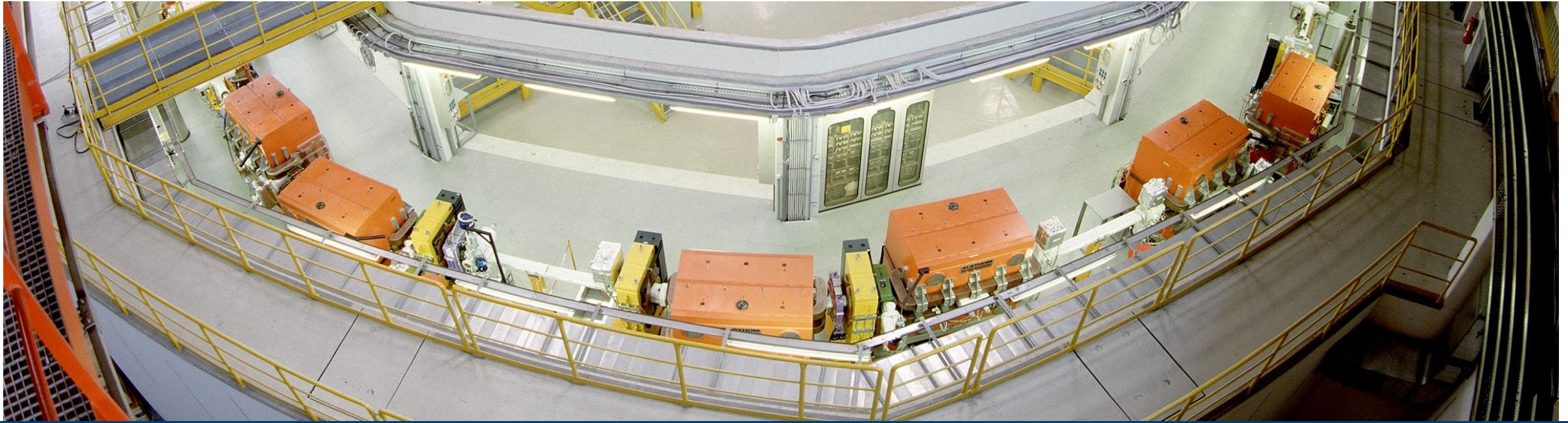




THE FIRST DIRECT MEASUREMENT OF THE DEUTERON ELECTRIC DIPOLE MOMENT AT THE COOLER SYNCHROTRON COSY



26th International Symposium on Spin Physics (SPIN2025)

23.09.2025 | ACHIM ANDRES



MATTER ANTIMATTER ASYMMETRY

- Big Bang: Equal amount of matter and antimatter

$$N_B = N_{\bar{B}}$$

- Early Universe:

Theory: $B + \bar{B} \rightarrow \gamma + \gamma + \dots$

Measurement: $B + \bar{B} \rightarrow \gamma + \gamma + \dots + B + \dots?$

- Today: Asymmetry between matter and antimatter

$$\eta = \frac{N_B - N_{\bar{B}}}{N_\gamma}$$

- Mismatch between expectation and measurement

$$\eta^{\text{SCM}} \approx 10^{-18} \quad \text{versus} \quad \eta^{\text{meas}} \approx 10^{-10}$$



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- Andrei Sakharov (1976)
 - Baryon Number Violation
 - **C and CP Violation**
 - Deviation from thermal equilibrium

ELECTRIC DIPOLE MOMENTS

- EDM is a **vectorial** property aligned with **the particles' spin**
- Magnetic Dipole Moment (MDM): $\vec{\mu} = \mu \cdot \vec{s}$ with $\mu = g \frac{q}{2m}$
- Electric Dipole Moment (EDM): $\vec{d} = d \cdot \vec{s}$ with $d = \eta^{\text{EDM}} \frac{q}{2mc}$

ELECTRIC DIPOLE MOMENTS AND CP VIOLATION

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$$H = -\vec{d} \cdot \vec{E} - \vec{\mu} \cdot \vec{B}$$

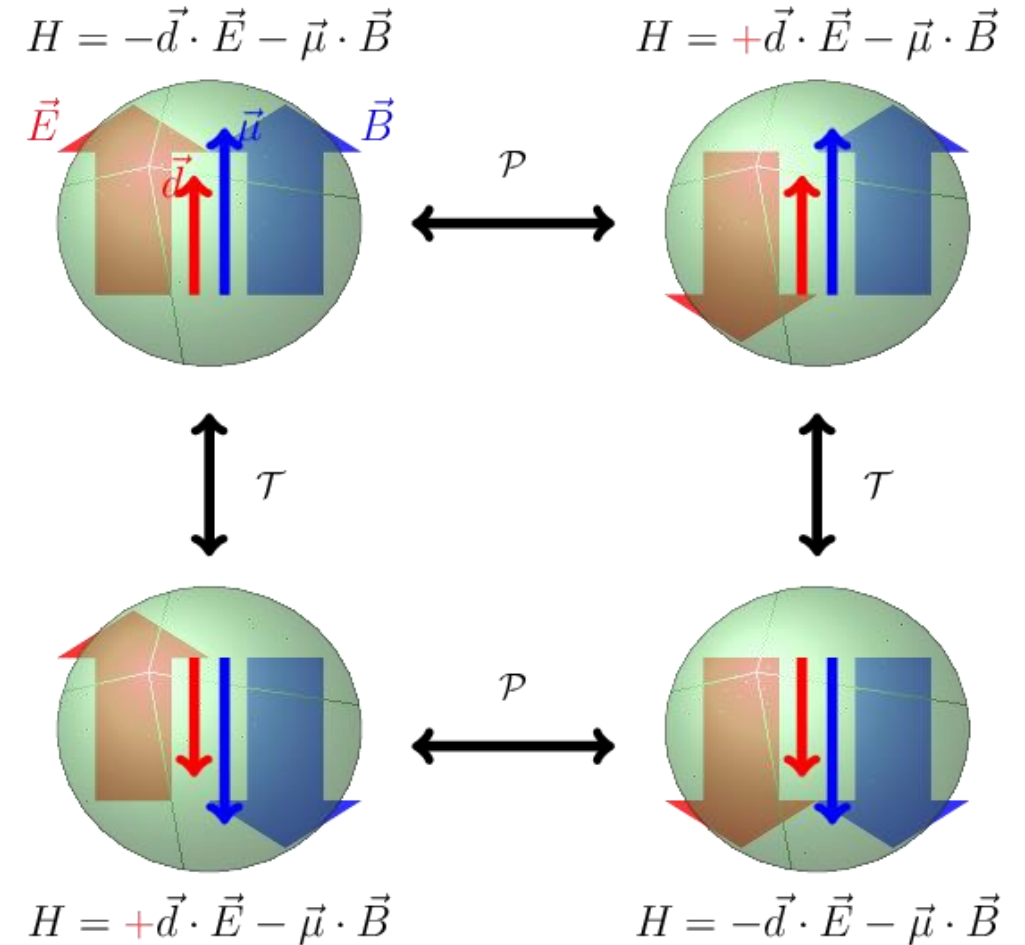
$$\text{Parity } P: H = +\vec{d} \cdot \vec{E} - \vec{\mu} \cdot \vec{B}$$

$$\text{Time } T: H = +\vec{d} \cdot \vec{E} - \vec{\mu} \cdot \vec{B}$$

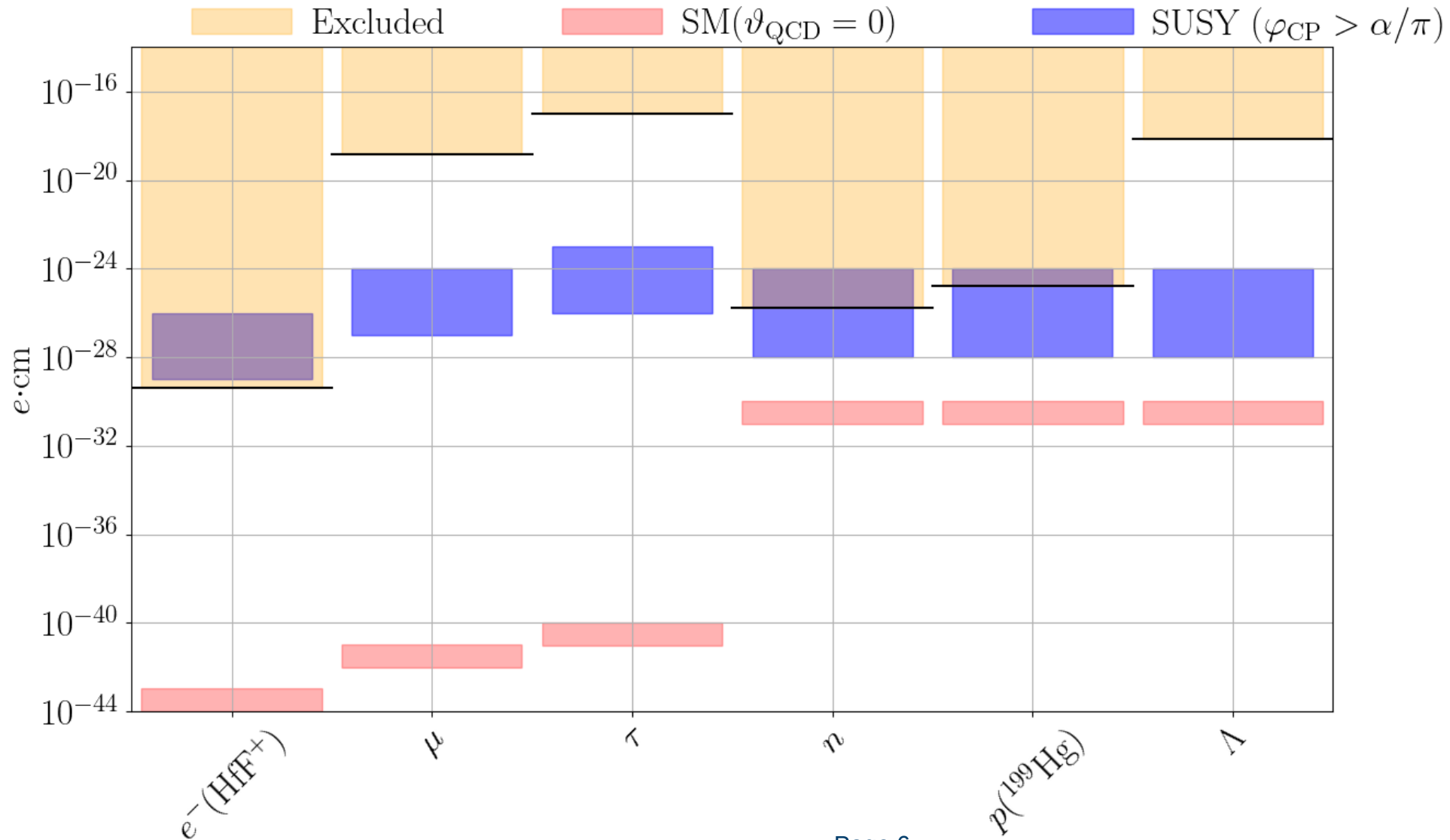
- According to CPT - Theorem:

$$T \text{ Violation} = CP \text{ Violation}$$

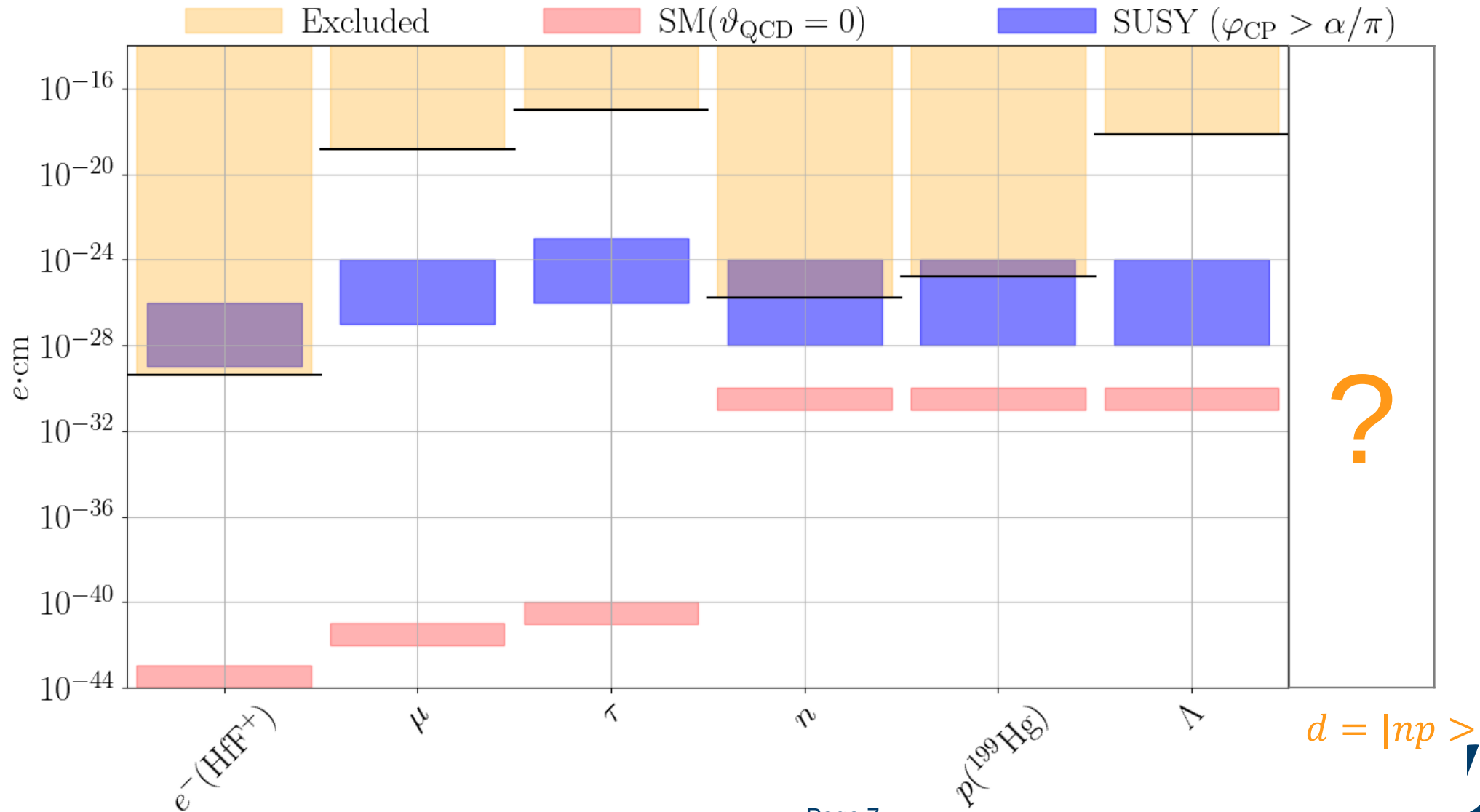
⇒ EDM violates both P and CP symmetry



PERMANENT EDM SEARCH



PERMANENT EDM SEARCH



SPIN DYNAMICS

Thomas – BMT Equation

$$\vec{d} = d \frac{\vec{s}}{|\vec{s}|} \text{ with } d = \eta^{\text{EDM}} \frac{q\hbar}{2mc}$$

- Measure the influence of the EDM on the spin motion

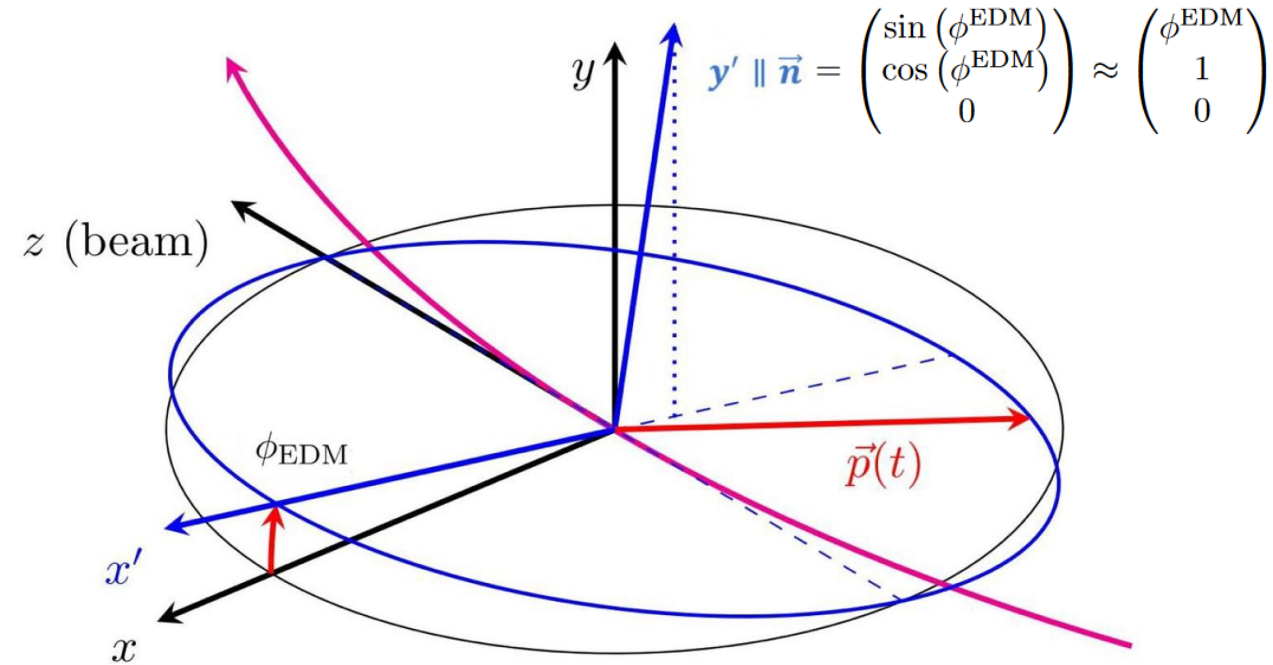
$$\frac{d\vec{S}}{dn} = (\vec{\Omega}_{\text{MDM}} + \vec{\Omega}_{\text{EDM}}) \times \vec{S}$$

$$\vec{\Omega}_{\text{MDM}} = -2\pi\gamma G \vec{e}_y \rightarrow f_{\text{MDM}} = 120 \text{ kHz}$$

$$\vec{\Omega}_{\text{EDM}} = \eta^{\text{EDM}} \gamma \beta \vec{e}_x$$

$$\tan\left(\frac{|\vec{\Omega}_{\text{EDM}}|}{|\vec{\Omega}_{\text{MDM}}|}\right) \approx \phi_{\text{EDM}} \approx -\eta^{\text{EDM}} \frac{2\beta}{G}$$

G_d	-0.143
G_μ	0.001



SPIN DYNAMICS

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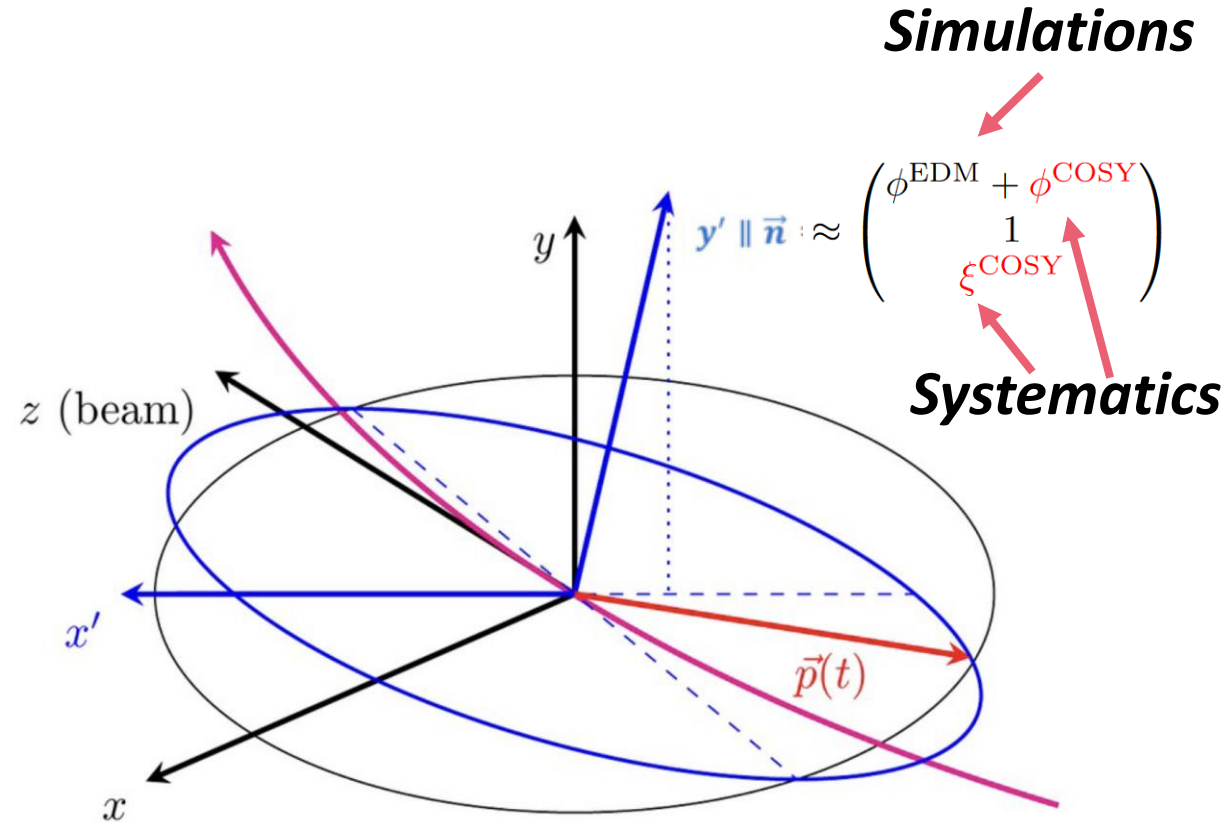
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- **Problem:** Ring **imperfections** (magnet misalignments,..)

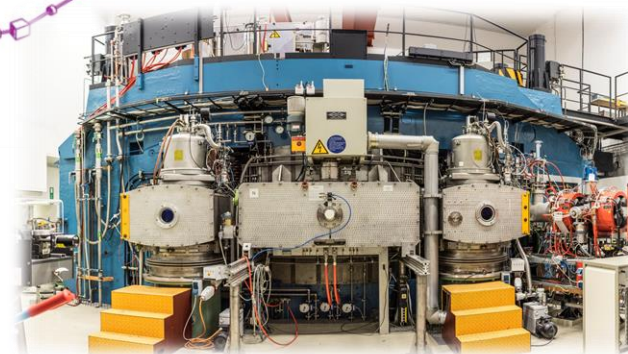
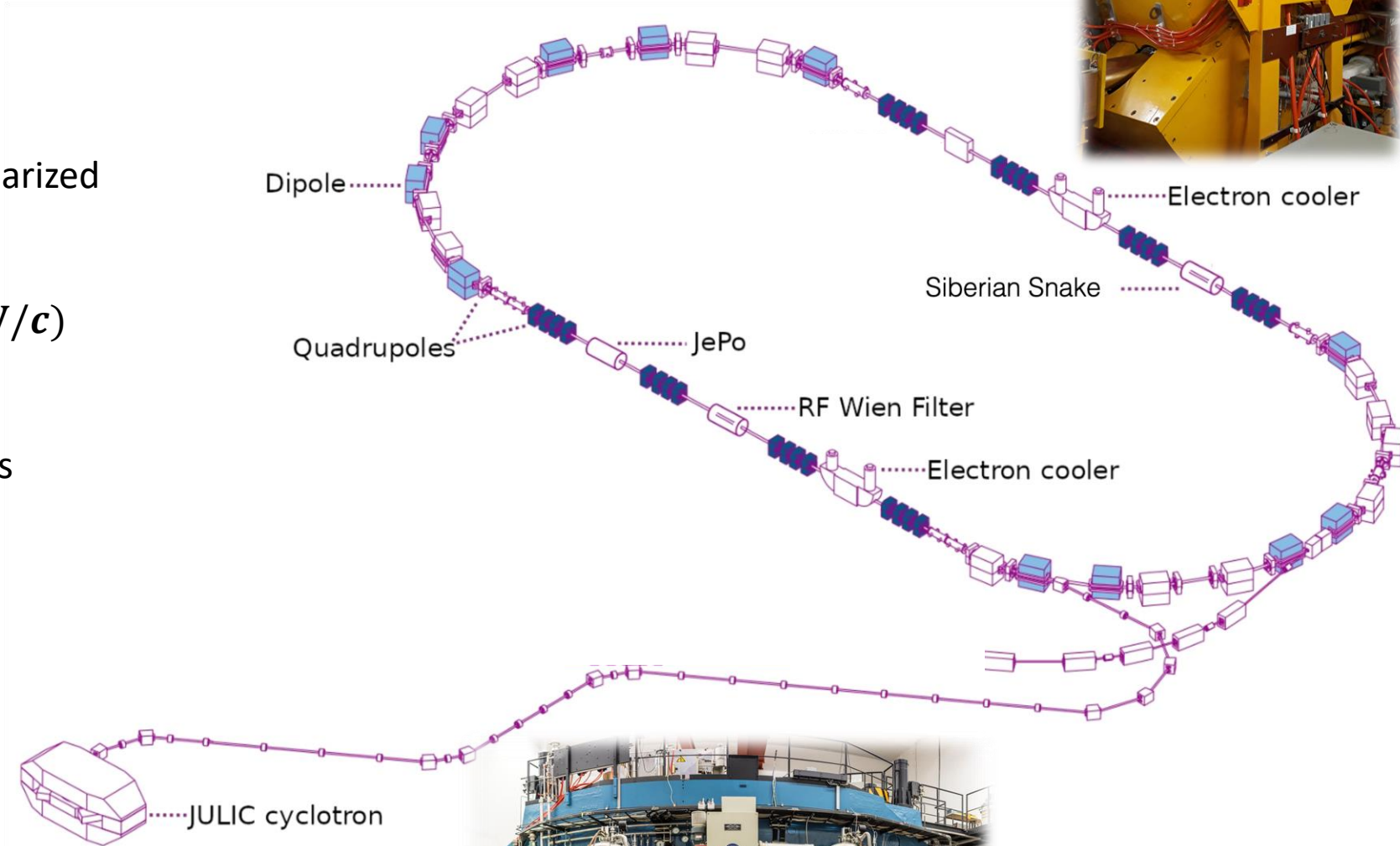
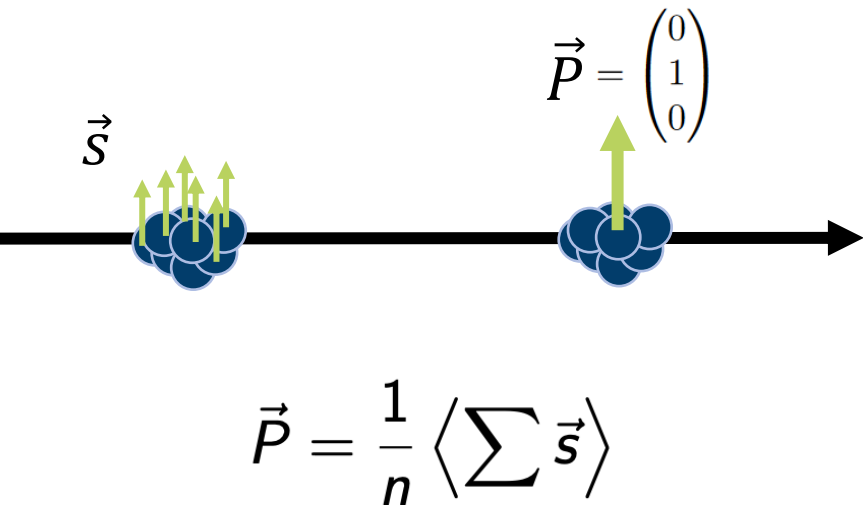
lead to rotations of $\hat{n}$ in **radial** (x) and **longitudinal** (z) direction



COSY - COOLER SYNCHROTRON (1993 – 2023)

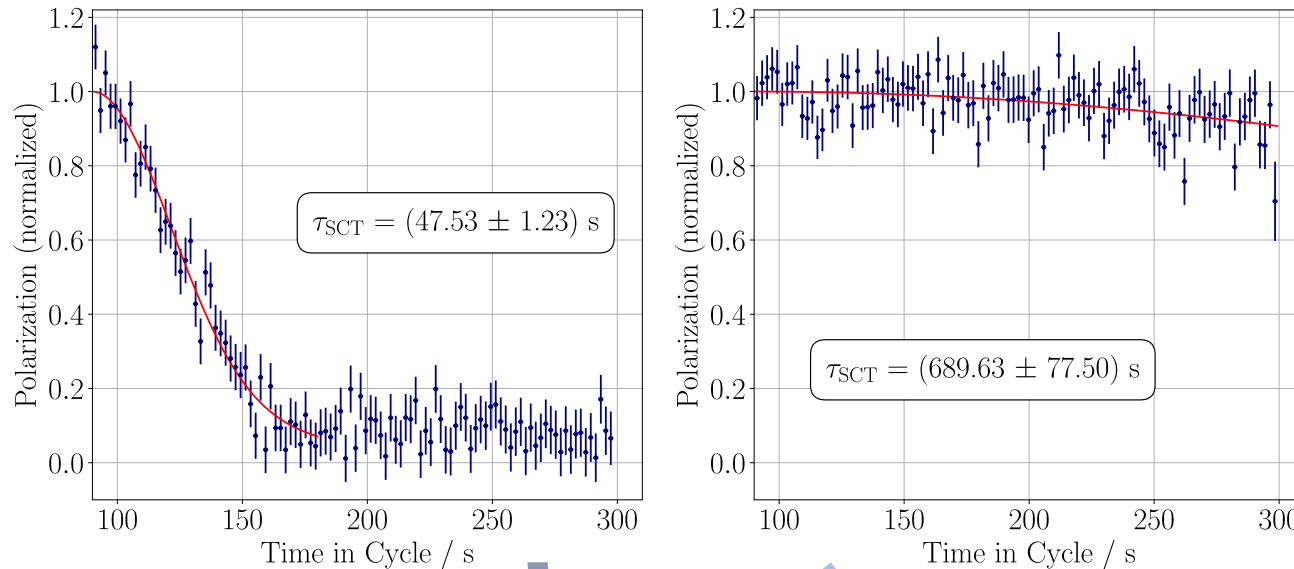
Overview

- Circumference 184 m
- Accelerate and Store **Polarized** / Unpolarized **Deuterons** and Protons
- $p = 0.3 - 3.7 \text{ GeV}/c$ ($p_d = 970 \text{ MeV}/c$)
- Excellent Beam Quality
- Hadron Physics / **Precision** Experiments

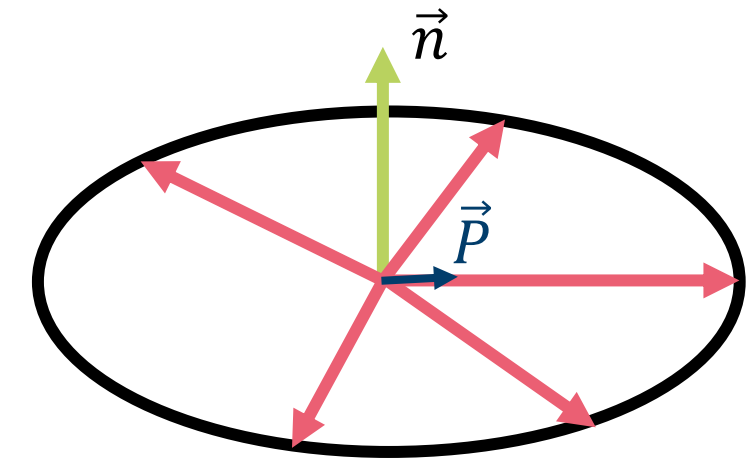
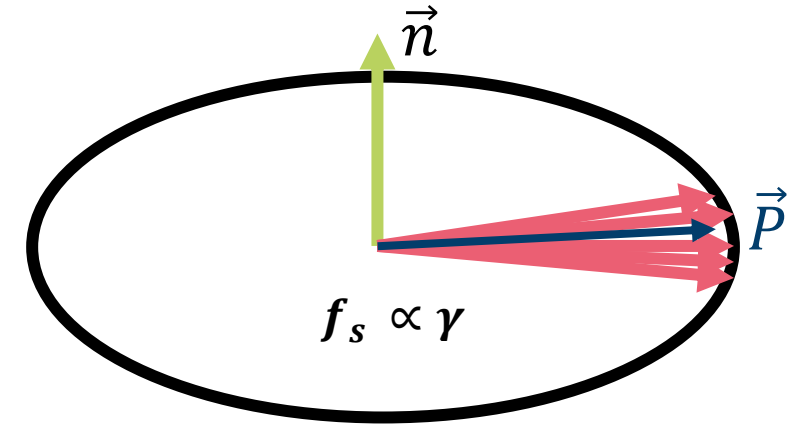


ACHIEVEMENTS AT COSY

- A **spread** in particles energy distribution leads to **decoherence** over time
- Measure 120 kHz spin tune precession to 10^{-10} in 100 s
- Development of polarization feed back system
- RF Wien filter (Single bunch spin manipulation)

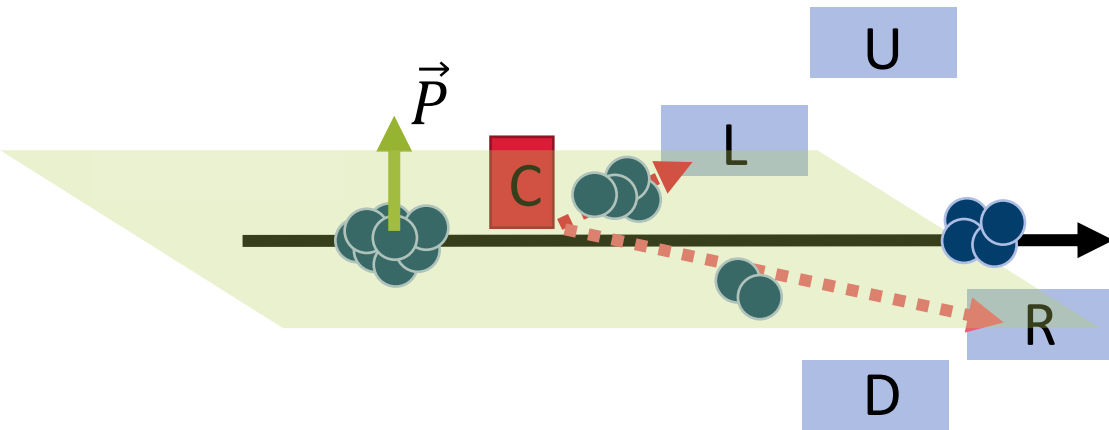


sextupole optimization



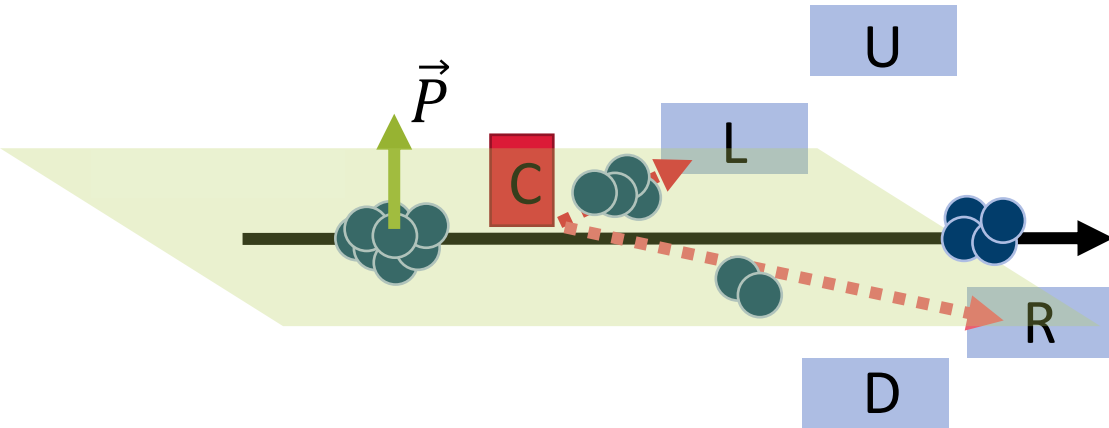
POLARIMETER

$$p_y \propto \frac{N_L - N_R}{N_L + N_R}$$



POLARIMETER

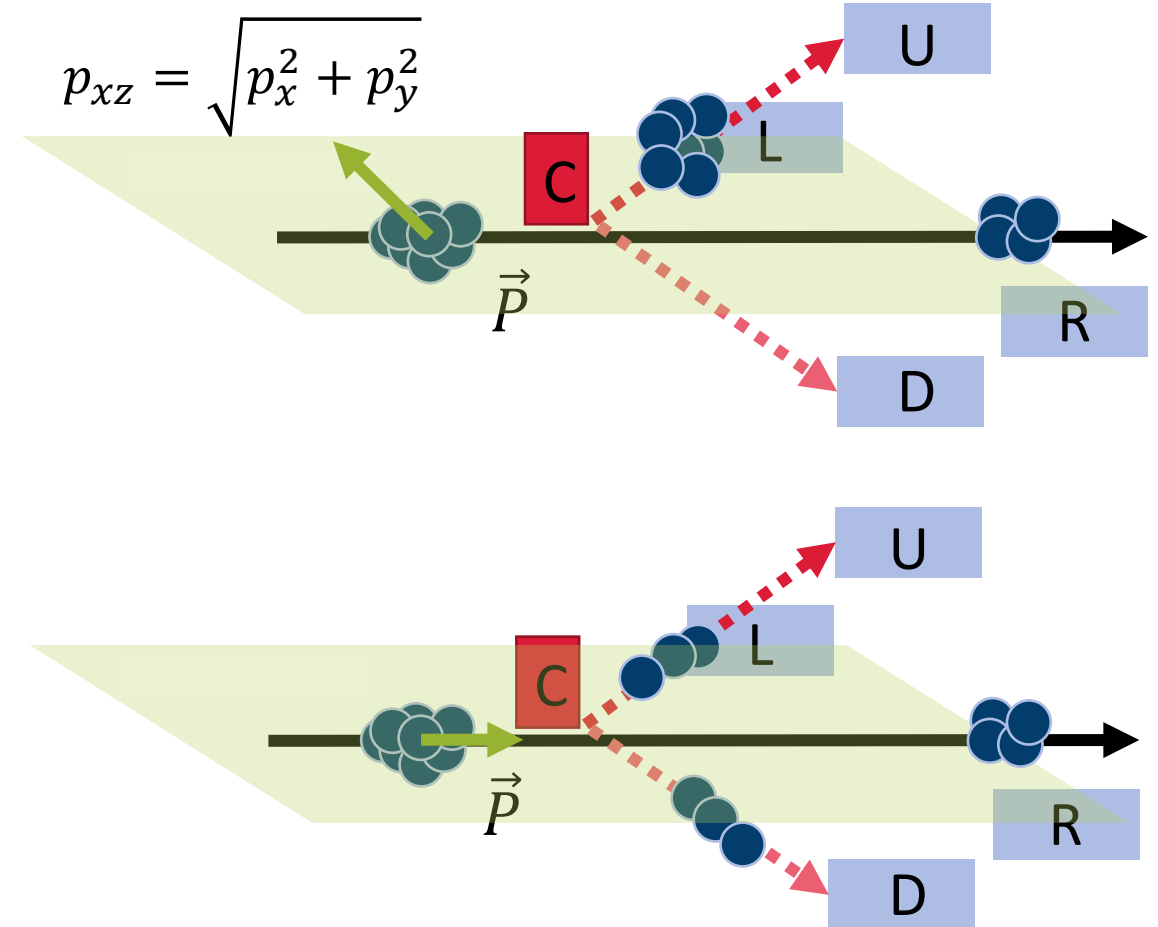
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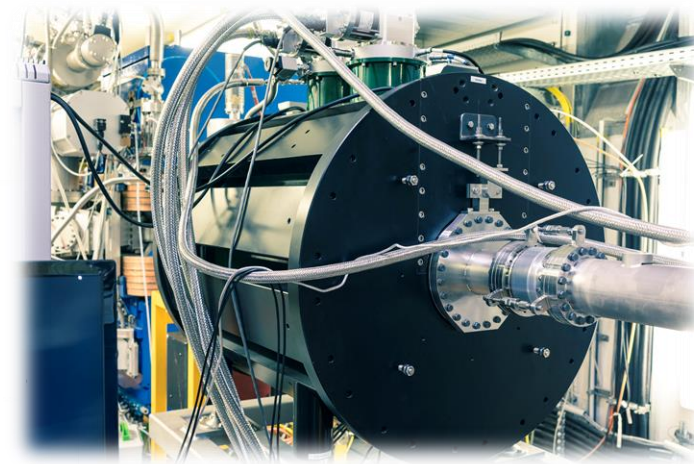
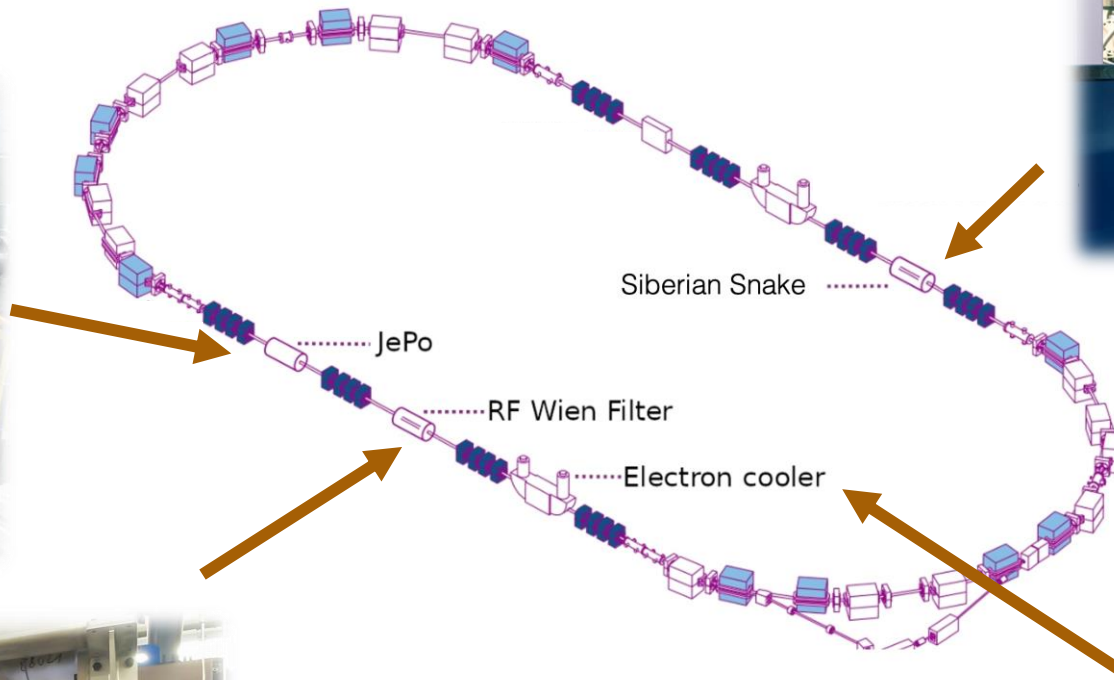
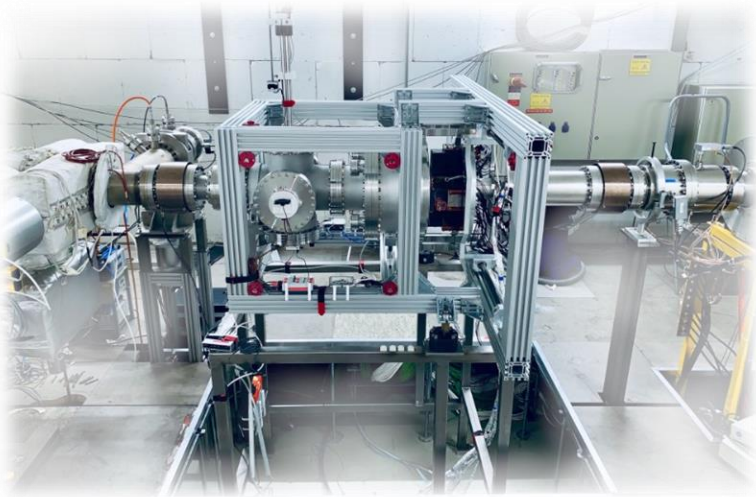
$$N_U \propto p_{xz} \cdot \sin(2\pi f_s t)$$

$$N_D \propto p_{xz} \cdot \sin(2\pi f_s t + \pi)$$

$$p_{xz} \propto \frac{N_U - N_D}{N_U + N_D}$$

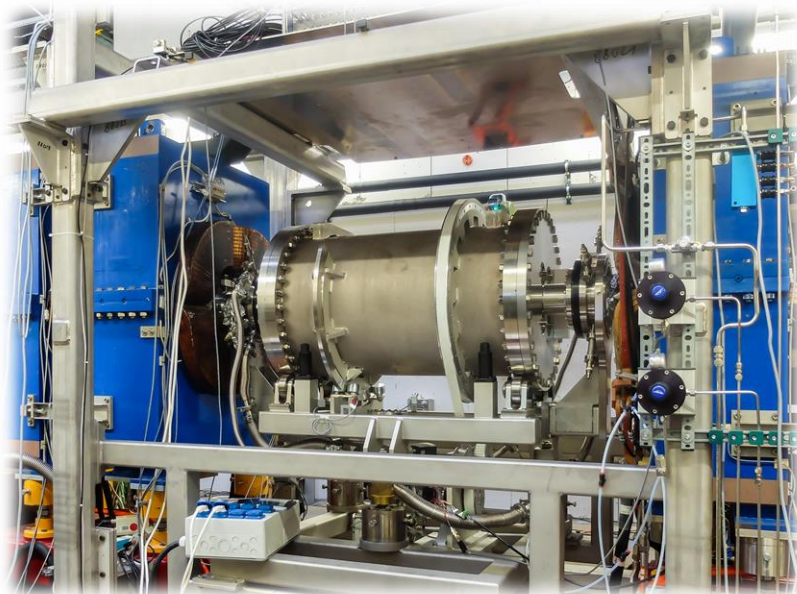


COSY



$\hat{n}_{z, \text{Snake}}$

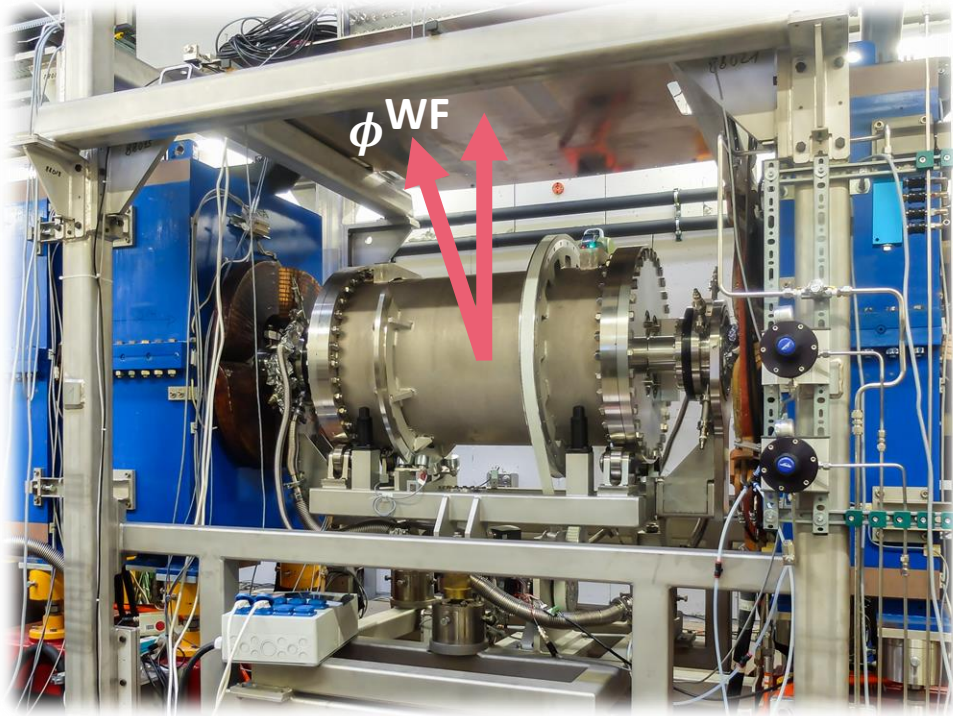
$\hat{n}_{z, \text{Sol}}$



$\hat{n}_{x, \text{WF}}$ & $\hat{n}_{z, \text{WF}}$

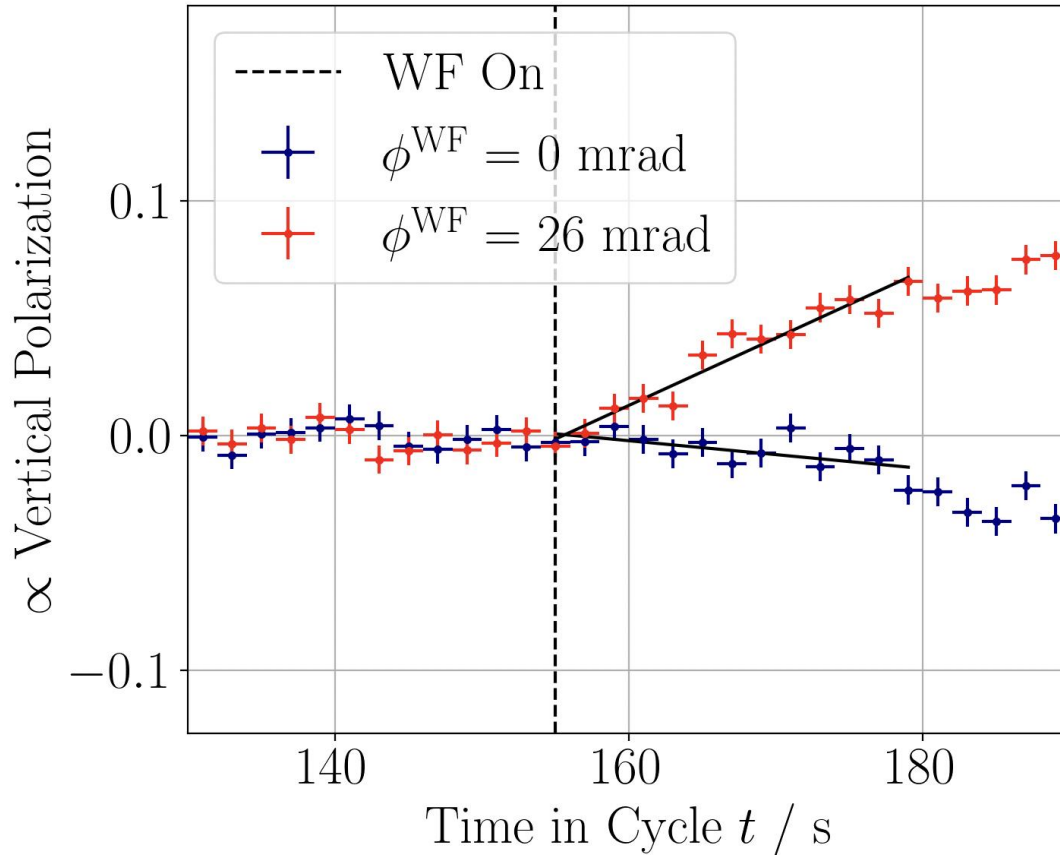
EDM signal!

RF WIEN FILTER

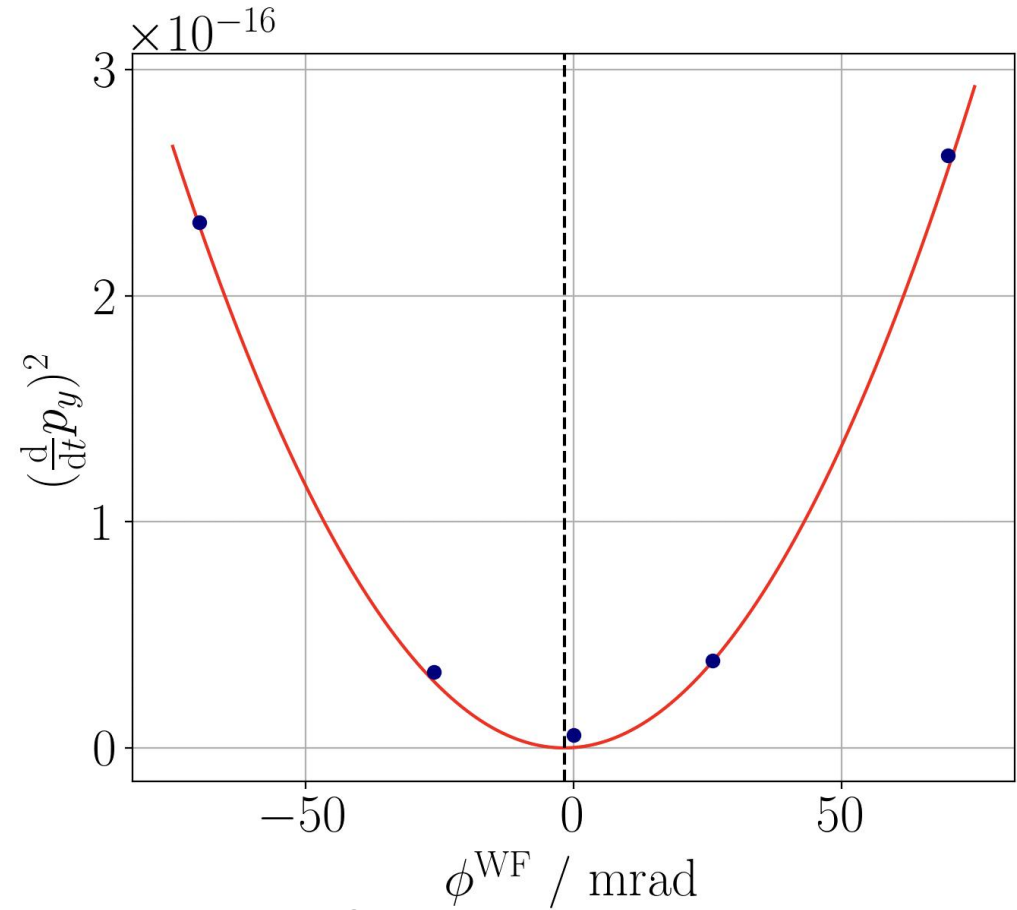


- Goal: **Measure n_x**
- $\vec{E} \perp \vec{B} \perp \text{Beam} \rightarrow \vec{F}_L = q \cdot (\vec{E} + \vec{v} \times \vec{B}) = 0$
- $\vec{B}$ - Field can be rotated around the beam pipe by ϕ^{WF}
- Needs to run on a harmonic of the spin precession frequency
$$f_{\text{WF}} = f_s$$
- $\vec{E} = \vec{E}_0 \cos(2\pi f_s + \phi)$ and $\vec{B} = \vec{B}_0 \cos(2\pi f_s + \phi)$
- Both frequencies need to have an adjustable phase relation

MEASUREMENT PRINCIPLE

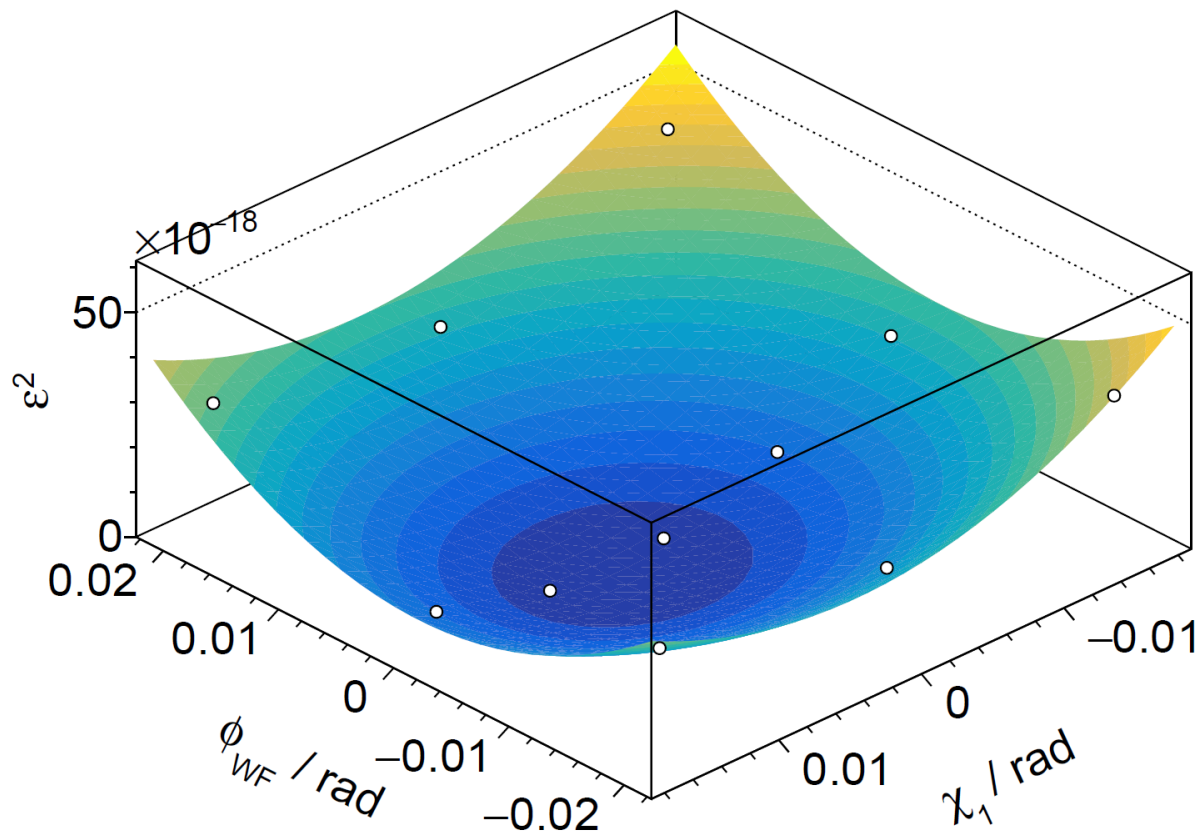
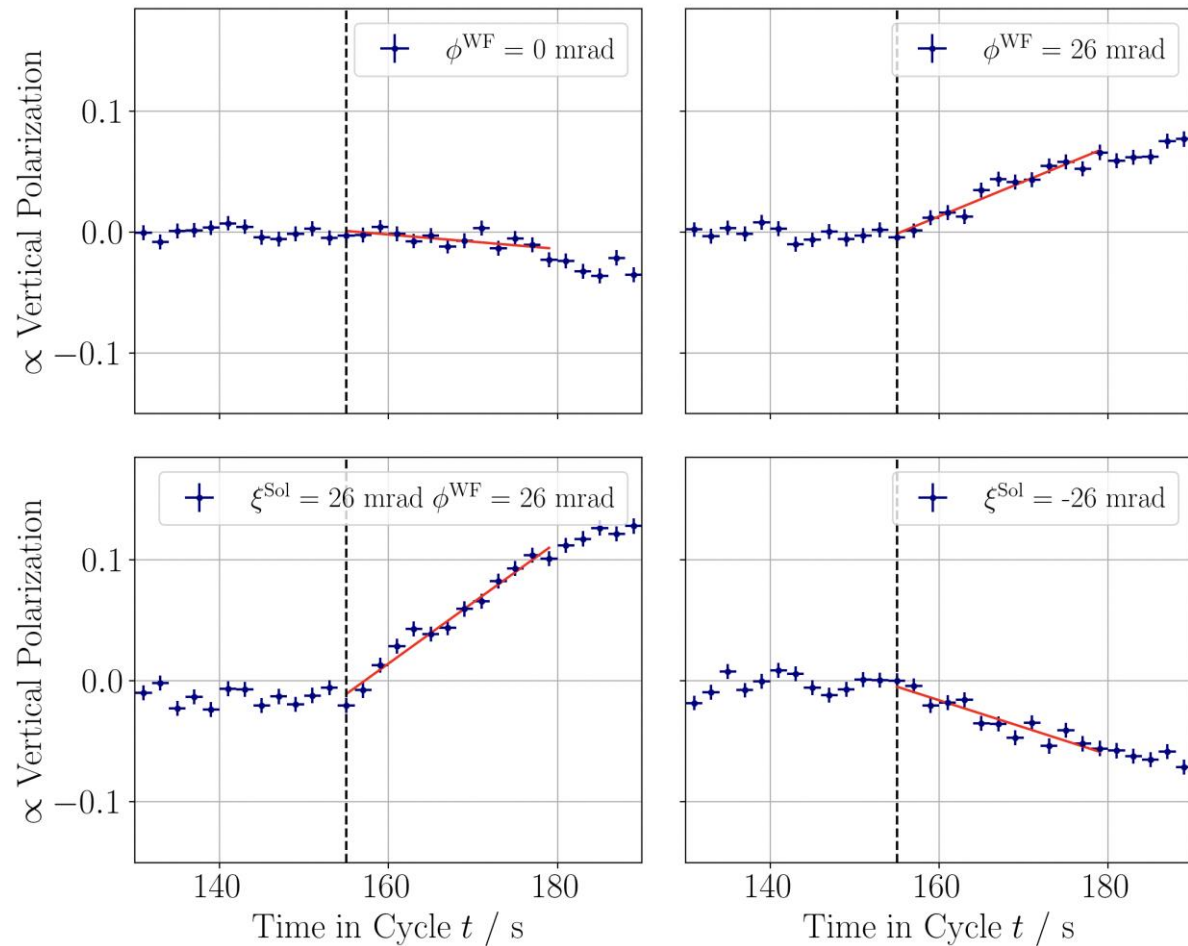


Build up rate $\frac{d}{dt} p_y(t)$



$$\left(\frac{d}{dt} p_y\right)^2 \propto |\vec{n} \times \vec{m}| \propto (n_x - \phi^{\text{WF}})^2 + \dots$$

MEASUREMENT PRINCIPLE

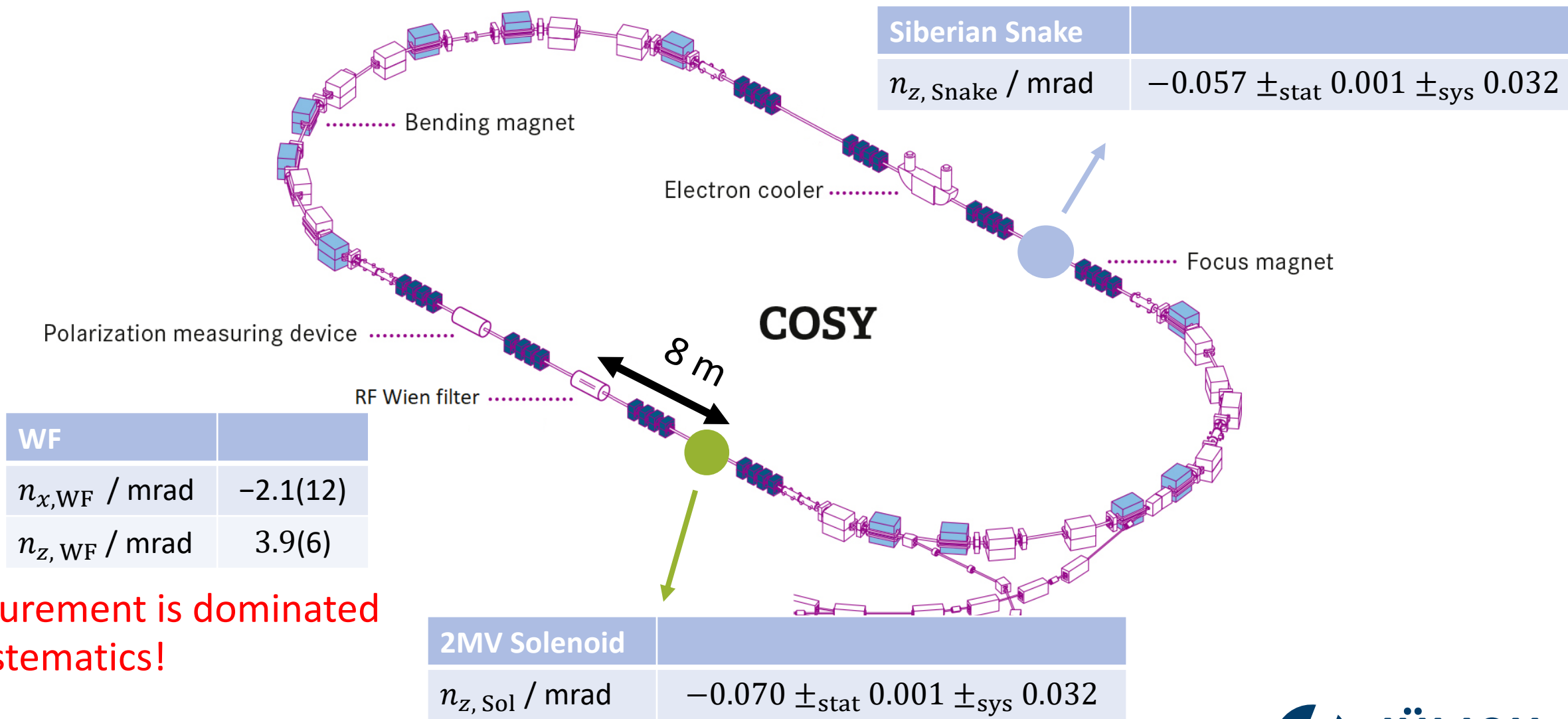


$$\left(\frac{d}{dt}p_y\right)^2 \propto |\vec{n} \times \vec{m}| \propto \left[(\mathbf{n}_x - \phi^{\text{WF}})^2 + (\mathbf{n}_z - \xi^{\text{Sol}})^2\right]$$

RESULTS

n_z : Systematics

n_x : **EDM** + Systematics



Measurement is dominated by systematics!

SPIN DYNAMICS

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$$\vec{d} = d \frac{\vec{s}}{|\vec{s}|} \text{ with } d = \eta^{\text{EDM}} \frac{q\hbar}{2mc}$$

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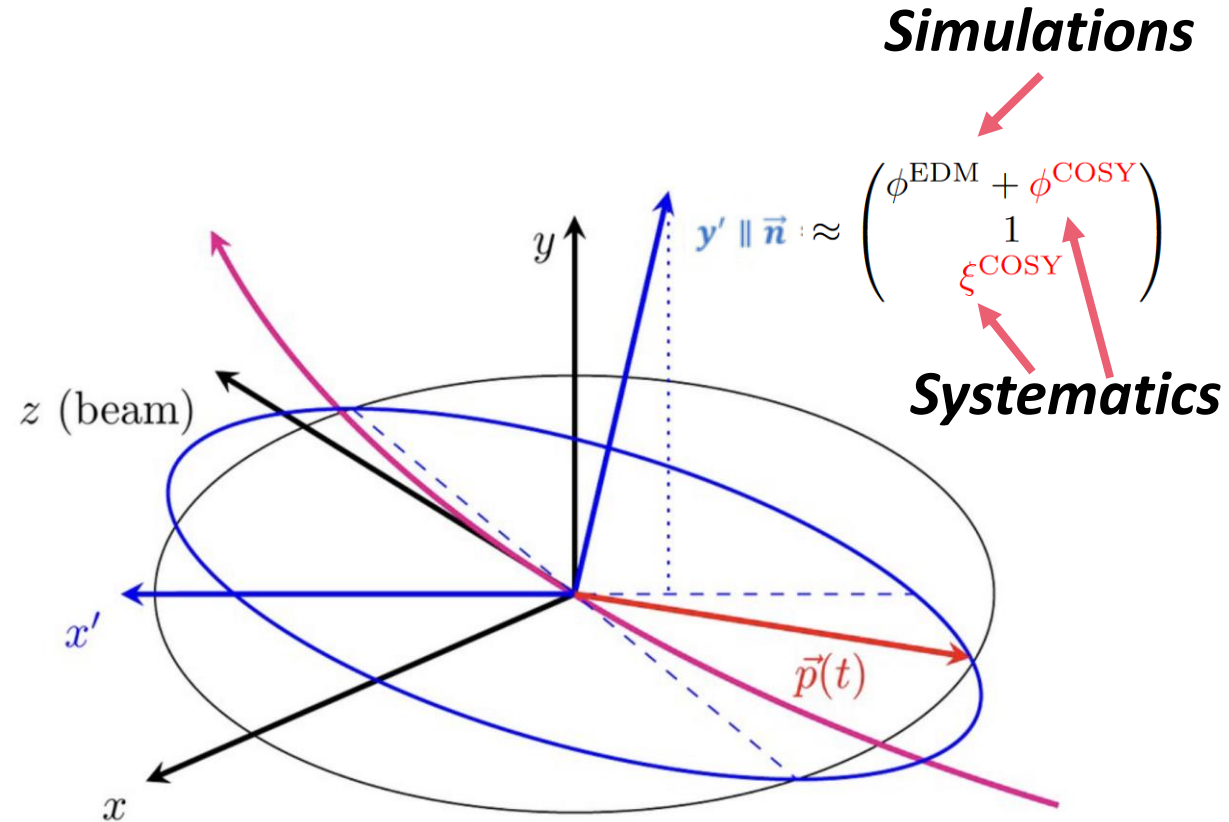
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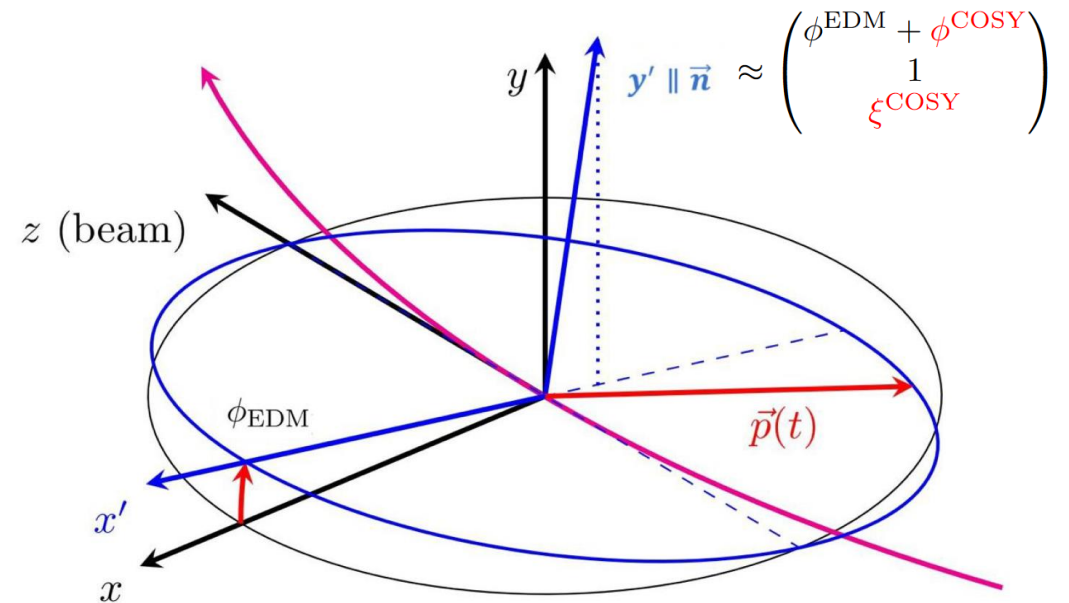
- $|d| < 3.0 \times 10^{-17} \text{ e} \cdot \text{cm}$ (95% C.L.) (Preliminary)



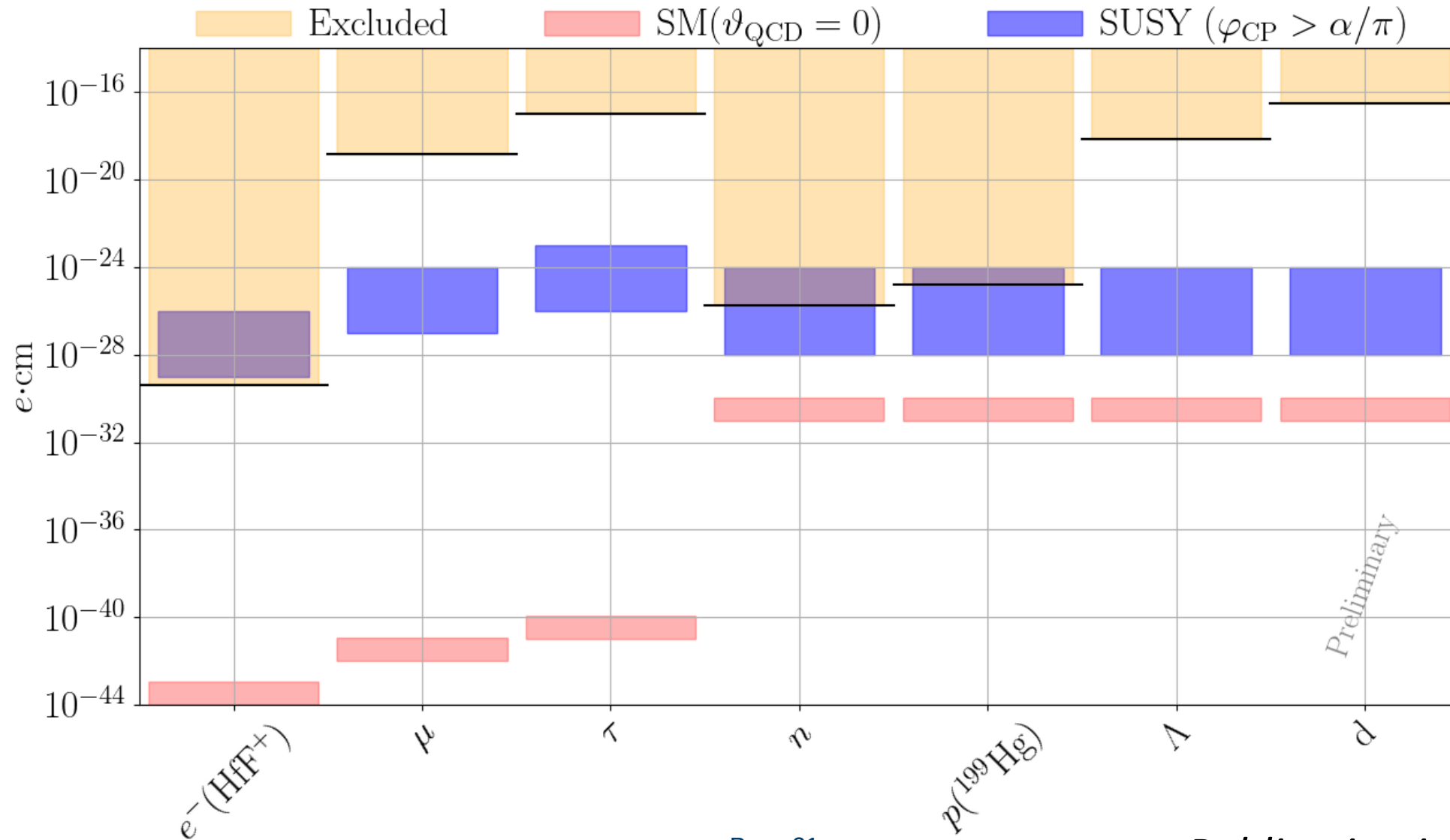
SUMMARY

- **Motivation:** EDMs as a source of CP violation and a problem solver
- **Goal:** Measure the **influence** of the deuteron EDM on the **beam polarization**
- While the method works, the data **cannot be interpreted correctly**
- We determine a **first limit** of the permanent deuteron EDM

$$|d| < 3.0 \times 10^{-17} \text{ e} \cdot \text{cm} \text{ (95\% C.L.) (Preliminary)}$$



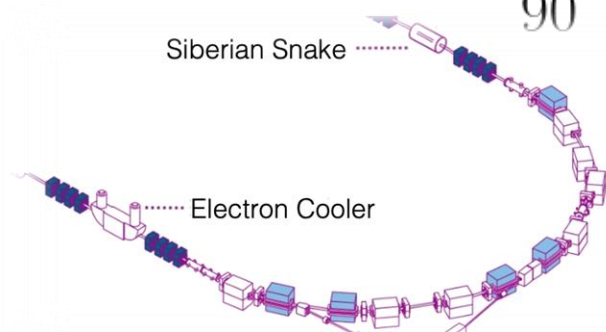
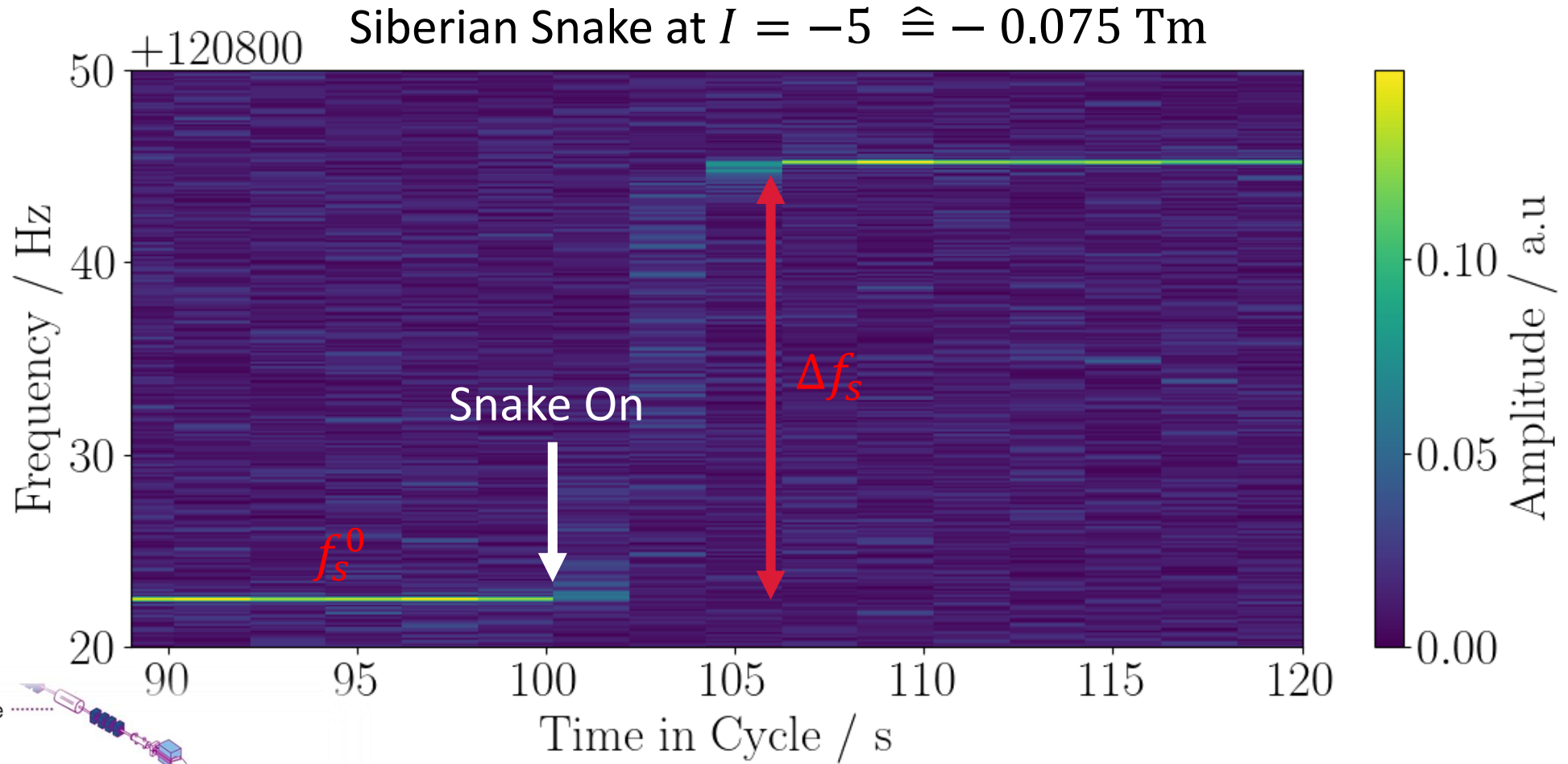
PERMANENT DEUTERON EDM SEARCH



BACK UP

INVARIANT SPIN AXIS AT THE SOLENOIDS

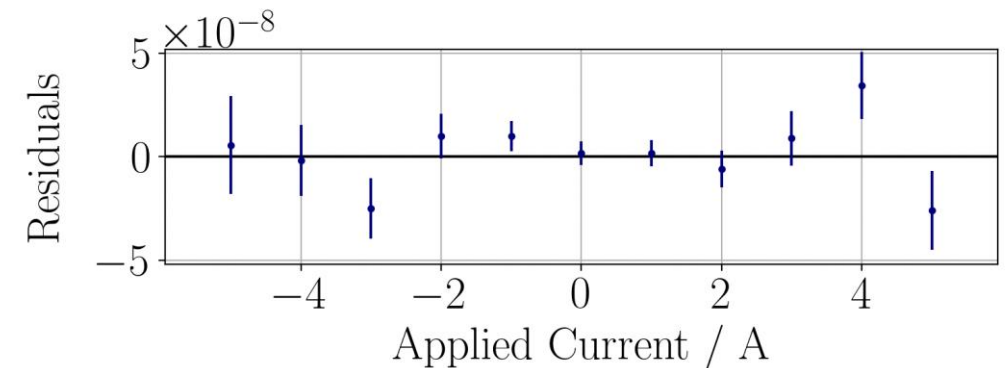
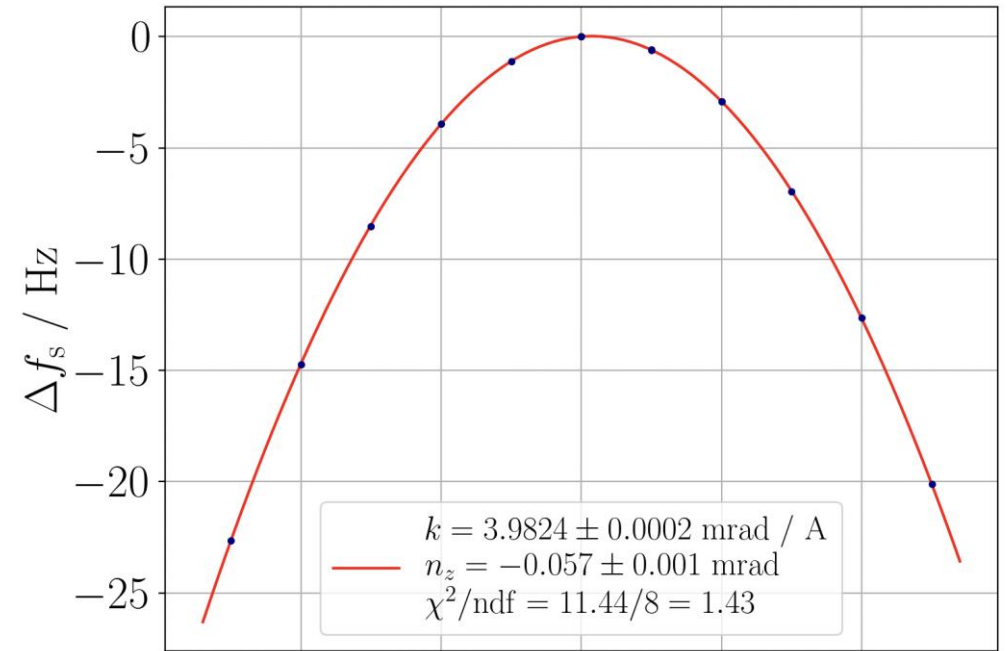
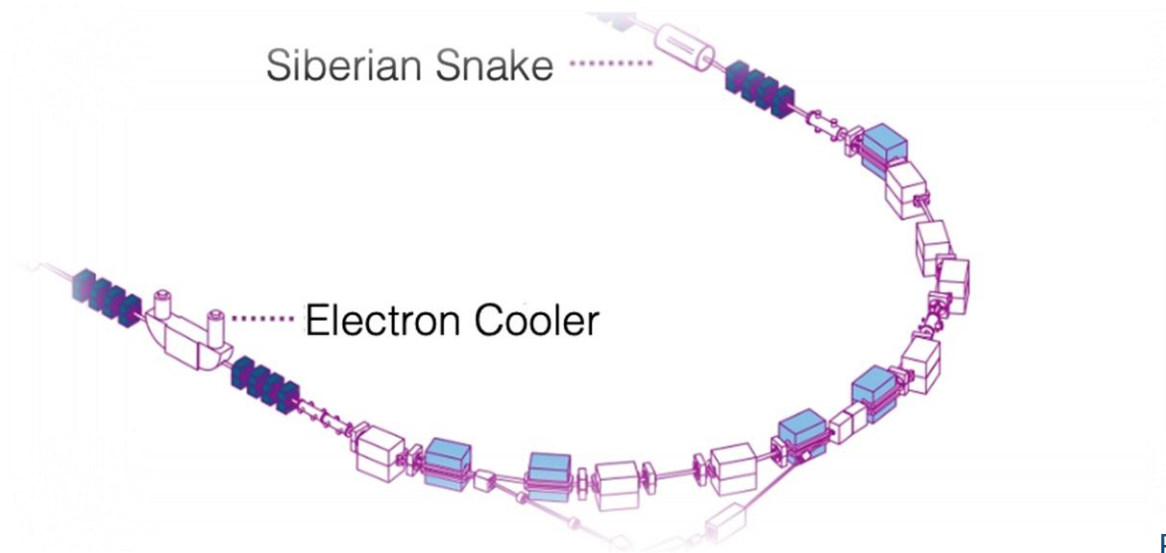
Methodology



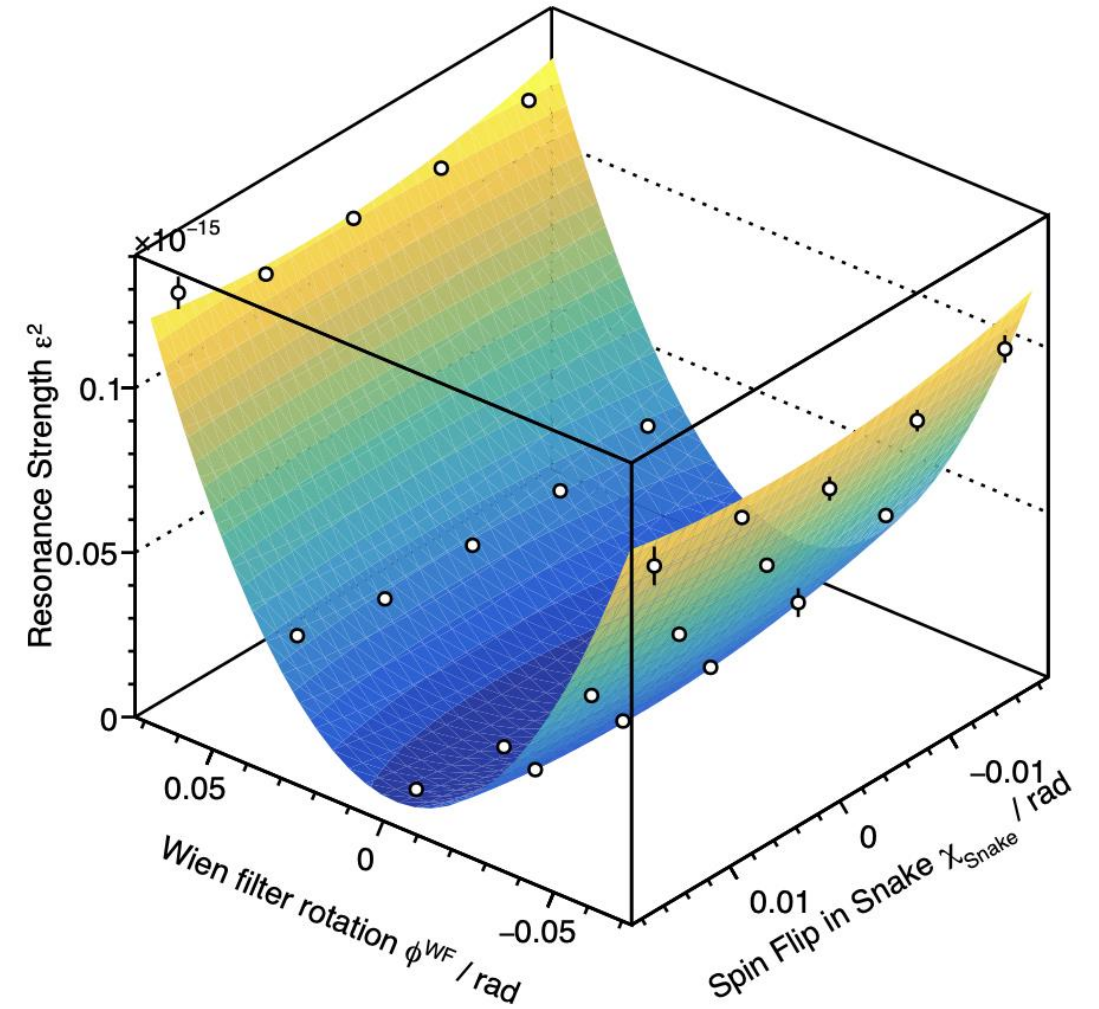
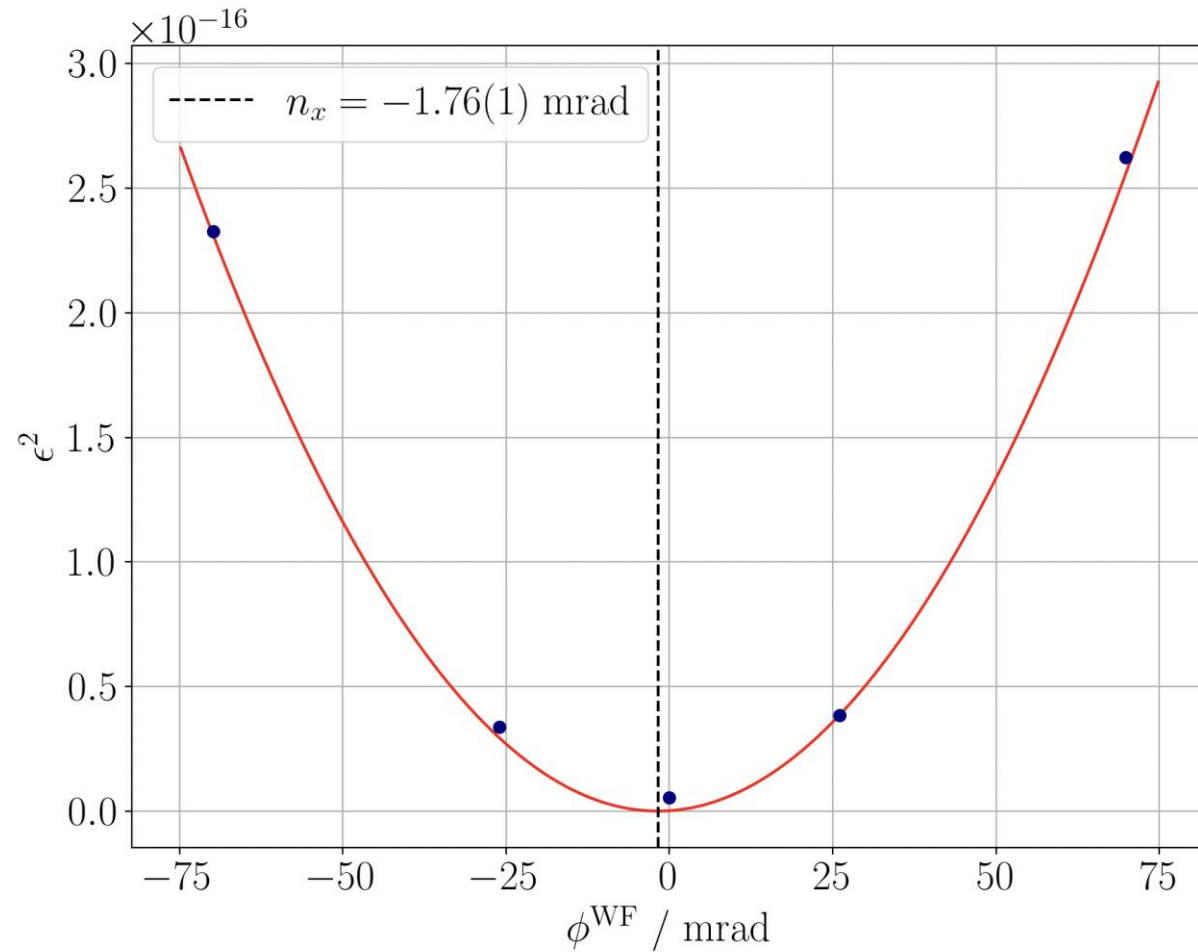
INVARIANT SPIN AXIS AT THE SOLENOIDS

$$\Delta f_s = \frac{f_{\text{rev}}}{4\pi} \left[\frac{\cos\left(\pi \frac{f_s^0}{f_{\text{rev}}}\right)}{2} k^2 I^2 + n_z k I \right]$$

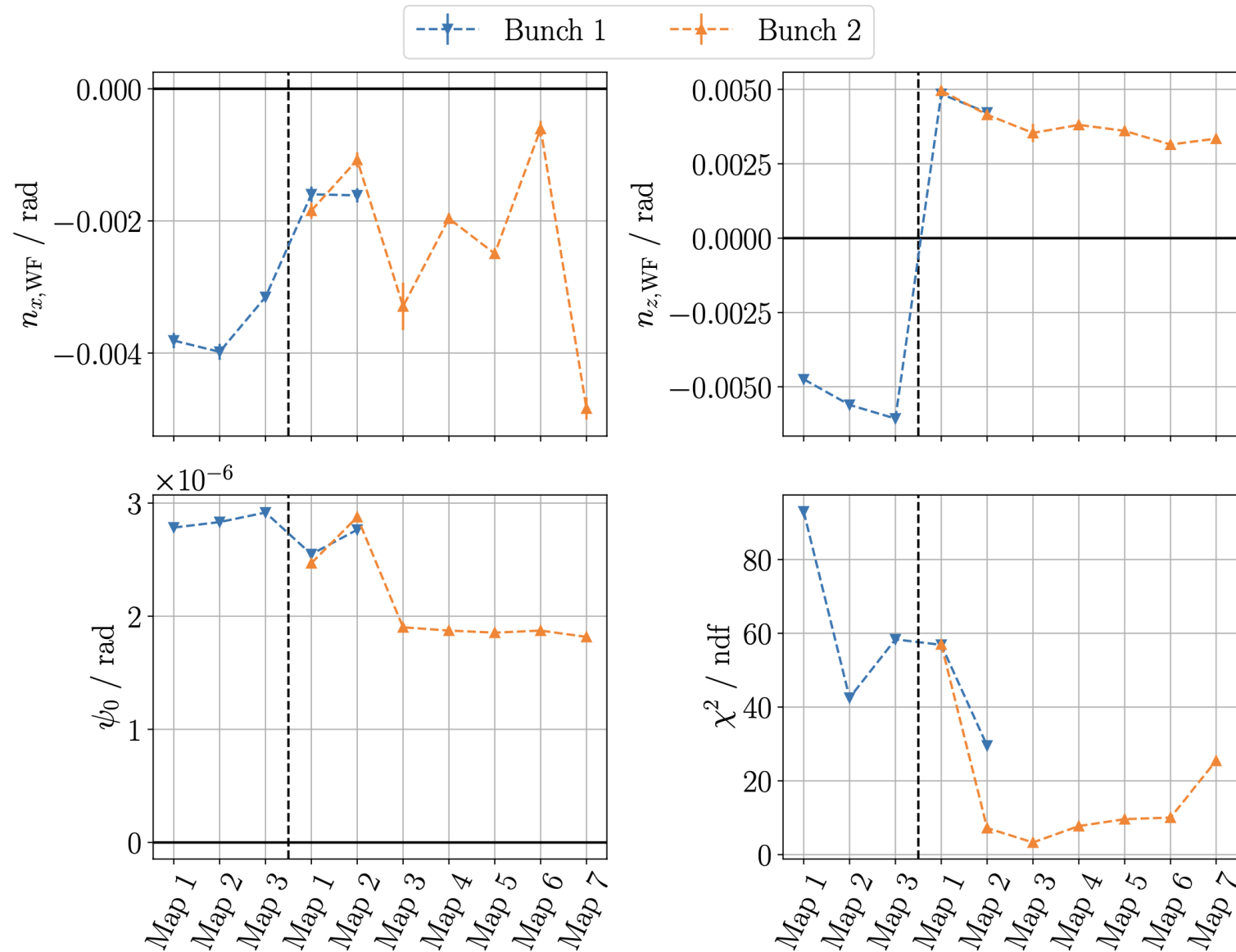
- k translates the coil current into a spin tilt angle
- n_z denotes the z component of the invariant spin axis



DETAILED LOOK INTO DATA

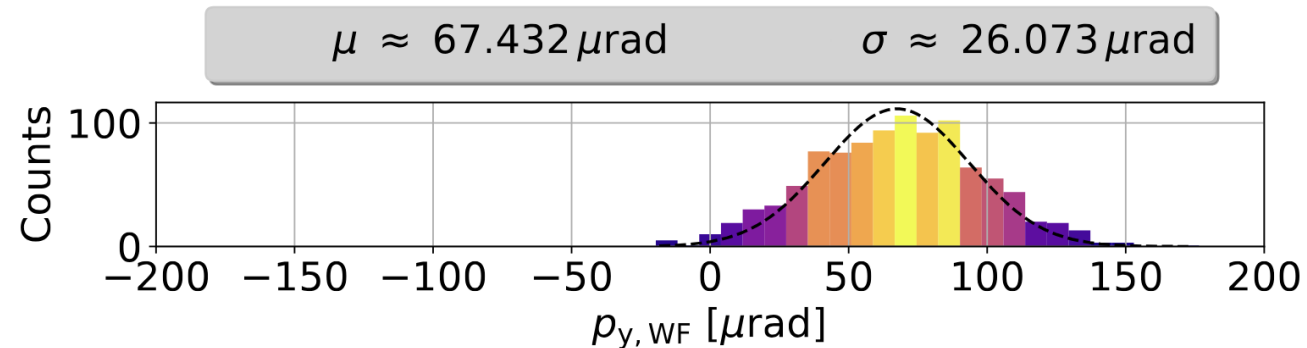
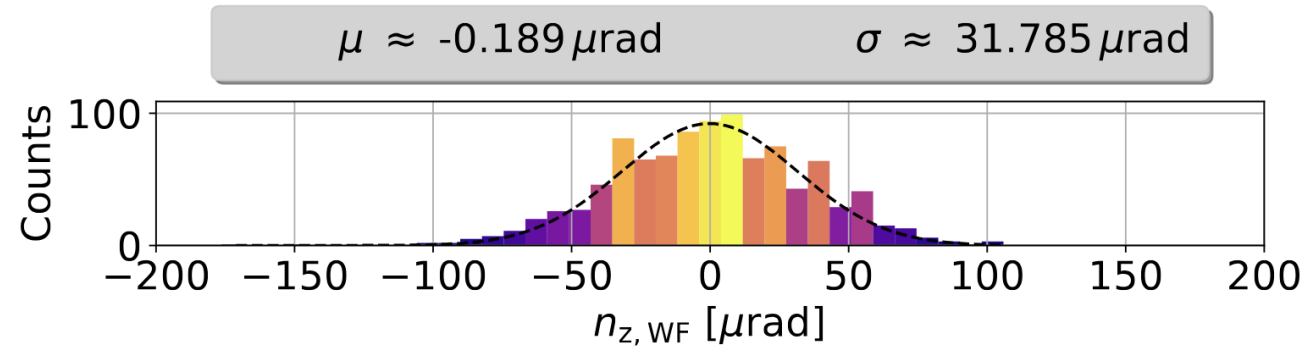
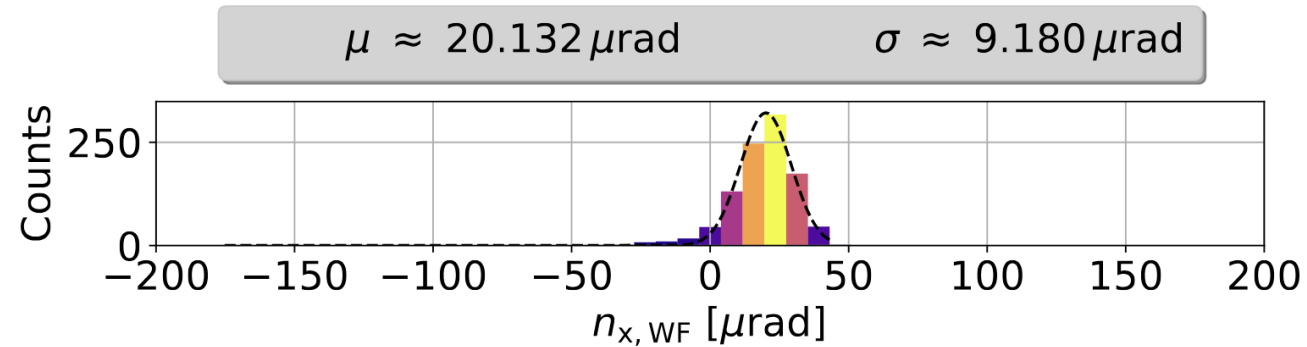


RESULTS



SYSTEMATIC ESTIMATES FROM SIMULATIONS

Max Vitz (PhD)



ELECTRIC DIPOLE MOMENTS

Axion Search

- Violation of symmetries was observed in the weak sector
- However: not sufficient
- **CP** violation in the strong sector

$$L_{\bar{\theta}_{\text{QCD}}} = -\bar{\theta}_{\text{QCD}} \frac{g_s^2}{64\pi^2} \varepsilon^{\mu\nu\alpha\beta} G_{\mu\nu}^a G_{\alpha\beta}^a$$

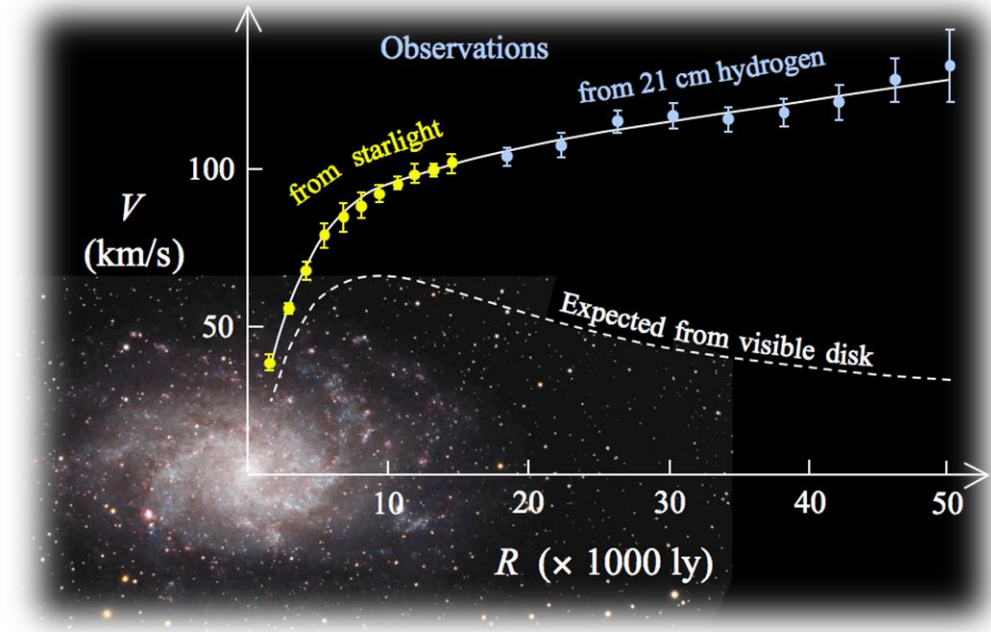
- Limits from neutron EDM measurements limit constrain

$$\bar{\theta}_{\text{QCD}} \leq 10^{-10} \Rightarrow \text{Strong CP Problem}$$

- Problem solver: **Axion** or **Axion Like Particles** (ALPs)

$$\vec{d} = d \frac{\vec{s}}{|\vec{s}|} \text{ with } d = d_{\text{DC}} + d_{\text{AC}} \cos(\omega_a + \phi_a) \text{ and } \omega_a = \frac{m_a c^2}{\hbar}$$

- Existence of an axion leads to an additional oscillating EDM component
- Axion could explain the strong CP - problem
- Axion are potential candidate for Dark Matter



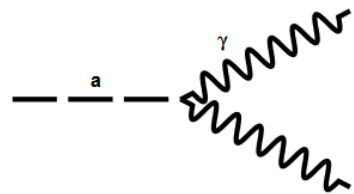
AXION SEARCH @ COSY

$$d = d_{\text{DC}} + d_{\text{AC}}^d \cos(\omega_a + \phi_a)$$

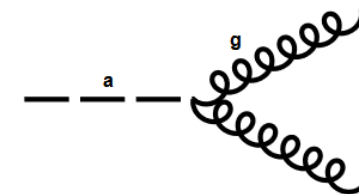
$$\omega_a = \frac{m_a c^2}{\hbar}$$

- Constraints for the axion gluon coupling:

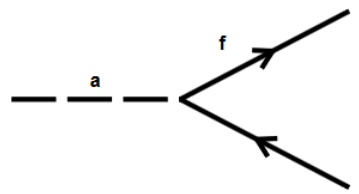
$$|g_{ad\gamma}| < 1.7 \times 10^{-7} \text{GeV}^{-2}$$

$$\mathcal{L} : -\frac{\alpha}{8\pi} \frac{C_\gamma}{f_a} a F_{\mu\nu} \tilde{F}^{\mu\nu}$$


studied by many experiments

$$-\frac{\alpha_s}{8\pi} \frac{C_G}{f_a} a G_{\mu\nu}^b \tilde{G}^{b,\mu\nu}$$


Electric Dipole Moment (EDM)

$$-\frac{1}{2} \frac{C_N}{f_a} \partial_\mu a \bar{\Psi}_f \gamma^\mu \gamma^5 \Psi_f$$


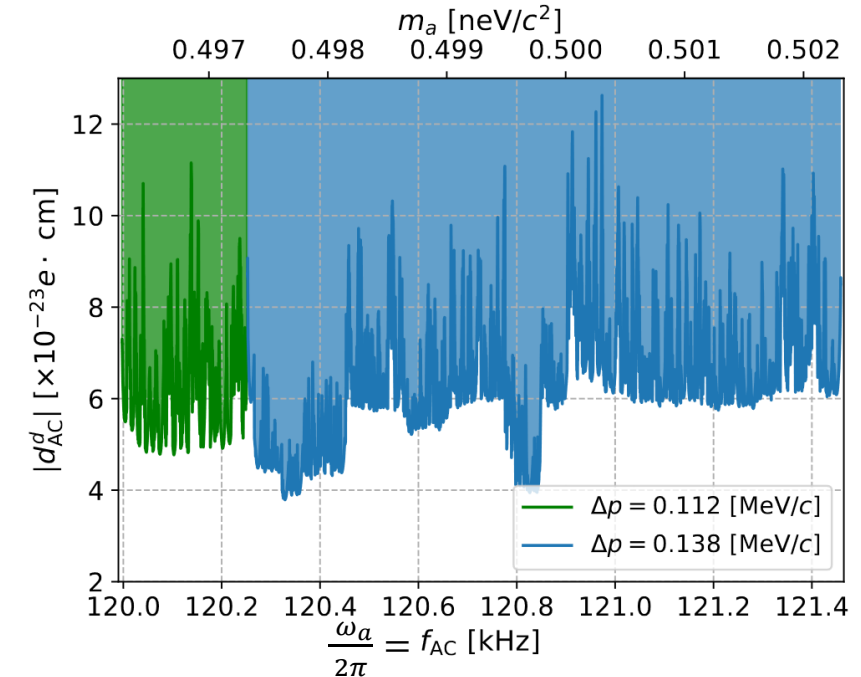
axion wind term

accessible in storage ring experiments with spin polarized beams

First Search for Axionlike Particles in a Storage Ring Using a Polarized Deuteron Beam

S. Karanth^{1,*}, E. J. Stephenson^{2,†}, S. P. Chang^{3,4}, V. Hejny⁵, S. Park⁴, J. Pretz^{5,6,7}, Y. K. Semertzidis^{3,4}, A. Wirzba^{5,8}, A. Wrońska¹, F. Abusaif^{6,5,9}, A. Aggarwal¹, A. Aksentev⁹, B. Alberdi^{6,5,*,††}, A. Andres^{6,5}, L. Barion¹⁰, I. Bekman^{5,‡}, M. Beyß^{6,5}, C. Böhme⁷, B. Breikreutz^{5,8}, C. von Byern^{6,5}, N. Canale¹⁰, G. Ciullo¹⁰, S. Dymov¹⁰, N.-O. Fröhlich^{5,‡}, R. Gebel^{5,11}, K. Grigoryev^{5,8}, D. Grzonka⁵, J. Hetzel⁵, O. Javakhishvili¹², H. Jeong¹³, A. Kacharava⁵, V. Kamedzhiev^{5,8}, I. Keshelashvili^{5,8}, A. Kononov¹⁰, K. Laihem^{6,5}, A. Lehrach^{5,7}, P. Lenisa¹⁰, N. Lomidze¹⁴, B. Lorentz¹¹, A. Magiera¹, D. Mchedlishvili^{14,19}, F. Müller^{6,5}, A. Nass⁵, N. N. Nikolaev^{15,16}, A. Pesce⁵, V. Poncza^{6,5}, D. Prasuhn^{5,8}, F. Rathmann⁵, A. Saleev¹⁰, D. Shergelashvili¹⁴, V. Shmakova^{10,8}, N. Shurkhno^{5,8}, S. Siddique^{6,5,8}, J. Slim^{6,11,‡}, H. Soltner¹⁷, R. Stassen⁵, H. Ströher^{5,7}, M. Tabidze¹⁴, G. Tagliente¹⁸, Y. Valdau^{5,8}, M. Vitz^{5,6}, T. Wagner^{5,6,8} and P. Wüstner¹⁷

(JEDI Collaboration)

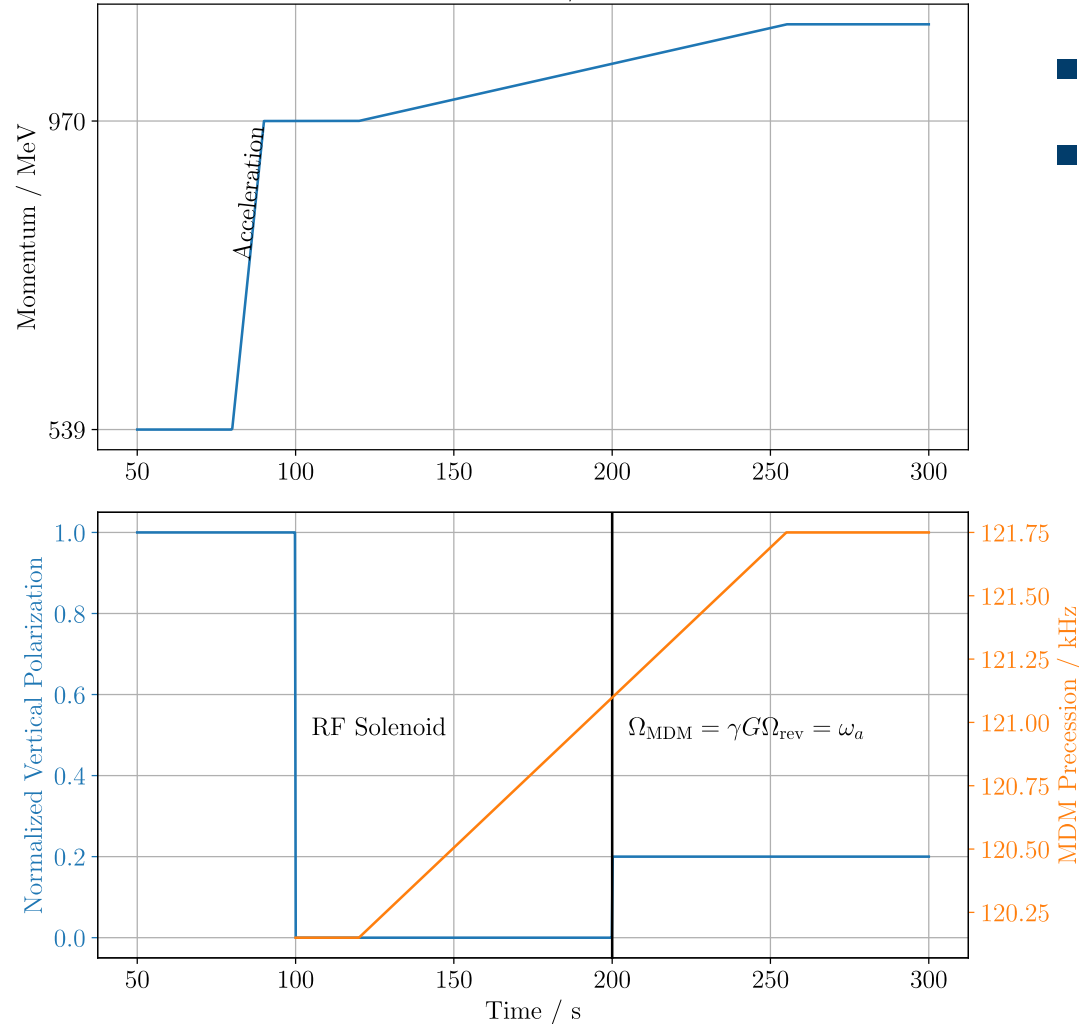


- 90% CL upper limit on the ALPs induced oscillating EDM
- Average of $|d_{\text{AC}}^d| < 6.4 \times 10^{-23} \text{e} \cdot \text{cm}$

AXION SEARCH

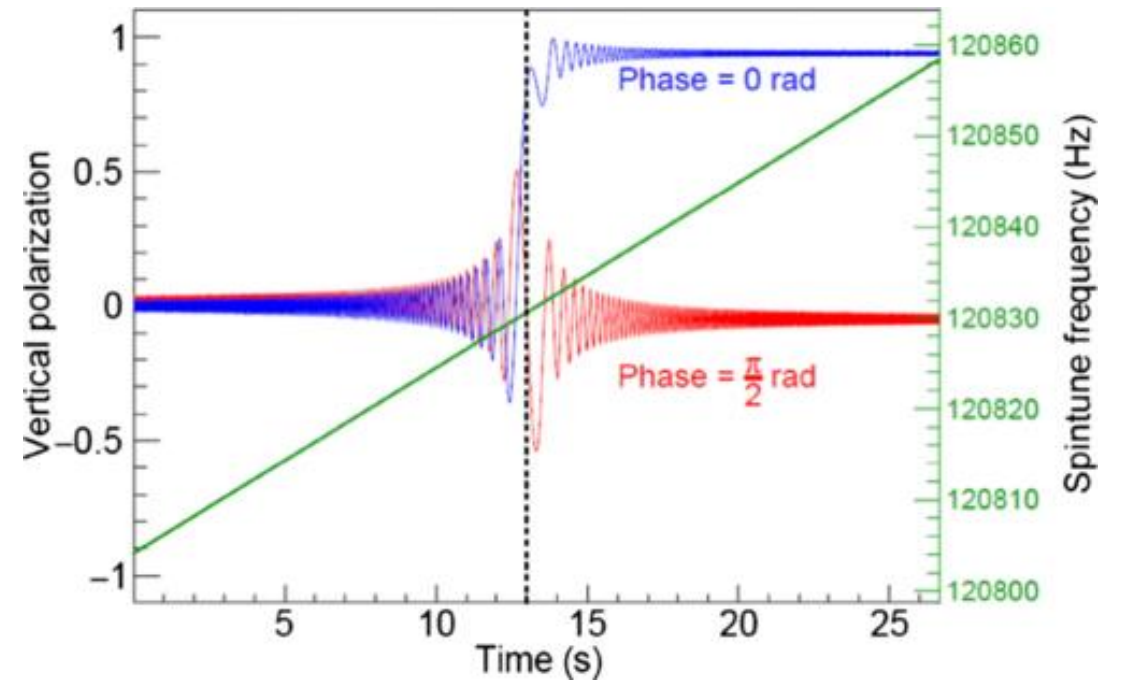
S. Karanth (PhD Work)

Measurement Principle / Expected Signal



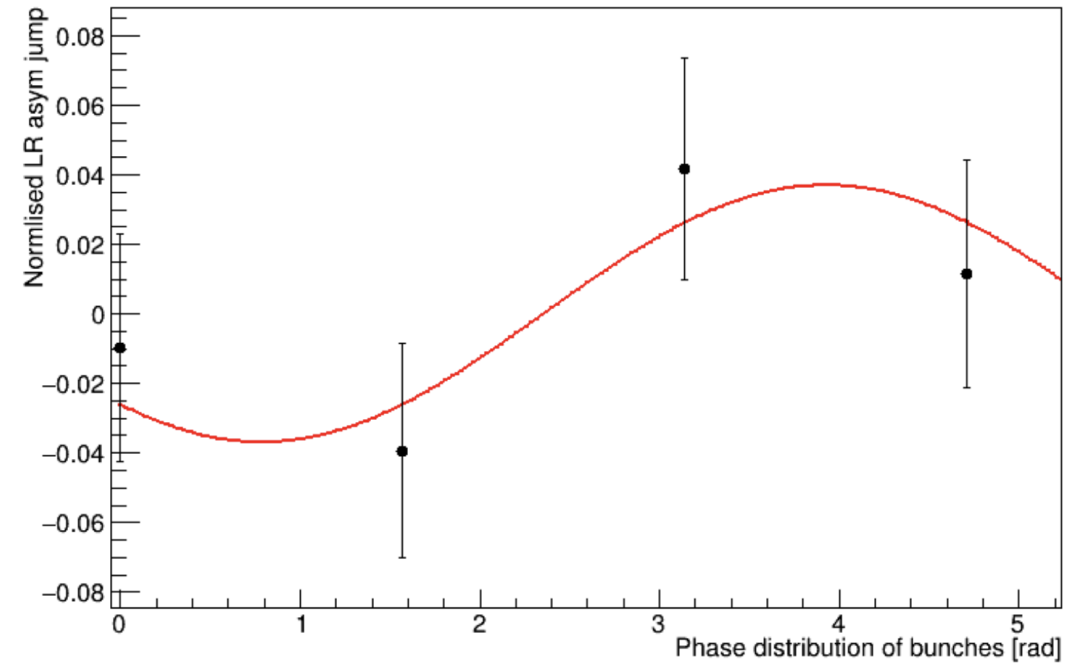
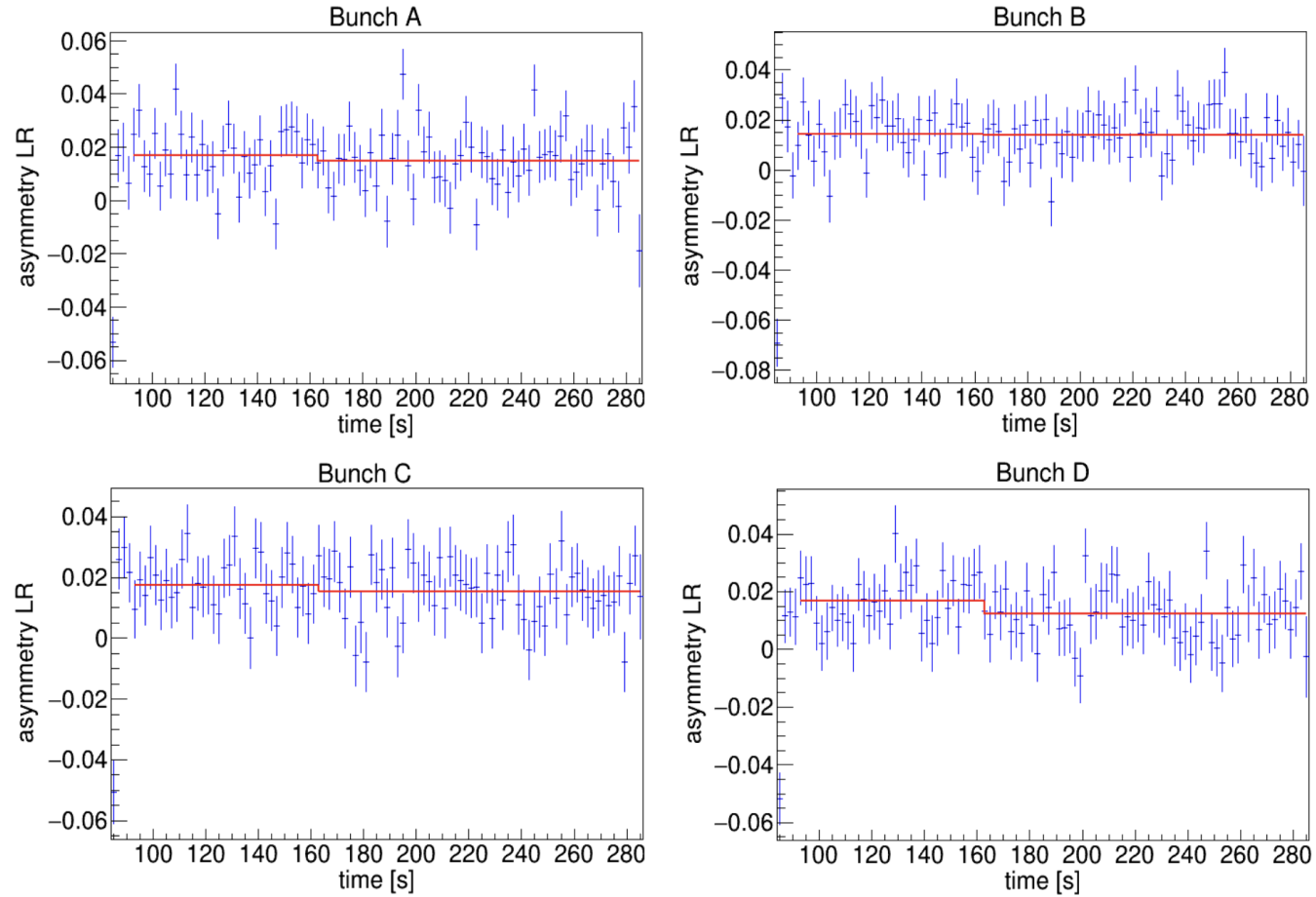
$$d = d_{\text{DC}} + d_{\text{AC}}^d \cos(\omega_a + \phi_a)$$

- Problem: Phase is unknown!
- Solution: Inject 4 bunches with different spin directionality!



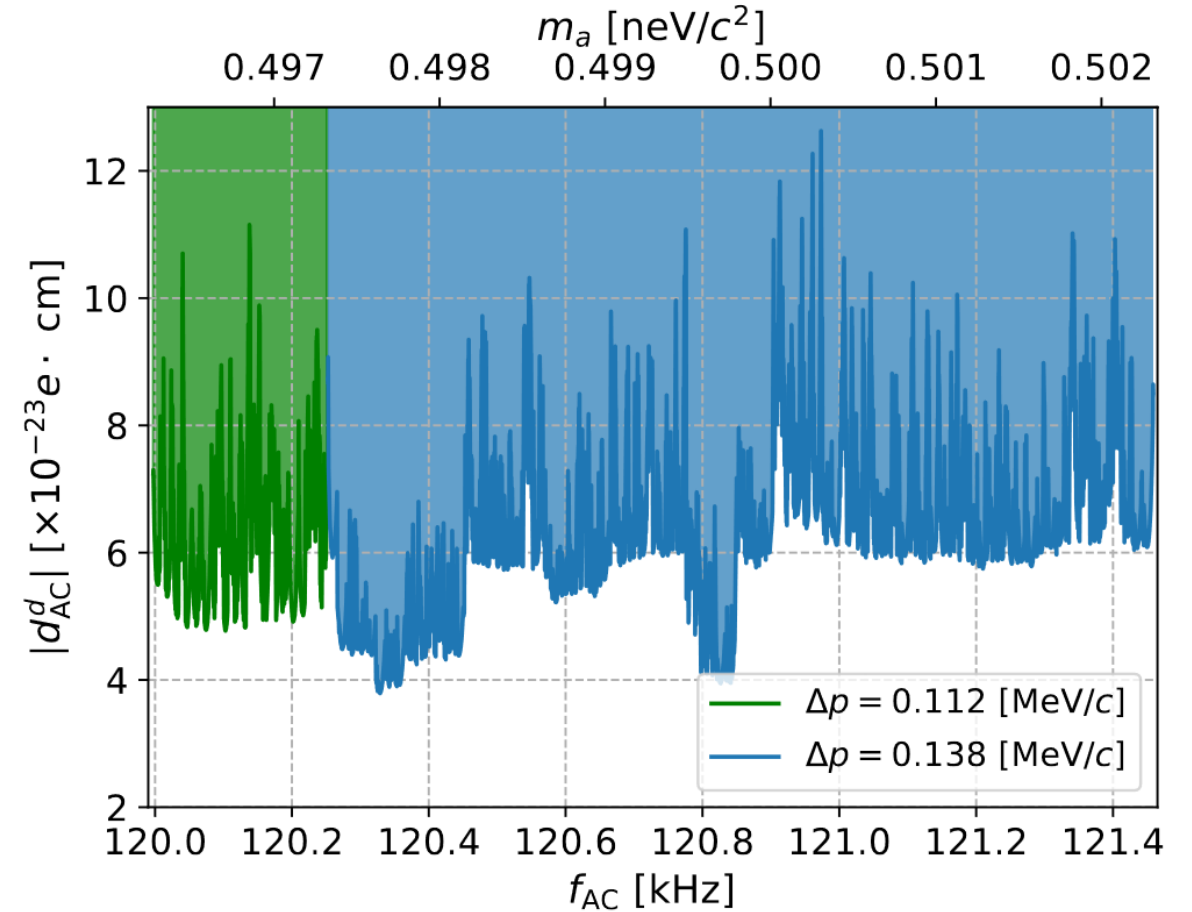
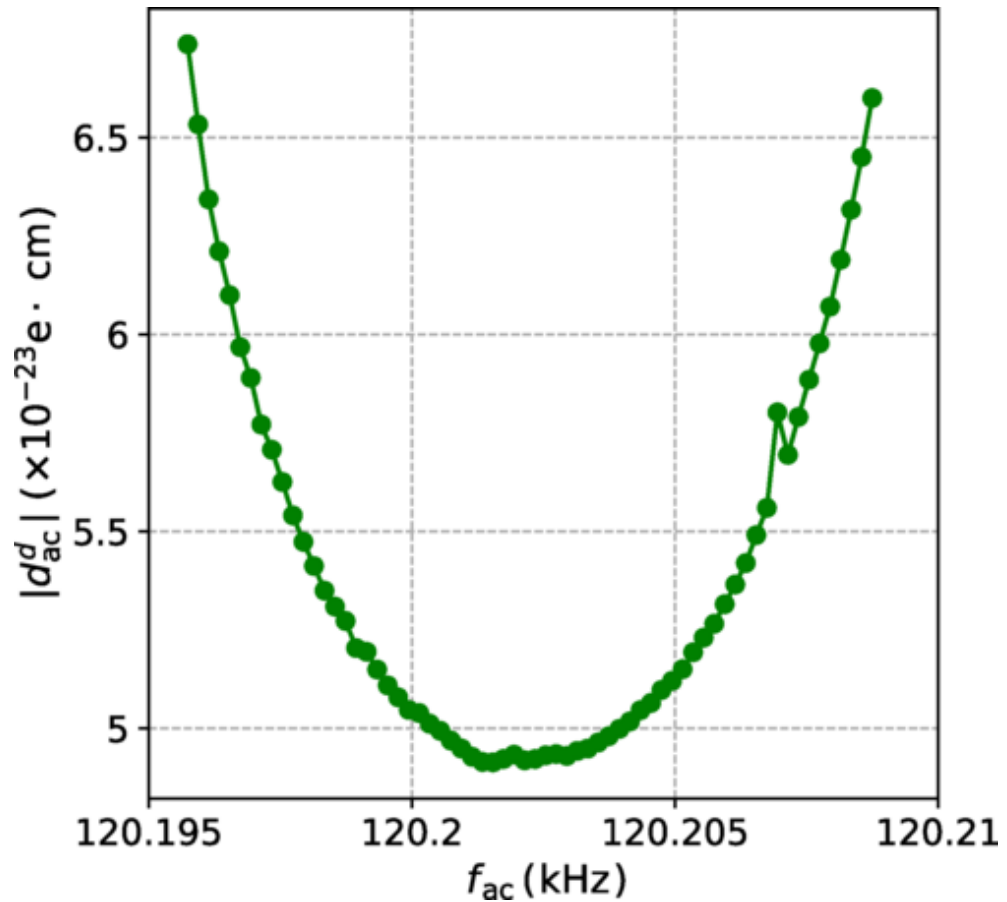
AXION SEARCH

S. Karanth (PhD Work)



AXION SEARCH

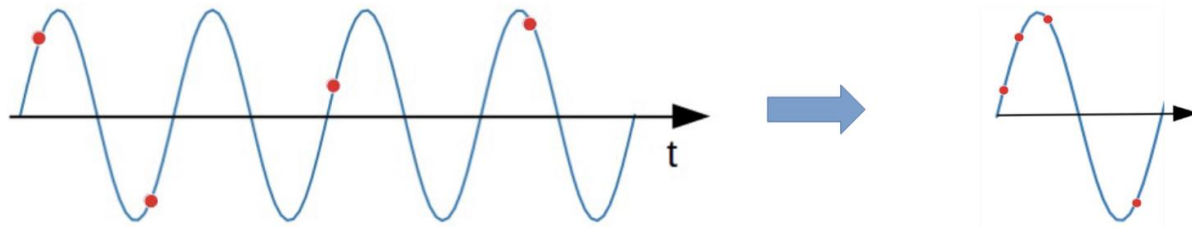
S. Karanth (PhD Work)



SPIN TUNE MEASUREMENT

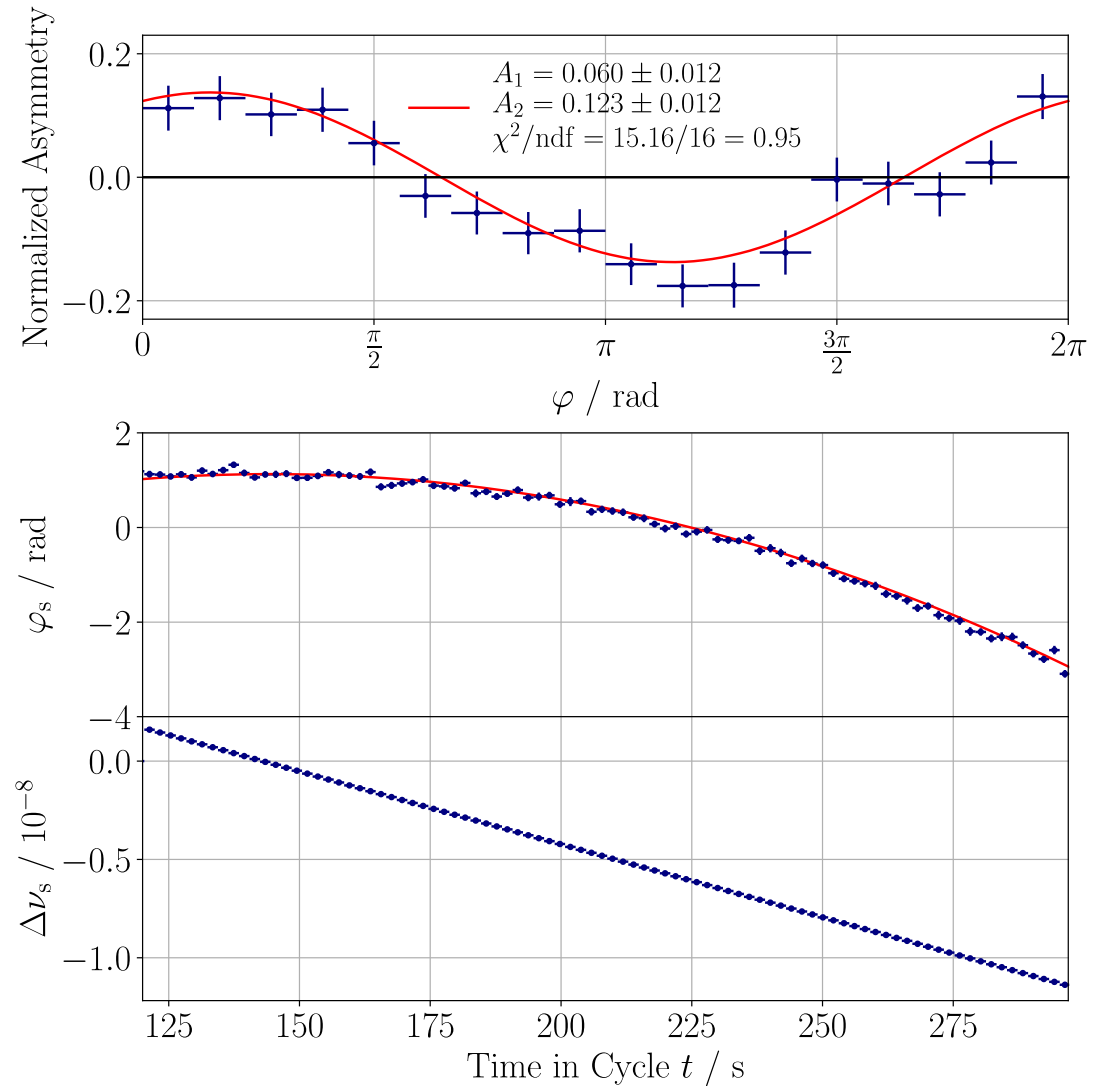
- Spin precesses with $f_s = \gamma G f_{\text{rev}} \approx 121 \text{ kHz}$
- Detector handles event rates $15000 \frac{1}{\text{s}}$
 - 1 hit per 10 precessions
 - No direct fit possible

- Assume a fixed frequency ν_s^{fixed}

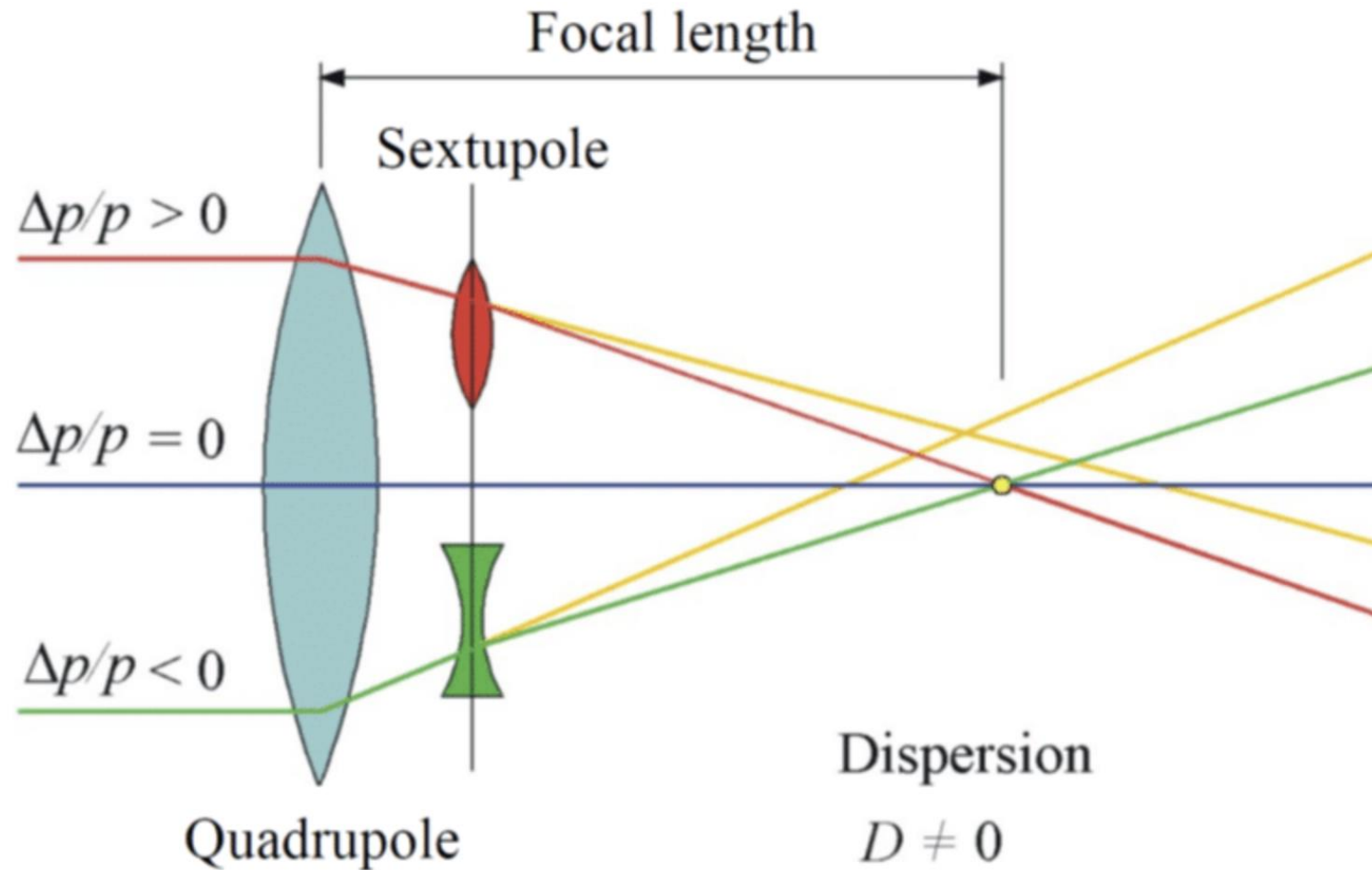


- The change of spin tune is given by

$$\nu_s(t) = \nu_s^{\text{fixed}} + \frac{1}{2\pi f_{\text{rev}}} \frac{d\varphi_s}{dt} = \nu_s^{\text{fixed}} + \Delta\nu_s(t)$$

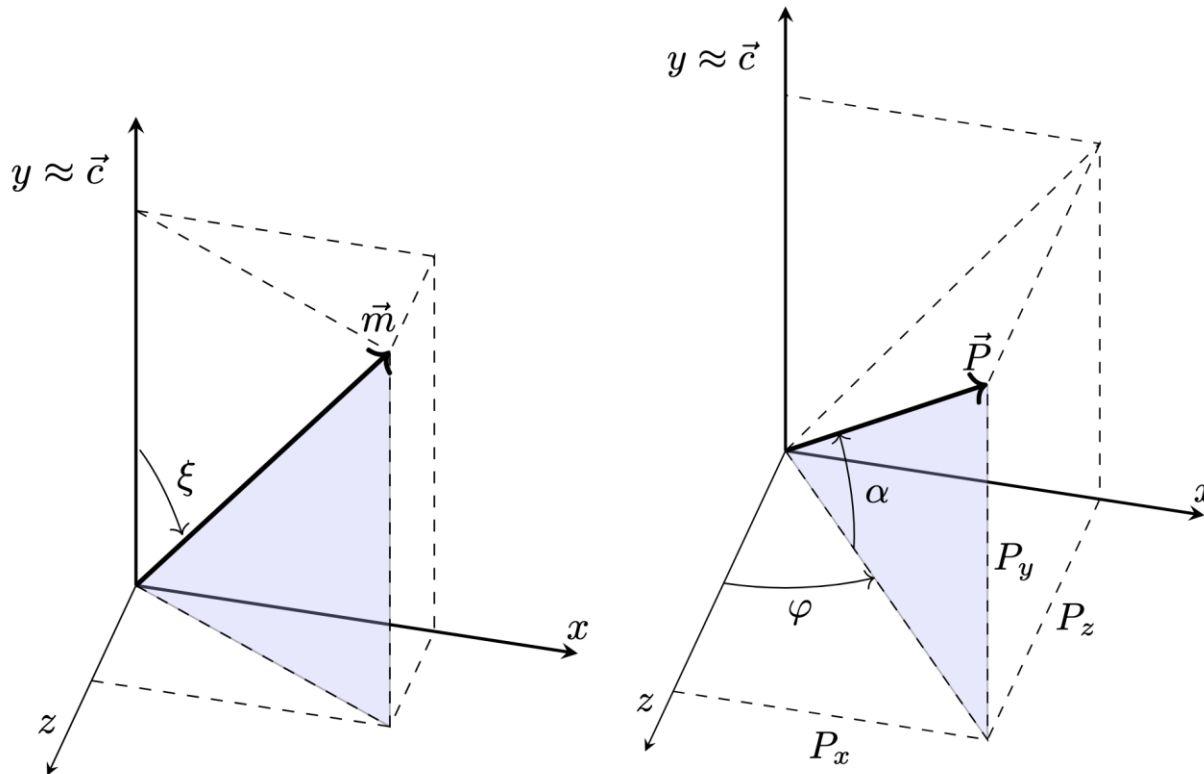


SEXTUPOLE CORRECTION FOR SCT ENHANCEMENT



PHASE FEEDBACK

- Tasks of the phase feedback:
 - Measures the spin precession frequency and adjusts the Wien filter frequency to it
 - Provides a fixed phase relation between both frequencies



$$\frac{d\alpha}{dn} = \frac{k}{2} \cos \varphi_{\text{rel}}$$

$$\frac{d\varphi_{\text{rel}}}{dn} = \frac{k}{2} (\sin \varphi_{\text{rel}} \tan \alpha + q)$$

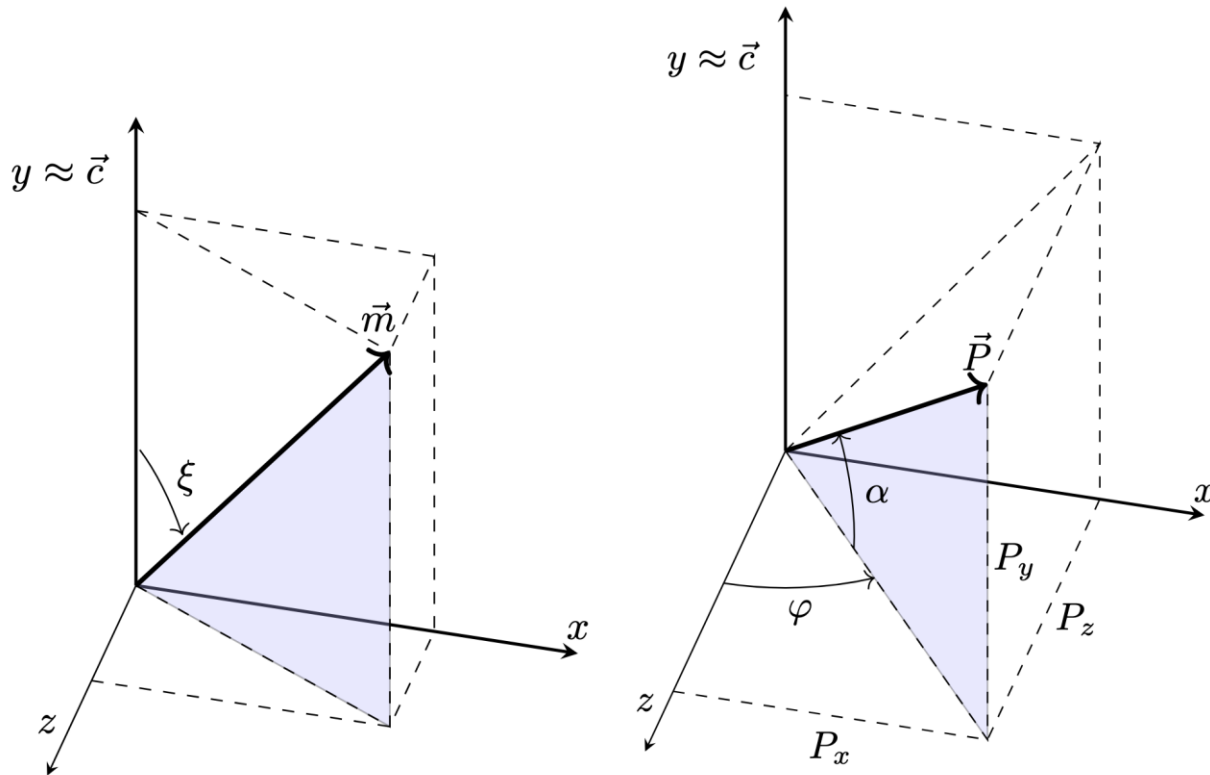
$$k = -\chi_0 \sin \xi = -4\pi\epsilon$$

$$q = \frac{4\pi\Delta f}{kf_{\text{rev}}}$$

$$\epsilon = \frac{f_v}{f_{\text{rev}}} = \frac{1}{4\pi} \chi_0 |\vec{c} \times \vec{m}| = \frac{1}{4\pi} \chi_0 \sin \xi$$

PHASE FEEDBACK

Slope Method



- Phase Feedback prevents the relative phase from changing

$$\frac{d\alpha}{dn} = \frac{k}{2} \cos \varphi_{\text{rel}}$$

$$\frac{d\varphi_{\text{rel}}}{dn} = \frac{k}{2} (\sin \varphi_{\text{rel}} \tan \alpha + q) \longrightarrow \frac{d\varphi_{\text{rel}}}{dn} = 0$$

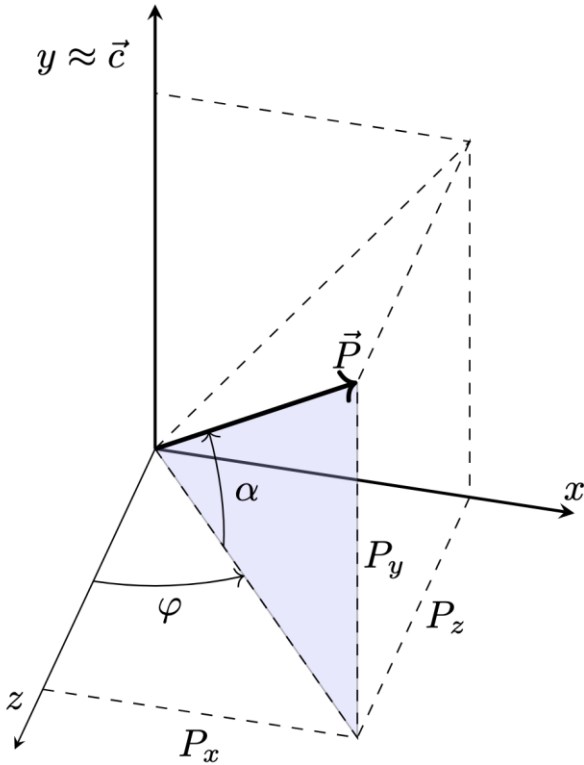
$$\frac{d\alpha}{dt} = 2\pi\epsilon f_{\text{rev}} \cos \varphi_{\text{rel}}$$

$$\epsilon = \frac{f_v}{f_{\text{rev}}} = \frac{1}{4\pi} \chi_0 |\vec{c} \times \vec{m}| = \frac{1}{4\pi} \chi_0 \sin \xi$$

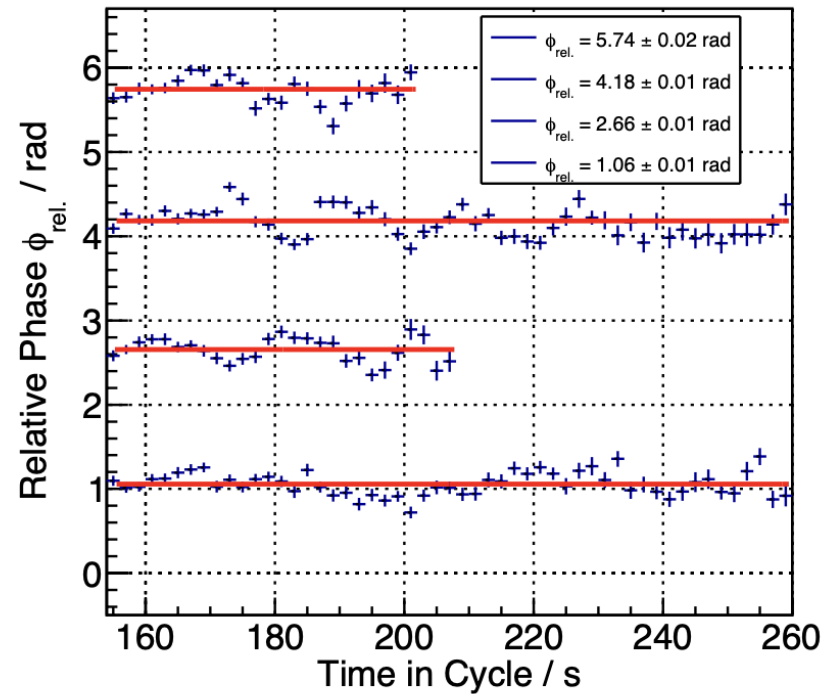
PHASE FEEDBACK

Slope Method

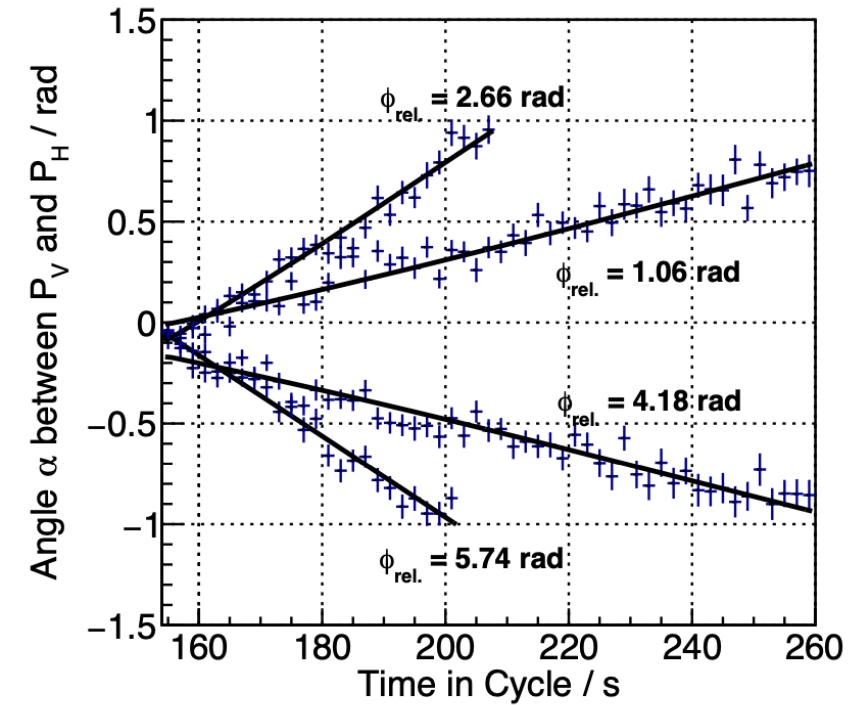
$$\epsilon = \frac{f_v}{f_{\text{rev}}} = \frac{1}{4\pi} \chi_0 |\vec{c} \times \vec{m}| = \frac{1}{4\pi} \chi_0 \sin \xi$$



$$\frac{d\varphi_{\text{rel}}}{dn} = 0$$

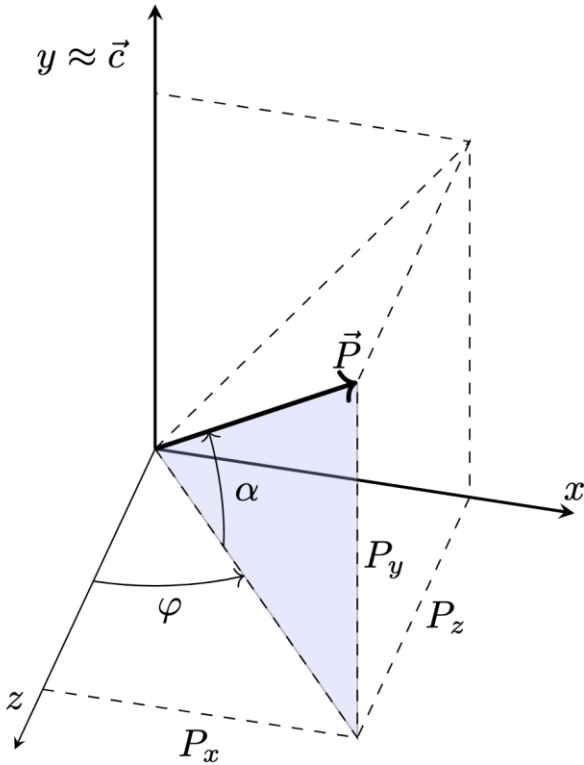


$$\frac{d\alpha}{dt} = 2\pi\epsilon f_{\text{rev}} \cos \varphi_{\text{rel}}$$



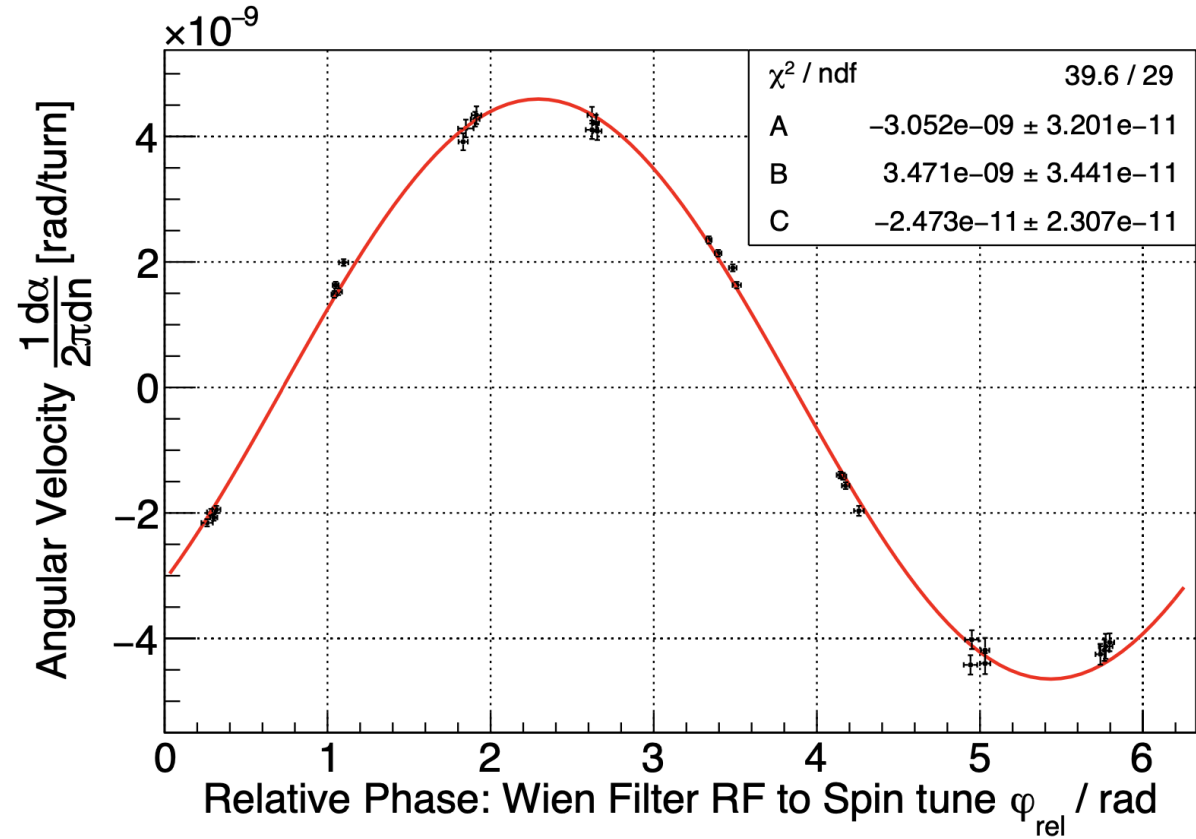
PHASE FEEDBACK

Slope Method



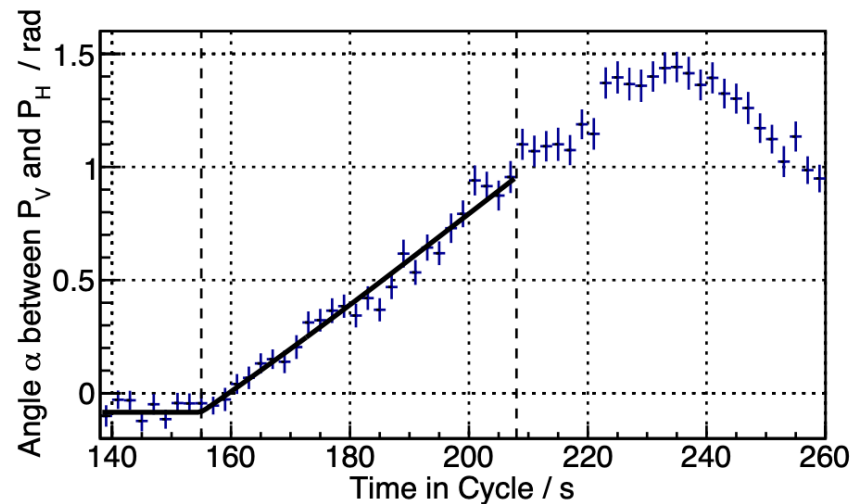
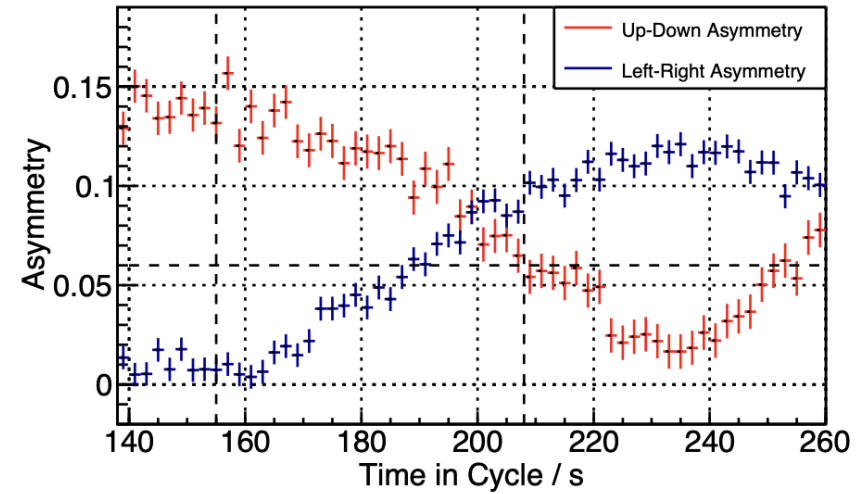
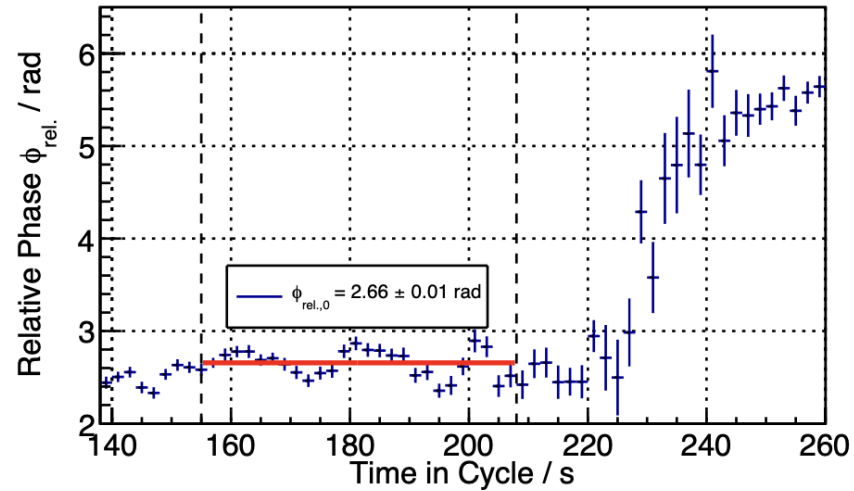
$$\epsilon = \frac{f_v}{f_{\text{rev}}} = \frac{1}{4\pi} \chi_0 |\vec{c} \times \vec{m}| = \frac{1}{4\pi} \chi_0 \sin \xi$$

$$\frac{d\alpha}{dt} = 2\pi \epsilon f_{\text{rev}} \cos \varphi_{\text{rel}}$$

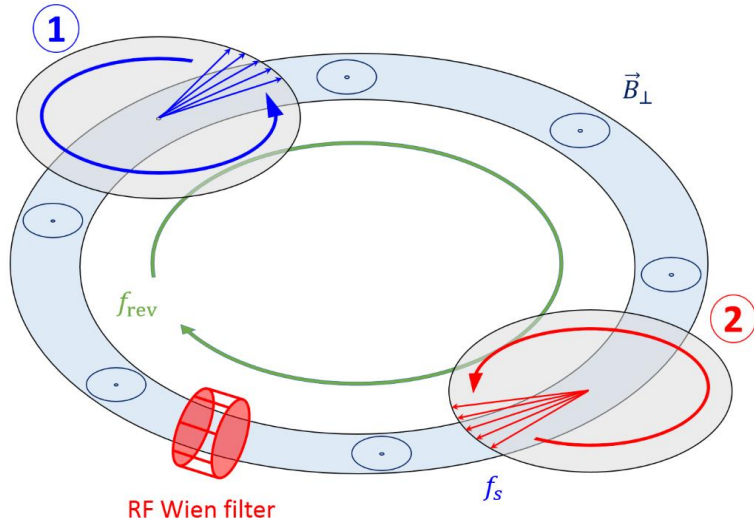


PHASE FEEDBACK

Slope Method - Limitations



PHASE FEEDBACK



Blue Bunch

$$\frac{d\alpha}{dn} = \frac{k}{2} \cos \varphi_{\text{rel}}$$

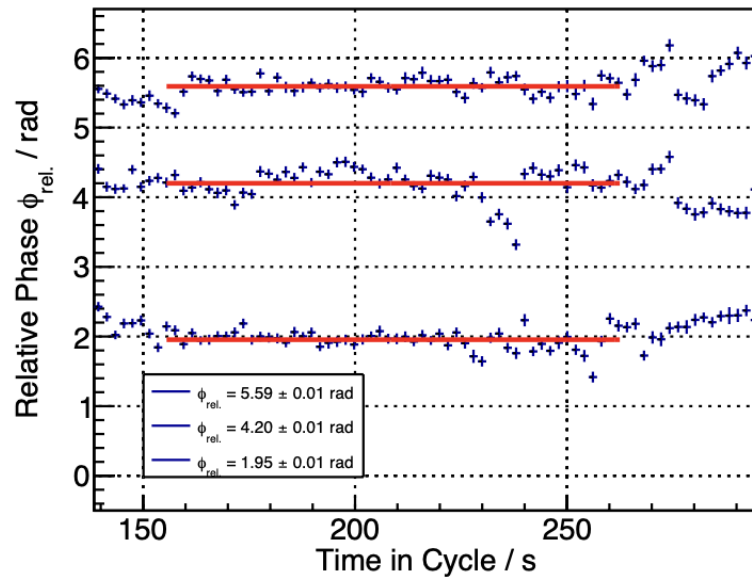
$$\frac{d\varphi_{\text{rel}}}{dn} = \frac{k}{2} (\sin \varphi_{\text{rel}} \tan \alpha + q) \rightarrow \frac{d\varphi_{\text{rel}}}{dn} = 0$$

Red Bunch

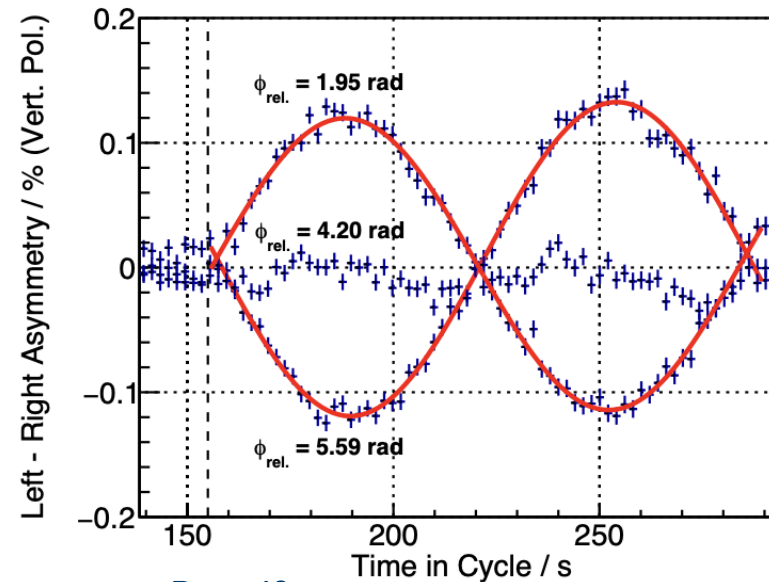
$$\frac{d\alpha}{dn} = \frac{k}{2} \cos \varphi_{\text{rel}}$$

$$\frac{d\varphi_{\text{rel}}}{dn} = \frac{k}{2} (\sin \varphi_{\text{rel}} \tan \alpha + q)$$

$$p_y(t) = \sin \alpha(t) = \cos \varphi_{\text{rel}}(0) \sin(2\pi \epsilon f_{\text{rev}} t).$$

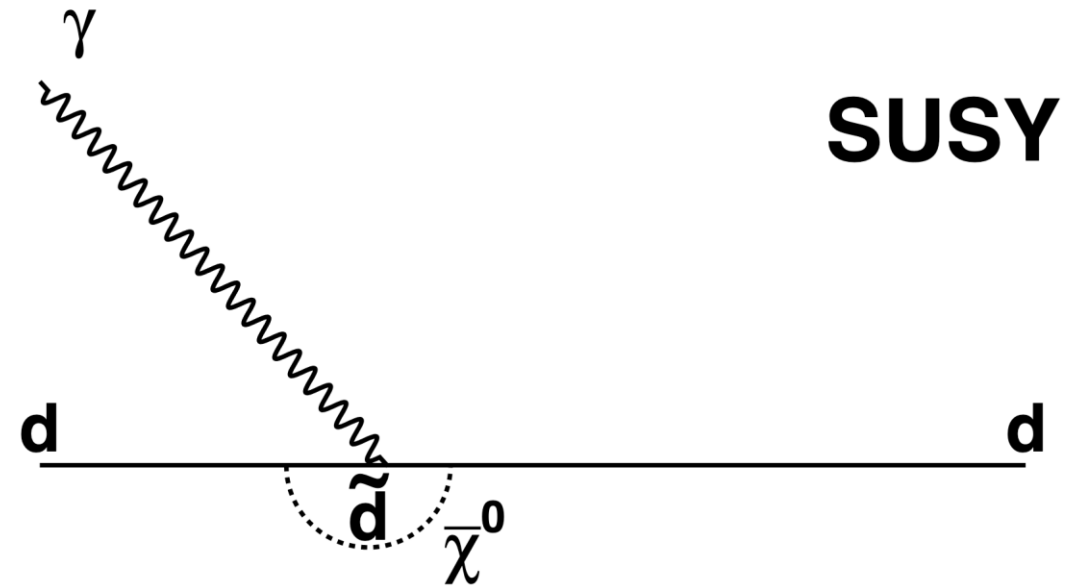
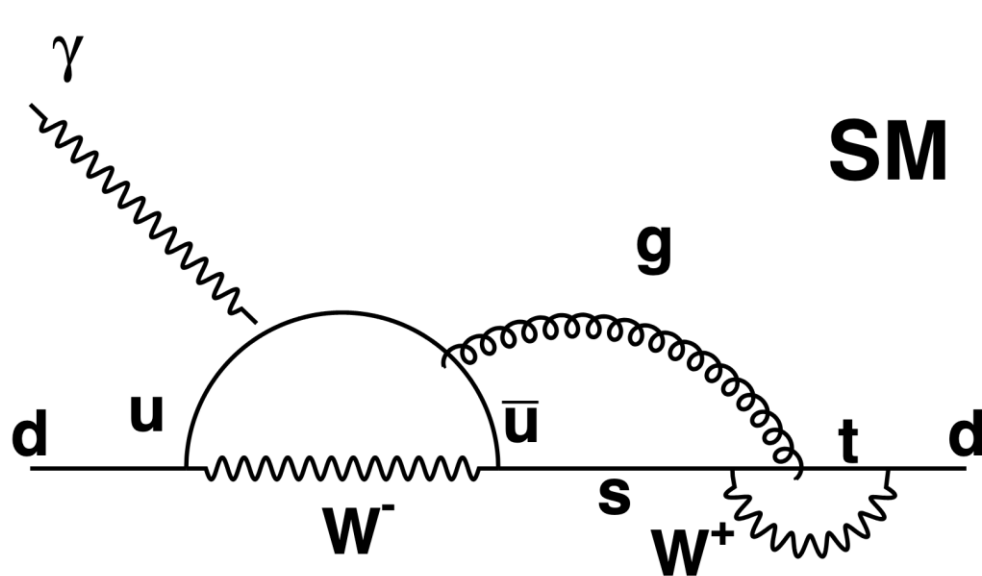


Blue Bunch



Red Bunch

EDM IN THE STANDARD MODEL AND SUSY



WHATS NEXT?

$$\frac{d\vec{S}}{dt} = (\vec{\Omega}_{\text{MDM}} + \vec{\Omega}_{\text{EDM}}) \times \vec{S},$$

$$\vec{\Omega}_{\text{MDM}} = -\frac{q}{m} \left[\left(G + \frac{1}{\gamma} \right) \vec{B} - \left(G + \frac{1}{\gamma + 1} \right) \vec{\beta} \times \frac{\vec{E}}{c} \right]$$

$$\vec{\Omega}_{\text{EDM}} = -\frac{q}{mc} \frac{\eta_{\text{EDM}}}{2} \left[\vec{E} + c\vec{\beta} \times \vec{B} \right].$$

Frozen Spin Condition

$$\vec{\Omega}_{\text{MDM}} - \vec{\Omega}_{\text{rev}} = -\frac{q}{m} \left[G\vec{B} - \left(G - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right] \stackrel{!}{=} 0.$$

For a pure magnetic ring (E=0)

$$G\vec{B}\gamma \stackrel{!}{=} 0 \quad \text{!}.$$

In an all – electric ring

$$\vec{\Omega}_{\text{MDM}} - \vec{\Omega}_{\text{rev}} = \frac{q}{m} \left(G - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \stackrel{!}{=} 0.$$

$$G - \frac{1}{\gamma^2 - 1} \stackrel{!}{=} 0 \Rightarrow p_{\text{magic}} = \frac{mc}{\sqrt{G}}.$$

Combined E – B Ring

$$B = E \cdot \frac{\beta^2 \gamma^2 G - 1}{c\beta \gamma^2 G}.$$

SPIN TUNE MAPPING

$$\frac{d\vec{S}}{d\theta} = \frac{d\vec{S}}{dt} \frac{dt}{d\theta} = \frac{1}{\omega_c} \vec{S} \times \vec{\Omega}_{\text{rs}} = -\frac{1}{\omega_c} \vec{\Omega}_{\text{rs}} \times \vec{S} = \vec{\omega} \times \vec{S}.$$

Spinor formalism

$$\psi = (u \ d)^T$$

$$\vec{S} \equiv \langle \psi | \vec{\sigma} | \psi \rangle = \psi^\dagger \vec{\sigma} \psi,$$

The TBMT equation of spin motion becomes

$$\frac{d\psi}{d\theta} = \frac{i}{2} (\vec{\sigma} \cdot \vec{\omega}) \psi = \frac{i}{2} \begin{pmatrix} \omega_z & \omega_x - i\omega_y \\ \omega_x + i\omega_y & -\omega_z \end{pmatrix} \psi.$$

$$\psi(\theta) = t_{\text{Ring}} \psi(\theta_0)$$

$$\mathbf{t}_{\text{Ring}} = e^{-i\pi\nu_s^0 \vec{\sigma} \cdot \vec{n}} = \cos(\pi\nu_s^0) I - i (\vec{\sigma} \cdot \vec{n}) \sin(\pi\nu_s^0)$$

$$\mathbf{t}_X = e^{-i\frac{\chi_X}{2} \vec{\sigma} \cdot \vec{k}} = \cos\left(\frac{\chi_X}{2}\right) I - i \left(\vec{\sigma} \cdot \vec{k}\right) \sin\left(\frac{\chi_X}{2}\right)$$

$$\mathbf{T} = \mathbf{t}_{\text{Ring}} \mathbf{t}_X$$

$$= \cos(\pi\nu_s^0) \cos\left(\frac{\chi_X}{2}\right) I - i \left(\vec{\sigma} \cdot \vec{k}\right) \sin\left(\frac{\chi_X}{2}\right) \cos(\pi\nu_s^0) \\ - i (\vec{\sigma} \cdot \vec{n}) \sin(\pi\nu_s^0) \cos\left(\frac{\chi_X}{2}\right) - (\vec{\sigma} \cdot \vec{n}) \left(\vec{\sigma} \cdot \vec{k}\right) \sin(\pi\nu_s^0) \sin\left(\frac{\chi_X}{2}\right)$$

$$\stackrel{\text{Eq. (C.3)}}{=} \cos(\pi\nu_s^0) \cos\left(\frac{\chi_X}{2}\right) I - (\vec{n} \cdot \vec{k}) I \sin(\pi\nu_s^0) \sin\left(\frac{\chi_X}{2}\right) \\ - i \vec{\sigma} \cdot \left[(\vec{n} \times \vec{k}) \sin(\pi\nu_s^0) \sin\left(\frac{\chi_X}{2}\right) + \vec{k} \sin\left(\frac{\chi_X}{2}\right) \cos(\pi\nu_s^0) + \vec{n} \sin(\nu_s^0) \cos\left(\frac{\chi_X}{2}\right) \right] \\ \stackrel{!}{=} \cos(\pi\nu_s(\chi_X)) I - i (\vec{\sigma} \cdot \vec{n}(\chi_X)) \sin(\pi\nu_s(\chi_X)).$$

$$\vec{n}(\chi_X) = \frac{1}{\sin(\pi\nu_s(\chi_X))} \cdot \left[(\vec{n} \times \vec{k}) \sin(\pi\nu_s^0) \sin\left(\frac{\chi_X}{2}\right) \right. \\ \left. + \vec{k} \sin\left(\frac{\chi_X}{2}\right) \cos(\pi\nu_s^0) + \vec{n} \sin(\nu_s^0) \cos\left(\frac{\chi_X}{2}\right) \right],$$

$$\cos(\pi\nu_s(\chi_X)) = \cos(\pi\nu_s^0) \cos\left(\frac{\chi_X}{2}\right) - (\vec{n} \cdot \vec{k}) \sin(\pi\nu_s^0) \sin\left(\frac{\chi_X}{2}\right).$$