Massive Gauge/Gravity Scattering Amplitudes and Massive Color-Kinematics Duality from Geometric and Topological Mass Generations

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IHEP, CAS, July 20, 2024

New Physics $\stackrel{?}{=}$ New Particles

New Physics = New Phenomena !

New Physics = New Principles !!

E.g., Special Relativity, Photo-electronic Effect, GR,
Spin, P and CP Violations, ……

Implications of Non-Discovery

- **Michelson-Morley Exp → No Aether!**
  But, it leads to New Revolation: → Special Relativity!

- **Discrepancy in Procession of Mercury's Perihelion**
  Using known Newton gravity Le Verrier postulated:
  → New Planet "Vulcan" (祝融星), but No discovery of it !
  → Real Solution: General Relativity !

- **Question and Challenge today:**
  After h(125), what does the Non-discovery of LHC imply ???

- **I am moderately (non)optimistic……**

Making of the Standard Model (123)

$$
\begin{aligned}
\mathcal{L} = \\
& -\frac{1}{4g'^4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4g^2}W^a_{\mu\nu}W^{\mu\nu a} - \frac{1}{4g_s^2}G^a_{\mu\nu}G^{\mu\nu a} \\
& +\bar{Q}_i i \not{D} Q_i + \bar{u}_i i \not{D} u_i + \bar{d}_i i \not{D} d_i + \bar{L}_i i \not{D} L_i + \bar{\ell}_i i \not{D} \ell_i \\
& + \left(Y_u^{ij} \bar{Q}_i u_j \tilde{H} + Y_d^{ij} \bar{Q}_i d_j H + Y_l^{ij} \bar{L}_i \ell_j H + c.c. \right) \\
& -\lambda (H^\dagger H)^2 + \lambda v^2 H^\dagger H + (D^\mu H)^\dagger D_\mu H
\end{aligned}
$$

No DM !

No Dark Energy !

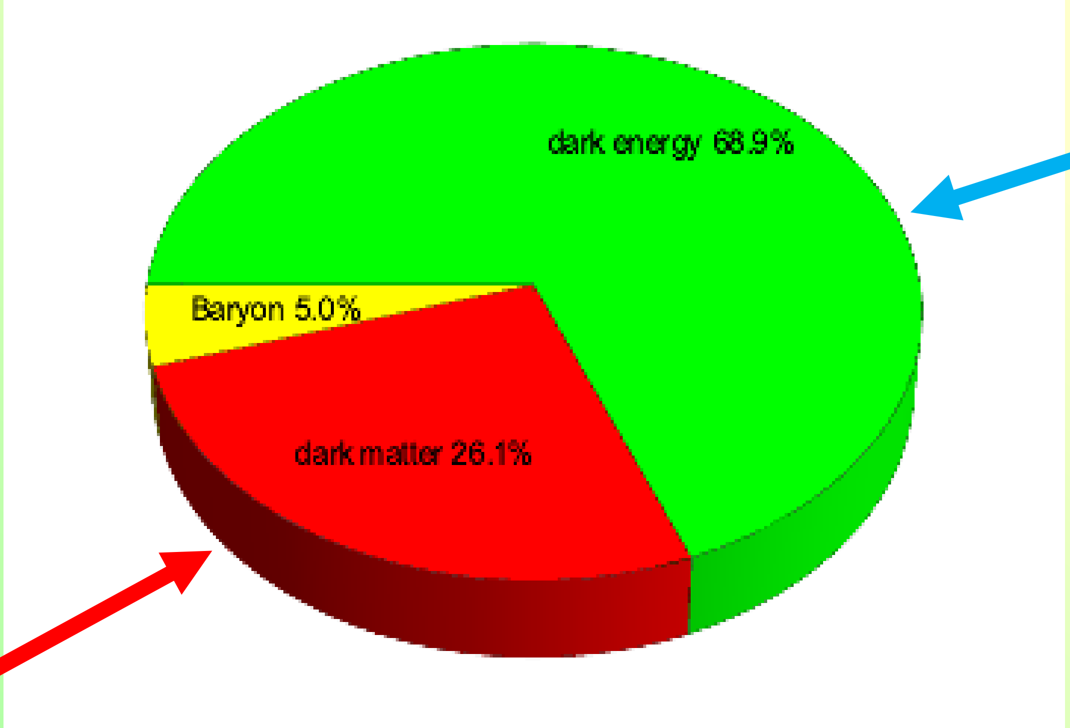
SM Structure seems complete, but ....

Recall: situation around 1900 –

→ **What is *Missing* in the SM ???**

Composition of the Universe

— Visible vs Dark Universe —



Making of the Standard Model (123)

$$\mathcal{L} =$$
$$-\frac{1}{4g'^4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4g^2} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4g_s^2} G^a_{\mu\nu} G^{\mu\nu a}$$
$$+\bar{Q}_i i \not{D} Q_i + \bar{u}_i i \not{D} u_i + \bar{d}_i i \not{D} d_i + \bar{L}_i i \not{D} L_i + \bar{\ell}_i i \not{D} \ell_i$$
$$+\left(Y_u^{ij} \bar{Q}_i u_j \tilde{H} + Y_d^{ij} \bar{Q}_i d_j H + Y_l^{ij} \bar{L}_i \ell_j H + c.c. \right)$$
$$-\lambda (H^\dagger H)^2 + \lambda v^2 H^\dagger H + (D^\mu H)^\dagger D_\mu H$$

SM Structure seems complete, but ....

→ **What is *Missing* in the SM ??**

No Gravity!

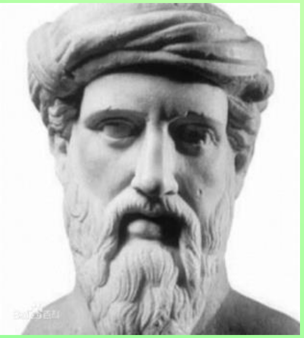
No Dark Energy !

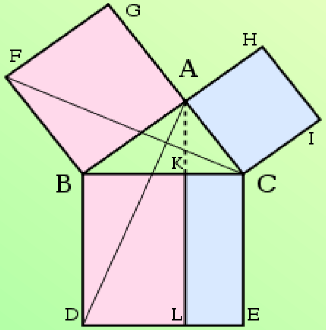
No DM !

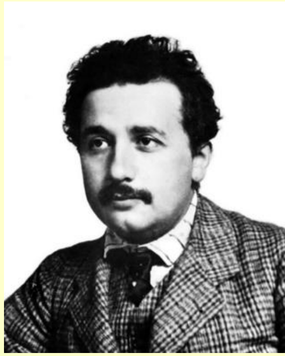
→→ **No full understanding on Quantum Gravity**
at both largest and smallest scales !!

→→ **UV-IR correspondence?**

Deep Relations: Importance of Equality & Square Law







- [EG] Pythagoras Theorem: $a^2 + b^2 = c^2$ → Fermat Last Theorem:
  $a^n + b^n \neq c^n$ $(n>2)$
- [SR] Mass-Energy Equivalence: $E = Mc^2$ and $P_\mu^{\ 2} = M^2c^2$
- [GR] Inertial Mass vs Gravitational-Mass Equivalence: $M_G = M_I$ → $g = a$
- Gravity Force from Gauge Force: (valid for $M = 0$ and $M \neq 0$?)

$$(\text{Gravity}) = (\text{Gauge Force})^2$$

$M = 0$
(Conventional works)

$M \neq 0$
(This Talk)

Challenges of Mass Generations

Nothing forbids Spin-0 Higgs Boson gets a huge mass from quantum corrections!
Quark/Lepton mass-scales are protected by P-violation and set by Higgs VEV through Yukawa coupling.

Fields	$m=0$		$m \neq 0$
$\mathbf{H}^0$ $(s=0)$	1 $(\lambda=0)$	$=$	1 $(\lambda=0)$
$\mathbf{A}^{\mu}$ $(s=1)$	$\mathbf{2}$ $(\lambda=\pm 1)$	$\neq$	$\mathbf{3}$ $(\lambda=0,\pm 1)$
$\mathbf{h}^{\mu\nu}$ $(s=2)$	$\mathbf{2}$ $(\lambda=\pm 2)$	$\neq$	$\mathbf{5}$ $(\lambda=0,\pm 1,\pm 2)$
(ψ_L,ψ_R) $(s=\frac{1}{2})$	$1+1$ $(\lambda=\frac{1}{2},-\frac{1}{2})$	$=$	$\mathbf{2}$ $(\lambda=\pm\frac{1}{2})$

Higgs Mechanism and Beyond

- "Higgs" Mechanism can be More General than Higgs Boson!

- It refers to spontaneously breaking a continuous local (gauge) symmetry while having the original interaction Lagrangian respect this symmetry.

- It converts would-be Goldstone boson(s) into extra components of the corresponding gauge particle(s) associated with broken group generator(s). →→ Generates Mass for gauge particle(s) .
  →→ Conserves Physical Degrees of Freedom !

Higgs Mechanism and Beyond

- "Higgs" Mechanism can be

  1). Conventional, 2). Geometric, 3). Topological

-- **Conventional Higgs Mechanism**:

SSB by Vacuum Expectation Value of Higgs Boson.

-- **Geomertric "Higgs" Mechanism**:

SSB by Compactification of Extra Dimension.

-- **Topological "Higgs" Mechanism**:

No SSB, Gauge-Invariant Mass Term,

Dynamical Conversion of Physical Degrees of Freedom.

Conventional vs Geometric vs Topological:

1). Higgs Interactions → Masses for Gauge Bosons, Fermions:

Gauge Boson Mass ~ v*g (vev*gauge-coupling),

Fermion Mass ~ v*y (vev*Yukawa-coupling).

2). Extra-dimension with Compactification → Geometric Mass (= n / R)

Geometric Mass for KK Gauge Bosons, KK Fermions

Geometric Mass for KK Gravitons

3). Topological Mass for Gauge Bosons and Gravitons:

From 3d Chern-Simons term,

CS Mass term is gauge-invariant,

Cause dynamical conversion of physical d.o.f.

Geometric Higgs Mechanism

References:

--- Y. F. Hang, HJH, PRD 105 (2022) 084005 arXiv:2106.04568 [hep-th] (90pp).

--- Y. F. Hang, HJH, Research (2022) 9860945 [arXiv:2207.11214 [hep-th]]

--- Y. Hang, W.W. Zhao, HJH, Y. Qiu, arXiv:2406.12713 [hep-th] (91pp)

--- Chivukula, Dicus, He, PLB 525 (2002) 175 [hep-ph/0111016]

--- Chivukula, He, PLB 532 (2002) 121 [hep-ph/0201164]

--- H.J. He, Int.J.Mod.Phys. A20(2005)3362, APS-2004 [hep-ph/0412113]

--- Chivukula, Dicus, He, Nandi, PLB 562 (2003) 109 [hep-ph /0302263]

Geometric Higgs Mechanism

- Kaluza-Klein (KK) Compactification of Extra Dimension leads to massive KK states, including KK Gauge Bosons and KK Gravitons. KK states obtain Geometric Masses by Compactification!

- KK Gauge Boson and KK Graviton obtain KK masses by spontaneous gauge symmetry breaking due to Compactification.
  → This is a new Geometric "Higgs" Mechanism (GHM).

- GHM was formulated for Massive KK Gauge Bosons at Lagrangian level and S-matrix level (by Chivukula-Dicus-He, PLB 532 (2002) 121); for Massive KK Gravitons at Lagrangian level (by Dolan-Duff, PRL-1985).

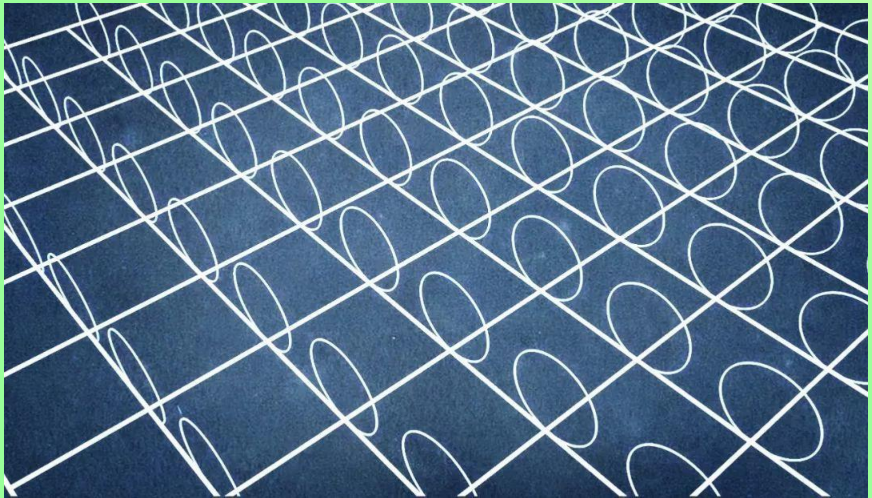
- Recently, GHM was formulated for KK Gravitons at S-Matrix level (by Hang & HJH, arXiv:2106.04568, 2207.11214, 2406.12713).

5d: Kaluza-Klein Compactification

- We are apparently living in (3+1)d spacetime.

- But our Universe could have Extra Dimensions beyond d = 4 , except that Extra Dimensions are all curled-up !

- Simplest case is a 5th dimension curled up on a circle S^1, called Kaluza-Klein (KK) Compactification (in 1920s)



Geometric Higgs Mechanism

- **Kaluza-Klein (KK) Compactification of Extra Dimension leads to massive KK states, including KK Gauge Bosons and KK Gravitons.** **KK states obtain Geometric Masses by Compactification!**

Define a 5d Yang-Mills theory **SU(*N*) :**

$$S_5 = \int d^5\hat{x} \, \frac{-1}{4} \hat{F}^a_{MN} \hat{F}^{aMN} \quad \text{(on Flat 5d for simplicity)}$$

where

$$M, N = 0, 1, 2, 3, 5, \quad \mu, \nu = 0, 1, 2, 3, \quad \hat{x} = (x^\mu, x^5),$$

$$\hat{A}^{aM} = (\hat{A}^{a\mu}, \hat{A}^{a5})_{(x^\mu, x^5)}$$

The 5th dimension is an interval $[0, L]$ with $L \equiv \pi R$,

Geometric Higgs Mechanism

➢ **We compactify a 5d SU(N) YM under orbifold S^1/Z_2:**

► Consider simplest Neumann Boundary Conditions (BCs) for $\hat{A}^{a\mu}$:

$$\partial_5 \hat{A}^a_\mu|_{x^5=0} = 0, \qquad \partial_5 \hat{A}^a_\mu|_{x^5=L} = 0,$$

and Dirichlet BCs for $\hat{A}^{a5}$,

$$\hat{A}^{a5}(x^\mu, 0) = 0, \qquad \hat{A}^{a5}(x^\mu, L) = 0.$$

► Kaluza-Klein Expansion,

$$\hat{A}^{a\mu}(\hat{x}) = \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} V_n^{a\mu}(x)\chi_n(x^5)$$

$$\hat{A}^{a5}(\hat{x}) = \frac{1}{\sqrt{L}} \sum_{n=0}^{\infty} V_n^{a5}(x)\tilde{\chi}_n(x^5)$$

with the Normalization Conditions,

$$\frac{1}{L}\int_0^L dx^5\, \chi_n \chi_m = \delta_{nm}\,, \qquad \frac{1}{L}\int_0^L dx^5\, \tilde{\chi}_n \tilde{\chi}_m = \delta_{nm}\,.$$

Geometric "Higgs" Mechanism in 5d

- **Under compactification of flat 5d, how do KK Masses arise? ---**

► Consider 5d Massless Gauge Fields $\hat{A}^{aM}(x^\mu, x^5)$,

$$\begin{cases} 0 \;=\; P^2 \;=\; p_\mu p^\mu + p_5 p^5 \;=\; p^2 - p_5^2 \\ p^5 \;=\; \dfrac{n}{R}\,, \qquad (n = 0, 1, 2, \cdots), \end{cases}$$

(After 5d compactification)

where p^5 is quantized due to BCs, and $p^2 = p_\mu p^\mu$ is the 4-momentum-squared in 4d.

► Hence, in 4d we can see the "KK Tower", as a unique consequence of 5d Compactification,

$$p^2 \;=\; p_5^2 \;=\; \frac{n^2}{R^2}\,, \quad (n = 0, 1, 2, \cdots), \quad \Rightarrow \quad \text{KK Tower !}$$

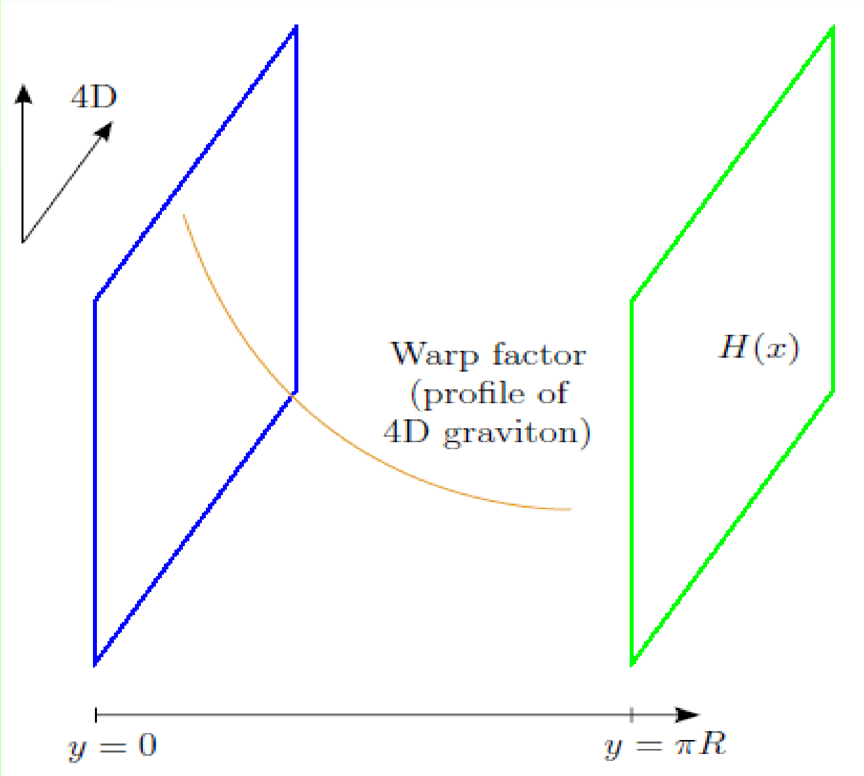
- **Masses Generation by KK Compactification:**

$\longrightarrow$ **Geometric "Higgs" Mechanism**

Chivukula, Dicus, HJH, PLB (2002) hep-ph/0111016

Warped 5d: Geometric "Higgs" Mechanism

- **Warped 5d RS1 under S^1/Z_2 :**



← **Randall-Sundrum, hep-ph/9905221**

- **Warped 5d metric:**

$$\mathrm{d}s^2 = e^{2A(y)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2,$$
$$\mathrm{d}s^2 = e^{2A(z)}(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2)$$

where $A(z) = -ky,$ $A(z) = -\ln(1+kz)\,,$ $k = \sqrt{-\Lambda/6}$ $\hat{g}_{MN} = e^{2A(z)}\hat{\eta}_{MN}$

Warped 5d: Geometric "Higgs" Mechanism

- **Warped 5d Gauge Theory (QCD_{5d}):**

$$\hat{\mathcal{L}}_{\text{YM}} = \sqrt{-\hat{g}}\left(-\frac{1}{4}\hat{g}^{MP}\hat{g}^{NQ}\hat{F}^a_{MN}\hat{F}^a_{PQ}\right),$$

$$\hat{\mathcal{L}}_{\text{GF}} = -\frac{e^{A(z)}}{2\xi}(\hat{\mathcal{F}}^a)^2, \quad \hat{\mathcal{F}}^a = \partial^\mu\hat{A}^a_\mu + \xi(A' + \partial_z)\hat{A}^a_5\,,$$

- **Boundary Conditions of orbifold S^1/Z_2 & KK Expansions:**

$$\partial_z\hat{A}^a_\mu(x,z)\Big|_{z=0,L} = 0\,, \quad \hat{A}^a_5(x,z)\Big|_{z=0,L} = 0\,,$$

$$\hat{A}^{a\mu}(x,z) = \frac{1}{\sqrt{L}}\sum_{n=0}^{\infty} A^{a\mu}_n(x)\,\mathsf{f}_n(z)\,,$$

$$\hat{A}^{a5}(x,z) = \frac{1}{\sqrt{L}}\sum_{n=1}^{\infty} A^{a5}_n(x)\,\tilde{\mathsf{f}}_n(z)\,.$$

Hang, Zhao, HJH, Qiu, (2024), arXiv:2406.12713

Geometric Higgs Mechanism vs KK ET

Hang, Zhao, He, Qiu, arXiv:2406.12713
H.-J. He, hep-ph/0412113

▶ KK Equivalence Theorem (KK-ET),

$$\mathscr{T}[A_L^{an}(P_n), A_L^{bm}(P_m), \cdots\cdots]$$

$$= C_{mod}\,\mathscr{T}[A_5^{an}(P_n), A_5^{am}(P_m), \cdots\cdots] + O\left(\frac{M_{n,m}}{E}\right),$$

$$C_{mod} = 1 + O(Loop)\,.$$

▶ This shows that A_L^{bn} remembers its Goldstone origin from A_5^{an} at high energies !

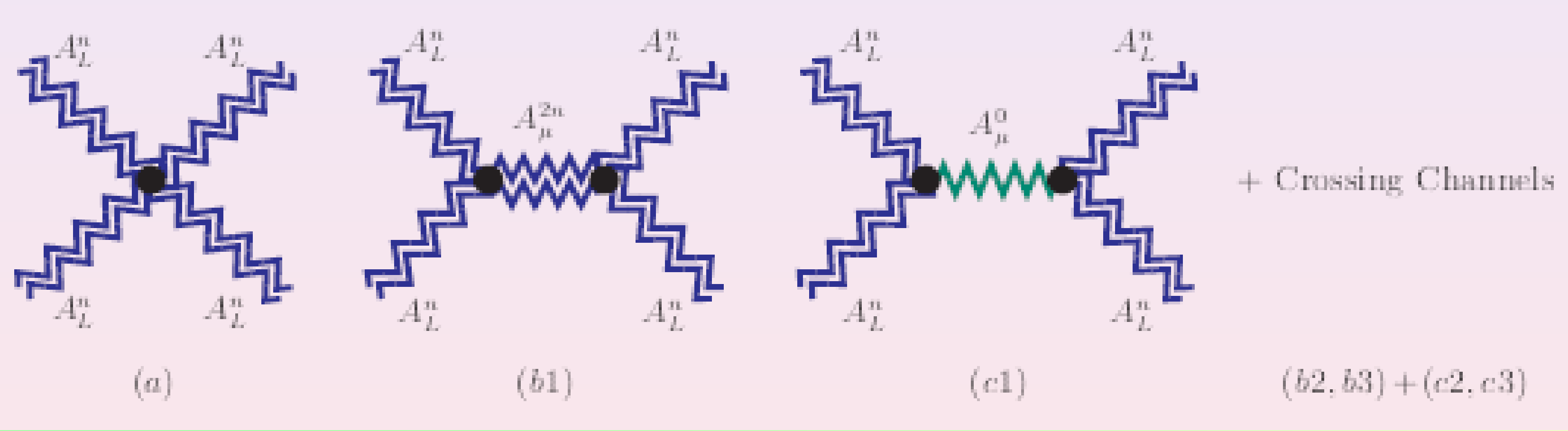
▶ This is the mathematical formulation/manifestation of Geometric Higgs Mechanism at the S-matrix level !

▶ Conversion: $A_n^{a5} \Rightarrow A_{nL}^{a\mu}$ with mass $M_{an} \overset{e.g.}{=} \dfrac{n}{R}$

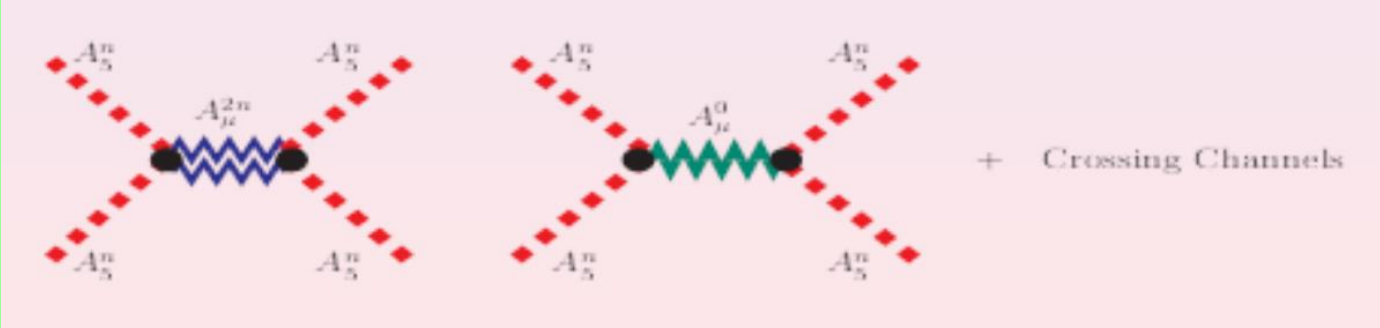
$\Rightarrow$ Geometric Mass Generation !

KK Gauge Amplitudes & E Cancellations

- For flat 5d under S^1/Z_2, Longitudinal KK gauge boson scattering via contact diagram and (s,t,u) diagrams with 0-mode and 2n-mode exchanges:



- KK Goldstone boson scattering via (s,t,u) diagrams with 0-mode and 2n-mode gauge boson exchanges:



KK Gauge Amplitude & E-Cancellations

- **For flat 5d under S^1/Z_2, Longitudinal KK gauge boson scattering via contact diagram and (s,t,u) diagrams with 0-mode and 2n-mode exchanges:**

► Scattering amplitude in M_n/E expansion: ($c = \cos\theta$, $x = E/M_n$)

Graph	$g^2 C^{eab} C^{ecd}$	$g^2 C^{eac} C^{edb}$	$g^2 C^{ead} C^{ebc}$
(a)	$6c(x^4 - x^2)$	$\frac{3}{2}(3-2c-c^2)x^4$	$\frac{-3}{2}(3+2c-c^2)x^4$
		$-3(1-c)x^2$	$+3(1+c)x^2$
$(b1)$	$-2c(x^4 - x^2)$		
$(c1)$	$-4cx^4$		
$(b2,3)$		$\frac{-1}{2}(3-2c+c^2)x^4$	$\frac{1}{2}(3+2c-c^2)x^4$
		$+3(1-c)x^2$	$-3(1+c)x^2$
$(c2,3)$		$(-3+2c+c^2)x^4$	$(3+2c-c^2)x^4$
		$-8cx^2$	$-8cx^2$
Sum	$-8cx^2$	$-8cx^2$	$-8cx^2 \;\Rightarrow\; 0$

KK Gauge Amplitude & E-Cancellations at 4-Point

- **Longitudinal KK gauge boson scattering via contact diagram and (s,t,u) diagrams with KK-mode exchanges:**

$$\mathcal{T}_0[A_L^{an} A_L^{bn} \to A_L^{cn} A_L^{dn}] = g^2 (\mathcal{C}_s \mathcal{K}_s^0 + \mathcal{C}_t \mathcal{K}_t^0 + \mathcal{C}_u \mathcal{K}_u^0),$$
$$\widetilde{\mathcal{T}}_0[A_5^{an} A_5^{bn} \to A_5^{cn} A_5^{dn}] = g^2 (\mathcal{C}_s \widetilde{\mathcal{K}}_s^0 + \mathcal{C}_t \widetilde{\mathcal{K}}_t^0 + \mathcal{C}_u \widetilde{\mathcal{K}}_u^0),$$

$$(\mathcal{C}_s, \mathcal{C}_t, \mathcal{C}_u) = (C^{abe} C^{cde}, C^{ade} C^{bce}, C^{ace} C^{dbe}),$$

- **Color Jacobi Identity:** $$\mathcal{C}_s + \mathcal{C}_t + \mathcal{C}_u = 0$$

- **E^4 and E^2 cancellations:** $$\sum_{j=0}^{\infty} a_{nnj}^2 = a_{nnnn}, \qquad \sum_{j=0}^{\infty} M_j^2 a_{nnj}^2 = \frac{4}{3} M_n^2 a_{nnnn}$$

$$\longrightarrow \sum_{j=0}^{\infty} \left(\frac{4}{3} - r_j^2\right) a_{nnj}^2 = 0$$

which can be verified for each given model, such as flat 5d model or RS1.

Hang, Zhao, He, Qiu, arXiv:2406.12713

KK Gauge Amplitude & GAET at 4-Point

- **Longitudinal KK gauge boson scattering via contact diagram and (s,t,u) diagrams with KK-mode exchanges:**

$$\mathcal{T}_0[A_L^{an} A_L^{bn} \to A_L^{cn} A_L^{dn}] = g^2 (\mathcal{C}_s \mathcal{K}_s^0 + \mathcal{C}_t \mathcal{K}_t^0 + \mathcal{C}_u \mathcal{K}_u^0),$$
$$\widetilde{\mathcal{T}}_0[A_5^{an} A_5^{bn} \to A_5^{cn} A_5^{dn}] = g^2 (\mathcal{C}_s \widetilde{\mathcal{K}}_s^0 + \mathcal{C}_t \widetilde{\mathcal{K}}_t^0 + \mathcal{C}_u \widetilde{\mathcal{K}}_u^0),$$

- **KK Gauge Theory Equivalence Theorem (GAET) at 4-point:**

$$\longrightarrow \quad \mathcal{K}_i^0 - \widetilde{\mathcal{K}}_i^0 = -2c_\theta \sum_{j=1}^{\infty} r_j^2 a_{nnj}^2 \,,$$

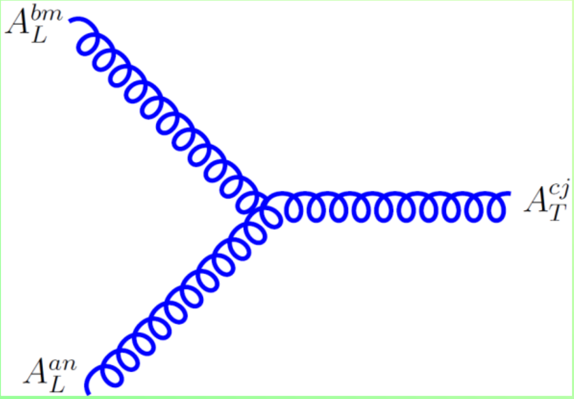
$$\longrightarrow \quad \mathcal{T}_0[4A_L^{an}] - \widetilde{\mathcal{T}}_0[4A_5^{an}] = -2c_\theta \sum_{j=1}^{\infty} r_j^2 a_{nnj}^2 \times (\mathcal{C}_s + \mathcal{C}_t + \mathcal{C}_u) = 0$$

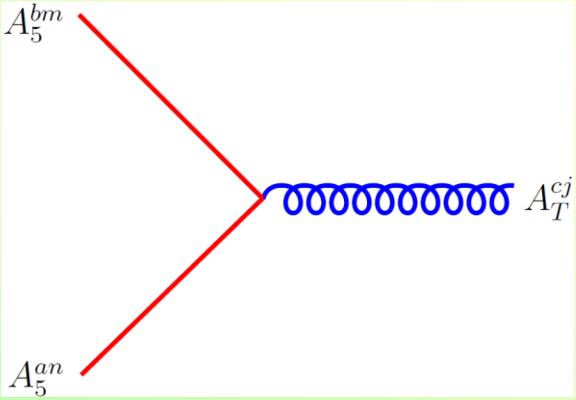
$$\longrightarrow \quad \mathcal{T}_0[A_L^{an} A_L^{bn} \to A_L^{cn} A_L^{dn}] \;=\; \widetilde{\mathcal{T}}_0[A_5^{an} A_5^{bn} \to A_5^{cn} A_5^{dn}]$$

Hang, Zhao, HJH, Qiu, arXiv:2406.12713

KK Gauge Amplitudes & E-Cancellations at 3-Point

- **3pt Longitudinal KK gauge boson amplitude ($A_L A_L A_T$) and KK Goldstone amplitude ($A_5 A_5 A_T$):**





$$\mathcal{T}_0[A_L^{an_1} A_L^{bn_2} A_\pm^{cn_3}] = -\mathrm{i} g f^{abc} (\epsilon_3 \cdot p_1) \frac{(M_{n_3}^2 - M_{n_1}^2 - M_{n_2}^2)}{M_{n_1} M_{n_2}} a_{n_1 n_2 n_3} ,$$

$$\widetilde{\mathcal{T}}_0[A_5^{an_1} A_5^{bn_2} A_\pm^{cn_3}] = -\mathrm{i} 2 g f^{abc} (\epsilon_3 \cdot p_1) \tilde{a}_{n_1 n_2 n_3}$$

Hang, Zhao, HJH, Qiu, arXiv:2406.12713

KK Gauge Amplitudes & E-Cancellations at 3-Point

- **3pt Longitudinal KK gauge boson amplitude ($A_L A_L A_T$) and KK Goldstone amplitude ($A_5 A_5 A_T$):**

$$\mathcal{T}_0[A_L^{an_1} A_L^{bn_2} A_\pm^{cn_3}] = -\mathrm{i} g f^{abc} (\epsilon_3 \cdot p_1) \frac{(M_{n_3}^2 - M_{n_1}^2 - M_{n_2}^2)}{M_{n_1} M_{n_2}} a_{n_1 n_2 n_3},$$

$$\widetilde{\mathcal{T}}_0[A_5^{an_1} A_5^{bn_2} A_\pm^{cn_3}] = -\mathrm{i}2 g f^{abc} (\epsilon_3 \cdot p_1) \tilde{a}_{n_1 n_2 n_3}$$

- **The most fundamental GAET:**

$$\longrightarrow \quad \mathcal{T}_0[A_L^{an_1} A_L^{bn_2} A_\pm^{cn_3}] = -\widetilde{\mathcal{T}}_0[A_5^{an_1} A_5^{bn_2} A_\pm^{cn_3}]$$

enforcing energy cancellation **$E^3 \rightarrow E^1$**

and it requires the coupling-mass condition:

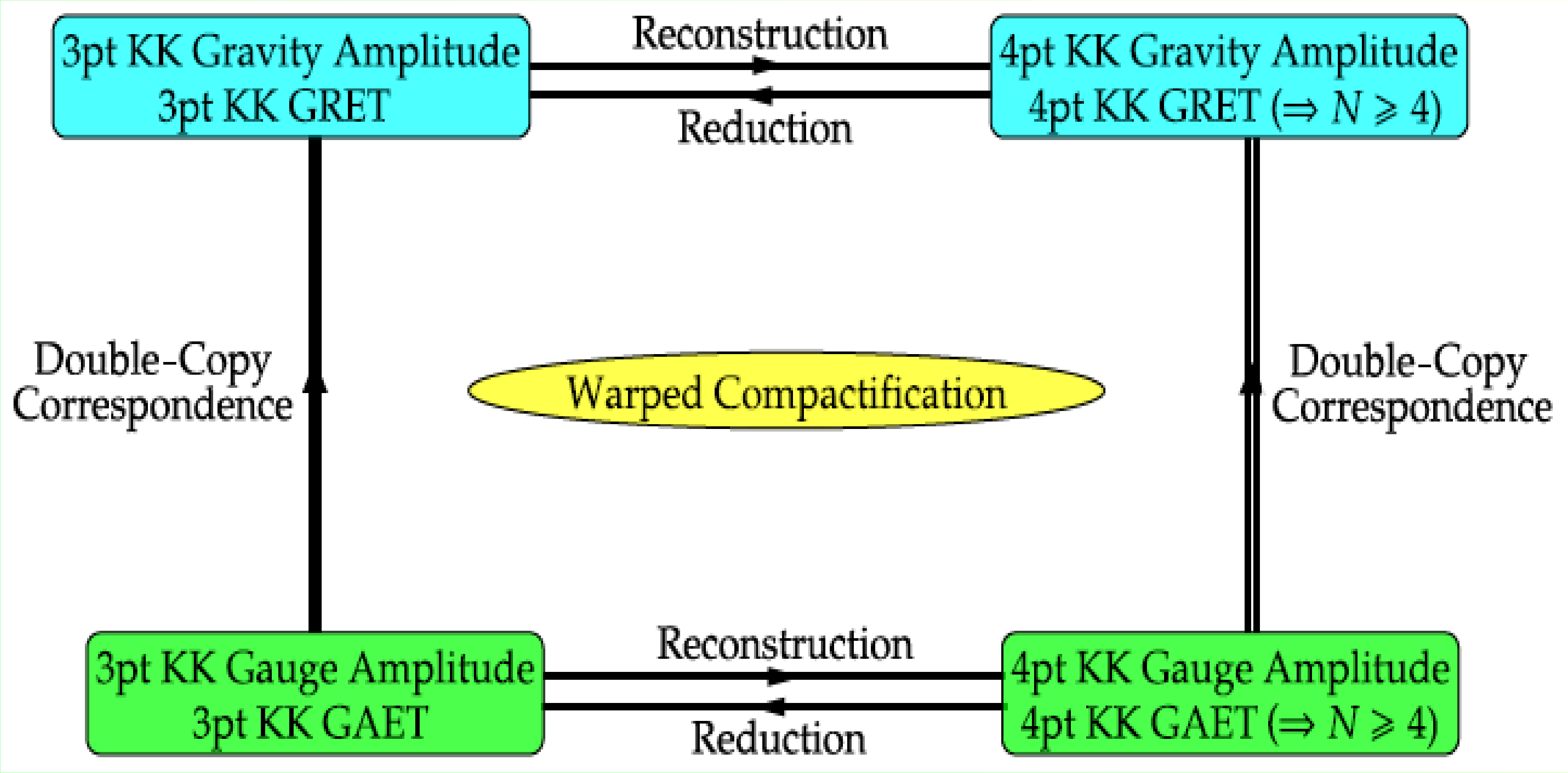
$$\longrightarrow \quad \left(M_{n_1}^2 + M_{n_2}^2 - M_{n_3}^2\right) a_{n_1 n_2 n_3} = 2 M_{n_1} M_{n_2} \tilde{a}_{n_1 n_2 n_3}$$

from which all N-point KK Amplitudes and GAET (N>3) can be derived !

Hang, Zhao, HJH, Qiu, arXiv:2406.12713

KK ET & Double-Copy Correspondance: 3-Point to 4-Point

- **Equivalence Theorem and Double-Copy Correspondences from 3pt KK amplitudes to 4pt KK amplitudes and from massive KK gauge amplitudes to massive KK gravitational amplitudes.**



Hang, Zhao, HJH, Qiu, arXiv:2406.12713

Geometric Higgs Mechanism for KK Graviton Mass Generation

Hang and HJH, PRD(2022), arXiv:2106.04568 (90pp)
Research (2022), arXiv:2207.11214
Hang, Zhao, HJH, Qiu, (2024), arXiv:2406.12713 (91pp)

Geometric Higgs Mechanism for KK Gravitons

- **5d Einstein-Hilbert (EH) Action:**

$$S_{\rm EH} = \int d^5x \, \frac{2}{\hat{\kappa}^2} \sqrt{-\hat{g}} \, \hat{R} \, ,$$

$$\hat{\kappa} = \sqrt{32\pi \hat{G}}$$

- **5d Compactification under orbifold S^1/Z_2 :**

$$\mathcal{L}_{\rm eff} = \sum_{j=0}^{\infty} \int_0^L dx^5 \, \hat{\kappa}^j \hat{\mathcal{L}}_j$$

$$\kappa = \frac{\hat{\kappa}}{\sqrt{L}} = \frac{2}{M_{\rm Pl}} \, ,$$

$$\kappa = \sqrt{32\pi G}$$

- **5d Graviton field is parametrized as:**

$$\hat{h}_{AB} = \begin{pmatrix} \hat{h}_{\mu\nu} + w\eta_{\mu\nu}\hat{\phi} & \hat{h}_{\mu 5} \\ \hat{h}_{5\nu} & \hat{\phi} \end{pmatrix},$$

$$\hat{\mathcal{V}}_\mu \equiv \hat{h}_{\mu 5}$$

$$\hat{\phi} \equiv \hat{h}_{55}$$

Geometric Higgs Mechanism for KK Gravitons

- Under S^1/Z_2 compactification, we impose the Boundary Conditions:

$$\partial_5 \hat{h}_{\mu\nu}\big|_{x^5=0,L} = 0\,, \qquad \partial_5 \hat{\phi}\big|_{x^5=0,L} = 0\,, \qquad \hat{h}_{\mu 5}\big|_{x^5=0,L} = 0$$

- 4d Quadratic KK Lagrangian contains Mixing Terms:

$$+\,2\mathbb{M}_n h_n \partial_\mu \mathcal{V}_n^\mu - 2\mathbb{M}_n h_n^{\mu\nu}\partial_\mu \mathcal{V}_{\nu,n} - \frac{3}{2}\mathbb{M}_n^2 h_n \phi_n - 3\mathbb{M}_n \partial_\mu \mathcal{V}_n^\mu \phi_n$$

which can be eliminated by R_ξ gauge fixing:

$$\mathcal{V}_n^\mu = h_n^{\mu 5}$$
$$\phi_n = h_n^{55}$$

$$\mathcal{L}_{\rm GF} = \int_0^L \mathrm{d}z\, \hat{\mathcal{L}}_{\rm GF} = -\sum_{n=0}^{\infty} \frac{1}{\xi_n}\Big[(\mathcal{F}_n^\mu)^2 + (\mathcal{F}_n^5)^2\Big],$$

$$\mathcal{F}_n^\mu = \partial_\nu h_n^{\mu\nu} - \left(1 - \frac{1}{2\xi_n}\right)\partial^\mu h_n + \frac{1}{\sqrt{2}}\xi_n \mathbb{M}_n \mathcal{V}_n^\mu\,,$$
$$\mathcal{F}_n^5 = \frac{1}{2}\mathbb{M}_n h_n - \sqrt{\frac{3}{2}}\,\xi_n \mathbb{M}_n \phi_n + \frac{1}{\sqrt{2}}\,\partial_\mu \mathcal{V}_n^\mu\,.$$

R_ξ Gauge-Fixing & KK Graviton Propagator

- **R_ξ Gauge Propagators:**

$$
\begin{aligned}
\mathcal{D}_{nm}^{\mu\nu\alpha\beta}(p) \;=\; & -\frac{\mathrm{i}\delta_{nm}}{2}\left\{ \frac{(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta})}{p^2+M_n^2} \right. \\
& +\frac{1}{3}\left[\frac{1}{p^2+M_n^2}-\frac{1}{p^2+(3\xi_n-2)M_n^2}\right]\left(\eta^{\mu\nu}-\frac{2p^\mu p^\nu}{M_n^2}\right)\left(\eta^{\alpha\beta}-\frac{2p^\alpha p^\beta}{M_n^2}\right) \\
& +\frac{1}{M_n^2}\left[\frac{1}{p^2+M_n^2}-\frac{1}{p^2+\xi_n M_n^2}\right](\eta^{\mu\alpha}p^\nu p^\beta+\eta^{\mu\beta}p^\nu p^\alpha+\eta^{\nu\alpha}p^\mu p^\beta+\eta^{\nu\beta}p^\mu p^\alpha) \\
& \left. +\frac{4p^\mu p^\nu p^\alpha p^\beta}{\xi_n M_n^4}\left(\frac{1}{p^2+\xi_n^2 M_n^2}-\frac{1}{p^2+\xi_n M_n^2}\right)\right\}, \qquad (2.
\end{aligned}
$$

Hang and HJH, 2106.04568, 2207.11214

KK Graviton Propagator in Feynman/Unitary Gauge

- **Feynman Gauge (ξ_n=1) Propagators take very simple form:**

$$\mathcal{D}_{nm}^{\mu\nu\alpha\beta}(p) = -\frac{\mathrm{i}\delta_{nm}}{2}\,\frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}}{p^2+M_n^2}$$

- **We also derive Unitary Gauge (ξ_n= ∞) Propagator:**

$$\mathcal{D}_{nm,\mathrm{UG}}^{\mu\nu\alpha\beta}(p) = -\frac{\mathrm{i}\delta_{nm}}{2}\,\frac{\bar{\eta}^{\mu\alpha}\bar{\eta}^{\nu\beta}+\bar{\eta}^{\mu\beta}\bar{\eta}^{\nu\alpha}-\frac{2}{3}\bar{\eta}^{\mu\nu}\bar{\eta}^{\alpha\beta}}{p^2+M_n^2}$$

where $\bar{\eta}^{\mu\nu} = \eta^{\mu\nu} + p^\mu p^\nu/M_n^2\,.$

Hang, HJH, (2021), 2106.04568, 2207.11214

KK Graviton Propagator vs vDVZ Discontinuity

- **KK Graviton propagator in $M_n \to 0$ limit:**

$$\mathcal{D}_{nm}^{\mu\nu\alpha\beta}(p) = -\frac{\mathrm{i}\delta_{nm}}{2}\left[\frac{(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta})}{p^2} - \frac{1-\xi_n}{p^4}(\eta^{\mu\alpha}p^\nu p^\beta+\eta^{\mu\beta}p^\nu p^\alpha+\eta^{\nu\alpha}p^\mu p^\beta\right.$$
$$\left.+\eta^{\nu\beta}p^\mu p^\alpha-2\eta^{\mu\nu}p^\alpha p^\beta-2\eta^{\alpha\beta}p^\mu p^\nu)-4(1-\xi_n)^3\frac{p^\mu p^\nu p^\alpha p^\beta}{p^6}\right],$$
$$= -\frac{\mathrm{i}\delta_{nm}}{2}\frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}}{p^2}, \qquad (\text{for } \xi_n=1).$$

It is free from vDVZ (van-Dam-Veltman-Zakharov) Discontinuity !

- **Compared with conventional massless graviton propagator:**

$$\mathcal{D}_{00}^{\mu\nu\alpha\beta}(p) = -\frac{\mathrm{i}}{2}\left[\frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}}{p^2}+(\xi-1)\frac{\eta^{\mu\alpha}p^\nu p^\beta+\eta^{\mu\beta}p^\nu p^\alpha+\eta^{\nu\alpha}p^\mu p^\beta+\eta^{\nu\beta}p^\mu p^\alpha}{p^4}\right]$$
$$= -\frac{\mathrm{i}}{2}\frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}}{p^2}, \qquad (\text{for } \xi=1).$$

KK Graviton Propagator vs vDVZ Discontinuity

- **KK Graviton propagator in $M_n \to 0$ limit:**

$$\mathcal{D}_{nm}^{\mu\nu\alpha\beta}(p) = -\frac{\mathrm{i}\delta_{nm}}{2}\left[\frac{(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta})}{p^2} - \frac{1-\xi_n}{p^4}(\eta^{\mu\alpha}p^\nu p^\beta+\eta^{\mu\beta}p^\nu p^\alpha+\eta^{\nu\alpha}p^\mu p^\beta \right.$$
$$\left. +\eta^{\nu\beta}p^\mu p^\alpha - 2\eta^{\mu\nu}p^\alpha p^\beta - 2\eta^{\alpha\beta}p^\mu p^\nu) - 4(1-\xi_n)^3\frac{p^\mu p^\nu p^\alpha p^\beta}{p^6}\right],$$
$$= -\frac{\mathrm{i}\delta_{nm}}{2}\frac{\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}}{p^2}, \qquad (\text{for } \xi_n = 1).$$

- **Compared with Fierz-Pauli massive graviton propagator:**

$$\mathcal{D}_{\rm PF}^{\mu\nu\alpha\beta}(p) = -\frac{\mathrm{i}}{2}\frac{\bar{\eta}^{\mu\alpha}\bar{\eta}^{\nu\beta}+\bar{\eta}^{\mu\beta}\bar{\eta}^{\nu\alpha}-\frac{2}{3}\bar{\eta}^{\mu\nu}\bar{\eta}^{\alpha\beta}}{p^2+M^2}$$

where $\bar{\eta}^{\mu\nu} = \eta^{\mu\nu} + p^\mu p^\nu/M^2$.

It has vDVZ discontinuity in $M_n \to 0$ limit!

KK Graviton Propagator vs vDVZ Discontinuity

- **5d KK Graviton conserves physical d.o.f in $M_n \to 0$ limit:**

$$5 \quad = \quad 2 \quad + \quad 2 \quad + \quad 1$$

$$\lambda = \quad 0, \pm 1, \pm 2 \qquad \pm 2 \qquad \pm 1 \qquad 0$$

where $\lambda = 0, \pm 1$ d.o.f arise from 3 KK Goldstones $(h_n^{\mu 5},\, h_n^{55})$.

- **But, the Massive Graviton of Fierz-Pauli gravity does not conserve physical d.o.f in $M_n \to 0$ limit:**

$$5 \;\neq\; 2$$

Geometric Higgs Mechanism via GRET

- **Gravitational Equivalence Theorem (GRET type-I):**

$$\mathcal{M}[h^L_{n_1},\cdots,h^L_{n_N},\Phi] = C_{\text{mod}}^{n_j m_j}\mathcal{M}[\phi_{m_1},\cdots,\phi_{m_N},\Phi] + O(\mathbb{M}_n/E_n\text{–suppressed})$$

$$C_{\text{mod}}^{n_j m_j} = C_{n_1 m_1}\cdots C_{n_N m_N} = \delta_{n_1 m_1}\cdots\delta_{n_N m_N} + O(\text{loop})$$

- **Gravitational Equivalence Theorem (GRET type-II):**

$$\mathcal{M}[h^{\pm 1}_{n_1},\cdots,h^{\pm 1}_{n_N},\Phi] = \hat{C}_{\text{mod}}^{n_j m_j}\mathcal{M}[\mathcal{V}^{\pm 1}_{m_1},\cdots,\mathcal{V}^{\pm 1}_{m_N},\Phi] + O(\mathbb{M}_n/E_n\text{–suppressed})$$

$$\hat{C}_{\text{mod}}^{n_j m_j} = \hat{C}_{n_1 m_1}\cdots\hat{C}_{n_N m_N} = (-\mathrm{i})^N\delta_{n_1 m_1}\cdots\delta_{n_N m_N} + O(\text{loop})$$

Hang, HJH, arXiv:2106.04568, 2207.11214
Hang, Zhao, HJH, Qiu, arXiv:2406.12713

Generalized Power Counting Method for KK Theories

- **Generalized Energy Power Counting for KK Theories:**

$$D_E \;=\; 2\mathcal{E}_{h_L} + (2L+2) + \sum_j \mathcal{V}_j\left(d_j - 2 + \tfrac{1}{2} f_j\right).$$

- **E-Power Counting of helicity-0 KK Graviton/Goldstone Amplitudes:**

$$D_E(Nh_n^L) = 2(N+L+1), \qquad D_E(N\phi_n) = 2(L+1),$$
$$\Longrightarrow \quad D_E(Nh_n^L) - D_E(N\phi_n) = 2N,$$

- **E-Power Counting of helicity-1 KK Graviton/Goldstone Amplitudes:**

$$D_E(Nh_n^{\pm 1}) = N + 2(L+1), \qquad D_E(N\mathcal{V}_n^{\pm 1}) = 2(L+1),$$
$$\Longrightarrow \quad D_E(Nh_n^{\pm 1}) - D_E(N\mathcal{V}_n^{\pm 1}) = N.$$

- **E-Power Counting of KK Gauge/Goldstone boson Amplitudes:**

$$D_E(NA_L^{an}) = 4, \qquad D_E(NA_5^{an}) = 4 - N - \bar{V}_3^{\min}$$
$$\Longrightarrow \quad D_E(NA_L^{an}) - D_E(NA_5^{an}) = N + \bar{V}_3^{\min}$$

Hang, HJH, arXiv:2106.04568, 2207.11214

E-Cancellations of KK Graviton Amplitude via GRET

- **Gravitational Equivalence Theorem (GET) identity:**

$$\mathcal{M}[h^L_{n_1}(k_1),\cdots,h^L_{n_N}(k_N),\Phi] = \mathcal{M}[\phi_{n_1}(k_1),\cdots,\phi_{n_N}(k_N),\Phi] + \sum_{1\leqslant j\leqslant N}\mathcal{M}[\{\tilde{\Delta}_{n_j},\phi_n\},\Phi],$$

- **Energy Power Counting:** $D_E[Nh^L_n] = 2(N+1)+2L\,,$

$$D_E[N\phi_n] = 2+2L\,. \qquad D_E[N\tilde{v}_n] = 2L\,,$$

- **We deduce h_L Amplitude has Large E-Cancellations:**

$$\longrightarrow \qquad D_E[Nh^L_n] - D_E[N\phi_n] = 2N$$

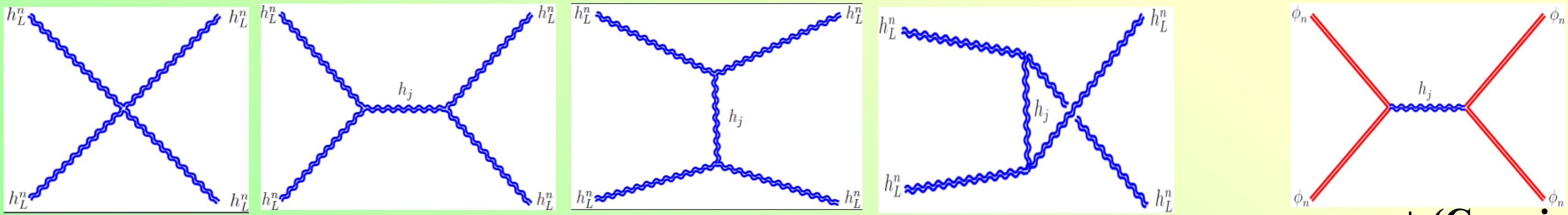
- **For N = 4 scattering at tree level, GET proves Large E-cancellation:**

$$\longrightarrow \qquad E^{10} \to E^2$$ **(by 10 - 2 = 8 powers!)**

Hang, HJH, arXiv:2106.04568, 2207.11214

KK Graviton/Goldstone Amplitudes and GRET

- **Scattering Amplitude of KK Gravitons and KK Goldstones:**



+ (Crossing)

- **We compute KK Goldstone Amplitude at LO and NLO:**

$$\mathcal{M}_0[4h_L^n] = \frac{\kappa^2 s\,(7+c_{2\theta})^2 \csc^2\theta}{2304} \sum_{j=1}^{\infty} \hat{r}_j^2 (\hat{r}_j^6 - 12\hat{r}_j^2 + 11)\,\alpha_{nnj}^2\,,$$

$$\widetilde{\mathcal{M}}_0[4\phi_n] = \frac{\kappa^2 s\,(7+c_{2\theta})^2 \csc^2\theta}{64} \sum_{j=0}^{\infty} \tilde{\beta}_{nnj}^2\,,$$

- **GRET:** $$\mathcal{M}_0[4h_L^n] = \widetilde{\mathcal{M}}_0[4\phi_n]\,,$$

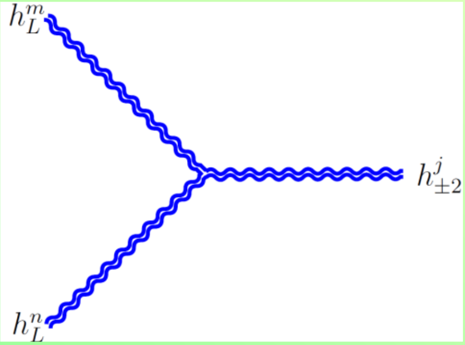
- **Sum Rule Condition:** $$\sum_{j=1}^{\infty} \hat{r}_j^2 (\hat{r}_j^6 - 12\hat{r}_j^2 + 11)\,\alpha_{nnj}^2 = 36 \sum_{j=0}^{\infty} \tilde{\beta}_{nnj}^2$$

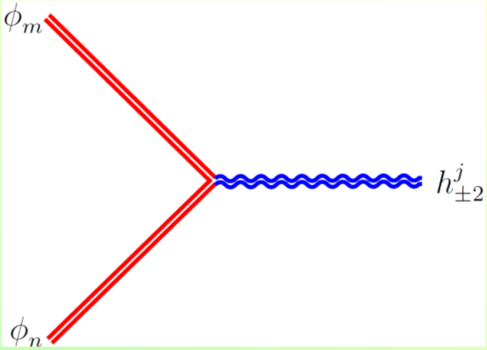
which can be proved from 3pt condition.

Hang, HJH, Zhao et al, arXiv:2406.12713, 2106.04568

KK Graviton Amplitudes & E-Cancellations at 3-Point

- **3pt Longitudinal KK graviton amplitude ($h_L h_L h_T$) and KK Goldstone amplitude ($\phi_n \phi_m h_T$) :**





$$\mathcal{M}_0[h^L_{n_1} h^L_{n_2} h^{\pm 2}_{n_3}] = \frac{\kappa\,(\epsilon_3 \cdot p_1)^2}{6\mathbb{M}^2_{n_1}\mathbb{M}^2_{n_2}} \left[2\mathbb{M}^2_{n_1}\mathbb{M}^2_{n_2} + \left(\mathbb{M}^2_{n_1} + \mathbb{M}^2_{n_2} - \mathbb{M}^2_{n_3}\right)^2\right] \alpha_{n_1 n_2 n_3}\,,$$

$$\widetilde{\mathcal{M}}_0[\phi_{n_1} \phi_{n_2} h^{\pm 2}_{n_3}] = \kappa(\epsilon_3 \cdot p_1)^2 \tilde{\beta}_{n_1 n_2 n_3}\,,$$

- **E Cancellations for above 3-point KK Graviton Amplitude: $E^6 \rightarrow E^2$**

Hang, Zhao, HJH, Qiu, arXiv:2406.12713

KK Graviton Amplitudes & E-Cancellations at 3-Point

- **3pt Longitudinal KK graviton amplitude ($h_L h_L h_T$) and KK Goldstone amplitude ($\phi_n \phi_m h_T$) :**

$$\mathcal{M}_0[h^L_{n_1} h^L_{n_2} h^{\pm 2}_{n_3}] = \frac{\kappa(\epsilon_3 \cdot p_1)^2}{6\mathbb{M}^2_{n_1}\mathbb{M}^2_{n_2}}\left[2\mathbb{M}^2_{n_1}\mathbb{M}^2_{n_2} + \left(\mathbb{M}^2_{n_1} + \mathbb{M}^2_{n_2} - \mathbb{M}^2_{n_3}\right)^2\right]\alpha_{n_1 n_2 n_3},$$

$$\widetilde{\mathcal{M}}_0[\phi_{n_1}\phi_{n_2} h^{\pm 2}_{n_3}] = \kappa(\epsilon_3 \cdot p_1)^2 \tilde{\beta}_{n_1 n_2 n_3},$$

- **The most fundamental GRET at 3-point:**

$$\longrightarrow \quad \mathcal{M}_0[h^L_{n_1} h^L_{n_2} h^{\pm 2}_{n_3}] = \widetilde{\mathcal{M}}_0[\phi_{n_1}\phi_{n_2} h^{\pm 2}_{n_3}],$$

which imposes the sum rule condition on KK masses & cubic couplings:

$$\longrightarrow \quad \left[\left(\mathbb{M}^2_{n_1} + \mathbb{M}^2_{n_2} - \mathbb{M}^2_{n_3}\right)^2 + 2\mathbb{M}^2_{n_1}\mathbb{M}^2_{n_2}\right]\alpha_{n_1 n_2 n_3} = 6\mathbb{M}^2_{n_1}\mathbb{M}^2_{n_2}\tilde{\beta}_{n_1 n_2 n_3}$$

fom which all N-point KK Amplitudes and GRET (N>3) can be derived !

- **E Cancellations (3-point) above:** **$E^6 \rightarrow E^2$**

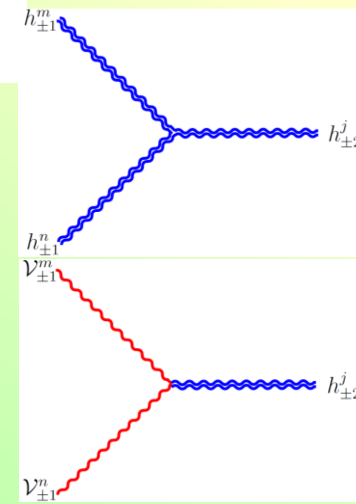
Hang, Zhao, HJH, Qiu, arXiv:2406.12713

KK Graviton Amplitudes & E-Cancellations at 3-Point

- **3pt amplitudes of helicity-1 KK gravitond ($h_1 h_1 h_2$) & KK Goldstones ($\mathcal{V}_1 \mathcal{V}_1 h_2$) :**

$$\mathcal{M}_0[h_{n_1}^{\pm 1} h_{n_2}^{\pm 1} h_{n_3}^{\pm 2}] = \frac{\kappa (\mathrm{M}_{n_3}^2 - \mathrm{M}_{n_1}^2 - \mathrm{M}_{n_2}^2)}{2\mathrm{M}_{n_1}\mathrm{M}_{n_2}} \Big[(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_1)^2 + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_2)(\epsilon_3 \cdot p_1) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot p_3)(\epsilon_3 \cdot p_1)\Big] \alpha_{n_1 n_2 n_3}, \quad ($$

$$\widetilde{\mathcal{M}}_0[\mathcal{V}_{n_1}^{\pm 1} \mathcal{V}_{n_2}^{\pm 1} h_{n_3}^{\pm 2}] = \kappa \Big[(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_1)^2 + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_2)(\epsilon_3 \cdot p_1) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot p_3)(\epsilon_3 \cdot p_1)\Big] \tilde{\alpha}_{n_1 n_2 n_3}.$$



- **The most fundamental GRET at 3-point:**

$$\longrightarrow \quad \mathcal{M}_0[h_{n_1}^{\pm 1} h_{n_2}^{\pm 1} h_{n_3}^{\pm 2}] = -\widetilde{\mathcal{M}}_0[\mathcal{V}_{n_1}^{\pm 1} \mathcal{V}_{n_2}^{\pm 1} h_{n_3}^{\pm 2}]$$

which requires the coupling condition:

$$\longrightarrow \quad \left(\mathrm{M}_{n_1}^2 + \mathrm{M}_{n_2}^2 - \mathrm{M}_{n_3}^2\right) \alpha_{n_1 n_2 n_3} = 2\mathrm{M}_{n_1}\mathrm{M}_{n_2} \tilde{\alpha}_{n_1 n_2 n_3}$$

fom which all N-point KK Amplitudes and GRET (N>3) can be derived !

- **E Cancellations in the above (3-point):** $\quad E^4 \rightarrow E^2$

Hang, Zhao, HJH, Qiu, 2406.12713

Schematic Summary:
ET vs Double-Copy and from 3pt to 4pt Amplitudes

$$
\begin{tikzcd}
\widetilde{\mathcal{T}}_0^{(4)}[A_5^n] & \xlongequal{\text{GAET}(4)} & \mathcal{T}_0^{(4)}[A_L^n] & \xRightarrow{\text{DC}_4} & \mathcal{M}_0^{(4)}[h_L^n] & \xlongequal{\text{GRET}(4)} & \widetilde{\mathcal{M}}_0^{(4)}[\phi_n] \\
\widetilde{\mathcal{T}}_0^{(3)}[A_5^n] & \xlongequal{\text{GAET}(3)} \arrow[u] & \mathcal{T}_0^{(3)}[A_L^n] & \xRightarrow{\text{DC}_3} \arrow[u] & \mathcal{M}_0^{(3)}[h_L^n] & \xlongequal{\text{GRET}(3)} \arrow[u] & \widetilde{\mathcal{M}}_0^{(3)}[\phi_n]
\end{tikzcd}
$$

- This can be further extended to the case of N>4 amplitudes.

Hang, Zhao, HJH, Qiu, 2406.12713

Gauge-Gravity Duality: Double-Copy
Massless vs Massive

Li, Hang, HJH, JHEP (2022), 2209.11191
Hang, Zhao, HJH, Qiu, (2024), 2406.12713
Hang, HJH, PRD (2022), 2106.04568, 2207.11214
Li, Hang, HJH, He, JHEP(2022), 2111.12042
Hang, HJH, Shen, JHEP(2022), 2110.05399;
Research (2023), 2406.13671

$$(\mathrm{GR}) = (\mathrm{QCD})^2$$

$$(\mathrm{Gravity}) = (\mathrm{Gauge\ Theory})^2$$

Gauge Forces vs Gravity Force

- **Deep Connection: Gauge vs Gravity forces**

$$\mathrm{GR} = (\text{Gauge Theory})^2$$

which is established for Massless gravitons/gauge bosons, by KLT and BCJ.

- **What happens to Massive Kaluza-Klein (KK) Theory of Gravity & Gauge Forces ? ?**

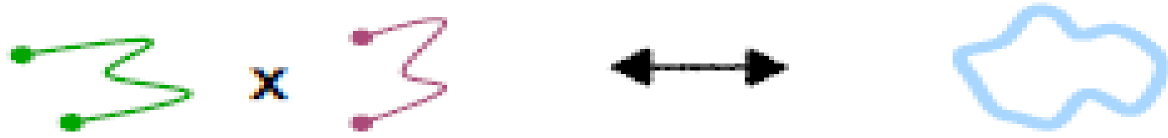
- **Kaluza-Klein Theory: extra 5th Dimension!**

  E.g., Super String Theory (10d) → SM+GR (4d)

  → 6 extra-dimensions under Compactifications !

Gauge vs Gravity Forces: KLT & Double-Copy

- **KLT Construction:** Massless Case Only -- Gravity = (Gauge Theory)^2



$$A^{(4)}_{closed} = -\pi\kappa^2 \sin(\pi\kappa_1\cdot\kappa_2) A^{(4)}_{open}(s,t)\ \bar{A}^{(4)}_{open}(s,u)$$

$$\begin{aligned} A^{(5)}_{closed} &= \pi\kappa^3 A^{(5)}_{open}(12345)\bar{A}^{(5)}_{open}(21435)\sin(\pi\kappa_1\cdot\kappa_2)\sin(\pi\kappa_3\cdot\kappa_4) \\ &+ \pi\kappa^3\ A^{(5)}_{open}(13245)\bar{A}^{(5)}_{open}(31425)\sin(\pi k_1\cdot k_3)\sin(\pi k_2\cdot k_4). \end{aligned}$$

$\downarrow$ Field theory limit

$$M_4^{\text{tree}}(1,2,3,4) = -i s_{12} A_4^{\text{tree}}(1,2,3,4)\ A_4^{\text{tree}}(1,2,4,3),$$

$$\begin{aligned} M_5^{\text{tree}}(1,2,3,4,5) &= i s_{12}s_{34} A_5^{\text{tree}}(1,2,3,4,5) A_5^{\text{tree}}(2,1,4,3,5) \\ &+ i s_{13}s_{24} A_5^{\text{tree}}(1,3,2,4,5)\ A_5^{\text{tree}}(3,1,4,2,5) \end{aligned}$$

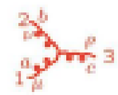
Double-Copy: KLT vs BCJ

- **for Amplitudes of Massless open/closed strings and for gauge/graviton bosons:**

1985: Kawai, Lewellen, Tye (KLT): "closed string amp=open-string amp^2"

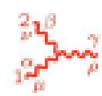
Field-theory limit:

Yang-Mills gauge theory:



$-g f^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic})$

Einstein gravity:



$i\kappa(\eta_{\mu\nu}(k_1-k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1-k_2)_\gamma + \text{cyclic})$

"square" of Yang-Mills vertex.

gravity

gauge theory color ordered

$M_4^{\text{tree}}(1,2,3,4) = -i s_{12} A_4^{\text{tree}}(1,2,3,4) A_4^{\text{tree}}(1,2,4,3)$

2008: Bern, Carrasco, Johansson (BCJ):

$$\mathscr{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

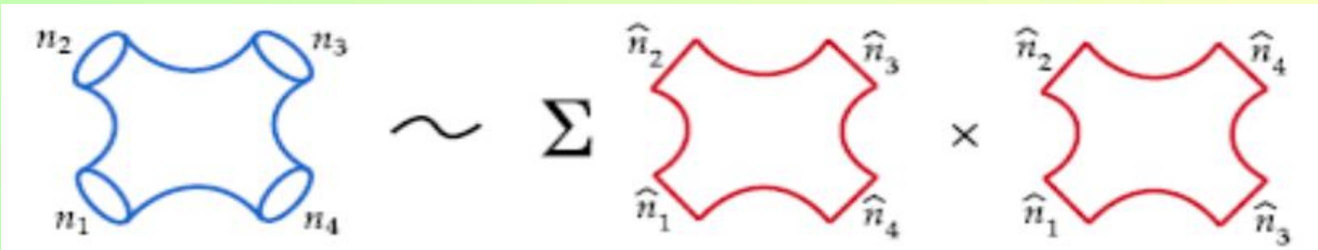
$$\mathscr{A}_4^{\text{tree}}\Big|_{c_i\to n_i} \equiv -i\mathscr{M}_4^{\text{tree}} = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

$$c_s + c_t + c_u = 0$$

$$n_s + n_t + n_u = 0$$

Gauge Forces vs Gravity Force

- **Massive Case:** **GR = (Gauge Theory)^2 ??**
- **What happens to massive Kaluza-Klein Theory ??**
- **First Principle Approach: Using KK bosonic string theory, we derived Massive KLT Relations between product of KK Open String Amplitudes and KK Closed String Amplitude. Taking Field Theory Limit, we derived Massive KLT Relations between product of KK Gauge Boson Amplitudes KK Graviton Amplitude:**



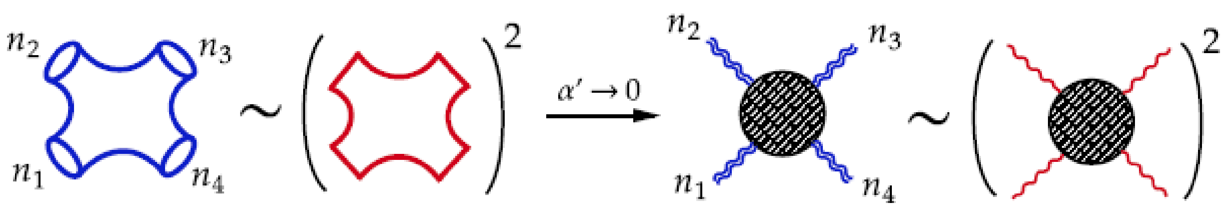
$$
\begin{aligned}
\mathcal{M}[1^n_L 2^n_L 3^n_L 4^n_L] = \frac{\kappa^2}{32} \sum_{\{a_j,b_j\}} \hat{\varrho}_{ab} \Big\{ & (s-4M_n^2)\, \mathcal{T}_{a_j}[1^{+n}2^{+n}3^{-n}4^{-n}]\, \mathcal{T}_{b_j}[1^{+n}2^{+n}4^{-n}3^{-n}] \\
& + s\, \mathcal{T}_{a_j}[1^{+n}2^{-n}3^{+n}4^{-n}]\, \mathcal{T}_{b_j}[1^{+n}2^{-n}4^{+n}3^{-n}] \\
& + s\, \mathcal{T}_{a_j}[1^{+n}2^{-n}3^{-n}4^{+n}]\, \mathcal{T}_{b_j}[1^{+n}2^{-n}4^{-n}3^{+n}] \Big\}
\end{aligned}
$$

Li, Hang, HJH, He, JHEP(2022), 2111.12042

KK Strings: Massive KLT Relations

➢ **We derive Extended Massive KLT-like relations:**

- Massive double copy? $\Rightarrow$ KK massive KLT relation for gauge/gravity $\Rightarrow$ Stringy generalization of KLT under toroidal compactification



- In $\mathbb{R}^{1,24} \times S^1$, $\{X^4, \cdots, X^{24}\}$ form a compact space small enough to have avoided experimental detection, $R_j = \mathcal{O}(M_{\rm Pl}^{-1}) \ll R_{25} \equiv R$
- Mass spectrums

$$M_{\rm cl}^2 = \frac{\mathfrak{n}^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2)\,, \qquad M_{\rm op}^2 = \left(\frac{\mathfrak{n} L + 2\pi w R}{2\pi\alpha'}\right)^2 + \frac{N-1}{\alpha'}\,, \quad \mathfrak{n} \in \mathbb{Z}$$

  - For (KK) gravitons/gauge bosons: $w = 0$ and $N = \widetilde{N} = 1$
  - Matching condition: setting $RL = 2\pi\alpha'$ in $M_{\rm op}^2$ and rescaling $(\alpha', R, L) \to \frac{1}{4}(\alpha', R, L)$ as well, such that $M_{\rm op}^2 = M_{\rm cl}^2$

Li, Hang HJH, He, JHEP (2022), 2111.12042

KK Strings: Massive KLT Relations

- **We derive Extended Massive KLT-like relations:**

- From KK string amplitude to KK gauge/graviton amplitude

$$
\boxed{
\begin{aligned}
\mathcal{A}_{\rm cl}^{(N)} &= g_{\rm cl}^{N-2}\left(\frac{32\pi}{\alpha'^3}\right)\left(\frac{1}{2}\right)^{\overline{N}/2}\left(-\frac{\pi\alpha'}{2}\right)^{N-3}\sum_{\{\lambda_j,\lambda'_j\}}\sum_{\{\mathrm{sgn}_j\}}\sum_{\{\alpha,\beta\}}\prod_{j=1}^{N}\Big\{\mathcal{C}^{\sigma_j}_{\lambda_j\lambda'_j}\,\mathcal{S}^{\alpha'}_{\rm ST}[\alpha|\beta]_{\hat{k}_1}\\
&\quad\times\mathcal{A}_{\rm op}^{(N)}[\zeta_j^{\lambda_j},\hat{k}_j|1,\alpha(N-3),N-1,N]\,\mathcal{A}_{\rm op}^{(N)}[\zeta_j^{\lambda'_j},\hat{k}_j|N-1,N,\beta(N-3),1]\Big\}\Big|_{\alpha'\to\frac{\alpha'}{4}}\\
&\Downarrow\ \alpha'\to 0\\
\mathcal{M}_{\rm GR}^{(N)} &= \left(\frac{\kappa}{4}\right)^{N-2}\left(\frac{1}{2}\right)^{\overline{N}/2}(-)^{N+1}\sum_{\{\lambda_j,\lambda'_j\}}\sum_{\{\mathrm{sgn}_j\}}\sum_{\{\alpha,\beta\}}\prod_{j=1}^{N}\Big\{\mathcal{C}^{\sigma_j}_{\lambda_j\lambda'_j}\,\mathcal{S}_{\rm FT}[\alpha|\beta]_{\hat{k}_1}\\
&\quad\times\mathcal{T}_{\rm YM}^{(N)}[\zeta_j^{\lambda_j},\hat{k}_j|1,\alpha(N-3),N-1,N]\,\mathcal{T}_{\rm YM}^{(N)}[\zeta_j^{\lambda'_j},\hat{k}_j|N-1,N,\beta(N-3),1]\Big\}
\end{aligned}
}
$$

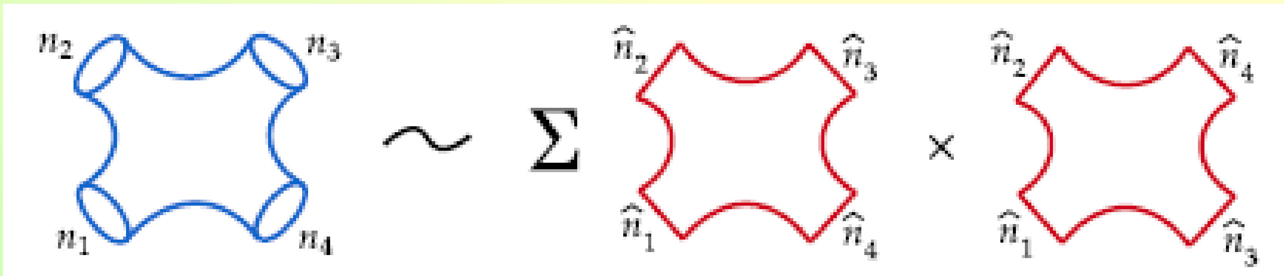
- Graviton polarization tensor can be decomposed to

$$
\zeta^{\sigma_j}_{\mu\nu} = \mathcal{C}^{\sigma_j}_{\lambda_j\lambda'_j}\zeta^{\lambda_j}_{\mu}\zeta^{\lambda'_j}_{\nu}\,; \qquad
\begin{array}{ll}
\mathcal{C}^{\pm 2}_{\pm 1,\pm 1} = 1 & \mathcal{C}^{\pm 1}_{\pm 1,\mathrm{L}} = \mathcal{C}^{\pm 1}_{\mathrm{L},\pm 1} = \sqrt{\frac{1}{2}}\\
\mathcal{C}^{\mathrm{L}}_{\pm 1,\mp 1} = \sqrt{\frac{1}{6}} & \mathcal{C}^{\mathrm{L}}_{\mathrm{L},\mathrm{L}} = \sqrt{\frac{2}{3}}
\end{array}
$$

Li, Hang HJH, He, JHEP (2022), 2111.12042

from KK String to KK Graviton/Gauge Boson

- **We derive Extended Massive KLT-like relations:**



- **New Massive Double-Copy for 4-point Gauge/Gravity Double-Copy (by taking field-theory limit $\alpha' \to 0$):**

$$\begin{aligned}\mathcal{M}[1^n_L 2^n_L 3^n_L 4^n_L] = \frac{\kappa^2}{32}\sum_{\{a_j,b_j\}} \widehat{\varrho}_{ab}\Big\{ & (s-4M_n^2)\,\mathcal{T}_{a_j}[1^{+n}2^{+n}3^{-n}4^{-n}]\,\mathcal{T}_{b_j}[1^{+n}2^{+n}4^{-n}3^{-n}] \\ & + s\,\mathcal{T}_{a_j}[1^{+n}2^{-n}3^{+n}4^{-n}]\,\mathcal{T}_{b_j}[1^{+n}2^{-n}4^{+n}3^{-n}] \\ & + s\,\mathcal{T}_{a_j}[1^{+n}2^{-n}3^{-n}4^{+n}]\,\mathcal{T}_{b_j}[1^{+n}2^{-n}4^{-n}3^{+n}]\Big\}\end{aligned}$$

Li, Hang HJH, He, JHEP (2022), 2111.12042

Massive Double-Copy vs Mass Spectral Condition

- Massless Double-Copy:

$$\mathcal{A} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}, \quad \rightarrow \quad \mathcal{M} = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

- Generalized Gauge Transformations: $n_i \rightarrow n_i + s_i \Delta,$

under which the Invariance is guaranteed:

$$\delta\mathcal{A} = (c_s + c_t + c_u)\Delta = 0$$
$$\delta\mathcal{M} = 2(n_s + n_t + n_u)\Delta + (s+t+u)\Delta^2 = 0$$

$$\mathcal{C}_s + \mathcal{C}_t + \mathcal{C}_u = 0$$
$$n_s + n_t + n_u = 0$$

- We consider the massive KK gauge boson and KK graviton amplitudes:

$$\mathcal{T}\left[A^{a\mathrm{n}_1}_{\lambda_1} A^{b\mathrm{n}_2}_{\lambda_2} A^{c\mathrm{n}_3}_{\lambda_3} A^{d\mathrm{n}_4}_{\lambda_4}\right] = g^2 \sum_j \frac{\mathcal{C}_j \mathcal{N}^{\mathrm{P}}_j(\lambda)}{s_j - M^2_{\mathrm{nn}_j}}$$

$$\mathcal{M}\left[h^{\mathrm{n}_1}_{\sigma_1} h^{\mathrm{n}_2}_{\sigma_2} h^{\mathrm{n}_3}_{\sigma_3} h^{\mathrm{n}_4}_{\sigma_4}\right] = -\frac{\kappa^2}{16} \sum_j \sum_{\lambda_k, \lambda'_k} \left(\prod_{k=1}^{4} C^{\sigma_k}_{\lambda_k \lambda'_k}\right) \frac{\mathcal{N}^{\mathrm{P}}_j(\lambda_k)\mathcal{N}^{\mathrm{P}}_j(\lambda'_k)}{s_j - M^2_{\mathrm{nn}_j}},$$

Li, Hang, HJH, JHEP, 2209.11191

Massive Double-Copy vs Mass Spectral Condition

- **For the massive KK graviton amplitude:**

$$\mathcal{M}\left[h^{\mathrm{n}_1}_{\sigma_1}h^{\mathrm{n}_2}_{\sigma_2}h^{\mathrm{n}_3}_{\sigma_3}h^{\mathrm{n}_4}_{\sigma_4}\right] = -\frac{\kappa^2}{16}\sum_j\sum_{\lambda_k,\lambda'_k}\Big(\prod_{k=1}^{4}C^{\sigma_k}_{\lambda_k\lambda'_k}\Big)\frac{\mathcal{N}^{\mathsf{P}}_j(\lambda_k)\mathcal{N}^{\mathsf{P}}_j(\lambda'_k)}{s_j - M^2_{\mathrm{nn}_j}}\,,$$

we impose the Generalized Gauge Transformation:

$$\mathcal{N}^{\mathsf{P}\,\prime}_j = \mathcal{N}^{\mathsf{P}}_j + \left(s_j - M^2_{\mathrm{nn}_j}\right)\times\Delta\,,$$

from which we deduce conditions:

$$\sum_j \mathcal{N}^{\mathsf{P}}_j = 0, \qquad \sum_j\left(s_j - M^2_{\mathrm{nn}_j}\right) = 0.$$

- **This leads to the 4-point Mass Spectral Condition:** → **Nontrivial !** → **Does Not always hold !**

$$\sum_{i=1}^{4} M^2_{\mathrm{n}_i} = M^2_{\mathrm{nn}_s} + M^2_{\mathrm{nn}_t} + M^2_{\mathrm{nn}_u}$$

Li, Hang, HJH, JHEP, 2209.11191

Massive Double-Copy vs Mass Spectral Condition

- Start from a general 4-point **Mass Spectral Condition**:

$$M_1^2 + M_2^2 + M_3^2 + M_4^2 \;=\; M_{12}^2 + M_{13}^2 + M_{14}^2$$

- We ask: **How to solve it and What is the solution ??**

- For the **flat 5d Toroidal Compactification** of S[1], we have:

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 = (n_1+n_2)^2 + (n_1+n_3)^2 + (n_1+n_4)^2$$

which leads to the condition:

$$n_1 + n_2 + n_3 + n_4 = 0$$

- This is just the **KK number (5d momentum) Conservation**!

- E.g., it does not hold for **5d Orbifold Compactification** or **5d Warped Compactification** ! (additional treatments needed.)

Li, Hang, HJH, JHEP, 2209.11191

Massive Double-Copy vs Mass Spectral Condition

- **Start from a general 4-point Mass Spectral Condition:**

$$M_1^2 + M_2^2 + M_3^2 + M_4^2 = M_{12}^2 + M_{13}^2 + M_{14}^2$$

- **We ask: How to solve it and What is the solution ???**

- **In fact, starting with 3 rather modest conditions:**

1). A massive theory contains at least 2 types of particles with Unequal masses.
2). There exists only a simple pole in each of (s, t, u) channels.
3). Each scattering amplitude should include the contributions from all 3 kinematic channels of (s, t, u).

we can prove:
Toroidal Compactification of flat Extra dimensons is the Unique Solution !

Li, Hang, HJH, JHEP, 2209.11191

Massive Double-Copy vs Mass Spectral Condition

- **General 4-point Mass Spectral Condition:**

$$M_1^2 + M_2^2 + M_3^2 + M_4^2 = M_{12}^2 + M_{13}^2 + M_{14}^2 \qquad \textbf{(1)}$$

- **Proof of the Unique Solution:**

We identified the group structure underlying condition (1) is a product of Integer Groups Z^r (with rank r) in the finitely generated Abelian group:

$$\mathcal{G} \cong \mathbb{Z}^r \oplus \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_s}, \qquad (r, s \in \mathbb{N})$$

- **Then, we proved that the unique solution to Condition (1) is given by** $M_{\{k\}}^2 = k^2 M_{\{1\}}^2\,,$ **for the case r = 1 and by** $\mathrm{M}_{\mathrm{n}}^2 = \mathrm{n}\overline{\mathrm{M}}^2\mathrm{n}^T$ **for the case r>1 .**

- **For the general case of r⩾1. By inspecting all the known consistent QFTs, we conclude that only the KK theories with δ (=r) extra dimensions under the toroidal compactification could have a mass spectrum behave exactly as above. These theories also hold the KK number conservation.**

Li, Hang, HJH, JHEP, 2209.11191

Topological Mass Generation & Scattering Amplitudes
Topological Equivalence Theorem & Double-Copy
for Chern-Simons Gauge & Gravity Theories

Hang, HJH, Shen JHEP(2022), 2110.05399
Research (2023), 2406.13671

3d Topological Massive Gauge & Gravity Theories

- **3d Chern-Simons (CS) Topological Massive Gauge Theories:**

$$\mathcal{L}_{\text{TMQED}} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}\widetilde{m}\,\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\,,$$

$$\mathcal{L}_{\text{TMYM}} = -\frac{1}{2}\text{tr}\mathbf{F}_{\mu\nu}^2 + \widetilde{m}\,\varepsilon^{\mu\nu\rho}\text{tr}\left(\mathbf{A}_\mu\partial_\nu\mathbf{A}_\rho - \frac{\text{i}2g}{3}\mathbf{A}_\mu\mathbf{A}_\nu\mathbf{A}_\rho\right),$$

- **CS mass is geometrized and is related to CS Level:**

$$n = 4\pi\widetilde{m}/g^2 \in \mathbb{Z}$$

- **3d CS Topological Massive Gravity Theories:**

$$S_{\text{TMG}} = -\frac{2}{\kappa^2}\int \text{d}^3x\left[\sqrt{-g}R - \frac{1}{2\widetilde{m}}\varepsilon^{\mu\nu\rho}\Gamma^{\alpha}{}_{\rho\beta}\left(\partial_\mu\Gamma^{\beta}{}_{\alpha\nu} + \frac{2}{3}\Gamma^{\beta}{}_{\mu\gamma}\Gamma^{\gamma}{}_{\nu\alpha}\right)\right]$$

Hang, HJH, Shen, JHEP, arXiv:2110.05399

Topological Mass Generation in 3d Chern-Simons

- **Equation of Motion of CS Gauge Bosons:**

$$\left[\eta^{\mu\nu}\partial^2+(\xi^{-1}-1)\partial^\mu\partial^\nu+\widetilde{m}\,\varepsilon^{\mu\rho\nu}\partial_\rho\right]A^a_\nu \,=\, 0\,,$$

$$(m\eta^{\mu\nu}\,-\,\mathrm{i}\mathfrak{s}\varepsilon^{\mu\rho\nu}p_\rho)\,\epsilon_\nu(p)\,=\,0$$

- **On-shell gauge boson has 1 Physical Polarization:**

$$\epsilon^\mu_{\mathrm{P}}(p)=\frac{1}{\sqrt{2}}(\bar{E}\beta,\,\bar{E}s_\theta+\mathrm{i}\mathfrak{s}c_\theta,\,\bar{E}c_\theta-\mathrm{i}\mathfrak{s}s_\theta)\,, \qquad \bar{E}=E/m$$

and 2 unphysical d.o.f with polarization vectors:

$$\epsilon^\mu_{\mathrm{T}}=(0,\,\mathrm{i}\mathfrak{s}c_\theta,\,-\mathrm{i}\mathfrak{s}s_\theta),\qquad \epsilon^\mu_{\mathrm{L}}=\bar{E}(\beta,\,s_\theta,\,c_\theta)\,.$$

- **We find a new decomposition:**

$$\epsilon^\mu_{\mathrm{P}}=\frac{1}{\sqrt{2}}(\epsilon^\mu_{\mathrm{T}}+\epsilon^\mu_{\mathrm{L}}) \longrightarrow A^a_{\mathrm{P}}=\frac{1}{\sqrt{2}}(A^a_{\mathrm{T}}+A^a_{\mathrm{L}})$$

Hang, HJH, Shen, JHEP, arXiv:2110.05399

Topological Mass Generation

- Conversion of physical d.o.f between m = 0 and m ≠ 0 :

$$A_{\mathrm{P}}^{a}=\frac{1}{\sqrt{2}}\left(A_{\mathrm{T}}^{a}+A_{\mathrm{L}}^{a}\right) \longrightarrow A_{\mathrm{T}}^{a}$$

- 2 orthogonal unphysical states:

$$\begin{aligned} A_{\mathrm{X}}^{a} &= \epsilon_{\mathrm{X}}^{\mu} A_{\mu}^{a}=\frac{1}{\sqrt{2}}\left(A_{\mathrm{T}}^{a}-A_{\mathrm{L}}^{a}\right), \\ A_{\mathrm{S}}^{a} &= \epsilon_{\mathrm{S}}^{\mu} A_{\mu}^{a}, \end{aligned}$$

- The physical d.o.f is conserved under m → 0 limit:

1 = 1

- Then, we ask:

$$\mathcal{T}\left[A_{\mathrm{P}}^{a_1}, \cdots, A_{\mathrm{P}}^{a_N}, \Phi\right] \stackrel{?}{=} \mathcal{T}\left[\tilde{A}_{\mathrm{T}}^{a_1}, \cdots, \tilde{A}_{\mathrm{T}}^{a_N}, \Phi\right]$$

Hang, HJH, Shen, JHEP, arXiv:2110.05399

Topological Mass Generation: TET

- **Indeed we can derive a new identity to connect the Scattering Amplitudes:**

$$\mathcal{T}[A_{\mathrm{P}}^{a_1},\cdots,A_{\mathrm{P}}^{a_N},\Phi] = \mathcal{T}[\tilde{A}_{\mathrm{T}}^{a_1},\cdots,\tilde{A}_{\mathrm{T}}^{a_N},\Phi] + \mathcal{T}_v\,,$$

$$\mathcal{T}_v = \sum_{j=1}^{N} \mathcal{T}[\tilde{v}^{a_1},\cdots,\tilde{v}^{a_j},\tilde{A}_{\mathrm{T}}^{a_{j+1}},\cdots,\tilde{A}_{\mathrm{T}}^{a_N},\Phi]\,,$$

- **Under high energy expansion, we derive, at S-matrix level, Topological Equivalence Theorem (TET):**

$$\mathcal{T}[A_{\mathrm{P}}^{a_1},\cdots,A_{\mathrm{P}}^{a_N},\Phi] = \mathcal{T}[\tilde{A}_{\mathrm{T}}^{a_1},\cdots,\tilde{A}_{\mathrm{T}}^{a_N},\Phi] + \mathcal{O}\!\left(\frac{m}{E}\right),$$

Hang, HJH, Shen, JHEP, arXiv:2110.05399

Generalized Power Counting for 3d Chern-Simons

- **Generalized Energy Power Counting for 3d CS physical gauge boson scattering amplitudes:**

$$D_E = (\mathcal{E}_{A_p} - \mathcal{E}_v) + (4 - \mathcal{E} - \overline{\mathcal{V}}_3) - L\,,$$

- **Large E Cancellations in N physical A_p amplitudes by TET:**

$$\Delta D_E = D_E[NA^a_P] - D_E[NA^a_T] = N$$

- **E-Power Counting of N physical Graviton Amplitudes:**

$$D_E = 2\mathcal{E}_{h_p} + (2 + \mathcal{V}_{d3} + L) = 3N + L$$

Hang, HJH, Shen, JHEP, arXiv:2110.05399

Topological Mass Generation: TET

- **Indeed we can derive a ST-type new identity:**

$$\mathcal{T}[A_{\mathrm{P}}^{a_1},\cdots,A_{\mathrm{P}}^{a_N},\Phi] \,=\, \mathcal{T}[\tilde{A}_{\mathrm{T}}^{a_1},\cdots,\tilde{A}_{\mathrm{T}}^{a_N},\Phi] \,+\, \mathcal{T}_v\,,$$
$$\mathcal{T}_v \,=\, \sum_{j=1}^{N}\mathcal{T}[\tilde{v}^{a_1},\cdots,\tilde{v}^{a_j},\tilde{A}_{\mathrm{T}}^{a_{j+1}},\cdots,\tilde{A}_{\mathrm{T}}^{a_N},\Phi]\,,$$

- **Topological Equivalence Theorem (TET) at S-matrix level:**

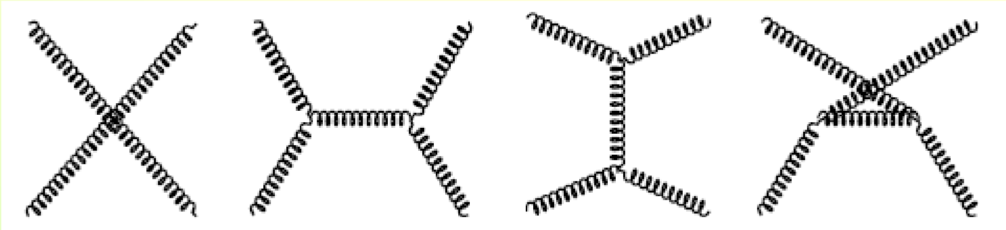
$$\mathcal{T}[A_{\mathrm{P}}^{a_1},\cdots,A_{\mathrm{P}}^{a_N},\Phi] \,=\, \mathcal{T}[\tilde{A}_{\mathrm{T}}^{a_1},\cdots,\tilde{A}_{\mathrm{T}}^{a_N},\Phi] + \mathcal{O}\left(\frac{m}{E}\right),$$

- **Energy Cancellations for *N* point gauge boson amplitudes:**

$$E^4 \,\rightarrow\, E^{4-N}$$

Hang and HJH, PRD, arXiv:2106.04568 + 2207.11214

E Cancellations of Gluon Amplitudes in CS QCD



Amplitude	$\times\bar{s}_0^2$	$\times\bar{s}_0^{3/2}$	$\times\bar{s}_0$	$\times\bar{s}_0^{1/2}$
$\mathcal{T}_{cs}$	$8s_\theta\,\mathcal{C}_s$	$\mathrm{i}32s_\theta\,\mathcal{C}_s$	$64c_\theta\,\mathcal{C}_s$	$\mathrm{i}64s_\theta\,\mathcal{C}_s$
$\mathcal{T}_{ct}$	$-(5+4c_\theta-c_{2\theta})\,\mathcal{C}_t$	$-\mathrm{i}8(2s_\theta-s_{2\theta})\,\mathcal{C}_t$	$-32(c_\theta-c_{2\theta})\,\mathcal{C}_t$	$-\mathrm{i}16(2s_\theta-5s_{2\theta})\,\mathcal{C}_t$
$\mathcal{T}_{cu}$	$(5-4c_\theta-c_{2\theta})\,\mathcal{C}_u$	$-\mathrm{i}8(2s_\theta+s_{2\theta})\,\mathcal{C}_u$	$-32(c_\theta+c_{2\theta})\,\mathcal{C}_u$	$-\mathrm{i}16(2s_\theta+5s_{2\theta})\,\mathcal{C}_u$
$\mathcal{T}_s$	$-8s_\theta\,\mathcal{C}_s$	$-\mathrm{i}56s_\theta\,\mathcal{C}_s$	$-192c_\theta\,\mathcal{C}_s$	$-\mathrm{i}368s_\theta\,\mathcal{C}_s$
$\mathcal{T}_t$	$(5+4c_\theta-c_{2\theta})\,\mathcal{C}_t$	$-\mathrm{i}8(s_\theta+s_{2\theta})\,\mathcal{C}_t$	$-32(3c_\theta+c_{2\theta})\,\mathcal{C}_t$	$-\mathrm{i}16(17s_\theta+5s_{2\theta})\,\mathcal{C}_t$
$\mathcal{T}_u$	$-(5-4c_\theta-c_{2\theta})\,\mathcal{C}_u$	$-\mathrm{i}8(s_\theta-s_{2\theta})\,\mathcal{C}_u$	$-32(3c_\theta-c_{2\theta})\,\mathcal{C}_u$	$-\mathrm{i}16(17s_\theta-5s_{2\theta})\,\mathcal{C}_u$
Sum	0	0	0	0

$$\mathcal{T}[A_{\mathrm{P}}^a A_{\mathrm{P}}^b \to A_{\mathrm{P}}^c A_{\mathrm{P}}^d] = \mathcal{T}[A_{\mathrm{T}}^a A_{\mathrm{T}}^b \to A_{\mathrm{T}}^c A_{\mathrm{T}}^d] + \mathcal{O}\left(\frac{m}{E}\right)$$

→ **E-Cancellations:** $E^4 \to E^0$

Hang, HJH, Shen, JHEP, arXiv:2110.05399

Topological Massive Gravity from Double-Copy

Hang, HJH, Shen, JHEP(2022), 2110.05399

- **Topological Massive Gravity in 3d:** (Deser, Jackiw, Templeton 1982)

$$S_{\mathrm{TMG}} = \frac{2}{\kappa^2}\int \mathrm{d}^3x \left[\sqrt{-g}R - \frac{1}{2\widetilde{m}}\varepsilon^{\mu\nu\rho}\Gamma^{\alpha}{}_{\rho\beta}\left(\partial_\mu \Gamma^{\beta}{}_{\alpha\nu} + \frac{2}{3}\Gamma^{\beta}{}_{\mu\gamma}\Gamma^{\gamma}{}_{\nu\alpha}\right)\right]$$

- **3d Massive Graviton has 1 physical degree of freedom.**

- **Power Counting on E dependence of N Physical Graviton Amplitude:**

$$D_E = 2\mathcal{E}_{h_{\mathrm{p}}} + (2 + \mathcal{V}_{d3} + L)$$

- **We deduce the E-dependence of N-point graviton amplitude:**

$$\longrightarrow \quad D_E^0 = 3\,\mathcal{E}_{h_p}\,.$$

- $\mathcal{E}_{h_{\mathrm{P}}}$: # of external $h_{\mathrm{P}} = (\zeta_{\mathrm{P}}^{\mu\nu} h_{\mu\nu})$
- $\mathcal{V}_{d3}$: # vertices containing three partial derivatives

Topological Massive Gravity from Double-Copy

Hang, HJH, Shen, JHEP(2022), 2110.05399

- **Topological Massive Gravity (TMG):**

$$S_{\rm TMG} = -\frac{2}{\kappa^2}\int \mathrm{d}^3x \left[\sqrt{-g}R - \frac{1}{2\widetilde{m}}\varepsilon^{\mu\nu\rho}\Gamma^{\alpha}{}_{\rho\beta}\left(\partial_\mu \Gamma^{\beta}{}_{\alpha\nu} + \frac{2}{3}\Gamma^{\beta}{}_{\mu\gamma}\Gamma^{\gamma}{}_{\nu\alpha}\right)\right],$$

- **Graviton Amplitude from Double-Copy:** ($\epsilon_{\mu\nu} = \epsilon_\mu \epsilon_\nu$)

$$\mathcal{M}[4h_{\rm P}] = \frac{\kappa^2}{16}\left[\frac{(\mathcal{N}'_s)^2}{s-m^2} + \frac{(\mathcal{N}'_t)^2}{t-m^2} + \frac{(\mathcal{N}'_u)^2}{u-m^2}\right]$$

- **Under high energy expansion:**

$$\mathcal{M}_0[4h_{\rm P}] = -\frac{\mathrm{i}\kappa^2}{2048}\, m\, s_0^{\frac{1}{2}}\,(494c_\theta + 19c_{3\theta} - c_{5\theta})\csc^3\theta$$

- **Energy Cancellations for 4 graviton scattering amplitudes:**

$$\mathcal{O}(E^{12}) \longrightarrow \mathcal{O}(E^{1})\,, \qquad (\text{for } \mathcal{E}_{h_{\rm P}} = 4 \text{ in 3d TMG})$$

Topological Massive Gravity from Double-Copy

Hang, HJH, Shen, JHEP, 2110.05399

- E Cancellations: $E^4 \to E^1$

Amplitude	$\times \bar{s}_0^2$	$\times \bar{s}_0^{3/2}$	$\times \bar{s}_0$
$\mathcal{M}_s$	$-\frac{99+28c_{2\theta}+c_{4\theta}}{1-c_{2\theta}}$	$-\mathrm{i}14(15c_\theta+c_{3\theta})\csc\theta$	$-\frac{2(75+326c_{2\theta}+47c_{4\theta})}{1-c_{2\theta}}$
$\mathcal{M}_t$	$\frac{99+28c_{2\theta}+c_{4\theta}}{4(1-c_\theta)}$	$\mathrm{i}(102+105c_\theta+70c_{2\theta}+7c_{3\theta}+4c_{4\theta})\csc\theta$	$\frac{75-107c_\theta+326c_{2\theta}+268c_{3\theta}+47c_{4\theta}+31c_{5\theta}}{1-c_{2\theta}}$
$\mathcal{M}_u$	$\frac{99+28c_{2\theta}+c_{4\theta}}{4(1+c_\theta)}$	$\mathrm{i}(-102+105c_\theta-70c_{2\theta}+7c_{3\theta}-4c_{4\theta})\csc\theta$	$\frac{75+107c_\theta+326c_{2\theta}-268c_{3\theta}+47c_{4\theta}-31c_{5\theta}}{1-c_{2\theta}}$
Sum	0	0	0

- Under high energy expansion:

$$\mathcal{M}_0[4h_\mathrm{P}] = -\frac{\mathrm{i}\kappa^2}{2048}\, m\, s_0^{\frac{1}{2}}\,(494c_\theta+19c_{3\theta}-c_{5\theta})\,\csc^3\theta$$

- Energy Cancellations for 4 graviton scattering amplitudes:

$E^4 \to E^0$ $\longrightarrow$ $\mathcal{O}(E^{12}) \longrightarrow \mathcal{O}(E^1)\,, \qquad (\text{for } \mathcal{E}_{h_\mathrm{P}} = 4 \text{ in 3d TMG})$

Topological Massive Gravity from Double-Copy

Hang, HJH, Shen, JHEP, 2110.05399

- Energy cancellations, $j \in (s, t, u)$

$$E^{12} \xrightarrow[\text{CK duality}]{\sim\mathcal{N}_j^2/s_j} E^{8} \xrightarrow[\text{Gauge trans.}]{\sim(\mathcal{N}_j')^2/s_j} E^{4} \xrightarrow{\sum_j \mathcal{M}_j} E^{1}$$

- LO amplitude: $\mathcal{M}_0[4h_\mathrm{P}] = \kappa^2 m s_0^{1/2} f(\theta) \sim \mathcal{O}(E^1)$

Topological Massive Gravity from Double-Copy

- **Energy Cancellations for 4 graviton scattering amplitudes:**

$$\mathcal{O}(E^{12}) \longrightarrow \mathcal{O}(E^{1}), \qquad (\text{for } \mathcal{E}_{h_\mathrm{p}} = 4 \text{ in 3d TMG})$$

- **Correspondence between 2 Energy Cancellations:**

$E^4 \to E^0$ → $E^{12} \to E^1$

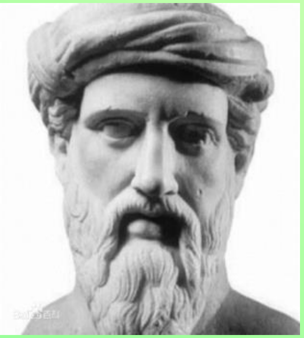
(3d CS Gauge Theory) **(3d CS Gravity Theory)**

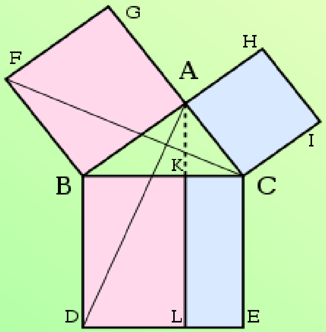
- **Graviton Amplitude has nontrivial 2-step E Cancellations:**

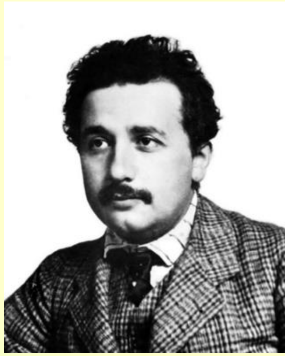
$$\mathbf{E^{12} \rightarrow E^{4}} \quad \textbf{and} \quad \mathbf{E^{4} \rightarrow E^{1}}$$

Hang, HJH, Shen, JHEP, arXiv:2110.05399

Deep Relations: Importance of Equality & Square Law







- [EG] Pythagoras Theorem: $a^2 + b^2 = c^2$ → Fermat Last Theorem:

  $a^n + b^n \neq c^n$ ($n>2$)
- [SR] Mass-Energy Equivalence: $E = Mc^2$ and $P_\mu^{\,2} = M^2c^2$
- [GR] Inertial Mass vs Gravitational-Mass Equivalence: $M_G = M_I$ → $g = a$
- Gravity Force from Gauge Force: (valid for $M = 0$ and $M \neq 0$)

$$(\text{Gravity}) = (\text{Gauge Force})^2$$

$M = 0$ (Conventional works) → ← $M \neq 0$ (This Talk)


→ Much more can be explored !
Thank You