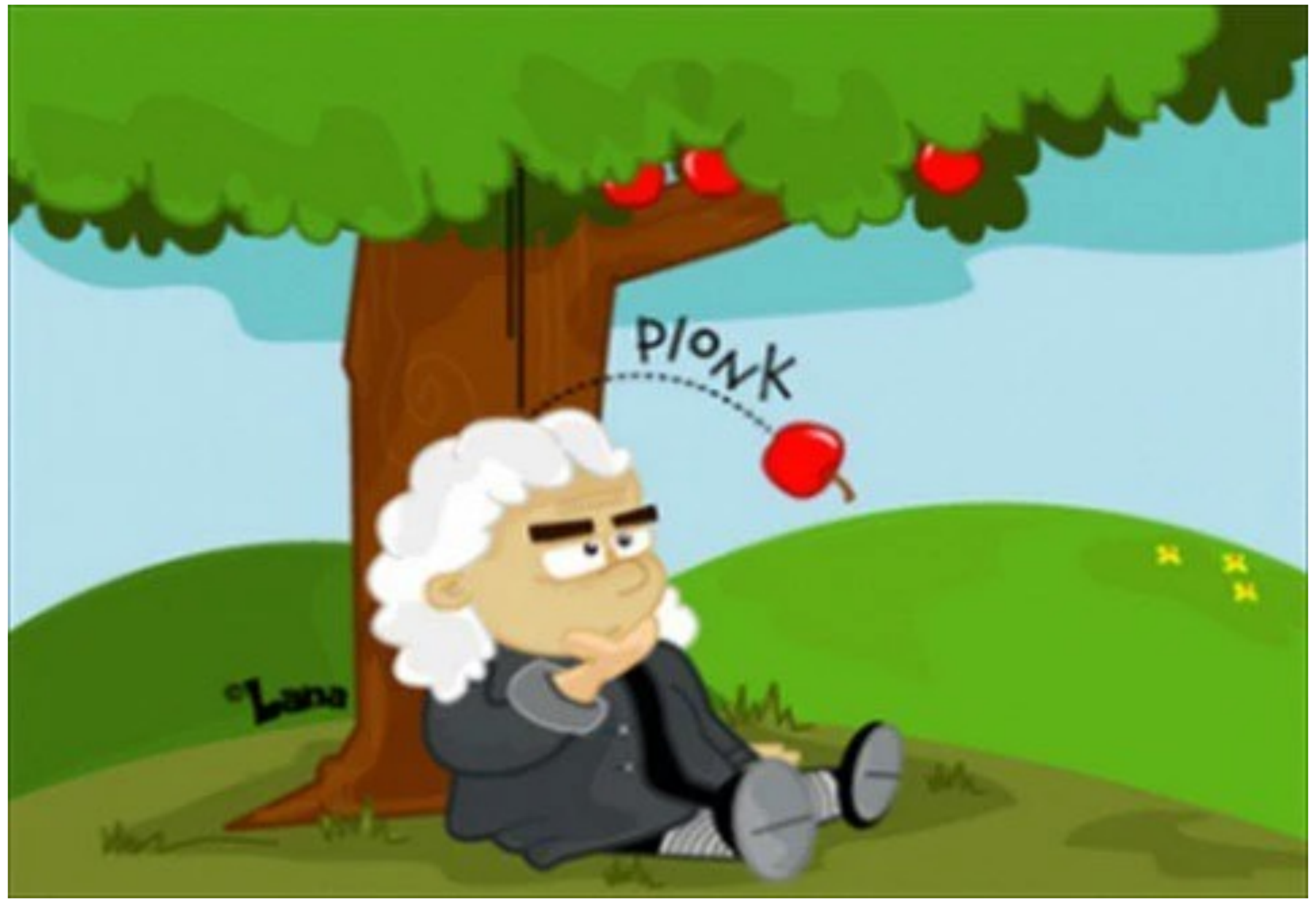
# The Classical Equations of Quantized Gauge Theories

**Surjeet Rajendran**

**With Anne-Katherine Burns, David E. Kaplan and Tom Melia**

# Classical and Quantum Particle Mechanics



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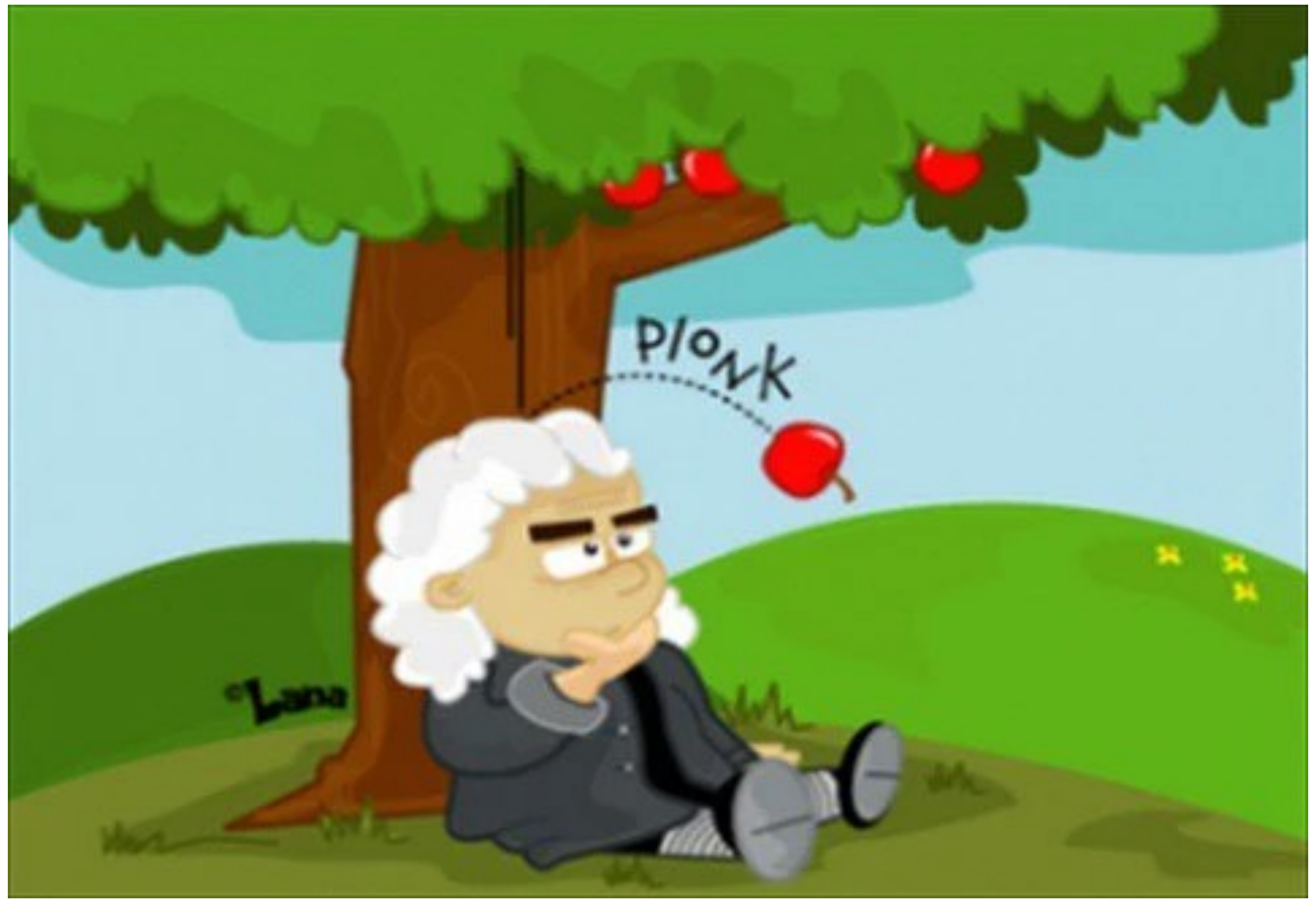

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$$i\frac{\partial|\Psi\rangle}{\partial t} = H|\Psi\rangle$$ ✓

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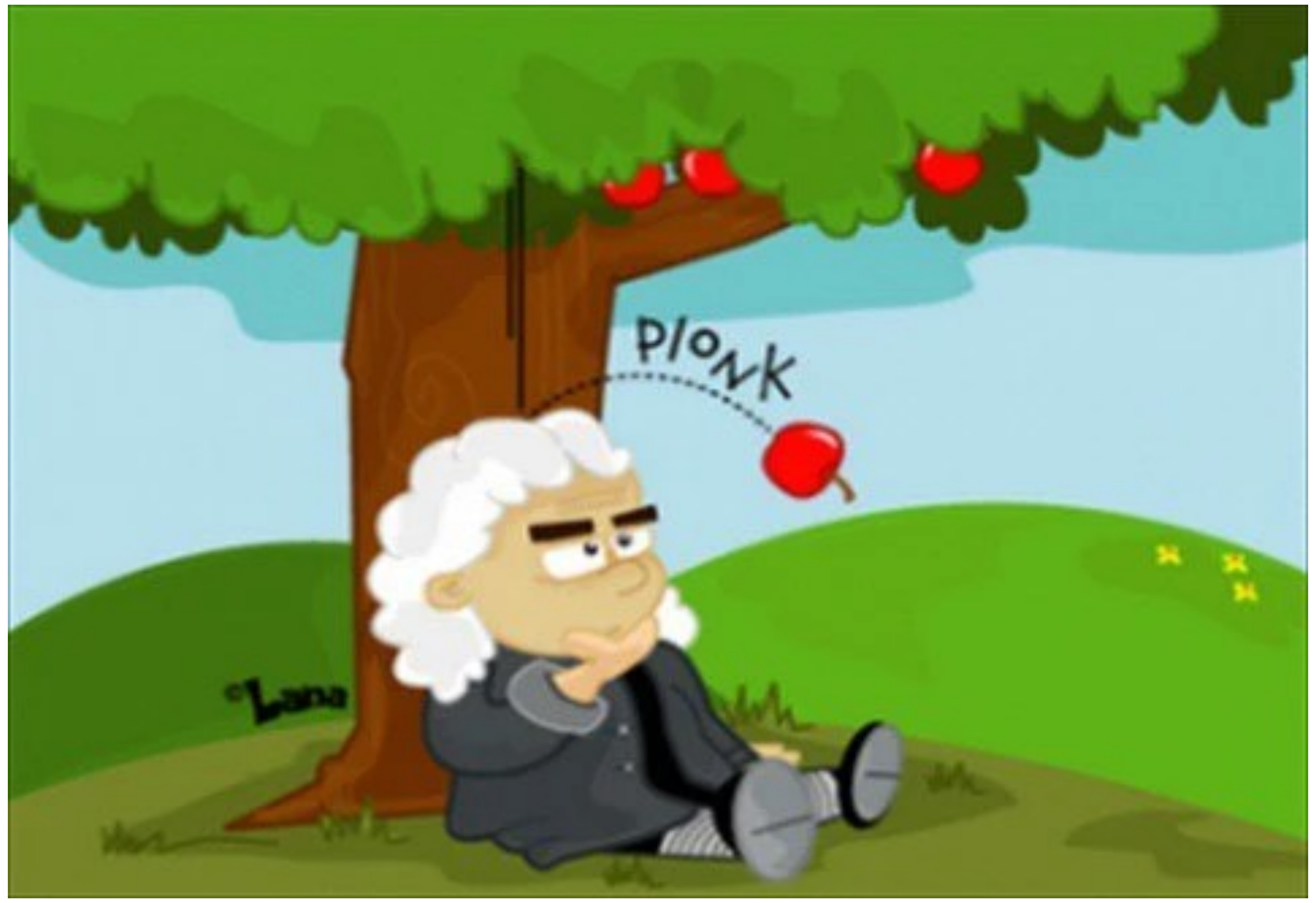

$$\xcancel{\vec{F} = m\vec{a}} \qquad \frac{d\langle\Psi|\hat{p}|\Psi\rangle}{dt} = -\langle\Psi|\frac{dV}{dx}|\Psi\rangle$$

$$\Uparrow$$

Ehrenfest's Theorem

$$i\frac{\partial|\Psi\rangle}{\partial t} = H|\Psi\rangle \qquad \checkmark$$

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**Classical Field Theory**
**(e.g. Klein Gordon)**

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**What about quantum mechanics?**

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$$Z = \int DA^\mu e^{iS_{gf}[A]} \implies \delta Z = \int DA^\mu \left( \frac{\delta S_{gf}}{\delta A^\mu} \right) \delta A^\mu = 0$$

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**????????**

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**First order ODE - can time evolve states that violate constraint**

**Does anything go wrong? i.e. massless photon?**

# Outline

1. Classical Mini Superspace Cosmology

2. Path Integral

3. Hamiltonian

4. Wheeler deWitt

5. Conclusions

# Mini Superspace Cosmology

# Homogeneous Universe

**Study homogeneous space-times in General Relativity**

$$g_{\mu\nu} = -N\left(t\right)^2 dt^2 + a\left(t\right)^2\left(dx^2 + dy^2 + dz^2\right)$$

**Couple this to homogenous sources of matter, for e.g. rolling scalar field ϕ**

$$S = \int dt \mathcal{L}\left(a\left(t\right), \dot{a}\left(t\right), N\left(t\right), \phi\left(t\right), \dot{\phi}\left(t\right)\right)$$

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**Classical Equations?**

**2 FRW + 1 Scalar Field**

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**2nd FRW Equation**

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**1st FRW Equation**
**(Hubble)**

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**1st FRW Equation (Hubble)**

**Classical Solutions are over-constrained (3 equations for 2 variables)**

**Solve: Klein Gordon + 2nd FRW with boundary condition from 1st FRW**

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Is this a problem?

# Gauge Invariant Physics

$$\langle \phi_f, a_f | T\left(t_2; t_1\right) | \phi_i, a_i \rangle$$

**Path integral defines the time evolution operator**

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**Perfectly Reasonable for the time evolution operator to be gauge dependent**

**We need gauge invariant physics. What does this mean?**

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$$|\phi_i, a_i\rangle \rightarrow \sum_{\phi_f, a_f} c_{\phi_f, a_f}(t_2, t_1) |\phi_f, a_f\rangle = \sum_{\phi_f, a_f} c_{\phi_f, a_f}\left(\tilde{t}_2, \tilde{t}_1\right) |\phi_f, a_f\rangle$$

$$|\Psi(t_1)\rangle = |\tilde{\Psi}\left(\tilde{t}_1\right)\rangle = |\Sigma\rangle \rightarrow |\Psi(t_2)\rangle = |\tilde{\Psi}\left(\tilde{t}_2\right)\rangle = |\Omega\rangle$$

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**Fix manifold (R¹).**

**Pick any choice of time co-ordinate on the manifold**

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**Moral: Gauge invariant physics from gauge dependent path integral**
**Reasonable since time is gauge choice!**

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**2nd FRW Equation**

**Heisenberg Picture**

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**Klein Gordon**

# Schwinger Dyson Equations

$$\langle \phi_f, a_f | T\left(t_2; t_1\right) | \phi_i, a_i \rangle = \int_{\phi(t_1)=\phi_i, a(t_1)=a_i, N(t_1)=N_1}^{\phi(t_2)=\phi_f, a(t_2)=a_f, N(t_2)=N_2} D\phi Da DN D\lambda e^{i\int_{t=t_1}^{t=t_2}\left(\mathcal{L}-\lambda\dot{N}\right)}$$

$$\phi \rightarrow \phi + \delta\phi, a \rightarrow a + \delta a, N \rightarrow N + \delta N$$

$$\langle \Psi | \partial_t \left(\frac{\partial \mathcal{L}}{\partial \dot{a}}\right) - \frac{\partial \mathcal{L}}{\partial a} | \Psi \rangle = 0$$

**2nd FRW Equation**

**Heisenberg Picture**

$$\langle \Psi | \partial_t \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} | \Psi \rangle = 0$$

**Klein Gordon**

$$\langle \Psi | \frac{\partial \lambda}{\partial t} - \frac{\partial \mathcal{L}}{\partial N} | \Psi \rangle = 0$$

**Tells you how λ evolves in the path integral - not 1st FRW Equation?????**

# Hamiltonian

# Hamiltonian Construction

$$g_{\mu\nu} = -N\left(t\right)^2 dt^2 + a\left(t\right)^2 \left(dx^2 + dy^2 + dz^2\right)$$

**Physical Degrees of freedom: a(t), ϕ(t)**
**Gauge Freedom: N(t)**

$$S = \int dt \mathcal{L}\left(a\left(t\right), \dot{a}\left(t\right), N\left(t\right), \phi\left(t\right), \dot{\phi}\left(t\right)\right)$$

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**Quantize These**

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**Quantize These**

$$\Pi_N = \frac{\partial \mathcal{L}}{\partial \dot{N}} = 0$$

**What to do with N?**

# Hamiltonian Construction

$$S = \int dt \mathcal{L}\left(a\left(t\right), \dot{a}\left(t\right), N\left(t\right), \phi\left(t\right), \dot{\phi}\left(t\right)\right)$$

## Construct Canonical Hamiltonian from this Lagrangian

$$H_N = N\, H_0\left(a, \Pi_a, \phi, \Pi_\phi\right)$$

# Hamiltonian Construction

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**Construct Canonical Hamiltonian from this Lagrangian**

$$H_N = N\, H_0\left(a, \Pi_a, \phi, \Pi_\phi\right)$$

**What is N? Different values of N yield different Hamiltonians**
**Different physics? Gauge Invariance?**

**If N is a non-trivial operator on the Fock space, no way to make physics gauge invariant**
**Possible choice: N is a c-number**
**But still, different choices of N yield different Hamiltonians!**

# Schrodinger Equation

$$i\frac{\partial|\chi\left(t\right)\rangle}{\partial t}=NH_0|\chi\left(t\right)\rangle$$

**N is a c-number, different choices of N yield different Hamiltonians**

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$$i\frac{1}{N}\frac{\partial|\chi(t)\rangle}{\partial t} = H_0|\chi(t)\rangle$$

$$d\tilde{t} = N dt$$

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**Different choices of N correspond to different choices of time co-ordinate.**
**Gauge invariant physics!**

$$\langle\phi_f, a_f|T(t_2;t_1)|\phi_i, a_i\rangle = \int_{\phi(t_1)=\phi_i, a(t_1)=a_i}^{\phi(t_2)=\phi_f, a(t_2)=a_f} D\phi Da e^{i\int_{t=t_1}^{t=t_2}\left(\tilde{\mathcal{L}}(N_0)\right)}$$

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**Consistent with Path Integral - you just pick N**

# Consequence of Schrodinger Equation

$$i\frac{\partial|\chi(t)\rangle}{\partial t} = NH_0|\chi(t)\rangle \implies \frac{d\langle\chi(t)|H_0|\chi(t)\rangle}{dt} = 0$$

**Identity, similar to Ehrenfest and Schwinger-Dyson**

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**Can Show:** $\langle\chi(t)|H_0|\chi(t)\rangle = \langle\chi(t)|a^3\frac{\partial\mathcal{L}}{\partial N}|\chi(t)\rangle$

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**Can Show:** $\langle\chi(t)|H_0|\chi(t)\rangle = \langle\chi(t)|a^3\frac{\partial\mathcal{L}}{\partial N}|\chi(t)\rangle$

Thus: $\frac{d\langle\chi|a^3\frac{\partial\mathcal{L}}{\partial N}|\chi\rangle}{dt} = 0$

**This is almost the 1st FRW equation - but not quite.**
**1st FRW equation only satisfied up to overall constant**

# Initial State

$$a = A + A^\dagger, \Pi_a = i\left(A - A^\dagger\right) \qquad \phi = B + B^\dagger, \Pi_\phi = i\left(B - B^\dagger\right)$$

## Create quantum states of a, ϕ

$$|\chi\rangle = f\left(A, A^\dagger, B, B^\dagger\right)|0>$$

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$$i\frac{\partial|\chi(t)\rangle}{\partial t} = NH_0|\chi(t)\rangle$$

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## Quantum Dynamics

$$i\frac{\partial|\chi\left(t\right)\rangle}{\partial t} = NH_0|\chi\left(t\right)\rangle$$

## First order ODE - no issue with time evolving

$$\langle\chi|a^3\frac{\partial\mathcal{L}}{\partial N}|\chi\rangle \neq 0$$

# Violating 1st FRW

$$\langle\chi|a^3\frac{\partial\mathcal{L}}{\partial N}|\chi\rangle = c$$

**Quantum Dynamics:** $$i\frac{\partial|\chi\left(t\right)\rangle}{\partial t} = NH_0|\chi\left(t\right)\rangle$$

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**Quantum Dynamics:** $$i\frac{\partial|\chi(t)\rangle}{\partial t} = NH_0|\chi(t)\rangle$$

## Implied Classical Dynamics

$$\langle\chi|\partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right)-\frac{\partial\mathcal{L}}{\partial\phi}|\chi\rangle = 0$$

**Klein Gordon**

$$\langle\chi|\partial_t\left(\frac{\partial\mathcal{L}}{\partial\dot{a}}\right)-\frac{\partial\mathcal{L}}{\partial a}|\chi\rangle = 0$$

**2nd FRW Equation**

$$\langle\chi|\frac{\partial\mathcal{L}}{\partial N}|\chi\rangle = \langle\chi|\frac{c}{a^3}|\chi\rangle$$

**1st FRW but with "Dark" Matter**

# Quantum “Dark” Matter

$$\langle\chi|a^3\frac{\partial\mathcal{L}}{\partial N}|\chi\rangle = c$$

$$a = A + A^\dagger, \Pi_a = i\left(A - A^\dagger\right) \qquad \phi = B + B^\dagger, \Pi_\phi = i\left(B - B^\dagger\right)$$

**Create quantum states of a, ϕ**
**Quantum Dynamics: Just 1 first order ODE (Schrodinger)**
**No reason to constrain initial state!**

**Failure manifests classically as “dark” matter - though no real particle excitation there. Conservation implies super-selection sector.**

**Can be positive or negative!**

# Wheeler deWitt

# Can we get all of Einstein's Equations?

$$i\frac{\partial|\chi(t)\rangle}{\partial t} = NH_0|\chi(t)\rangle$$

$$\text{Want } \langle\chi|a^3\frac{\partial\mathcal{L}}{\partial N}|\chi\rangle = 0$$

**Hamiltonian changes with N**

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**Universe is in an energy eigenstate**

**But… Time???**

# Path Integral Version

$$\langle \phi_f, a_f | T\left(t_2; t_1\right) | \phi_i, a_i \rangle = e^{iP(\phi_f, a_f, t_2; \phi_i, a_i, t_1; N_0)}$$

**Gauge fixed path integral implies we lose 1st FRW equation**

**Path Integral explicitly depends upon random gauge choice $N_0$**

**Path integral obviously not gauge invariant**

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Integrate over all $N_0$

Unsurprisingly, this yields infinity

Demanding gauge invariant path integral implies only possible states are static (Wheeler deWitt)

# Conclusions

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**Opinion A**

**Opinion B**

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**Gauge Dependent Time Evolution With Gauge Invariant Physics**

**1st FRW Equation only true up to constant**

**Quantum “Dark” Matter**

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Only static states…no time!

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**All Equations Hold…but….**

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**Full General Relativity? Electromagnetism?**

# Quantizing Gauge Theories

## Yukawa Theory

$$S \supset \int d^4x \, \lambda(x) \, \phi \bar{\Psi} \Psi$$

$$\text{Do Not } \frac{\partial S}{\partial \lambda} = 0$$

$$\text{Do Not } \phi \bar{\Psi} \Psi |\chi\rangle = 0$$

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**Why?**

$$\pi_\lambda = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0$$

**λ not d.o.f.**

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**????????**

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$$S_{MS} = \int dt\, \mathcal{L}\left(a\left(t\right), \dot{a}\left(t\right), N\left(t\right)\right) \qquad S_{EM} = \int dt \mathcal{L}\left(\vec{A}, \dot{\vec{A}}, A_0\right) \qquad S_{GR} = \int dt\, \mathcal{L}\left(g_{ij}, \dot{g_{ij}}, g_{0\mu}\right)$$

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**Can show that for these states, still get massless photon and graviton - the only physical aspect of gauge invariance we actually care about**

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Can show that for these states, still get massless photon and graviton - the only physical aspect of gauge invariance we actually care about

Further, by suitable redefinitions, can show that the c-number coupling constants are not observable in linear quantum mechanics

# General Relativity

$$g_{\mu\nu} = g_{0\mu} dt dx^{\mu} + g_{ij} dx^{i} dx^{j}$$

**$g_{0\mu}$ do not have conjugate momenta - fixed c-number functions**

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**Classically:**

**$G_{\mu\nu} = T_{\mu\nu} + H_{\mu\nu}$**

**With: $H_{\mu\nu} = H_{0\mu}$**

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**New cosmological observables**

# Electromagnetism

$$A_\mu = A_0 dt + A_i dx^i$$

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**(Gauss's Law)** $\partial_\mu F^{\mu 0} = J^0$

$\partial_\mu F^{\mu i} = J^i$ **(Ampere's Law)**

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## Classically

$$\partial_\mu F^{\mu\nu} = J^\nu + J^\mu_d$$

With: $J^\mu_d = \left(J^0\left(x\right), 0, 0, 0\right) \implies \frac{d}{dt}\left(J^0_d\right) = 0$

# Electromagnetism

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**$A_0$ does not have a conjugate momentum - fixed c-number function**

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$$\partial_0 \partial_\mu F^{\mu 0} = 0$$

## Classically

$$\partial_\mu F^{\mu\nu} = J^\nu + J_d^\mu$$

$$\text{With: } J_d^\mu = \left(J^0\left(x\right), 0, 0, 0\right) \implies \frac{d}{dt}\left(J_d^0\right) = 0$$

**New cosmological observables**

# Backup

# Classical and Quantum Gauge Theories

**Classical Field Theory**
**(e.g. General Relativity)**

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0$$

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**Not the Same**
**????????**

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**Quantum Supremacy**

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✓

# Hamiltonian Construction

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{a}}, \Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

**Quantize These**

$$[a, \Pi_a] = [\phi, \Pi_\phi] = i$$

$$a = A + A^\dagger, \Pi_a = i\left(A - A^\dagger\right) \qquad \phi = B + B^\dagger, \Pi_\phi = i\left(B - B^\dagger\right)$$

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**Fully non-linear General Relativity - no "free" theory with "free" kinetic term**

**But, can still construct Hilbert space with Fock states of A, B - these are operator level statements independent of kinetic terms of the theory**

$$A|0\rangle = 0, A^\dagger|0\rangle = |1\rangle \text{ etc.}$$