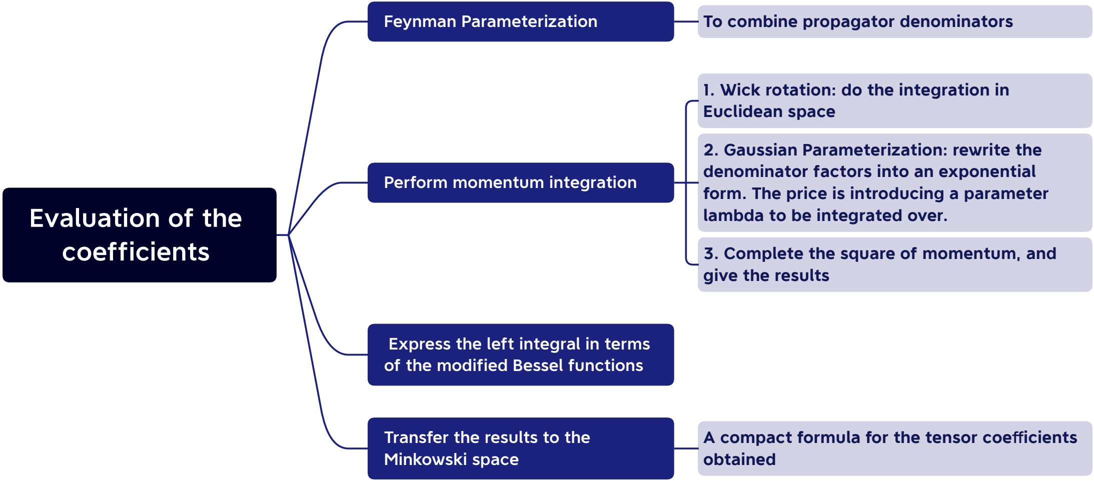
$$\widehat{T}^{N,\mu} = \sum_{i=1}^{N-1} p_i^\mu \widehat{T}_i^N + n^\mu \widehat{T}_N^N \,,$$

$$\widehat{T}^{N,\mu\nu} = g^{\mu\nu}\widehat{T}_{00}^N + \sum_{i,j=1}^{N-1} p_i^\mu p_j^\nu \widehat{T}_{ij}^N + \sum_{i=1}^{N-1}\{pn\}_i^{\mu\nu}\widehat{T}_{iN}^N + n^\mu n^\nu \widehat{T}_{NN}^N \,,$$

$$\begin{aligned}\widehat{T}^{N,\mu\nu\rho} &= \sum_{i=1}^{N-1}\{gp\}_i^{\mu\nu\rho}\widehat{T}_{00i}^N + \{gn\}^{\mu\nu\rho}\widehat{T}_{00N}^N + \sum_{i,j,k=1}^{N-1} p_i^\mu p_j^\nu p_k^\rho \widehat{T}_{ijk}^N + \sum_{i,j=1}^{N-1}\{ppn\}_{ij}^{\mu\nu\rho}\widehat{T}_{ijN}^N \\ &+ \sum_{i=1}^{N-1}\{pnn\}_i^{\mu\nu\rho}\widehat{T}_{iNN}^N + n^\mu n^\nu n^\rho \widehat{T}_{NNN}^N \,.\end{aligned}$$

Evaluation of the coefficients

- Technical steps:



Evaluation of the coefficients

❑ By the application of Feynman parameterization, the one-loop tensor integrals can be rewritten as

$$\widetilde{T}^{N,\mu_1\cdots\mu_P} = \sum_{\mathbf{n}\neq 0}\int_0^1 \mathrm{d}\mathcal{X}_N \left\{ \frac{1}{i}\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-il_k\cdot k} \frac{k^{\mu_1}\cdots k^{\mu_P}}{[(k+\mathcal{P}_N)^2 - \mathcal{M}_N^2 + i0^+]^N} \right\} .$$

☞ Here, the abbreviation $\int_0^1 \mathrm{d}\mathcal{X}_N \equiv \Gamma(N)\int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{N-1} x_2 \cdots x_{N-1}^{N-2}$ has been used, and x_i are the Feynman parameters.

☞ The recursive relations of $\mathcal{P}_N$ and $\mathcal{M}_N^2$ are

$$\begin{aligned}
\mathcal{P}_{j+1} &= x_j \mathcal{P}_j + (1-x_j)p_j \ , \quad \mathcal{P}_1 = p_0 \ , \\
\mathcal{Q}_{j+1}^2 &= x_j \mathcal{Q}_j^2 + (1-x_j)(m_{j+1}^2 - p_j^2) \ , \quad \mathcal{Q}_1^2 = m_1^2 - p_0^2 \ , \\
\mathcal{M}_{j+1}^2 &= \mathcal{Q}_{j+1}^2 + \mathcal{P}_{j+1}^2 \ ,
\end{aligned}$$

with $p_0 = 0$ and $j = 1, \cdots, N-1$.

Evaluation of the coefficients

- To perform the momentum integration, it's more convenient that do the integration in Euclidean space, and then by making use of Wick rotation

$$\Big\{\cdots\Big\}_E = (-1)^N \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} e^{il_k\cdot k_E} \frac{k_E^{\mu_1}\cdots k_E^{\mu_P}}{[(k_E+\mathcal{P}_N^E)^2+\mathcal{M}_N^{E,2}]^N} \ .$$

  - The definition is $k_E^\mu \equiv (k^0, \vec{k})$ with $k^0 = ik_E^0$ and $\vec{k} = \vec{k}_E$.
  - Same for all the other momenta .
  - In Euclidean space, the metric tensor is $\delta_{\mu\nu} = \mathrm{diag}(1,1,1,1)$.

- Gaussian parameterization is used to rewrite the denominator factors into an exponential form as

$$\Big\{\cdots\Big\}_E = \frac{(-1)^N}{\Gamma(N)} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \int_0^\infty \mathrm{d}\lambda \lambda^{N-1} \{k_E^{\mu_1}\cdots k_E^{\mu_P}\} e^{-\lambda[(k_E+\mathcal{P}_N^E)^2+\mathcal{M}_N^{E,2}]+il_k\cdot k_E} \ .$$

Evaluation of the coefficients

- Complete the square for k_E in the exponential factors and shift it to $\bar{k}_E = k_E + \mathcal{P}_N^E - \frac{il_k}{2\lambda}$, and we get

$$\left\{\cdots\right\}_E = \frac{(-1)^N}{\Gamma(N)} e^{-il_k\cdot\mathcal{P}_N^E} \int_0^\infty \mathrm{d}\lambda \lambda^{N-1} e^{-\lambda \mathcal{M}_N^{E,2} - \frac{l_k^2}{4\lambda}} \times \int \frac{\mathrm{d}^d\bar{k}_E}{(2\pi)^d} \{[\bar{k}_E + \frac{il_k}{2\lambda} - \mathcal{P}_N^E]^{\mu_1} \cdots [\bar{k}_E + \frac{il_k}{2\lambda} - \mathcal{P}_N^E]^{\mu_P}\} e^{-\lambda \bar{k}_E^2} .$$

  - Owing to the domain of the momentum integral is symmetric about zero, the terms with odd numbers of $\bar{k}_E$'s vanish.
  - The terms with even numbers of $\bar{k}_E$'s can be reformulated by utilizing the following identity

$$\bar{k}_E^{\mu_1} \cdots \bar{k}_E^{\mu_{2s}} = \frac{1}{2^s (d/2)_s} \{\delta \cdots \delta\}^{\mu_1 \cdots \mu_{2s}} (\bar{k}_E^2)^s ,$$

with the Pochhammer symbol $(d/2)_s = \Gamma(\frac{d}{2} + s)/\Gamma(\frac{d}{2})$.

Evaluation of the coefficients

- the momentum integrations over $\bar{k}_E$ and $\mathcal{P}_N'^E \equiv \frac{il_k}{2\lambda} - \mathcal{P}_N^E = \frac{inL}{2\lambda} - \mathcal{P}_N^E$.

$$\Big\{\cdots\Big\}_E = \sum_{s=0}^{[P/2]} \frac{(-1)^N e^{-il_k\cdot\mathcal{P}_N^E}}{2^s(4\pi)^{d/2}\Gamma(N)} \int_0^\infty \mathrm{d}\lambda \{\underbrace{\delta\cdots\delta}_{s}\mathcal{P}_N'^E\cdots\mathcal{P}_N'^E\}^{\mu_1\cdots\mu_P} \lambda^{N-s-\frac{d}{2}-1} e^{-\lambda\mathcal{M}_N^{E,2}-\frac{l_k^2}{4\lambda}} \, .$$

- Pull the rank-P tensor out of the λ integral

$$\Big\{\cdots\Big\}_E = \sum_{s=0}^{[P/2]}\sum_{r=0}^{P-2s} \frac{(-1)^N e^{-il_k\cdot\mathcal{P}_N^E}}{2^s(4\pi)^{d/2}\Gamma(N)} \{\underbrace{\delta\cdots\delta}_{s}\mathcal{P}_N^E\cdots\mathcal{P}_N^E\underbrace{n\cdots n}_{r}\}^{\mu_1\cdots\mu_P}$$
$$\times\left(\frac{iL}{2}\right)^r(-1)^{P-2s-r}\int_0^\infty \mathrm{d}\lambda\lambda^{N-s-r-\frac{d}{2}-1}e^{-\lambda\mathcal{M}_N^{E,2}-\frac{l_k^2}{4\lambda}} \, .$$

Evaluation of the coefficients

- Express the λ integral in terms of the modified Bessel functions and change the equation to the Minkowski space

$$\widetilde{T}^{N,\mu_1,\cdots,\mu_P} = \sum_{\mathbf{n}\neq 0} \sum_{s=0}^{[P/2]} \sum_{r=0}^{P-2s} \frac{(-1)^{N+P-s-r}}{2^s (4\pi)^{d/2}\Gamma(N)} \left(\frac{iL}{2}\right)^r \int_0^1 \mathrm{d}\mathcal{X}_N \{\underbrace{g\cdots g}_{s} \mathcal{P}_N^E \cdots \mathcal{P}_N^E \underbrace{n\cdots n}_{r}\}^{\mu_1\cdots\mu_P}$$

$$\times\, e^{il_k\cdot\mathcal{P}_N} \mathcal{K}_{N-s-r-\frac{d}{2}}\left(\frac{|\mathbf{n}^2 L^2|}{4}, \mathcal{M}_N^2\right) .$$

  - ☞ $\mathcal{P}_N$ can be expressed as

$$\mathcal{P}_N = \sum_{j=1}^{N-1} X_N^j p_j \, , \quad X_N^j = \begin{cases} x_{N-1}\cdots x_{j+1}(1-x_j) & \text{for } N-1 \geq j+1 \\ 1-x_j & \text{otherwise} \end{cases}$$

Evaluation of the coefficients

- By inserting the expression of $\mathcal{P}_N$, one get

$$\widetilde{T}^{N,\mu_1,\cdots,\mu_P} = \sum_{\mathbf{n}\neq 0} \sum_{s=0}^{[P/2]} \sum_{r=0}^{P-2s} \sum_{\substack{i_{2s+1}=1 \\ \cdots \\ i_{P-2s-r}=1}}^{N-1} \{\underbrace{g\cdots g}_{s}\, p\cdots p\, \underbrace{n\cdots n}_{r}\}^{\mu_1\mu_2\cdots\mu_P}_{i_{2s+1},\cdots,i_{P-2s-r}} \frac{(-1)^{N+P-s-r}}{(4\pi)^{d/2}2^s}\left(\frac{iL}{2}\right)^r$$

$$\times \int_0^1 \mathrm{d}X_N X_N^{i_{2s+1}} \cdots X_N^{i_{P-2s-r}}\ e^{il_k\cdot\mathcal{P}_N} \mathcal{K}_{N-s-r-\frac{d}{2}}\left(\frac{|\mathbf{n}|^2 L^2}{4}, \mathcal{M}_N^2\right) \,,$$

with $\int \mathrm{d}X_N \equiv \frac{1}{\Gamma(N)} \int_0^1 \mathrm{d}\mathcal{X}_N = \int_0^1 \mathrm{d}x_1 \cdots \int_0^1 \mathrm{d}x_{N-1} x_2 \cdots x_{N-1}^{N-2}$.

- A **general expression** for the coefficients reads

$$\widehat{T}_{\underbrace{0\cdots 0}_{2s}\, i_{2s+1}\cdots i_{P-2s-r}\,\underbrace{N\cdots N}_{r}} = \frac{2}{(4\pi)^{d/2}} \frac{(-1)^{N+P-s-r}}{2^s}\left(\frac{iL}{2}\right)^r \int_0^1 \mathrm{d}X_N X_N^{i_{2s+1}} \cdots X_N^{i_{P-2s-r}}\ e^{iL\, n\cdot\mathcal{P}_N}$$

$$\times \left(\frac{|\mathbf{n}|^2 L^2}{4\mathcal{M}_N^2}\right)^{\frac{N-s-r-d/2}{2}} K_{|N-s-r-\frac{d}{2}|}(|\mathbf{n}| L \mathcal{M}_N) \,.$$

☞ The Lorentz invariance is broken by $n\cdot\mathcal{P}_N$.

3. Reduction of Tensor Coefficients in CM frame

Center-of-Mass frame

- It is convenient to compute FVC in the rest frame or in the **CM frame**, where the net three momentum is zero.

$$l_k \cdot p_i = 0 \Longleftrightarrow n \cdot p_i = 0 \ , \qquad i = 1, \cdots, N-1 \ .$$

  - e.g. elastic two-body forward scattering at threshold, mass renormalization in the rest frame are satisfied by this condition.

- This condition lead to the $\widetilde{L}^{\mu_1 \cdots \mu_P}$ tensors with odd n-vectors vanish. And then the dependence on $\mathbf{n}$ of the rank-P tensor can be relieved

$$\sum_{\mathbf{n}\neq 0} n^{\mu_1} \cdots n^{\mu_{2t}} F(n^2) = \frac{1}{2^t (d_s/2)_t} \{h \cdots h\}^{\mu_1 \cdots \mu_{2t}} \sum_{\mathbf{n}\neq 0} (n^2)^t F(n^2) \ ,$$

  - The auxiliary tensor $h_{\mu\nu}$ is defined as $h_{\mu\nu} \equiv g_{\mu\nu} - \bar{h}_\mu \bar{h}_\nu = \mathrm{diag}(0, -1, -1, -1)$ with $\bar{h}_\mu = (1, 0, 0, 0)$, which serves to eliminate the zero-th component of the vector.
  - The rank-P tensor is irrelevant of $\mathbf{n}$, and enable us to perform the sum over $\mathbf{n}$ in advance.

Tensor coefficients of FVC integrals in CM frame

- The tensor decomposition of the FVC integrals

$$\widetilde{T}^{N,\mu_1\cdots\mu_P}=\sum_{s=0}^{[\frac{P}{2}]}\sum_{t=0}^{[\frac{P-2s}{2}]}\sum_{\substack{i_{2s+1}=1\\ \cdots\\ i_{P-2s-2t}=1}}^{N-1}\{\underbrace{g\cdots g}_{s}p\cdots p\underbrace{h\cdots h}_{t}\}^{\mu_1\mu_2\cdots\mu_P}_{i_{2s+1},\cdots,i_{P-2s-2t}}\widetilde{T}^{N}_{\underbrace{0\cdots0}_{2s}i_{2s+1}\cdots i_{P-2s-2t}\underbrace{N\cdots N}_{2t}}\,.$$

- The $\mathbf{n}$-independent coefficients are

$$\widetilde{T}^{N}_{\underbrace{0\cdots0}_{2s}i_{2s+1}\cdots i_{P-2s-2t}\underbrace{N\cdots N}_{2t}}=\frac{1}{2^t(d_s/2)_t}\sum_{\mathbf{n}\neq0}\left[(n^2)^t\widehat{T}^{N}_{\underbrace{0\cdots0}_{2s}i_{2s+1}\cdots i_{P-2s-2t}\underbrace{N\cdots N}_{2t}}\right]\,.$$

- Now the equation relies merely on n^2, then the triple sum can be replaced by a single sum $n_s\equiv n_1^2+n_2^2+n_3^2$ once the multiplicity $\vartheta(n_s)$ for a given n_s takes into account.

$$\widetilde{T}^{N}_{\underbrace{0\cdots0}_{2s}i_{2s+1}\cdots i_{P-2s-2t}\underbrace{N\cdots N}_{2t}}=\frac{(-1)^t}{2^t(d_s/2)_t}\sum_{n_s>0}\left[\vartheta(n_s)n_s^t\widehat{T}^{N}_{\underbrace{0\cdots0}_{2s}i_{2s+1}\cdots i_{P-2s-2t}\underbrace{N\cdots N}_{2t}}\right]\,.$$

Passarino-Veltman reduction

- In the CM frame, every two n-vectors are replaced by an auxiliary tensor $h_{\mu\nu}$, and the PV reduction is still valid.
- The essence of PV reduction is to establish algebraic relations between the tensor coefficients, by means of contracting the tensor integrals with external momenta $p_{i\mu}$ and the metric tensor $g_{\mu\nu}$, and lead to reduction of tensor rank and cancellation of denominators.

PV reduction of one-point tensor integrals

- For **one-point tensor integrals**, they can only be contracted by the metric tensor, and then the recurrence relations

$$\big[(d-1)+2(t-1)\big]\widetilde{A}_{\underbrace{0\cdots0}_{2s}\underbrace{1\cdots1}_{2t}} + \big[d+2s+4(t-1)\big]\widetilde{A}_{\underbrace{0\cdots0}_{2s+2}\underbrace{1\cdots1}_{2t-2}} = m_1^2\widetilde{A}_{\underbrace{0\cdots0}_{2s}\underbrace{1\cdots1}_{2t-2}} \ .$$

  - Specifically, the relations of one-point tensor integrals are, i.e.

$$d\widetilde{A}_{00} + (d-1)\widetilde{A}_{11} = m_1^2\widetilde{A}_0 \ ,$$

$$(d+2)\widetilde{A}_{0000} + (d-1)\widetilde{A}_{0011} = m_1^2\widetilde{A}_{00} \ .$$

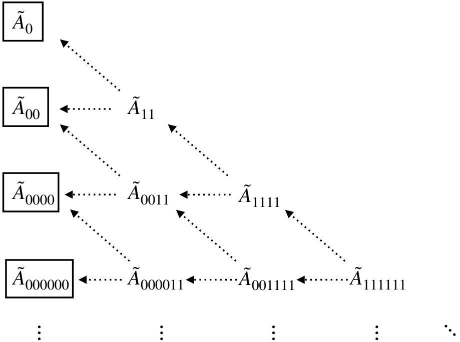
  - All the relations can either be checked numerically or be verified by the recurrence relations of the modified Bessel functions $K_z(Y)$.
  - All the one-loop FVC integrals can be reduced to a linear combination of $\widetilde{A}_{\underbrace{0\cdots0}_{2s}}$.

$$\widetilde{A}_{\underbrace{0\cdots0}_{2s}\underbrace{1\cdots1}_{2t}} = \sum_{i=0}^{t}\left\{\frac{[m_1^2]^{t-i}}{\prod_{j=1}^{t}a(j)}\sum_{\substack{i_1=0\\ \cdots\\ i_t=0}}^{1}\left[\delta_{i,\sum_{j=1}^{t}i_j}\prod_{j=1}^{t}[b(j)]^{i_j}\right]\widetilde{A}_{\underbrace{0\cdots0}_{2(s+i)}}\right\} \ ,$$

where $a(j) = (d-1)+2(j-1)$, $b(j) = -[d+2s+4(j-1)]$, and δ is the Kronecker delta.

PV reduction of one-point tensor integrals

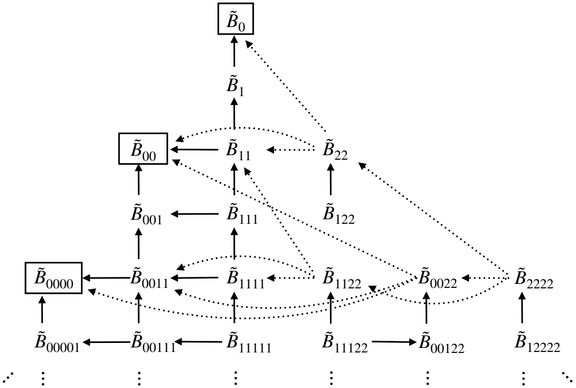
- Schematic roadmap for PV reduction of one-loop FVC tensor integrals



> ☞ **Dashed lines** : represent simplification operations by the recursive use of the recurrence relations.
>
> ☞ The $\widetilde{A}_0$, $\widetilde{A}_{00}$, $\widetilde{A}_{0000}$, etc, can be adopted as the tensor basis.

PV reduction of two-point tensor integrals

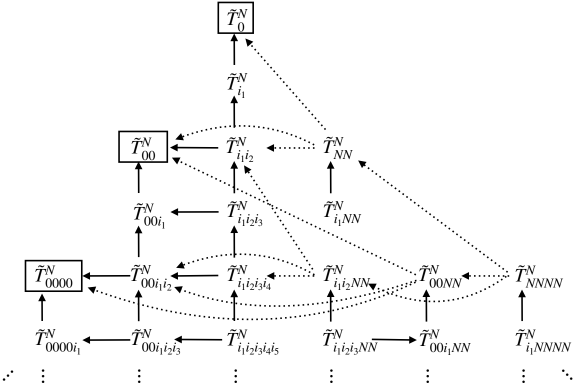
- Schematic roadmap for PV reduction of two-point FVC tensor integrals



- **Dashed lines** : the number of subscripts "2" is reduced by recursively utilizing the relation deduced by contracting the $g_{\mu\nu}$.
- **Solid lines** : the indices "1" can be eliminated by making use of the relation obtained by contracting of the external momentum $p_{1\mu}$.
- Like the case for one-point integrals, the tensor coefficients only with even numbers of "0" survive.

PV reduction of N-point tensor integrals

- Schematic roadmap for PV reduction of N-point FVC tensor coefficients



> - **Dashed lines** : by recursively utilizing the relation deduced by contracting the $g_{\mu\nu}$.
> - **Solid lines** : by making use of the relation obtained by contracting of the external momentum $p_j^{\mu_1}$.
> - The **boxed coefficients** are chosen as the tensor basis.

- It is a first attempt and only aim at finding out the feasibility of PV reduction and the existence of a tensor basis for the one-loop integrals at finite volume.

4. A Pedagogic Example of Application

The FVC of nucleon mass

- Leading one-loop Feynman diagrams contributing the nucleon mass

(a) (b)

- The self-energy of the nucleon can be expressed as

$$\Sigma(\not{p}, \not{n}) = \sum_{\mathbf{n} \neq 0} \left[\mathcal{A} + \not{p}\mathcal{B} + \not{n}\mathcal{C} \right]$$

  - $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are functions of the scalar products of the external momentum and the unit space-like vectors.
  - The occurrence of the third term is due to the introduction of spatial boundary conditions of the finite volume.

The FVC of nucleon mass

- The self-energy functions for (a)

$$\mathcal{A}_a = \frac{3g_A^2 m_N}{4F_\pi^2}\left\{s\widehat{B}_0 + 2s\widehat{B}_1 + d\widehat{B}_{00} + s\widehat{B}_{11} + n^2\widehat{B}_{22} - 2n\cdot p\left[\widehat{B}_2 + \widehat{B}_{12}\right]\right\},$$

$$\mathcal{B}_a = \frac{3g_A^2}{4F_\pi^2}\left\{s\widehat{B}_1 + 2s\widehat{B}_{11} + 2d\widehat{B}_{00} + (d+2)\widehat{B}_{001} + s\widehat{B}_{111} + n^2(2\widehat{B}_{22} + \widehat{B}_{122}) - 2n\cdot p\left[\widehat{B}_2 + 2\widehat{B}_{12} + \widehat{B}_{112}\right]\right\},$$

$$\mathcal{C}_a = \frac{3g_A^2}{4F_\pi^2}\left\{s\widehat{B}_2 - (d+2)\widehat{B}_{002} - s\widehat{B}_{112} - n^2\widehat{B}_{222} + 2n\cdot p\widehat{B}_{122}\right\},$$

- ☞ $s = p^2$.
- ☞ g_A is the axial coupling constant, F_π is the pion decay constant, and m_N denotes the nucleon mass in the chiral limit.

The FVC of nucleon mass

- In the CM frame, one has $\bar{u}(p)\not{n}u(p) = 0$. And the self-energy functions for (a) can be simplified to

$$\mathcal{A}_a = \frac{3g_A^2 m_N}{4F_\pi^2}\left\{s\widetilde{B}_0 + 2s\widetilde{B}_1 + d\widetilde{B}_{00} + s\widetilde{B}_{11} + (d-1)\widetilde{B}_{22}\right\} ,$$

$$\mathcal{B}_a = \frac{3g_A^2}{4F_\pi^2}\left\{s\widetilde{B}_1 + 2s\widetilde{B}_{11} + 2d\widetilde{B}_{00} + (d+2)\widetilde{B}_{001} + s\widetilde{B}_{111} + (d-1)\left[2\widetilde{B}_{22} + \widetilde{B}_{122}\right]\right\} .$$

- The form is by making use of PV reduction

$$\mathcal{A}_a(L) = \frac{3g_A^2 m_N}{4F_\pi^2}\left\{\widetilde{A}_0(m_N^2; L) + M_\pi^2 \widetilde{B}_0(m_N^2, m_N^2, M_\pi^2; L)\right\} ,$$

$$\mathcal{B}_a(L) = \frac{1}{m_N}\mathcal{A}_a(L) .$$

where M_π is the pion mass and L is the size of the spatial cubic box.

The FVC of nucleon mass

- The self-energy functions for (b)

$$
\begin{aligned}
\mathcal{A}_b(L) = -\frac{h_A^2}{3F_\pi^2 m_\Delta}\Big\{&(m_\Delta^2 - m_N^2 + 3M_\pi^2)\widetilde{A}_0(M_\pi^2; L) - (m_\Delta^2 + m_N^2 - M_\pi^2)\widetilde{A}_0(m_\Delta^2; L) \\
&+ \lambda(m_\Delta^2, m_N^2, M_\pi^2)\widetilde{B}_0(m_N^2, m_\Delta^2, M_\pi^2; L)\Big\} , \\
\mathcal{B}_b(L) = \frac{h_A^2}{6F_\pi^2 m_\Delta^2 m_N^2}\Big\{&\lambda(m_\Delta^2, m_N^2, M_\pi^2)\widetilde{A}_0(m_\Delta^2; L) - [(m_\Delta^2 - M_\pi^2)^2 - m_N^4 + 4m_N^2 M_\pi^2]\widetilde{A}_0(M_\pi^2; L) \\
&+ 4m_N^2[\widetilde{A}_{00}(m_\Delta^2; L) - \widetilde{A}_{00}(M_\pi^2; L)] \\
&+ \lambda(m_\Delta^2, m_N^2, M_\pi^2)(m_\Delta^2 + m_N^2 - M_\pi^2)\widetilde{B}_0(m_N^2, m_\Delta^2, M_\pi^2; L)\Big\} ,
\end{aligned}
$$

  - Källén function $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$.
  - h_A is the coupling constant of the $\pi N\Delta$ interaction, and m_Δ is the mass of the Δ resonance in the chiral limit.

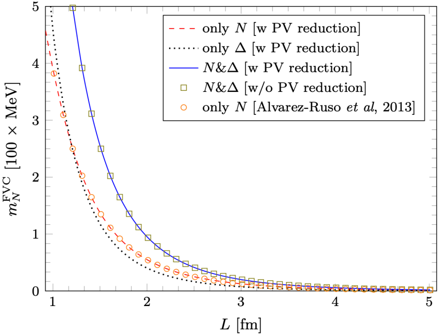
A pedagogic example of application

- The expression of the FVC on the nucleon mass

$$m_N^{\mathrm{FVC}}(L) = \left[\mathcal{A}(L) + m_N \mathcal{B}(L)\right]$$

with $\mathcal{A}(L) = \mathcal{A}_a(L) + \mathcal{A}_b(L)$ and $\mathcal{B}(L) = \mathcal{B}_a(L) + \mathcal{B}_b(L)$.

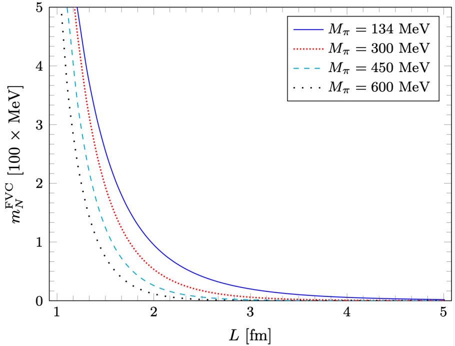
- FVC to the nucleon mass



- ☞ The validity of the PV reduction for the FVC tensor coefficients is explicitly verified.
- ☞ The result of diagram (a) is identical to the one given in Ref. [L. Alvarez-Ruso, *et al*, PRD 88, 054507 (2013)].
- ☞ The contributions of the nucleon and delta loops are comparable with each other, which implies the importance of the Δ resonance in the estimation of FVC to the nucleon mass.

A pedagogic example of application

- The L-dependence of the nucleon mass with different pion mass.



- ☞ For a given finite size L, the larger the pion mass is, the smaller the FVC become.
- ☞ The effect of FVC on the nucleon mass becomes negligible when $M_\pi L \gtrsim 3$.

5. Summary and Outlook

Summary and Outlook

- A systematical formulation of one-loop tensor integrals for FVC is achieved.
- A compact formula for the tensor coefficients in the decomposition has been derived, which is suitable for numerical computations.
- CM frame: the tensor coefficients can be simplified by means of PV reduction.
- An example is given to illustrate the application of our formulation.

- The formulation pave a path for efficient computations of FVC. (e.g. can be readily implemented in FeynCalc.)
- Chiral extrapolation of Lattice QCD results with FVC and precise extraction of physical quantities.
- Generalize to two-loop integrals.

Thank you very much!