Finite-volume effects in the chiral limit

Akaki Rusetsky, University of Bonn
in collaboration with: U.-G. Meißner and F. Müller

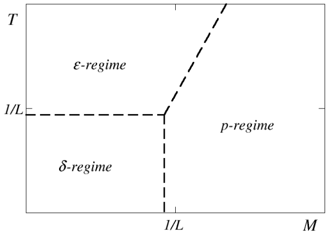
The 4th China Lattice Quantum Chromodynamics Workshop, Changsha, October 11-15, 2024

Plan

- QCD at a finite temperature/volume: different regimes
- The zero mode
- Naive power counting and perturbation theory
- Threshold expansion: recovering power counting
- Renormalization
- Conclusions & outlook

QCD at a finite temperature/volume: different regimes

- Scales:
  - Chiral symmetry breaking scale F
  - Goldstone boson mass M
  - Temperature $T = 1/L_t$
  - Box size L
- Hadrons fit the box, EFT applicable: FL, FL_t
- Reordering of perturbation series:
  - p-regime: $M \sim T \sim 1/L$
  - ϵ-regime: $M \ll T \sim 1/L$
  - δ-regime: $M \sim T \ll 1/L$
- δ-regime: applications
  - QCD
  - Condensed matter physics



Leutwyler (1987)

p-regime: Chiral Perturbation Theory in a finite volume

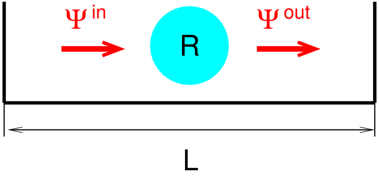
- Lagrangian:

$$\mathscr{L} = \frac{F^2}{4}\langle \partial_\mu U \partial_\mu U^\dagger + \chi U^\dagger + U\chi^\dagger \rangle + \mathscr{L}^{(4)} + \mathscr{L}^{(6)} + \cdots$$

- Discretized momenta in a box:
  - Three-momenta: $\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{n}\,, \quad \boldsymbol{n} \in \mathbb{Z}^3$
  - Matsubara frequencies: $k_0 = \frac{2\pi}{L_t} n_0\,, \quad n_0 \in \mathbb{Z}$
- Applications:
  Exponentially suppressed finite-volume corrections to the masses, coupling constants, matrix elements,. . .

p-regime, $ML \gg 1$

- Description of the scattering processes: scale separation



- Non-relativistic EFT: $R \sim 1/M$ and $p \sim 1/L \quad \hookrightarrow \quad p/M \ll 1$
- Lagrangian: $\mathscr{L}_{NR} = \phi^\dagger \left(i\partial_t - M + \frac{\nabla^2}{2M} + \cdots \right) \phi + c_0 \phi^\dagger \phi^\dagger \phi\phi + \cdots$
  - Antiparticles integrated out
  - Particle number is conserved
- Two- and three-body quantization condition, decay amplitudes,. . .

ϵ-regime

- Scales: $M \ll 1/L$ and $M \ll 1/L_t$
- Zero mode at:

$$n_\mu = (n_0, \boldsymbol{n}) = 0\,, \qquad M \to 0$$

- A consistent perturbative expansion through:

  Introducing a collective variable (net magnetization) and separating $n_\mu = 0$ mode by using the Faddeev-Popov trick (Hasenfratz and Leutwyler, 1989)

δ-regime

- Asymmetric box: $L \ll L_t$
- Chiral limit at zero temperature: $M \to 0$, $L_t \to \infty$ and L finite
- Zero mode at $\boldsymbol{n} = 0$ and arbitrary n_0

$$\int d^3\boldsymbol{x}\mathscr{L} = \frac{F^2L^3}{2}\sum_{n,a}\left(|\dot{q}_n^a|^2 - w_n^2|q_n^a|^2\right) , \qquad w_n = \left(M^2 + \left(\frac{2\pi}{L}\right)^2 \boldsymbol{n}^2\right)^{1/2}$$

The amplitude small $|q_n^a| \ll 1$ for $F^2L^3M \gg 1$, violated at $M \to 0$ (Leutwyler, 1987)

$\hookrightarrow$ The zero mode is non-perturbative

Separating zero mode

- Non-linear $O(N)$ σ-model at $M \to 0$:

$$\mathscr{L} = \frac{F^2}{2} \partial_\mu S^\alpha \partial_\mu S^\alpha + \cdots$$

$$Z = \int \mathscr{D}S \delta(S^2 - 1) \exp\left(-A[S]\right)$$

- Time-dependent net magnetization as a collective variable:

$$1 = \int \mathscr{D}m \delta^N \left(m^\alpha(t) - \frac{1}{L^3} \int d^3\boldsymbol{x} S^\alpha(\boldsymbol{x}, t) \right)$$

$$\begin{aligned} m^\alpha(t) &= m(t) e^\alpha(t) \qquad e^2(t) = 1 \\ \mathscr{D}m &= \prod_t m^{N-1}(d) dm(t) de(t) \end{aligned}$$

Spontaneous symmetry breaking

- Choosing direction:

$$e^{\alpha}(t) = \Omega^{\alpha\beta}(t) n^{\beta}\,, \qquad n = (1, 0, \cdots, 0)$$

$$\Omega^{00}(t) = e^{0}(t)\,, \quad \Omega^{i0}(t) = -\Omega^{0i}(t) = e^{i}(t)\,, \quad \Omega^{ij}(t) = \delta^{ij} - \frac{e^{i}(t) e^{j}(t)}{1 + e^{0}(t)}$$

- Not unambiguous: $\Omega \to \Omega \Sigma^{T}$, where $\Sigma^{T} n = n$
- Redefining field variables:

$$S^{\alpha}(x) = \Omega^{\alpha\beta}(t) R^{\alpha}(x)\,, \qquad R = \left(\sqrt{1 - \boldsymbol{R}^2}, \boldsymbol{R}\right)$$

$$Z = \int \mathscr{D}\boldsymbol{R} \int \mathscr{D}e\, \delta(e^2 - 1) \delta^{N-1}\left(\int d^3\boldsymbol{x} \boldsymbol{R}(\boldsymbol{x}, t)\right) \exp\left(-A[\Omega R]\right)$$

Propagator of the fast mode

$$\langle R^i(x) R^j(y) \rangle = \frac{\delta^{ij}}{F^2} \int \frac{dk^0}{2\pi} \frac{1}{L^3} \sum_{\boldsymbol{k} \neq 0} \frac{e^{ik(x-y)}}{k_0^2 + \boldsymbol{k}^2}, \qquad i, j = 1, \cdots N-1$$

↪ Zero mode is absent, owing to the condition

$$\int d^3\boldsymbol{x} R^i(\boldsymbol{x}, t) = 0$$

↪ Zero mode becomes the collective variable $e^{\alpha}(t)$

Finite-volume spectrum of the free Hamiltonian

$$\mathscr{L}_0 = \underbrace{\frac{F^2}{2}\,\dot{e}^\alpha(t)\dot{e}^\alpha(t)}_{\text{rigid rotator}} + \underbrace{\frac{F^2}{2}\,\partial_\mu \boldsymbol{R}(x)\partial_\mu \boldsymbol{R}(x)}_{\text{oscillator}}$$

- Two-point function of the zero mode

$$\langle e^\alpha(t)e^\alpha(0)\rangle = \sum_n e^{-\varepsilon_n|t|}\langle 0|e^\alpha(0)|n\rangle\langle n|e^\alpha(0)|0\rangle = e^{-\varepsilon_1|t|}$$

$$\varepsilon_n = \frac{n(n+N-2)}{2\Theta}\,, \qquad \Theta = F^2L^3 \ \text{(moment of inertia)}$$

$\hookrightarrow$ There are no massless excitations in a chiral limit in a finite volume (Leutwyler, 1987)

$$\varepsilon_1 = \frac{N-1}{2F^2L^3}$$

The spectrum in the moving frame

$$D(\boldsymbol{p}, t) = \int d^3\boldsymbol{x} e^{-i\boldsymbol{p}\boldsymbol{x}} \langle S^\alpha(x) S^\alpha(0) \rangle \sim e^{-E(\boldsymbol{p})t}\,, \qquad t > 0$$

- If $\boldsymbol{p} = 0$, then $E(\boldsymbol{p}) = \varepsilon_1$
- If $\boldsymbol{p} \neq 0$, then $E(\boldsymbol{p}) = |\boldsymbol{p}|$
- Counting: $\boldsymbol{p} = O(1/L)$ and $\varepsilon_n = O(1/L^3)$

- The relativistic dispersion law

$$\sqrt{\boldsymbol{p}^2 + m^2} \simeq |\boldsymbol{p}| + \frac{m^2}{2|\boldsymbol{p}|} + \cdots$$

... is not fulfilled, ε_1 cannot be interpreted as the pion mass in a finite volume!

Wick's theorem for the zero mode

$$\langle e^\alpha(t_1)e^\beta(t_2)\rangle = \int \mathscr{D}e\,\delta(e^2-1)\exp\left(-\frac{\Theta}{2}\int d\tau\,\dot{e}^\sigma(\tau)\dot{e}^\sigma(\tau)\right)e^\alpha(t_1)e^\beta(t_2)$$
$$= \frac{\delta^{\alpha\beta}}{N}\,e^{-\varepsilon_1|t_1-t_2|}$$

- If the unperturbed Lagrangian describes rigid rotator, Wick's theorem does not work

$$\langle e^{\alpha_1}(t_1)e^{\alpha_2}(t_2)e^{\alpha_3}(t_3)e^{\alpha_4}(t_4)\rangle \neq \sum_{\text{perm } ijkl}\langle e^{\alpha_i}(t_i)e^{\alpha_j}(t_j)\rangle\langle e^{\alpha_k}(t_k)e^{\alpha_l}(t_l)\rangle$$

$\hookrightarrow$ Insert full set of eigenstates between operators, evaluate matrix elements

The Lagrangian and the counting rules

- The Lagrangian used in the calculations at NNLO

$$\mathscr{L} = \frac{F^2}{2}\,\partial_\mu S^\alpha \partial_\mu S^\alpha - \ell_1 \partial_\mu S^\alpha \partial_\mu S^\alpha \partial_\nu S^\beta \partial_\nu S^\beta - \ell_2 \partial_\mu S^\alpha \partial_\nu S^\alpha \partial_\mu S^\beta \partial_\nu S^\beta$$

- Counting rules

$$e^\alpha \sim O(1)\,, \qquad \boldsymbol{R} \sim O(1/L)$$

$$\partial_t e^\alpha = O(1/L^3)\,, \qquad \partial_\mu \boldsymbol{R} \sim O(1/L^2)$$

$\hookrightarrow$ Different power counting to the fast and zero modes!

Green functions

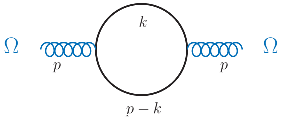
$$
\begin{aligned}
\langle S^\alpha(x) S^\beta(y) \cdots \rangle &= \int \mathscr{D}\boldsymbol{R}\,\delta^{N-1}\left(\int d^3\boldsymbol{x}\boldsymbol{R}(\boldsymbol{x},t)\right) \mathscr{D}e\,\delta(e^2-1) \\
&\times \exp\left(-\frac{F^2}{2}\int d^4z\,\partial_\mu\boldsymbol{R}(z)\partial_\mu\boldsymbol{R}(z) - \frac{\Theta}{2}\int d\tau\,\dot{e}^\alpha(\tau)\dot{e}^\alpha(\tau)\right) \\
&\times \left\{1 - \frac{1}{1!}\int d^4u\,\mathscr{L}_{\text{int}}(u) + \cdots\right\}\left(S^\alpha(x) S^\beta(y) \cdots\right)
\end{aligned}
$$

- Evaluate path integrals over $\boldsymbol{R}$, using Feynman diagrammatic technique,
- Calculate matrix elements containing $\Omega^{\alpha\beta}$

Breakdown of the counting rules

- Different scaling of momenta:

$$p_0 \sim 1/L \quad \text{(fast mode)}, \qquad p_0 \sim 1/L^3 \quad \text{(zero mode)}$$



$$I = \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\boldsymbol{k} \neq 0} \frac{1}{(k_0^2 + \boldsymbol{k}^2)((p_0 - k_0)^2 + \boldsymbol{k}^2)}$$

- In order to restore power counting, a field transformation $\Omega \to \Omega \Sigma^T$ was used, which removes the ΩRR vertex from the Lagrangian (Hasenfratz & Niedermayer, 1992)
- Systematic power counting at higher orders?

Threshold expansion

- Expand integrands of all Feynman integrals:

$$I = \int \frac{dk_0}{2\pi} \frac{1}{L^3} \sum_{\boldsymbol{k} \neq 0} \frac{1}{(k_0^2 + \boldsymbol{k}^2)^2} \Bigg(1 + \underbrace{\frac{2 p_0 k_0 - k_0^2}{k_0^2 + \boldsymbol{k}^2}}_{\text{power corrections in } 1/L} + \cdots \Bigg)$$

- Reproduces the results of Hasenfratz & Niedermayer (1992) up to and including NNLO
- No field transformations are necessary. The result does not change under field redefinitions
- Can be systematically pursued at any order in the $1/L$ expansion

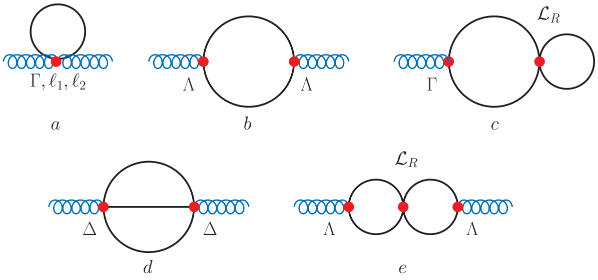
Calculations at the NNLO: the Lagrangian

$$\mathscr{L}_{\mathrm{int}} = \frac{F^2}{2}\Gamma^{ij}R^iR^j + \frac{F^2}{2}\Lambda^{ij}R^i\dot{R}^j - F^3\Delta^i\boldsymbol{R}^2\dot{R}^i + \mathscr{L}_R - 4\ell_1 b_1^{ij}\dot{R}^i\dot{R}^j - 2\ell_2 b_2^{ij}\dot{R}^i\dot{R}^j + \cdots$$

where

$$\begin{aligned}
\Gamma^{ij} &= -\delta^{ij}\dot{e}^{\alpha}\dot{e}^{\alpha} + \dot{\Omega}^{\alpha i}\dot{\Omega}^{\alpha j} \\
\Lambda^{ij} &= \dot{\Omega}^{\alpha i}\Omega^{\alpha j} - \dot{\Omega}^{\alpha j}\Omega^{\alpha i} \\
\Delta^{i} &= \dot{\Omega}^{\alpha 0}\Omega^{\alpha i} - \dot{\Omega}^{\alpha i}\Omega^{\alpha 0} \\
b_1^{ij} &= \dot{\Omega}^{\alpha 0}\Omega^{\alpha i}\dot{\Omega}^{\beta 0}\Omega^{\beta j} \\
b_2^{ij} &= \dot{\Omega}^{\alpha 0}\dot{\Omega}^{\alpha 0}\Omega^{\beta i}\Omega^{\beta j} + \dot{\Omega}^{\alpha 0}\dot{\Omega}^{\beta 0}\Omega^{\alpha i}\Omega^{\beta j} \\
\mathscr{L}_R &= (\boldsymbol{R}\partial_\mu\boldsymbol{R})^2
\end{aligned}$$

Calculations at the NNLO: the diagrams



- Agrees with Hasenfratz & Niedermayer (1992) up to and including NNLO:

$$\Theta = F^2 L^3 \left(1 + \frac{C_1}{(FL)^2} + \frac{C_2}{(FL)^4} + \frac{C_3}{(FL)^4} \ln(FL)\right)$$

Calculations of the matrix elements: zero mode

$$\int \mathscr{D}e\,\delta(e^2-1)\exp\left(-\frac{\Theta}{2}\int d\tau\,\dot{e}^\alpha(\tau)\dot{e}^\alpha(\tau)\right)O_1(e(t_1))\cdots O_m(e(t_m))$$

$$= \sum_{n_1\cdots n_{m-1}} e^{-\varepsilon_{n_1}(t_1-t_2)-\cdots-\varepsilon_{n_{m-1}}(t_{m-1}-t_m)}\langle 0|O_1|n_1\rangle\cdots\langle n_{m-1}|O_m|0\rangle$$

for $t_1 > t_2 > \cdots > t_m$.

- Matrix elements are expressed through integrals over hyperspheric functions

$$\langle n|O|n'\rangle = \int d\Omega_N\, Y_n^*(e)O(e)Y_{n'}(e)$$

where

$$e^0 = \cos\theta_1\,, \qquad e^1 = \sin\theta_1\cos\theta_2\,, \cdots$$

$$d\Omega_N = (\sin\theta_1)^{N-2}\cdots\sin\theta_{N-2}d\theta_1\cdots d\theta_{N-2}d\varphi$$

Renormalization

- Do the counterterms that are already present in the Lagrangian, remove all ultraviolet divergences despite splitting off the zero mode? Yes!

$\hookrightarrow$ Divergences arise only in the Feynman integrals of the fast modes which have the same structure as in the infinite volume ✓

$\hookrightarrow$ The sums over the rotator eigenstates are all convergent ✓

Conclusions & outlook

- A consistent perturbation theory has been formulated in the δ-regime
  - No need for the field transformation. The result is invariant under field redefinitions
  - Power-counting rules are restored by using the threshold expansion
- Outlook: Finite-volume artifacts in various lattice models of the condensed-matter physics:
  - Undoped antiferromegnets
- Outlook: Hole-doped antiferromagnets
  - Similar to Baryon Chiral Perturbation Theory, interplay of multiple scales