UV complete 4-derivative scalar field theory

Bob Holdom

UNIVERSITY OF TORONTO

International Workshop on New Opportunities for Particle Physics
July 2024
Institute of High Energy Physics, Chinese Academy of Sciences

the QFT

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(\Box + m^2)\partial^\mu\phi + \lambda_3(\partial_\mu\phi\partial^\mu\phi)\Box\phi + \lambda_4(\partial_\mu\phi\partial^\mu\phi)^2$$

- 4-derivatives, both in the interaction terms and the kinetic terms
- dimensionless real scalar field $\phi(x)$ and dimensionless couplings λ_3 and λ_4
- shift symmetry $\phi \to \phi + c$
- m^2 breaks the classical scale invariance

proxy for quantum quadratic gravity (QQG)

- Einstein action is supplemented with terms quadratic in curvature, and these terms bring in 4-derivatives
- 4-derivatives in kinetic and interaction terms
- both theories are renormalizable
- the shift symmetry is playing the role of coordinate invariance of the gravity theory
- the $m^2\partial_\mu\phi\,\partial^\mu\phi$ is playing the role of the Einstein term
- at low energies this term dominates; left with a normal massless field with non-renormalizable interactions

UV completeness

- both theories are UV complete, and so we can see what happens at energies much higher than m
- also refer to this as the $m \to 0$ limit
- ultra-Planckian energies in the case of gravity
- the story of four derivatives is similar for the two theories
- will focus on the simpler theory

ghost

- propagator has massive pole with abnormal sign — implies negative norm state
- with correct quantization this does not violate stability or unitarity
- all perturbative states have positive energy
- S-matrix unitarity holds, that is $S\mathbb{1}S^\dagger = \mathbb{1}$, where $\mathbb{1} = \sum_X \frac{|X\rangle\langle X|}{\langle X|X\rangle}$ reflects the negative norms

optical theorem

- the optical theorem can be directly verified in perturbation theory by keeping track of minus signs
- the LHS is imag part of forward scattering amplitude, and its calculation is affected by any wrong-sign propagators
- the RHS is a scattering process into on-shell final states, and this is affected by any negative norms among these states
- thus the LHS and RHS of the optical theorem are both affected in such a way that it remains satisfied

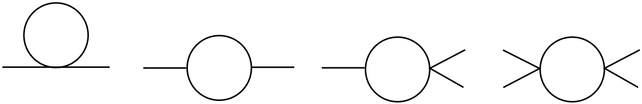
Born rule is the problem

- due to negative norms, probabilities can be negative or greater than unity
- but Born rule is a separate aspect of QM — and thus could be modified independently of unitarity (a different talk)
- here we shall assume standard Born rule and consider positivity constraints in the high energy limit

β-functions

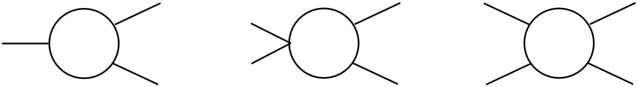
- treat the renormalization of $\partial_\mu\phi\Box\partial^\mu\phi$ term as a standard wave function renormalization (2023)

$$\frac{d\lambda_3}{d\ln\mu} = -\frac{5}{4\pi^2}(\lambda_4\lambda_3 + \frac{3}{4}\lambda_3^3)$$
$$\frac{d\lambda_4}{d\ln\mu} = -\frac{5}{4\pi^2}(\lambda_4^2 + \lambda_4\lambda_3^2)$$

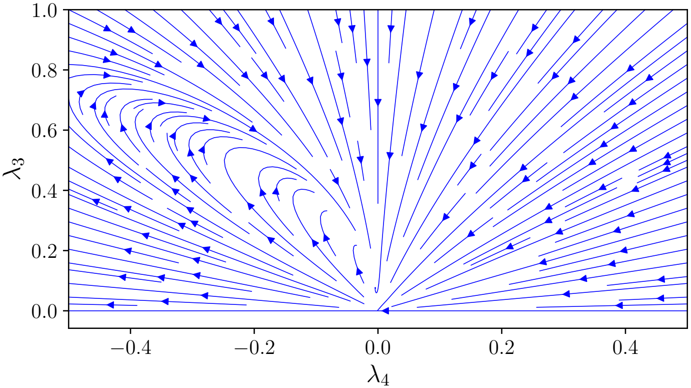


curiosities

- first diagram (tadpole) naively contributes to running of m^2, but it does not have $ln(p/\mu)$ dependence, and so does not
- the following diagrams are finite



renormalization group flow



- arrows point to the UV
- mostly asymptotic freedom in UV
- some flows also show asymptotic freedom in IR

a new mass scale

- flow towards IR stops when the energy scale drops below m; this is the transition to the low energy theory
- for sufficiently small m, the flow towards the IR can result in large couplings
- this can create a new mass scale through dimensional transmutation
- in gravity this can be the origin of the Planck mass

preview

- describe a simplified method for calculating in the high energy limit (2024)
- original method involves decomposing ϕ into two degrees of freedom
- with simplified method there appears to be only one degree of freedom at high energies
- calculate LHS and RHS of optical theorem, and also a differential cross section, as functions of λ_3 and λ_4
- positivity picks out the allowed region on the RG flow plane

a related issue

- four derivative interaction terms produce diverging amplitudes at large momenta
- using original method we found that cancellations take place at the level of differential cross sections
- then inclusive diff cross sections have good high energy behaviour (as in QQG 2022)
- the simplified method with effectively one degree of freedom clarifies what is happening

mass derivative

- four derivative propagator $G^{(4)}(p^2, m^2)$ can be written in terms of the Feynman propagator

$$G^{(2)}(p^2, m^2) = \frac{1}{p^2 - m^2 + i\varepsilon}$$

as

$$G^{(4)}(p^2, m^2) = -\frac{G^{(2)}(p^2, m^2) - G^{(2)}(p^2, 0)}{m^2}$$

- thus in the $m \to 0$ limit (high energy limit)

$$\lim_{m\to 0} G^{(4)}(p^2, m^2) = \lim_{m\to 0}\left(-\frac{d}{dm^2}\right)G^{(2)}(p^2, m^2)$$

LHS of optical theorem

- the imaginary part of a forward scattering amplitude $\mathcal{A}_{i\to i}$ is extracted by using

$$\mathrm{Im}(G^{(2)}(p^2, m^2)) = -i\pi\delta(p^2 - m^2).$$

- the analog for $G^{(4)}$ in the $m \to 0$ limit is

$$\lim_{m\to 0} \mathrm{Im}(G^{(4)}(p^2, m^2)) = -i\pi \lim_{m\to 0}(-\frac{d}{dm^2})\delta(p^2 - m^2)$$

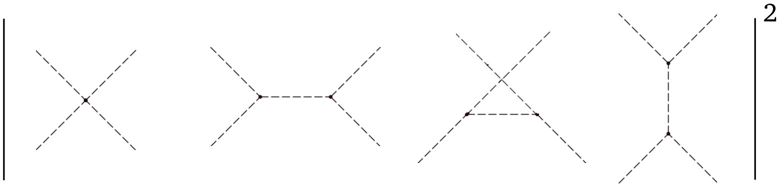
- the additional operation, $-\lim_{m\to 0}\frac{d}{dm^2}$, also works on the RHS of the optical theorem

RHS of optical theorem

- each on-shell particle in a final state f should be assigned its own dummy mass m_j
- $|\mathcal{A}_{i\to f}|^2$ will depend on the values of these m_j's via the on-shell conditions
- contribution from final state f takes the form

$$\lim_{\{m_j\}\to 0} \prod_{j=1}^{n} \left(-\frac{d}{dm_j^2}\right) |\mathcal{A}_{i\to f}(m_1..m_n)|^2$$

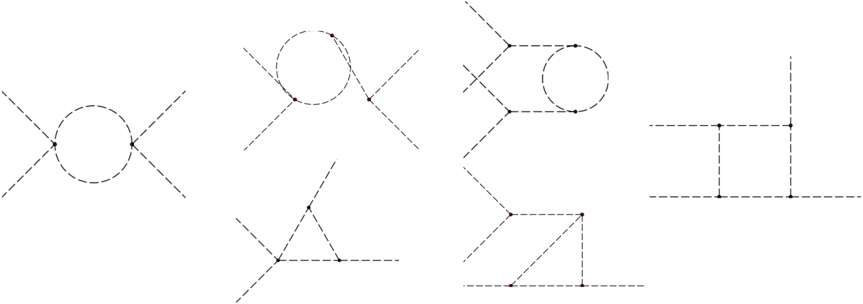
$|\mathcal{A}_{i\to f}|^2$ for $\phi\phi \to \phi\phi$



- initial state is fixed e.g. two massless particles
- new method reproduces the usual sum over the $\phi\phi$ final states in $m \to 0$ limit

now the LHS of optical theorem

- imaginary part of the forward scattering amplitude
- various diagrams are of order λ_4^2, $\lambda_4 \lambda_3^2$ or λ_3^4



result for optical theorem

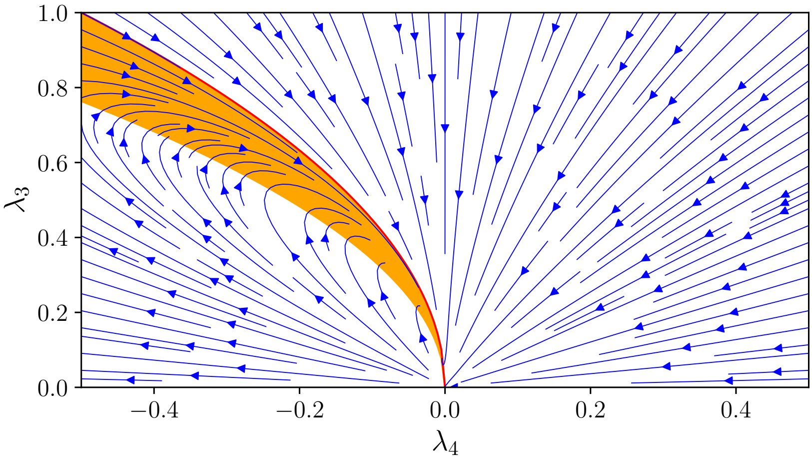
- LHS and RHS calculated independently

$$\text{LHS} = \text{RHS} = \frac{s^2}{6\pi}(6\lambda_3^4 + 19\lambda_3^2\lambda_4 + 14\lambda_4^2)$$

- RHS naively goes like s^4, being the square of amplitudes that go like s^2
- is reduced to s^2 behaviour because of $-\frac{d}{dm^2}$ applied twice
- RHS of optical theorem must be positive since it is related to a total cross section

the positivity constraint

- RHS is negative for $-\frac{6}{7} < \lambda_4/\lambda_3^2 < -\frac{1}{2}$
- this region is shaded orange, and the red line is $\lambda_4 = -\frac{1}{2}\lambda_3^2$



below red line

- all flows below this line will eventually enter the orange region in the UV
- thus all such flows are forbidden
- the allowed flows are on or to the right of the red line
- these couplings are asymptotically free in the UV and can become strong in the IR

meaning of red line

- the red line marks the boundary between two sets of flows that are qualitatively different
- on red line and with $m = 0$, Lagrangian becomes a square

$$\mathcal{L} = -\frac{1}{2}(\Box\phi - \lambda_3 \partial_\mu \phi \partial^\mu \phi)^2$$

- LHS = RHS vanishes on red line
- is the theory trivial on red line? (see below)

two degrees of freedom?

- consider the two fields constructed from ϕ,

$$\psi_1 = \frac{1}{m^2}(\Box + m^2)\phi$$
$$\psi_2 = \frac{1}{m^2}\Box\phi$$

- when expressed in terms of ψ_1 and ψ_2 the kinetic term of the Lagrangian becomes

$$-\frac{m^2}{2}\psi_1\Box\psi_1 + \frac{m^2}{2}\psi_2(\Box + m^2)\psi_2$$

- ψ_1 and ψ_2 are the two fields of definite mass (0 and m) and definite norm (+ and $-$)

one degree of freedom

- but we also have $\phi = \psi_1 - \psi_2$
- thus $\phi = \psi_1 - \psi_2$ is the only combination that appears in interaction terms
- meanwhile the operation $-\lim_{m\to 0} \frac{d}{dm^2}$ accounts for the difference in propagation of the two fields
- we can calculate with just a single massive degree of freedom and then apply the $-\lim_{m\to 0} \frac{d}{dm^2}$ operations

apply to initial as well as final states of $\phi\phi \to \phi\phi$

- need to find the dependence on the set of four masses m_j coming from the on-shell constraints
- before taking the four m_j^2-derivatives, would diverge as $\sim (s^2)^2/s$ for large s
- a term $\sim m_1^2 m_2^2 m_3^2 m_4^2/s$ is needed to survive
- in the end we have a differential cross section that behaves like $1/s$ at large s times a function of the scattering angle

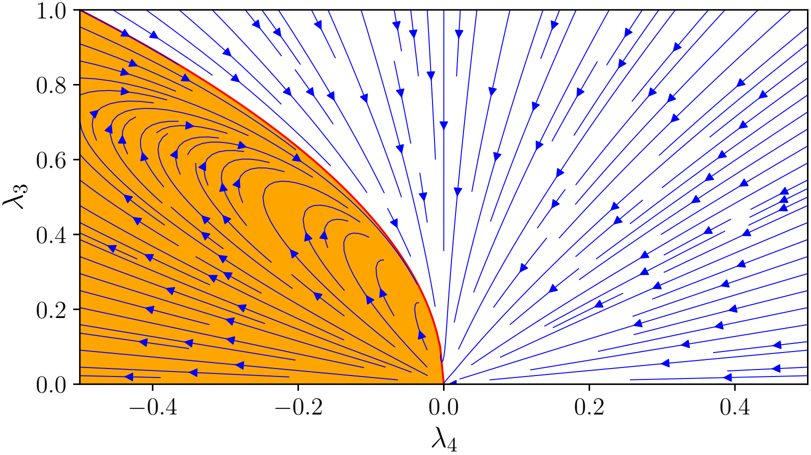
result

- the differential cross section for $\phi\phi \to \phi\phi$ scattering at high energies

$$\frac{d\sigma}{d\Omega} = \Big(\left(\lambda_3^4 - 4\lambda_4^2\right) \sin(\theta)^6 + 24\lambda_4^2 \sin(\theta)^4 + \left(-48\lambda_3^4 - 96\lambda_3^2\lambda_4\right) \sin(\theta)^2 + 64\lambda_3^4 + 128\lambda_3^2\lambda_4 \Big) / (16\pi^2 \sin(\theta)^4 s)$$

- this result is positive definite for any θ as long as $\lambda_4 \geq -\frac{1}{2}\lambda_3^2$
- this is the same constraint as before!

the $\lambda_4 \geq -\frac{1}{2}\lambda_3^2$ constraint



- positivity has again picked out the running couplings that flow to strong coupling in the infrared

on red line

- interesting special case on the red line:

$$\frac{d\sigma}{d\Omega} = \frac{3\lambda_4^2}{2\pi^2 s} \quad \text{when } \lambda_4 = -\frac{1}{2}\lambda_3^2$$

- simple, but does not vanish, and so theory is not completely trivial on red line

- calculating the $m \to 0$ limit may be different from the $m \equiv 0$ theory

what happens at lower energies?

- then massive ghost state ψ_2 can be distinguished from the massless normal state ψ_1
- three possibilities:

1. ghost state is not an asymptotic state because of decay (Lee-Wick, Donoghue-Menezes)

2. ghost state is not an asymptotic state because of strong interactions

3. ghost state remains an asymptotic state (Kubo and Kugo)
    - but theory may support an inner product whereby the norm of this state is positive

gauged extension

- consider another example of how 4-derivative kinetic terms can enable good high energy behavior: an interacting $U(1)$ gauge field

- the Stuckelberg representation of a massive $U(1)$ gauge field is

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 V_\mu V^\mu,$$

$$V_\mu = A_\mu + \partial_\mu \phi,$$

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- adding interaction terms, such as $\lambda_3 V_\mu V^\mu \, \partial_\mu V^\mu + \lambda_4 (V_\mu V^\mu)^2$, produce amplitudes that grow like s^2/m^4

gauged extension

- thus the interacting theory can only be a low energy effective theory (Kribs, Lee, Martin 2022)

- to UV complete the Stuckelberg theory with interactions, we need a 4-derivative kinetic term here as well

- leads to gauged extension of our scalar theory

$$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}V_\mu(\Box + m^2)V^\mu + \lambda_3 V_\mu V^\mu\, \partial_\mu V^\mu + \lambda_4(V_\mu V^\mu)^2$$

- starting point for a renormalizable theory of a $U(1)$ gauge-boson that is not only massive but is also interacting

gauge symmetry

- our previous shift symmetry has been promoted into a gauge symmetry, $\phi \to \phi - \alpha$ and $A_\mu \to A_\mu + \partial_\mu \alpha$, such that V_μ is invariant
- add standard gauge fixing term to the Lagrangian,

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}\mathcal{G}^2 - \omega^*(\mathbf{s}\mathcal{G})$$

- $\mathbf{s}$ implements BRST transformation and ω^* is the antighost
- choose $\mathcal{G} = \partial_\mu A^\mu - \xi(\Box + m^2)\phi$

cancellations

- this causes the terms mixing A_μ and $\partial_\mu \phi$ to cancel

$$
\begin{aligned}
\mathcal{L}^{(2)} = & -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} A_\mu(\Box + m^2)A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\
& + \frac{1}{2}\partial_\mu \phi(\Box + m^2)\partial^\mu \phi - \frac{1}{2}\xi((\Box + m^2)\phi)^2 \\
& - \omega^* \left(\partial^2 - \xi(\Box - m^2)\right) \omega
\end{aligned}
$$

combined propagator

- define $1/\zeta = 1 + 1/g^2$

$$
\begin{aligned}
& -i\langle 0|T(A_\mu + \partial_\mu \phi)(A_\nu + \partial_\nu \phi)|0\rangle \\
& = -\frac{1}{p^2 - \zeta m^2}\left(\zeta g_{\mu\nu} + (1-\zeta)\frac{p_\mu p_\nu}{p^2 - m^2}\right)
\end{aligned}
$$

- combined propagator is independent of the gauge parameter due to the gauge invariance of $A_\mu + \partial_\mu \phi$

- the physical parameter ζ varies over $0 \le \zeta \le 1$ as $0 \le g^2 \le \infty$

four degrees of freedom

- usual decomposition of the standard massive gauge-boson propagator

$$-\frac{1}{p^2 - m^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{1}{m^2}\frac{p_\mu p_\nu}{p^2}$$

- instead we have

$$-\frac{\zeta}{p^2 - \zeta m^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) - \frac{p_\mu p_\nu}{p^2(p^2 - m^2)}$$

- first term propagates two transverse degrees of freedom
- second term propagates two longitudinal degrees of freedom

$2 \to 2$ scattering in high energy limit

- calculate various exclusive differential cross sections for A_μ's and ϕ's and then add together to get the physical result
- the various exclusive differential cross sections are separately well-behaved at high energies, all falling like $1/s$
- we have achieved good high energy behaviour, necessary for a UV complete theory
- but not quite, the $U(1)$ gauge coupling is not asymptotically free

now look at positivity

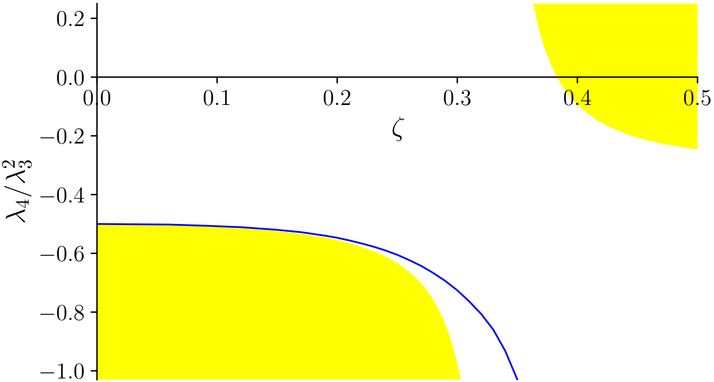
- consider the exclusive differential cross section for $\phi\phi \to \phi\phi$; now has contribution from photon exchange

- this is the only exclusive differential cross section that has the $\sin(\theta)^{-4}$ dependence, and near $\theta \approx 0$ it is

$$\frac{d\sigma}{d\Omega} = \frac{4{\lambda_3}^2\left(\left(\zeta^2 - 3\zeta + 1\right){\lambda_3}^2 - 6\zeta\lambda_4 + 2\lambda_4\right)}{\pi^2\theta^4 s}$$

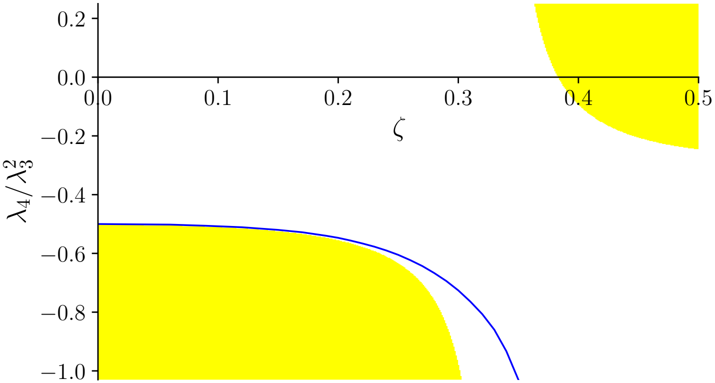
constraint that this pole be positive

- the yellow region is ruled out



full positivity constraint

- sum of all the exclusive differential cross sections
  $\to$ must be above the blue line



conclusion

- considered 4 derivative scalar field theory as proxy for quantum quadratic gravity
- scattering of the effectively one scalar degree of freedom displays good high energy behavior
- positivity constrains $\lambda_4/\lambda_3^2 \geq -1/2$; a natural boundary on the RG flow diagram
- opens up a renormalizable and interacting $U(1)$ gauge theory