原行星盘与行星形成暑期学校, 北京, 2024/07/23

Advanced Topic:

# Magnetohydrodynamics: introduction

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# General references

- Landau & Lifshitz, *Fluid Mechanics (Vol. 6 of Course of Theoretical Physics)*, 1987, 2nd English Edition
- Shu, F. H., *The Physics of Astrophysics, Vol. 2: Gas dynamics*, 1992, University Science Books
- Ogilvie, G. I., *Lecture notes on Astrophysical Fluid Dynamics*, 2016, Journal of Plasma Physics, vol. 82, Cambridge Univ. Press
- Spruit, H. C., *Essential Magnetohydrodynamics for Astrophysics*, 2013, arXiv:1301.5572
- Kulsrud, R., *Plasma Physics for Astrophysics*, 2005, Princeton University Press

Note: here we only consider Newtonian (non-relativistic) fluid dynamics

# Outline

- Astrophysical fluids as plasmas
- The MHD formulation
- Conservation laws and physical interpretation
- Generalized Ohm's law, and limitations of MHD
- MHD waves
- MHD instabilities (examples)

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# What is a plasma?

**Plasma is a state of matter comprising of fully/partially ionized gas.**

Lightening

The restless Sun

Crab nebula

A plasma is generally quasi-neutral and exhibits collective behavior.

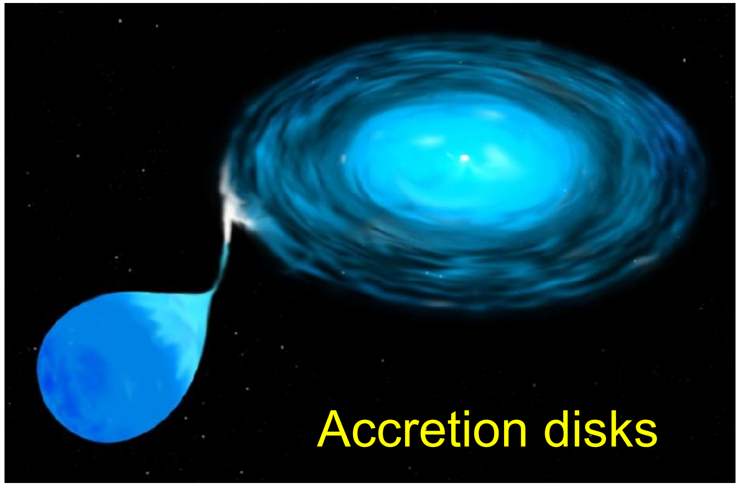
Net charge density averages to zero above microscale (i.e., Debye length).

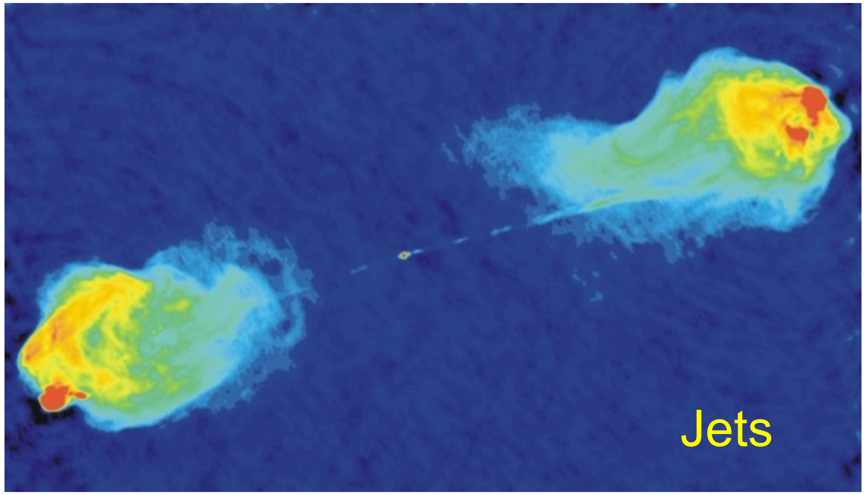
particles interact with each other at long-range through electro-magnetic fields (plasma waves).

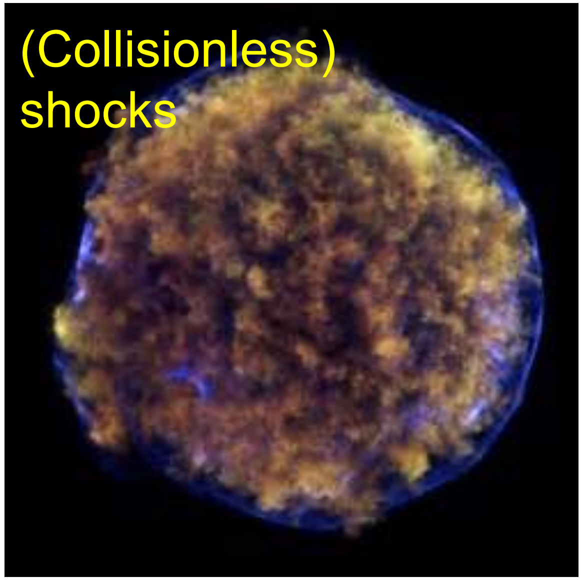
# Why plasma astrophysics?

- More than 99.9% of observable matter in the universe is plasma.
- Magnetic fields play vital roles in many astrophysical processes.
- Plasma astrophysics allows the study of plasma phenomena at extreme regions of parameter space that are in general inaccessible in the laboratory.

# Astrophysical applications




# Comparison of plasma and gas phases

| Property | Gas | Plasma |
|---|---|---|
| Electrical conductivity | Very low | Usually very high (effectively infinite in most cases) |
| Independently acting species | Usually, one | Two or three (electrons, ions and sometimes neutrals) |
| Velocity distribution | Maxwellian (due to frequent collisions) | Often non-Maxwellian: many plasmas are collisionless |
| Interactions | Binary collisions | Collective: organized motion by interacting with long-range electromagnetic fields in the form of plasma waves. |

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# Magneto-hydrodynamics (MHD)

MHD couples Maxwell's equations with hydrodynamics to describe the macroscopic behavior of highly conducting fluid such as plasmas.

Ideal MHD involves several important approximations, as we list below.

1. Flow velocity is very non-relativistic.

   Can be relaxed to formulate *relativistic MHD*.

2. Electric conductivity is so high that can be considered as infinite.

   Can be relaxed to formulate *non-ideal MHD*.

3. Low-frequency, long-wavelength.

   This is the key to the fluid description of plasmas (besides collisionalities).

We first formulate MHD equations from the above approximations, and then justify them (for 2 and 3).

# Reduction of Maxwell's equations

We focus on the non-relativistic regime where flow velocity v<<c.

Starting from Maxwell's equations:

[1] $\nabla \cdot \boldsymbol{E} = 4\pi\rho_e$ , (implied from [4])

[2] $\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$ , ⇨ $\frac{E}{B} \sim \frac{L}{cT} \sim \frac{V}{c}$

[3] $\nabla \cdot \boldsymbol{B} = 0$ , (implied from [2])

[4] $\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}$ . Displacement current can be dropped.

~B/L ~E/cT~ $(V^2/c^2)$ (B/L)

⇨ $\boldsymbol{J} = \frac{c}{4\pi}\nabla \times \boldsymbol{B}$

The system can adjust its current adjusted instantaneously to match field configuration.

# The induction equation

B field evolves according to the induction equation:

$$\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}$$

But what determines $\boldsymbol{E}$?

Under the assumption that the fluid is infinitely conducting:

$$\boldsymbol{E}' = 0$$

(in fluid rest frame)

This leads to **ideal MHD**.

The relation between E and E' is given by a Lorentz transformation:

$$\boldsymbol{E}' = \frac{\boldsymbol{E} + (\boldsymbol{v}/c) \times \boldsymbol{B}}{\sqrt{1 - v^2/c^2}} \approx \boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$$

where again we have assumed v<<c.

With E'=0, the induction equation becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

# Momentum equation

A conducting fluid is further subject to the Lorentz force:

$$\boldsymbol{F}_{\mathrm{EM}} = \frac{1}{c}\boldsymbol{J} \times \boldsymbol{B} + \rho_e \boldsymbol{E}$$

~$B^2/L$ ~$E^2/L$~$(V^2/c^2)B^2/L$

However, the electric force about a factor $(V/c)^2$ smaller and can be dropped.

Therefore, the MHD momentum equation simply reads:

$$\begin{aligned}\rho(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla)\boldsymbol{v} &= -\nabla P - \nabla \Phi + \frac{1}{c}\boldsymbol{J} \times \boldsymbol{B} \\ &= -\nabla P - \nabla \Phi + \frac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\end{aligned}$$

# Summary: ideal MHD equations

Continuity equation (unchanged):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

Momentum equation (now includes the Lorentz force):

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)\boldsymbol{v} = -\nabla P + \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{c} + \rho \boldsymbol{g}$$

Induction equation (new addition):

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

Thermal energy equation (unchanged in the current form):

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \nabla)s = 0 \quad \text{or} \quad \frac{\partial P}{\partial t} + \boldsymbol{v} \cdot \nabla P + \gamma P \nabla \cdot \boldsymbol{v} = 0$$

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# Momentum conservation

With the addition of the Lorentz force, we note

$$\frac{1}{c}\boldsymbol{J}\times\boldsymbol{B} = \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B} - \nabla\left(\frac{B^2}{8\pi}\right)$$

$$= \nabla\cdot\left(\frac{\boldsymbol{BB}}{4\pi} - \frac{B^2}{8\pi}\mathsf{I}\right)$$

magnetic tension

magnetic pressure

Momentum conservation becomes

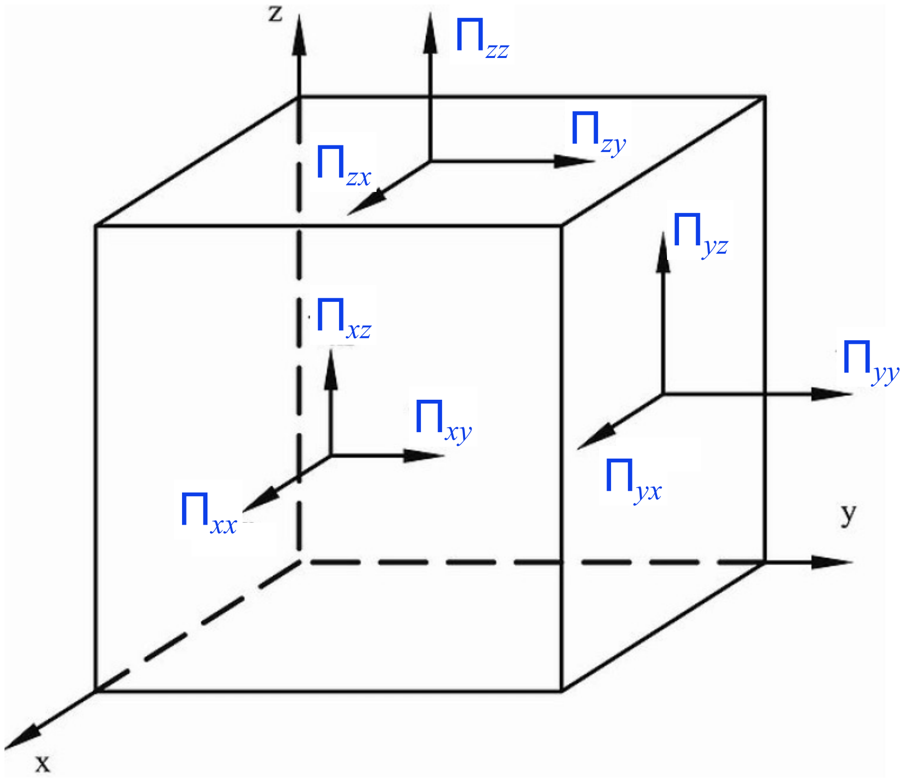
$$\frac{\partial\rho\boldsymbol{v}}{\partial t} + \nabla\cdot(\rho\boldsymbol{vv} + \Pi) = \rho\boldsymbol{g}$$

where the stress tensor is $\Pi \equiv \left(P + \frac{B^2}{8\pi}\right)\mathsf{I} - \frac{\boldsymbol{BB}}{4\pi}$

total pressure

One may further include viscous stress introduced earlier, and gravitational stress (for self-gravitating system) depending on application.

# Understanding the stress tensor




- Imagine you have an infinitesimally small box.
- Forces are exerted on each face from the outside volume.
- The forces on each side have components in 3 directions.
- The stress tensor then includes all 9 quantities needed to describe these forces.

# More on magnetic tension and pressure

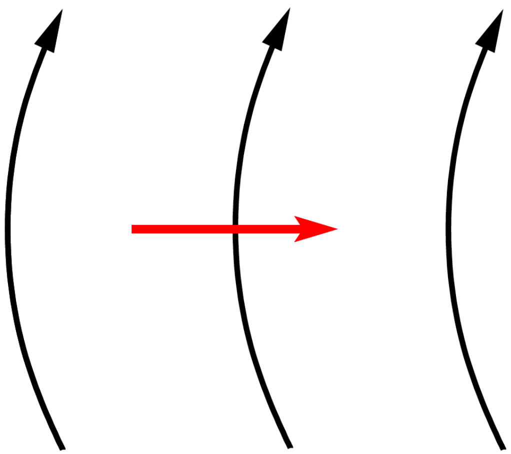
The Lorentz force is perpendicular to B, but magnetic pressure looks like an isotropic pressure. A better way to decompose the Lorentz force is as follows.

$$\frac{\boldsymbol{J}\times\boldsymbol{B}}{c} = \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} = \frac{\boldsymbol{B}\cdot\nabla\boldsymbol{B}}{4\pi} - \nabla\left(\frac{B^2}{8\pi}\right)$$

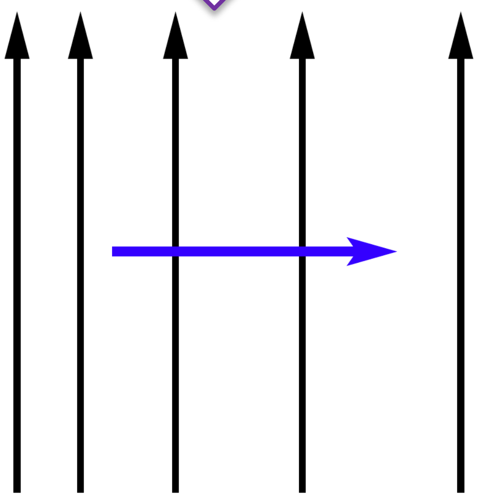
$$= \boldsymbol{\kappa}\frac{B^2}{4\pi} - \nabla_\perp\left(\frac{B^2}{8\pi}\right)$$

Curvature:

$$\boldsymbol{\kappa} \equiv \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}} = -\frac{\mathbf{R}}{R^2}$$



Magnetic tension

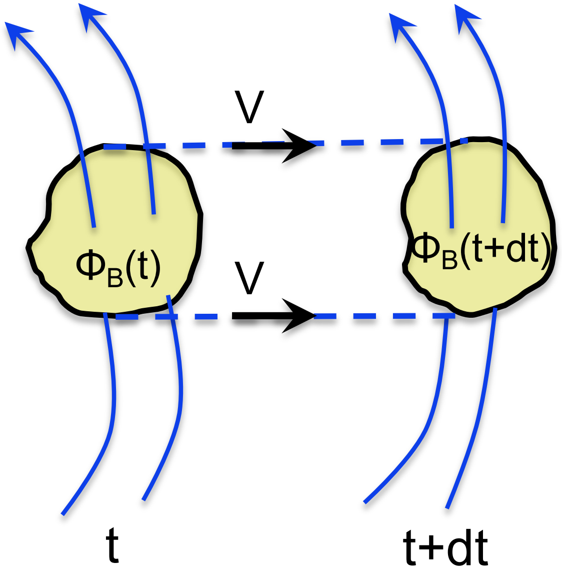


Magnetic pressure

# Magnetic flux conservation

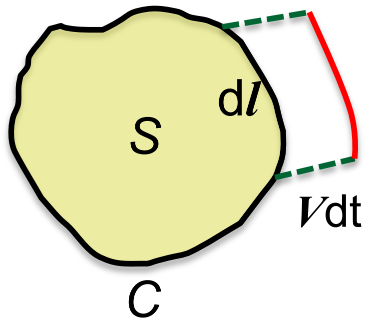
The induction equation: $$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

already implies B flux conservation. More importantly, it implies:




The B flux through a co-moving fluid loop is constant (known as *Alfvén's theorem*).

Proof: $$\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$$



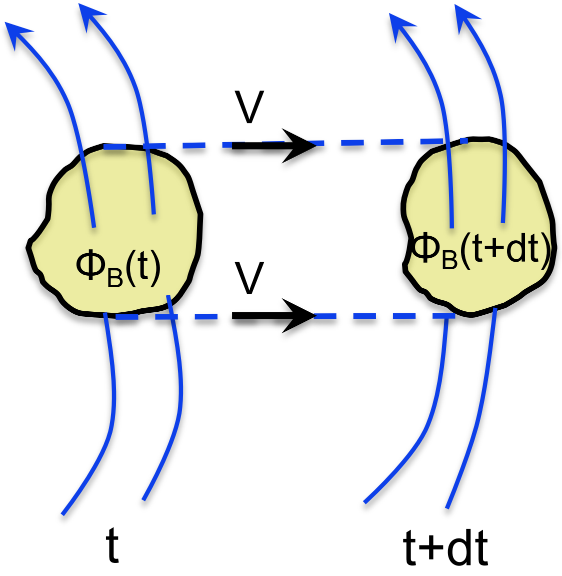

$$\frac{D\Phi}{Dt} = \int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} + \oint_C \boldsymbol{B} \cdot (\boldsymbol{V} \times d\boldsymbol{l})$$

$$= -c \int_S \nabla \times \boldsymbol{E} \cdot d\boldsymbol{S} - \oint_C (\boldsymbol{V} \times \boldsymbol{B}) \cdot d\boldsymbol{l}$$

$$= -c \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} + c \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

# Flux freezing: physical meaning

In ideal MHD, the magnetic field are plasma are "*frozen-in*" to each other.




- The plasma can not move across B field lines.
- If two plasma elements are initially connected by a field line, they will remain connected.
- Magnetic topology is preserved in ideal MHD.

Physically: charged particles are tied to field lines as they gyrate.

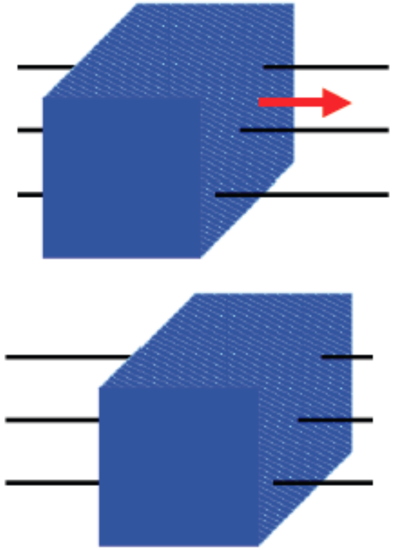
The frozen-in condition can break in non-ideal circumstances.

In particular, *magnetic reconnection* is a process that breaks magnetic field topology. It generally involves kinetic effects beyond MHD (in collisionless plasmas) and/or dissipation by resistivity.

We will briefly address this later.

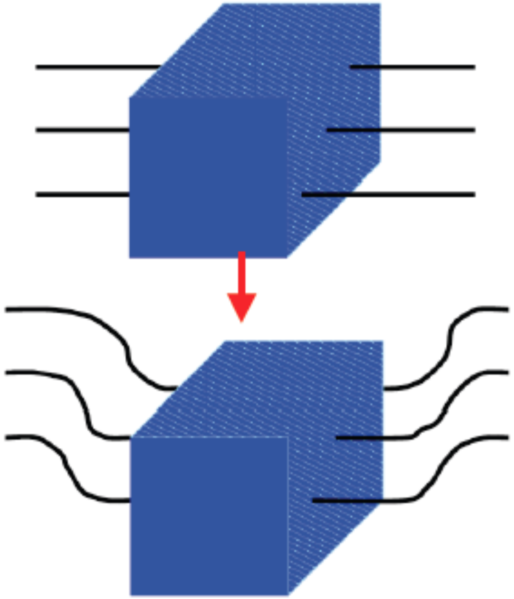
# Flux freezing with weak/strong field

Strong field: matter move along field lines (beads on a wire).



$$\frac{|B|^2}{8\pi} >> P_{gas} + \rho|\mathbf{v}|^2$$

Weak field: field lines are forced to move with the gas.



$$\frac{|B|^2}{8\pi} << P_{gas} + \rho|\mathbf{v}|^2$$

Strength of the B field is commonly characterized by the plasma $\beta$ parameter:

$$\beta \equiv \frac{P_{\rm gas}}{P_{\rm mag}} = \frac{8\pi P_{\rm gas}}{B^2}$$

# Energy conservation

We can rewrite the energy conservation equation

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2+\Phi+\epsilon\right)+\frac{B^2}{8\pi}\right]+\nabla\cdot\left[\rho\boldsymbol{v}\left(\frac{1}{2}v^2+\Phi+\epsilon\right)+P\boldsymbol{v}+\frac{c\boldsymbol{E}\times\boldsymbol{B}}{4\pi}\right]=0$$

Using $\boldsymbol{E}=-(\boldsymbol{v}/c)\times\boldsymbol{B}$ , we arrive at:

$$\boxed{\frac{\partial e}{\partial t}+\nabla\cdot(e\boldsymbol{v}+\Pi\cdot\boldsymbol{v})=0}$$

It has exactly the same form as HD energy equation except that we have updated

the energy density: $$e\equiv\rho\left(\frac{1}{2}v^2+\Phi+\epsilon\right)+\frac{B^2}{8\pi}$$

and the stress tensor: $$\Pi\equiv\left(P+\frac{B^2}{8\pi}\right)\mathsf{I}-\frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}$$

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# Generalized Ohm's law

We relax the assumption of infinite conductivity, which helps address the conditions under which ideal MHD fails.

The plasma is generally made of electrons and ions. Ions carry almost all the mass, representing the bulk plasmas. Here separate out the electrons (treated as a fluid).:

$$n_e m_e \frac{D\boldsymbol{v}_e}{Dt} = -\nabla P_e - e n_e \left( \boldsymbol{E} + \frac{\boldsymbol{v}_e}{c} \times \boldsymbol{B} \right) - n_e m_e \frac{\boldsymbol{v}_e - \boldsymbol{v}}{\tau_c}$$

inertia

pressure gradient

Lorentz force

"collision" with the bulk plasma

Being the lightest, $e^-$ almost instantly respond to EM fields (to avoid huge acceleration)
=> ignore inertia

⇨ The balance among the other terms determine the E field.

# Generalized Ohm's law

Electric field is found to be:

$$\boldsymbol{E} = -\frac{\nabla P_e}{en_e} - \frac{\boldsymbol{v}_e}{c} \times \boldsymbol{B} - m_e \frac{\boldsymbol{v}_e - \boldsymbol{v}}{e\tau_c}$$

$$= -\frac{\nabla P_e}{en_e} - \frac{\boldsymbol{v}}{c} \times \boldsymbol{B} - \frac{\boldsymbol{v}_e - \boldsymbol{v}}{c} \times \boldsymbol{B} - m_e \frac{\boldsymbol{v}_e - \boldsymbol{v}}{e\tau_c}$$

Note: $\boldsymbol{J} = -en_e(\boldsymbol{v}_e - \boldsymbol{v}_i)$

We arrive at the generalized Ohm's law:

$$\boldsymbol{E} = -\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} - \frac{\nabla P_e}{en_e} + \frac{1}{en_e c}\boldsymbol{J} \times \boldsymbol{B} + \frac{m_e}{e^2 n_e \tau_c}\boldsymbol{J}$$

$e^-$ pressure gradient — Hall term — resistivity

# Generalized Ohm's law

Are additional terms in the generalized Ohm's law important?

$$\boldsymbol{E} = -\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} - \underline{\frac{\nabla P_e}{e n_e} + \frac{1}{e n_e c}\boldsymbol{J} \times \boldsymbol{B}} + \frac{m_e}{e^2 n_e \tau_c}\boldsymbol{J}$$

One can show that these two terms are relevant only at microscopic scales:

$$d_i \sim \frac{c}{\omega_{pi}} = \sqrt{\frac{m_i c^2}{4\pi e^2 n_i}} \approx 230\ \mathrm{km}\left(\frac{n_i}{\mathrm{cm}^{-3}}\right)$$ (ion inertial length)

While we have ignored electron inertia, one can also show that that term is relevant at even smaller scales:

$$d_e \sim \frac{c}{\omega_{pe}} = \sqrt{\frac{m_e c^2}{4\pi e^2 n_e}} \approx 5.3\ \mathrm{km}\left(\frac{n_e}{\mathrm{cm}^{-3}}\right)$$ (electron inertial length)

This gives *one example* that MHD applies only on macroscopic scales.

# Applicability and limitations of MHD

- MHD applies to timescales much longer than

$$\tau \gg \omega_{pe}^{-1}, \omega_{pi}^{-1}, \Omega_{ce}^{-1}, \Omega_{ci}^{-1}$$

e⁻/ion plasma frequencies — e⁻/ion cyclotron frequencies

- MHD applies to length scales much larger than

$$L \gg \lambda_D, r_{Le}, r_{Li}, c/\omega_{pe}, c/\omega_{pi}$$

Debye length — e⁻/ion Larmor radii — e⁻/ion inertial lengths

- MHD requires the particle distribution function to be (at least approximately) isotropic and Maxwellian with $T_i = T_e$.

# Resistive MHD

We are left with the standard Ohm's law: $\boldsymbol{J} = \sigma \boldsymbol{E}' = \sigma \left( \boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c} \right)$

where electric conductivity $\sigma \equiv \frac{e^2 n_e \tau_c}{m_e}$ .

The induction equation now reads

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \frac{c}{\sigma} \nabla \times \boldsymbol{J}$$

$$-\frac{c^2}{4\pi\sigma} \nabla \times (\nabla \times \boldsymbol{B})$$

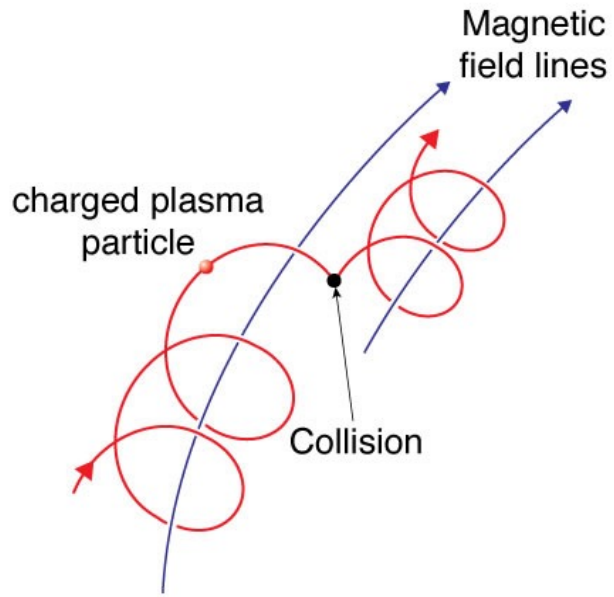
which finally becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

where Ohmic resistivity is given by $\eta \equiv \frac{c^2}{4\pi\sigma} = \frac{m_e c^2}{4\pi e^2 n_e \tau_c}$

# Resistive MHD

- Resistivity breaks the frozen-in condition, allowing filed lines to slide through the plasma.




Physically, this is due to particle collisions:

- Resistivity leads to energy dissipation (Ohmic heating).

Energy conservation becomes:

$$\frac{\partial e}{\partial t} + \nabla \cdot \left( e\boldsymbol{v} + \Pi \cdot \boldsymbol{v} + \frac{c\boldsymbol{E}_O \times \boldsymbol{B}}{4\pi} \right) = 0$$

where $\boldsymbol{E}_O \equiv \frac{4\pi\eta}{c^2}\boldsymbol{J}$

with the associated Ohmic heating: $\rho T \frac{Ds}{Dt} = \boldsymbol{E}_O \cdot \boldsymbol{J}$

# When is resistivity important?

Order of magnitudes from the induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}$$

~B/T ~BV/L ~$\eta$B/$L^2$

Define the magnetic Reynolds number:

$$\mathrm{Re}_M \equiv \frac{VL}{\eta}$$

resistivity is dominant when $\mathrm{Re}_M$~1 or less.

In most astrophysical systems, this number is huge:

In the solar interior, $\mathrm{Re}_M \sim 10^{16}$

In a 1pc parcel of the ISM, $\mathrm{Re}_M \sim 10^{18}$

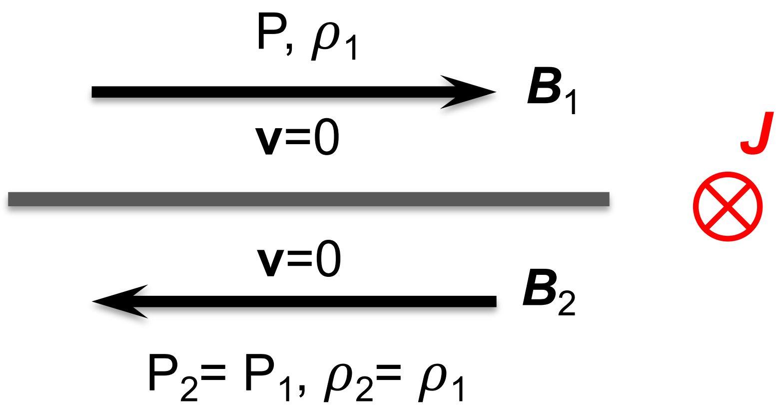
⇒ Flux freezing is valid.

Even with large $\mathrm{Re}_M$ for the bulk flow, resistivity can still play important roles at small scales (e.g., reconnection, discontinuities, turbulent dissipation).

The resistivity term is important in weakly ionized gas (to be discussed tomorrow).

# Current sheet

Current sheet is a special example of tangential discontinuity.

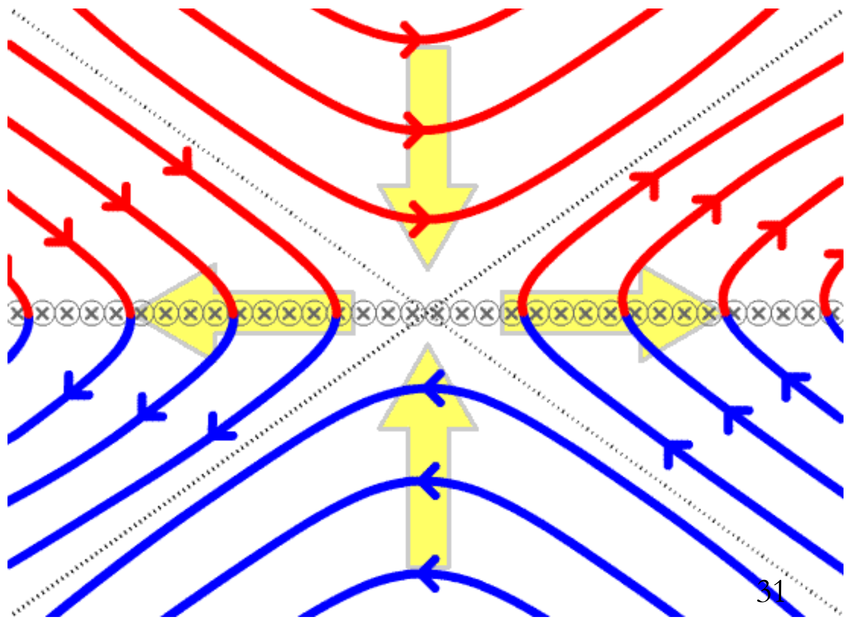



Strong current in the current sheet can lead to strong dissipation via resistivity.

The configuration is also subject to MHD instabilities (e.g., tearing mode and drift-kink).

This configuration is the prototype for studying magnetic reconnection.

Leading to change of field topology with rapid dissipation of magnetic energy.



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# MHD waves

We consider perturbations on top of a static homogeneous plasma with uniform field $\mathbf{B}_0$, and start with linearized MHD equations (subscript 1 for perturbed quantities).

We then decompose all perturbed quantities into Fourier modes in the form of $e^{\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$ to derive the dispersion relation.

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla\cdot\boldsymbol{v}_1 = 0\ ,$$
$$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} = -\nabla P_1 + \frac{(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0}{4\pi}\ ,$$
$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0)\ ,$$
$$\frac{\partial P_1}{\partial t} + \gamma P_0 \nabla\cdot\boldsymbol{v}_1 = 0$$

⇒

Independent from others (in adiabatic MHD) => entropy wave ($v_1$=$P_1$=B1=0)

$$-\omega\rho_1 + \rho_0 \boldsymbol{k}\cdot\boldsymbol{v}_1 = 0$$

$$-\omega\rho_0\boldsymbol{v}_1 = -P_1\boldsymbol{k} + \frac{(\boldsymbol{k}\times\boldsymbol{B}_1)\times\boldsymbol{B}_0}{4\pi}\ ,$$
$$-\omega\boldsymbol{B}_1 = \boldsymbol{k}\times(\boldsymbol{v}_1\times\boldsymbol{B}_0)\ ,$$
$$\omega P_1 + \gamma P_0 \boldsymbol{k}\cdot\boldsymbol{v}_1 = 0$$

Total of 8 equations (adiabatic MHD), with 7 degrees of freedom (since $\nabla\cdot\boldsymbol{B}_1 = 0$):

7 waves (1+2x3). *For isothermal MHD, there are 6 waves (2x3).*

# MHD waves

Without loss of generality, may take $\boldsymbol{B}_0$ to be along the $z$ direction, and let the angle between $\boldsymbol{k}$ and $z$ to be $\theta$.

$$-\omega\rho_0\boldsymbol{v}_1 = -P_1\boldsymbol{k} + \frac{(\boldsymbol{k}\times\boldsymbol{B}_1)\times\boldsymbol{B}_0}{4\pi} \ ,$$

$$-\omega\boldsymbol{B}_1 = \boldsymbol{k}\times(\boldsymbol{v}_1\times\boldsymbol{B}_0) \ ,$$

$$\omega P_1 + \gamma P_0\boldsymbol{k}\cdot\boldsymbol{v}_1 = 0$$

After some algebra, we obtain

$$-\omega^2\boldsymbol{v}_1 = -c_s^2(\boldsymbol{k}\cdot\boldsymbol{v}_1)\boldsymbol{k} - v_A^2\{\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{v}_1\times\boldsymbol{e}_z)]\}\times\boldsymbol{e}_z$$

where $c_s = \sqrt{\gamma P_0/\rho_0}$ is the adiabatic sound speed as usual.

$v_A \equiv \dfrac{B_0}{\sqrt{4\pi\rho_0}}$ is the Alfvén speed whose meaning will become clear shortly.

This is a linear equation for $\mathbf{v}_1$, whose solution gives the dispersion relation.

# Alfvén waves

The linear equations permit an incompressible mode which is unique in MHD.

$$-\omega^2 \boldsymbol{v}_1 = -c_s^2(\boldsymbol{k}\cdot\boldsymbol{v}_1)\boldsymbol{k} - v_A^2\{\boldsymbol{k}\times[\boldsymbol{k}\times(\boldsymbol{v}_1\times\boldsymbol{e}_z)]\}\times\boldsymbol{e}_z$$

For incompressible mode, $\boldsymbol{k}\cdot\boldsymbol{v}_1 = 0$

Nobel prize (1970)

The above equation reduces to:

$$-\omega^2 \boldsymbol{v}_1 = -k_z^2 v_A^2 \boldsymbol{v}_1$$

acceleration    restoring force:

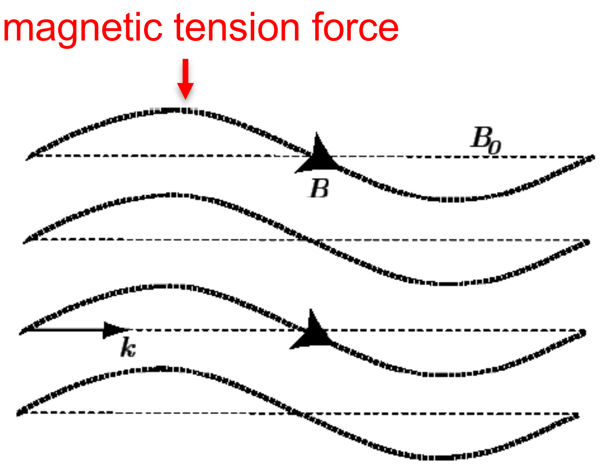
Lorentz force (magnetic tension)

This gives the dispersion relation for Alfvén waves:

$$\omega = \pm \boldsymbol{k}\cdot\boldsymbol{v}_A$$

← along $\boldsymbol{B}_0$.

The physics is analogous to waves on a string:



# Magnetosonic waves

There are two compressible modes with a dispersion relation:

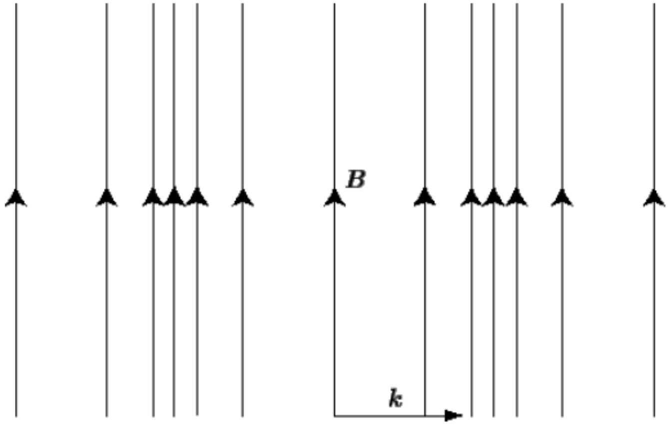
$$\frac{\omega^2}{k^2} = \frac{(c_s^2 + v_A^2)}{2} \pm \left[ \left( \frac{c_s^2 + v_A^2}{2} \right)^2 - c_s^2 v_A^2 \cos^2\theta \right]^{1/2}$$

They are know as fast (+) and slow (-) magnetosonic waves.

Analogous to sound waves modified by a B field.

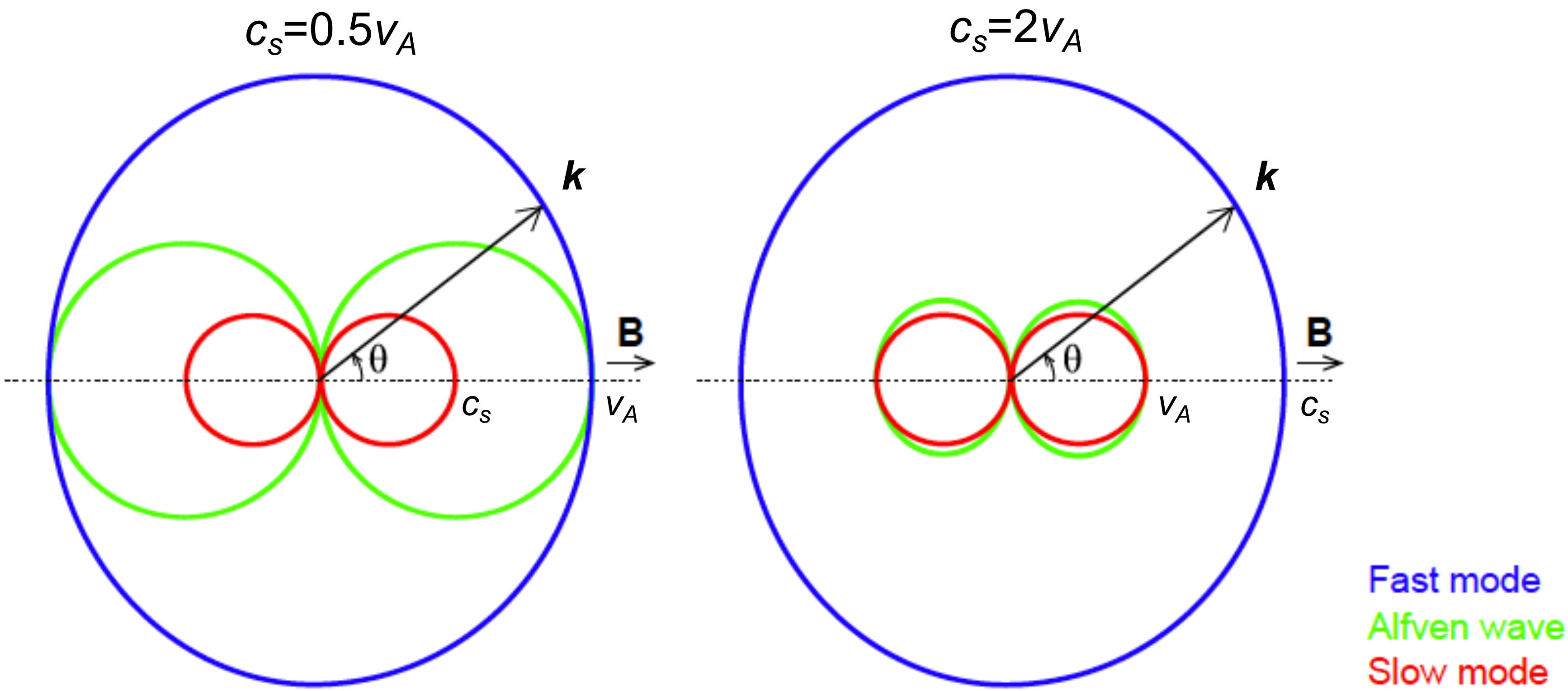
The restoring force include contributions from both thermal and magnetic pressure. Roughly speaking:

- In a slow wave, the two effects are out of phase.
- In a fast wave, the two effects are in phase.




The general behaviors are complex and are better visualized in Friedrichs diagrams.

# Friedrichs diagram




The slow wave can not propagate orthogonally.

The fast wave propagate quasi-isotropically.

# Outline

- Astrophysical fluids as plasmas
- The MHD formulation
- Conservation laws and physical interpretation
- Generalized Ohm's law, and limitations of MHD
- MHD waves
- MHD instabilities (examples)

# Fluid/MHD instability: guiding principles


Consider a Lagrangian perturbation with displacement vector $\boldsymbol{\xi}$ on top of an equilibrium configuration.

In response to the perturbation there is a force $\boldsymbol{F}(\boldsymbol{\xi})$.

This configuration is unstable if $\boldsymbol{\xi} \cdot \boldsymbol{F}(\boldsymbol{\xi}) > 0$

Namely, the force encourages the displacements, making the perturbation grow.

# Fluid/MHD instability: linear analysis

In general, one needs to conduct standard linear analysis:

One can either adopt the Eulerian approach as we did before in linear waves, or the Lagrangian approach described by a displacement $\boldsymbol{\xi}$ that satisfies

$$\delta \boldsymbol{v} = \frac{\partial \xi(\boldsymbol{r}, t)}{\partial t}$$

With linearized equations, we seek for solutions of the form:

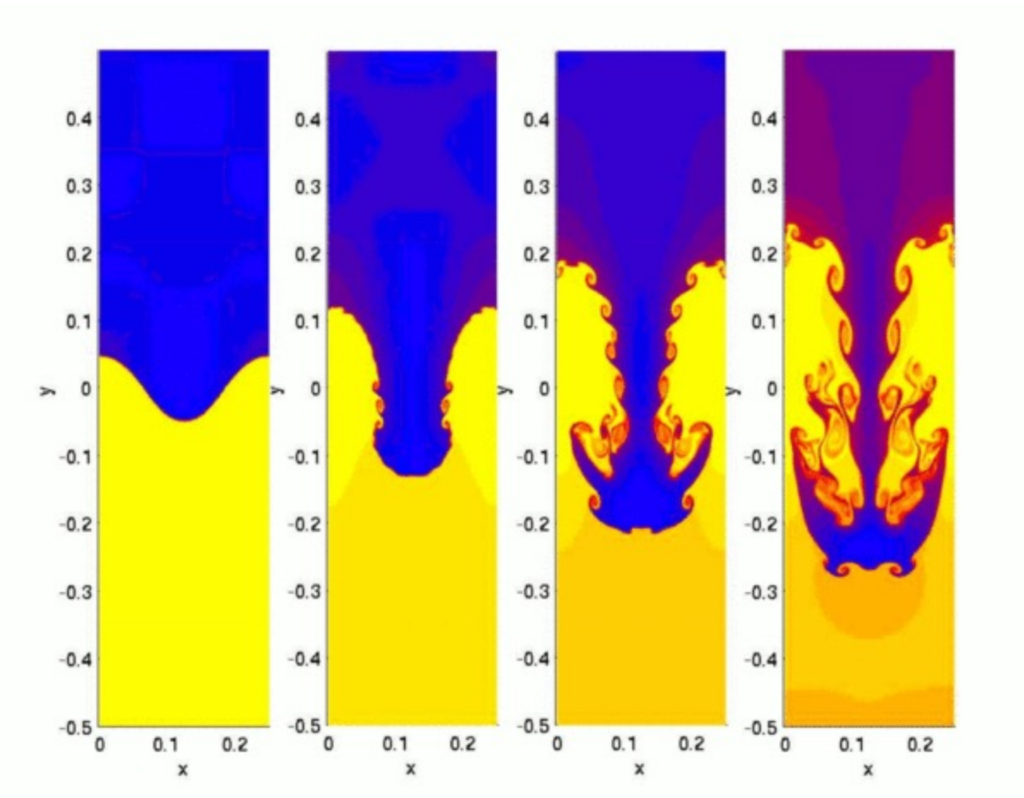
$$\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(\mathbf{r}) e^{i\omega t}$$

↖ an eigenfunction (e.g., a Fourier mode)

One can prove that in ideal MHD, solutions in $\omega^2$ is always real:

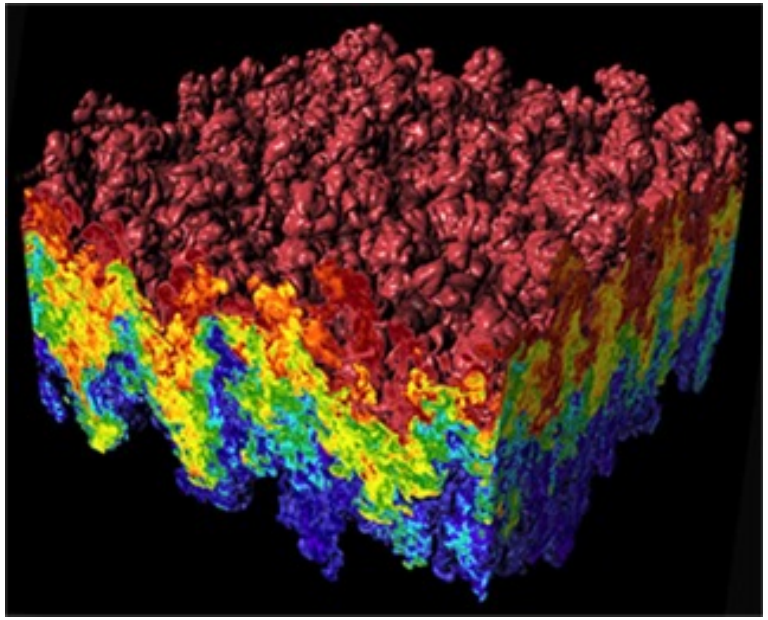
- If $\omega^2>0$ for all solutions, then the equilibrium is stable (oscillatory).
- If $\omega^2<0$ for some solutions, then the equilibrium is unstable (exponential growth).

# Rayleigh-Taylor instability




Unstable because heavy fluid is placed above light fluid.


It occurs in supernova explosions and subsequent evolution of the remnant



Can lead to substantial mixing

# Parker instability

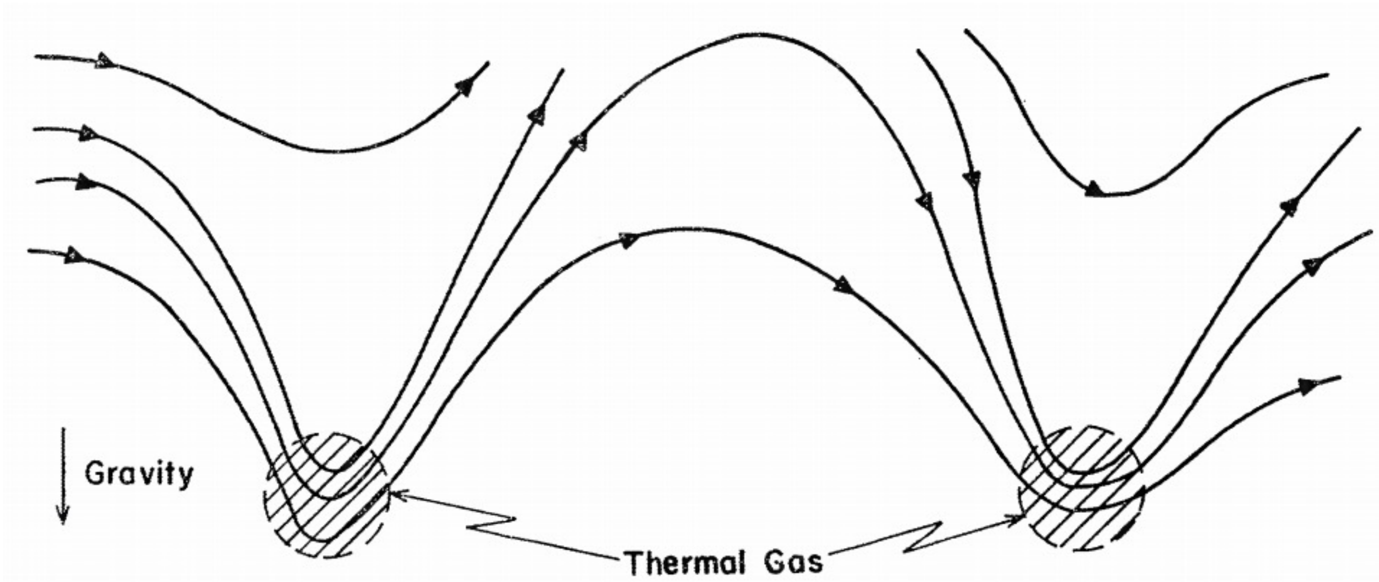
Parker 1966

Starting from a magneto-hydrostatic equilibrium in the galactic disk:

$$\frac{d}{dz}\left(p + P + \frac{B^2}{8\pi}\right) = -\rho(z)\, g(z)$$

gas pressure — CR pressure — B field pressure

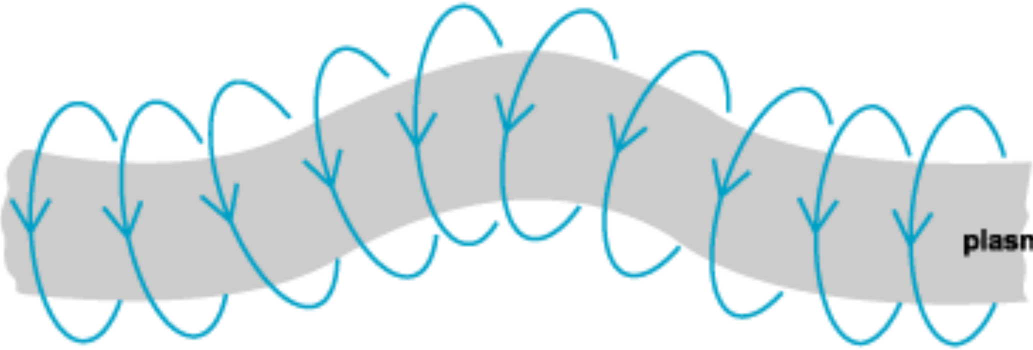
This configuration can be unstable at long wavelength due to magnetic buoyancy.




It may be responsible for the formation of molecular cloud.

# Kink instability

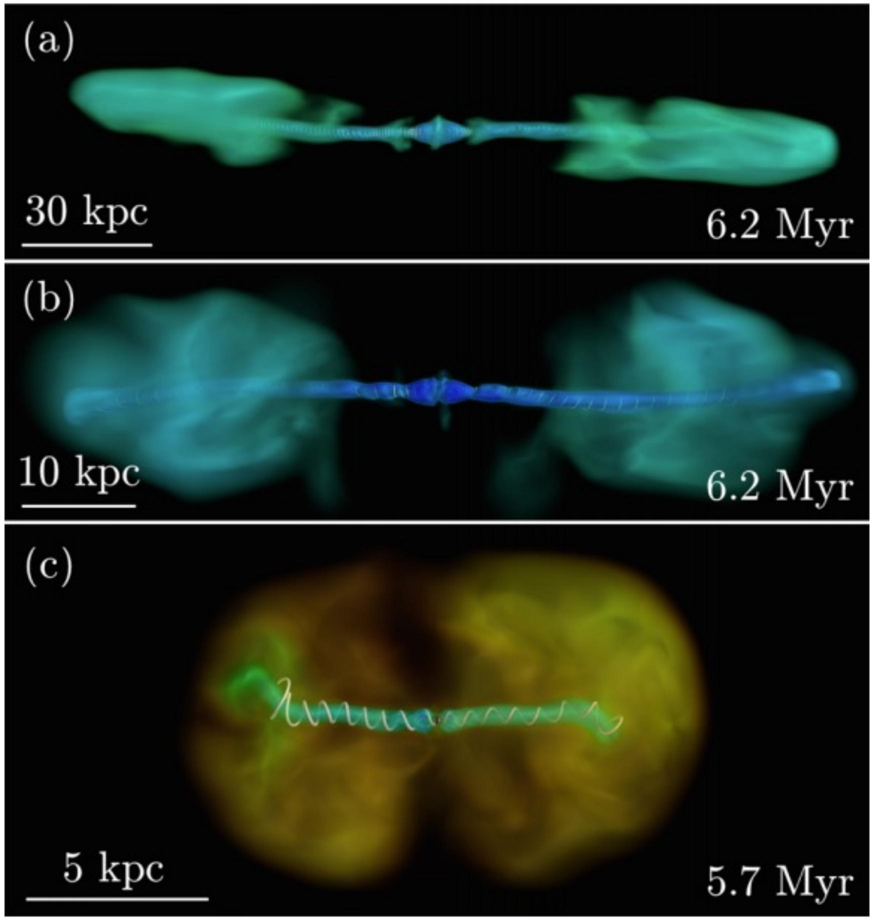
lower B pressure




higher B pressure

The distortion above leads to a difference in magnetic pressure that further enhances the distortion.

Adding an axial field is stabilizing which offers a restoring force from magnetic tension.




Tchekhovskoy & Bromberg, 2016

Kink instability is closely related to the stability of astrophysical jets.

# Magnetorotational instability (MRI)

- Rayleigh criterion for unmagnetized rotating disks:

Unstable if: $$\frac{d(\Omega R^2)}{dR} < 0$$ (Rayleigh, 1916)

Confirmed experimentally (Ji et al. 2006).

All astrophysical disks should be stable against this criterion.

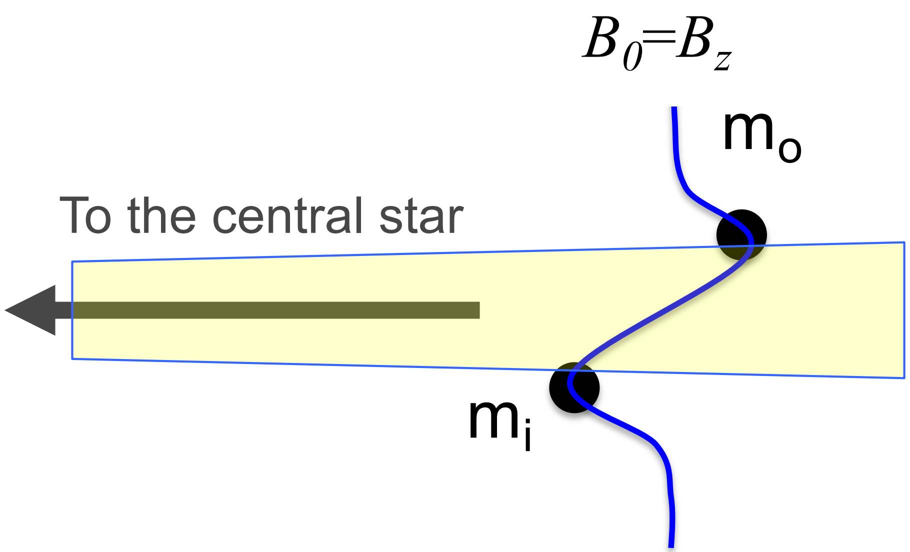
- Including (a vertical, well-coupled) magnetic field qualitatively changes the criterion (even as B->0):

Unstable if: $$\frac{d\Omega}{dR} < 0$$ Velikhov (1959), Chandrasekhar (1960), Balbus & Hawley (1991)

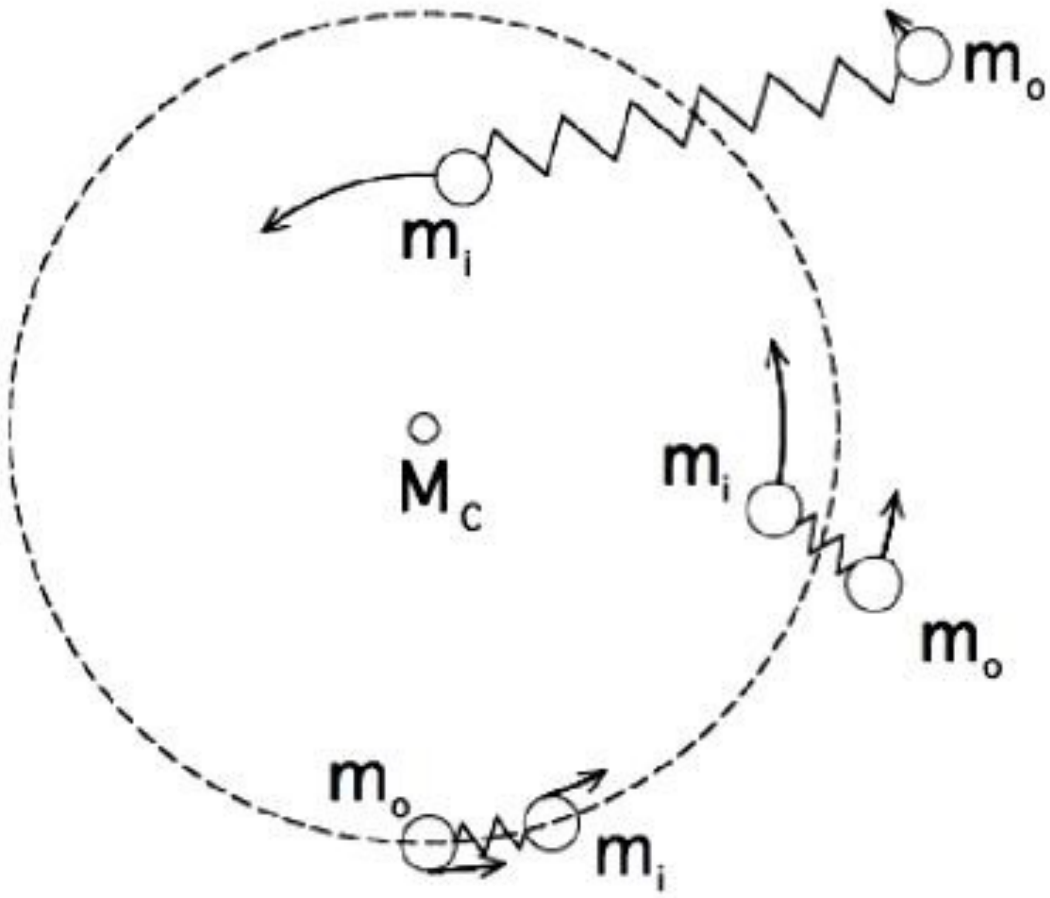
All astrophysical disks should be unstable!

# Magnetorotational instability (MRI)

Edge on view:

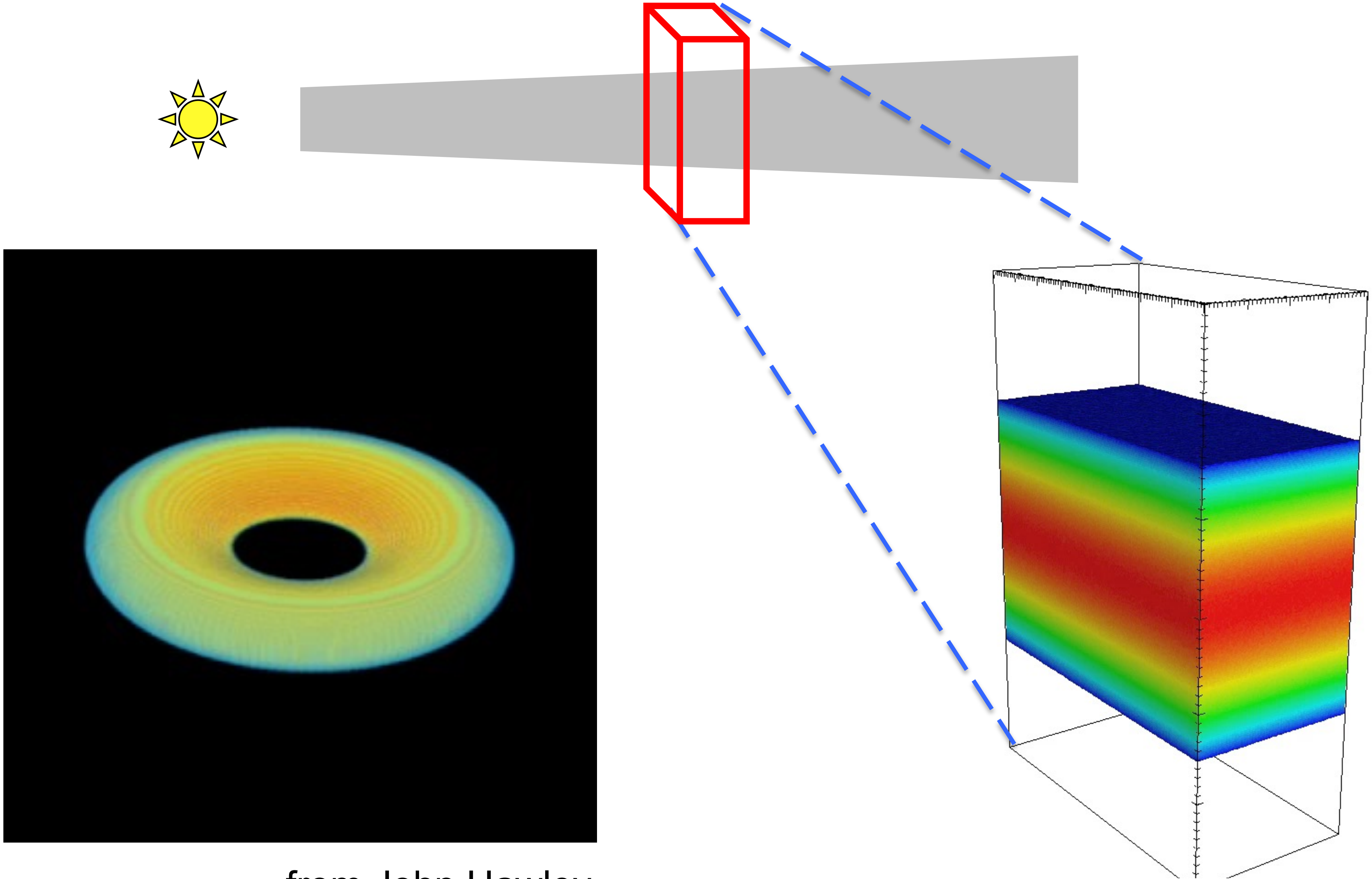



Face on view:




Magnetic tension force behaves like a spring.

# Local and global simulations of the MRI



from John Hawley

# Summary

- Plasmas are ubiquitous in astrophysical systems.
- MHD, the fluid description of plasmas, is a long-wavelength, low-frequency limit approximation.
- Lorentz force consists of magnetic tension and pressure.
- In ideal MHD, magnetic flux is frozen-in to the fluid, whereas resistivity breaks this condition.
- Three types of MHD waves: slow/fast magnetosonic (compressible) waves, and the Alfvén wave (incompressible).
- There are a wide variety of HD/MHD instabilities with substantial astrophysical significance.

# Importance to preserve divergence of B

- For multi-dimensions numerical schemes, there is no guarantee that divergence of B is kept zero, due to truncation error.

Consequence:

spurious parallel acceleration

$$\boldsymbol{J} \times \boldsymbol{B} = -\nabla \cdot \left( \frac{B^2}{8\pi} \mathbf{I} - \frac{\boldsymbol{B}B}{4\pi} \right) - \frac{(\nabla \cdot \boldsymbol{B})\boldsymbol{B}}{4\pi}$$

Divergence error can accumulate, leading to inconsistent results over long term.

In some cases, it can lead to numerical instabilities and make the code crash…

# Techniques to preserve divergence of B

- Divergence cleaning:

Powell's 8-wave scheme (Powell, 1999):

Add source terms to momentum/induction equations to advect magnetic monopoles away. But: can give the wrong shock jump conditions.

Projection method (Brackbil & Bams, 1980):

Solve a Poisson equation for the "magnetic charge": $\Delta\Phi = \nabla \cdot \boldsymbol{B}$

Then clean the divergence field: $\boldsymbol{B} \rightarrow \boldsymbol{B} - \nabla\Phi$

But: very expensive to solve elliptic PDE, and may smooth discontinuities in B.

Dedner's scheme (Dedner et al. 2002): introducing a general Lagrangian multiplier, transporting div(B) errors away. Reasonably robust in most cases.

- Use vector potential (usually used in finite-difference codes, e.g., Pencil)

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{v} \times \boldsymbol{B} \, , \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Div(B)=0 by construction, but need hyper-resistivity for stabilization.

# Constrained transport (CT)

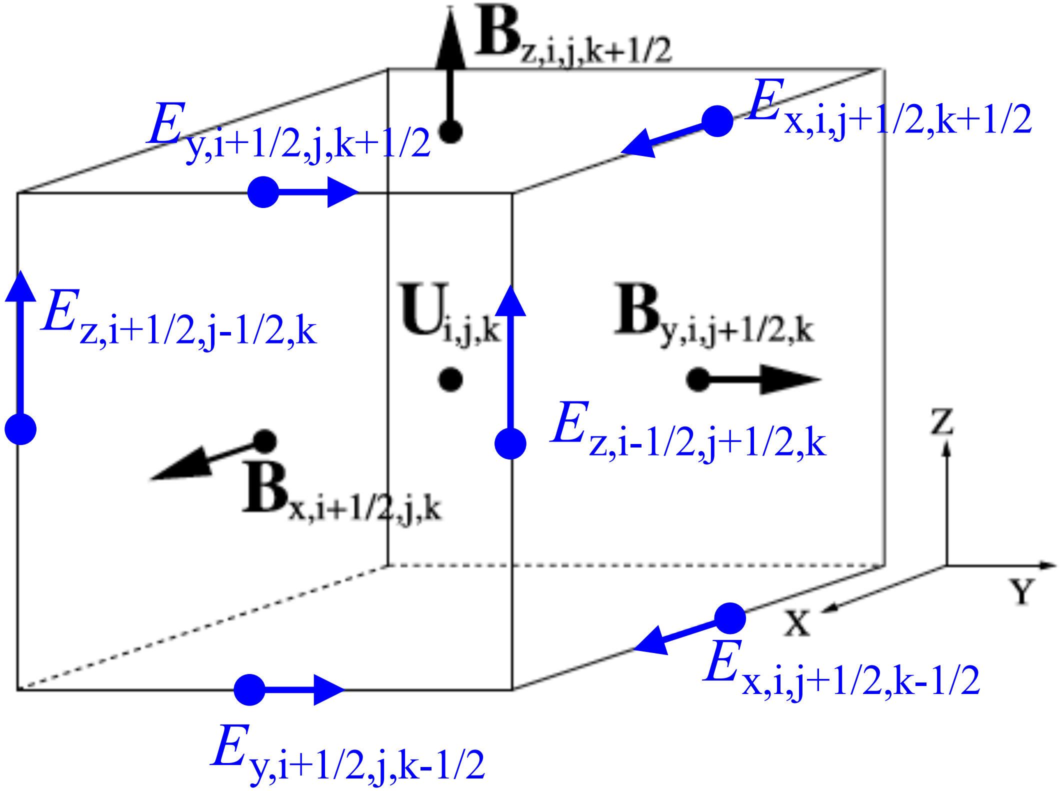
(Evans & Hawley, 1988)

Magnetic fields defined at face-center, area-averaged:

$$(B_x)_{i+1/2,j,k} = \frac{1}{\Delta y \Delta z} \int_S B_x(y,z)\,dydz$$

Electromotive forces (vxB) defined at edges, line-averaged:

$$(E_x)_{i,j+1/2,k-1/2} = \frac{1}{\Delta x \Delta t} \int E_x(x)\,dxdt$$




Evolve magnetic field via Stoke's law:

$$\frac{\partial}{\partial t} \int_S \boldsymbol{B} \cdot d\boldsymbol{S} = - \int_L \boldsymbol{E} \cdot d\boldsymbol{l}$$

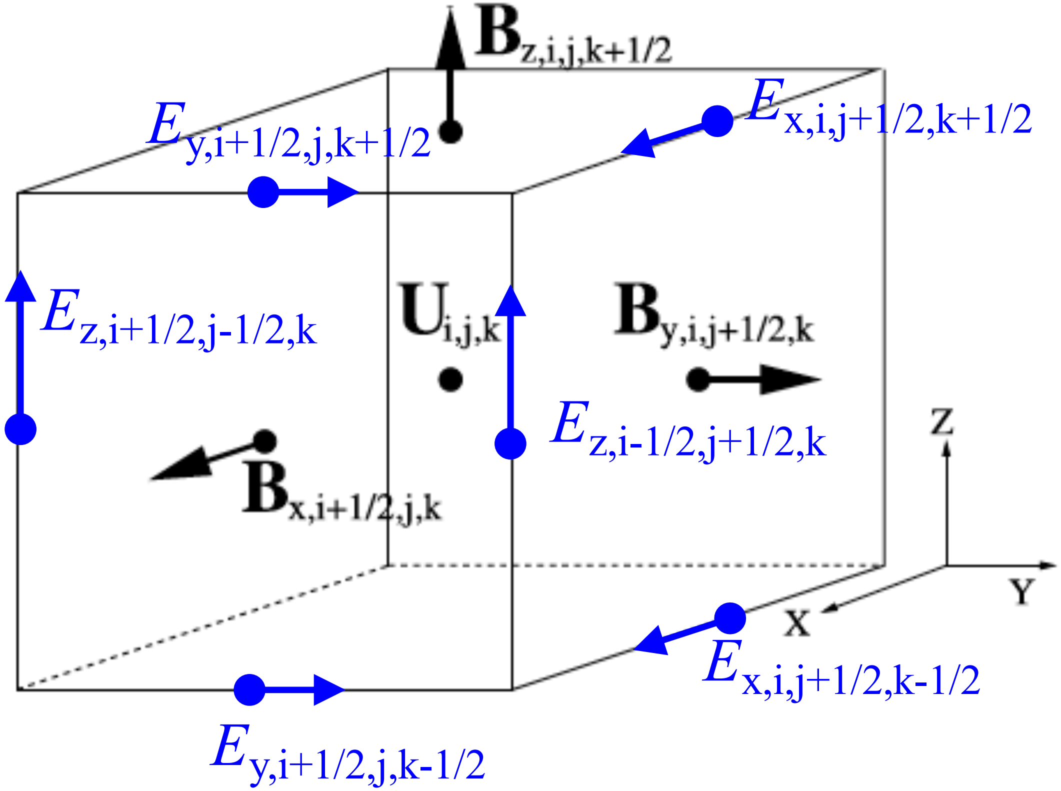
$$B^{n+1}_{x,i+1/2,j,k} = B^n_{x,i+1/2,j,k} - \frac{\Delta t}{\Delta y}(E^{n+1/2}_{z,i-1/2,j+1/2,k} - E^{n+1/2}_{z,i-1/2,j-1/2,k}) + \frac{\Delta t}{\Delta z}(E^{n+1/2}_{y,i-1/2,j,k+1/2} - E^{n+1/2}_{z,i-1/2,j,k-1/2})$$

These equations are exact: no approximations.

# Constrained transport (CT)

Div (B)=0 is preserved to machine accuracy:

$$\nabla \cdot \boldsymbol{B} = \frac{B_{x,i+1/2,j,k} - B_{x,i-1/2,j,k}}{\Delta x} + \frac{B_{y,i,j+1/2,k} - B_{x,i,j-1/2,k}}{\Delta y} + \frac{B_{z,i,j,k+1/2} - B_{z,i,j,k-1/2}}{\Delta z}$$




Updates in Div(B) corresponds to differences in the EMFs that cancel exactly.

Main challenge: construct electric fields at cell edges (3D) or corners (2D).

By arithmetic averaging the EMFs returned from the Riemann solvers (at face centers), the EMFs are not properly upwinded.

Need to reconstruct the EMF at the corners (Gardiner & Stone, 2005).