Recent Progress on Inclusive Quarkonium Production and Polarization

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PRD Letter (2022); JHEP (2023)	Nora Brambilla, Hee Sok Chung, Antonio Vairo & XPW
PRD Letter (2025)	Zexuan Chu, Jinhui Chen, XPW & Hongxi Xing
PRD Letter (2025)	Nora Brambilla, Mathias Butenschön & XPW

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Outlines

- Introduction and Review
- Potential NRQCD in Inclusive Quarkonium Production
- How Well Does NRQCD Factorization Work at NLO?
- Summary and Outlook

Quarkonium: A multi-scale problem

- Quarkonium: Excellent probe of PDFs, GPDs, TMDs, QGP, nucleon helicity structure and so on. Referred as the QCD version of hydrogen atom – The simplest QCD system.

- Quarkonium system involves following typical scales (besides $\Lambda_{\rm QCD}$)
  - m_Q, heavy-quark mass scale, $m_c \sim 1.5$ GeV, $m_b \sim 4.75$ GeV;
  - $m_Q v$, heavy-quark momentum;
  - $m_Q v^2$, heavy-quark kinetic energy and binding energy;
  - v is the typical heavy-quark velocity in the quarkonium rest frame,
    $v^2 \simeq 0.25$ for charmonium; $v^2 \simeq 0.1$ for bottomonium.

- Quarkonium system is considered as nonrelativistic system, nonrelativistic QCD (NRQCD) is the default effective theory describing quarkonium system. Caswell & Lepage, PLB 167 (1986) 437-442

Nonrelativistic QCD (NRQCD) factorization

- NRQCD factorization is the most prominent approach to describe quarkonium production
  Bodwin, Braaten & Lepage, PRD 51, 1125 (1995), $3000+$ citations.

$$\sigma_{\mathcal{Q}+X} = \sum_n \hat{\sigma}\left(ij \to Q\bar{Q}(n) + X\right) \langle \mathcal{O}^{\mathcal{Q}}(n) \rangle, \tag{1}$$

  with $n = {}^{2S+1}L_J^{[1/8]}$, representing the quantum state of the heavy quark anti-quark pair.
- $\hat{\sigma}$, short-distance-coefficients (SDCs), α_s expansion,
- $\langle \mathcal{O}^{\mathcal{Q}}(n) \rangle$, long-distance-matrix-elements (LDMEs), supposed to be universal, v^2 expansion.
- Typically, the ${}^3S_1^{[1]}$ ${}^3S_1^{[8]}$, ${}^1S_0^{[8]}$, ${}^3P_J^{[8]}$ channels are considered in perturbative computations.
- NRQCD factorization for inclusive quarkonium production is a conjecture, not proved yet.
- Violations of NRQCD factorization for inclusive processes were found in specific cases, for instance, processes involving two or more quarkonia. He, Kniehl & **XPW**, PRL 121, (2018) 17, 172001

NRQCD long-distance-matrix elements (LDMEs)

Spin-1 S-wave quarkonium (V) LDMEs definitions:

$$\langle\mathcal{O}^V({}^3S_1^{[1]})\rangle = \langle\Omega|\chi^\dagger\sigma^i\psi\mathcal{P}_{V(\boldsymbol{P}=\boldsymbol{0})}\psi^\dagger\sigma^i\chi|\Omega\rangle, \tag{2a}$$

$$\langle\mathcal{O}^V({}^3S_1^{[8]})\rangle = \langle\Omega|\chi^\dagger\sigma^i T^a\psi\Phi_\ell^{\dagger ab}\mathcal{P}_{V(\boldsymbol{P}=\boldsymbol{0})}\Phi_\ell^{bc}\psi^\dagger\sigma^i T^c\chi|\Omega\rangle, \tag{2b}$$

$$\langle\mathcal{O}^V({}^1S_0^{[8]})\rangle = \langle\Omega|\chi^\dagger T^a\psi\Phi_\ell^{\dagger ab}\mathcal{P}_{V(\boldsymbol{P}=\boldsymbol{0})}\Phi_\ell^{bc}\psi^\dagger T^c\chi|\Omega\rangle, \tag{2c}$$

$$\begin{aligned}\langle\mathcal{O}^V({}^3P_0^{[8]})\rangle = {}& \frac{1}{3}\langle\Omega|\chi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})T^a\psi\Phi_\ell^{\dagger ab}\mathcal{P}_{V(\boldsymbol{P}=\boldsymbol{0})} \\ & \times \Phi_\ell^{bc}\psi^\dagger(-\tfrac{i}{2}\overleftrightarrow{\boldsymbol{D}}\cdot\boldsymbol{\sigma})T^c\chi|\Omega\rangle,\end{aligned} \tag{2d}$$

$\mathcal{P}_{V(\boldsymbol{P})} = \sum_{X}|V+X\rangle\langle V+X|$, $\Phi_\ell = P\exp[-ig\int_0^\infty d\lambda\,\ell\cdot A^{\mathrm{adj}}(\ell\lambda)]$ is the path-ordered Wilson line that ensures the gauge invariance.

- CSLDMEs can be related to wavefunction squared at the origin $|R(0)|^2$.
- Unclear how to calculate CO LDMEs using lattice QCD.
- CO LDMEs are determined through fitting with experimental data.

Heavy quark spin symmetry (HQSS) and relations between LDMEs

- For the spin-1 S-wave quarkonium V ($J/\psi, \Upsilon ...$), based on HQSS, we have

$$\langle \mathcal{O}^V(^3P_J^{[8]})\rangle = (2J+1)\langle \mathcal{O}^V(^3P_0^{[8]})\rangle(1+\mathcal{O}(v^2)). \tag{3}$$

So, for each spin-1 S-wave quarkonium V, we have 3 independent frequently used color-octet LDMEs $\langle \mathcal{O}^V(^3S_1^{[8]})\rangle$, $\langle \mathcal{O}^V(^1S_0^{[8]})\rangle$, $\langle \mathcal{O}^V(^3P_0^{[8]})\rangle$.

- Relations between the LDMEs of η_c and J/ψ due to HQSS,

$$\langle \mathcal{O}^{\eta_c}(^1S_0^{[1]}/^1S_0^{[8]})\rangle = \frac{1}{3}\langle \mathcal{O}^{J/\psi}(^3S_1^{[1]}/^3S_1^{[8]})\rangle(1+\mathcal{O}(v^2)), \tag{4}$$

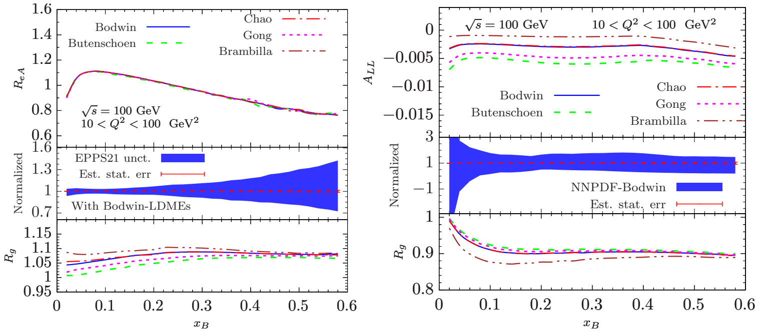
$$\langle \mathcal{O}^{\eta_c}(^3S_1^{[8]})\rangle = \langle \mathcal{O}^{J/\psi}(^1S_0^{[8]})\rangle(1+\mathcal{O}(v^2)), \tag{5}$$

$$\langle \mathcal{O}^{\eta_c}(^1P_1^{[8]})\rangle = 3\langle \mathcal{O}^{J/\psi}(^3P_0^{[8]})\rangle(1+\mathcal{O}(v^2)). \tag{6}$$

We will use the above relations between J/ψ and η_c to perform $J/\psi, \eta_c$ combined fit to constrain the 3 J/ψ color-octet LDMEs later.

Constraining nucleon gluon PDFs and helicity: J/ψ-tagged DIS at EIC

Chu, Chen, **XPW** & Xing, PRD 111 (2025)1, L011501



- R_{eA}: nuclear modification factor; A_{LL}: double longitudinal spin asymmetry; R_g: gluon contribution.
- J/ψ-tagged DIS at future EIC is ideal for constraining nucleon PDFs and helicity.
- Predictions based on NRQCD factorization, but how well does NRQCD factorization work?

A recent review: Testing universality of the LDMEs

Progress in Particle and Nuclear Physics 142 (2025) 104162

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Review

Physics case for quarkonium studies at the Electron Ion Collider

Table 3.1
Phenomenological comparison of a selection of existing J/ψ-η_c LDME extractions at NLO in α_s. The cut on the J/ψ transverse momentum, applied in each fit, is indicated in parentheses in the third column. This cut is applied because all but the first fit badly fail to account for the low-P_T data.

Acronym	Reference	J/ψ hadropr.	J/ψ photopr. and e^+e^-	J/ψ polar. in hadropr.	η_c hadropr. ($P_T > 6.5$ GeV)
BK11	Butenschön et al. [110–113]	✓($P_T > 3$ GeV)	✓	✗	✗
H14	Chao et al. + η_c [120]	✓($P_T > 6.5$ GeV)	✗	✓	✓
Z14	Zhang et al. [121]	✓($P_T > 6.5$ GeV)	✗	✓	✓
G13	Gong et al. [115]	✓($P_T > 7$ GeV)	✗	✓	✗
C12	Chao et al. [114]	✓($P_T > 7$ GeV)	✗	✓	✗
B14	Bodwin et al. [86]	✓($P_T > 10$ GeV)	✗	✓	✗
pNRQCD′	Brambilla et al. [116,122]	✓($P_T > 9$ GeV)	✗	✓	✗
pNRQCD	Brambilla et al. [116,122]	✓($P_T > 15$ GeV)	✗	✓	✓

J/ψ LDMEs fittings

Table: Selected representative fitting results in units of 10^{-2} GeV^3.

Group	$\langle\mathcal{O}^{J/\psi}(^3S_1^{[8]})\rangle$	$\langle\mathcal{O}^{J/\psi}(^1S_0^{[8]})\rangle$	$\langle\mathcal{O}^{J/\psi}(^3P_0^{[8]})\rangle/m_c^2$
Chao et al. set 1	0.05	7.4	0
Chao et al. set 2	1.11	0	1.89
Butenschön et al.	0.168 $\pm$ 0.046	3.04 $\pm$ 0.35	$-$ 0.404 $\pm$ 0.072
Zhang et al.	1.0 $\pm$ 0.3	0.74 $\pm$ 0.3	1.7 $\pm$ 0.5
Bodwin et al.	$-$ 0.713 $\pm$ 0.364	11 $\pm$ 1.4	$-$ 0.312 $\pm$ 0.151
Feng et al.	0.117 $\pm$ 0.058	5.66 $\pm$ 0.47	0.054 $\pm$ 0.005

- Fittings are based on NLO calculations, which are complicated, NNLO are infeasible in near future.
- Dramatically different LDME sets are fitted, but none of them can well describe all the data, challenging the LDME universality.

Score card of fittings

Table: Tests of the J/ψ LDMEs fits from high p_T pp, and low p_T γp, $e^+e^-, \gamma\gamma$ data.

Group	pp (p_T in fit)	pp (pol.)	pp (η_c)	$J/\psi + Z$	e^+e^-	γp	$\gamma\gamma$
Chao et al. set 1	✓ ($p_T > 7$GeV)	✓	✗	-	✗	✗	-
Chao et al. set 2	✓ ($p_T > 7$GeV)	✓	✓	-	✗	✗	-
Butenschön et al.	✓ ($p_T > 3$GeV)	✗	✗	✗	✓	✓	✗
Zhang et al. $+\eta_c$	✓ ($p_T > 7$GeV)	✓	✓	-	✗	✗	-
Bodwin et al.	✓ ($p_T > 10$GeV)	✓	✗	✗	✗	✗	-
Feng et al.	✓ ($p_T > 7$GeV)	✓	✗	-	✗	✗	-

- All the fits fail to describe the hadroproduction data with $p_T < 7$ GeV, except for Butenschön et al.
- Simplest explanation: NRQCD factorization fails at relatively low p_T (< 7GeV).
- It has been pointed out, NRQCD factorization fails for $\gamma p, e^+e^-$ at the end-point regions, $z \to 1$, $E_{J/\psi} \to E^{\max}_{J/\psi}$, respectively.

  Beneke, Rothstein & Wise, PLB 408, 373 (1997)

Spin-1 S-wave LDMEs in pNRQCD

- Based on potential NRQCD (pNRQCD), we have (up to $\mathcal{O}(1/N_c^2, v^2)$ corrections),
  Brambilla, Chung, Vairo & **XPW**, PRD105, L111503 (2022); JHEP 03 (2023) 242

$$\langle \mathcal{O}^V({}^3S_1^{[1]})\rangle = 2N_c \times \frac{3|R_V^{(0)}(0)|^2}{4\pi}, \tag{7a}$$

$$\langle \mathcal{O}^V({}^3S_1^{[8]})\rangle = \frac{1}{2N_c m^2}\frac{3|R_V^{(0)}(0)|^2}{4\pi}\mathcal{E}_{10;10}, \tag{7b}$$

$$\langle \mathcal{O}^V({}^1S_0^{[8]})\rangle = \frac{1}{6N_c m^2}\frac{3|R_V^{(0)}(0)|^2}{4\pi}c_F^2\mathcal{B}_{00}, \tag{7c}$$

$$\langle \mathcal{O}^V({}^3P_0^{[8]})\rangle = \frac{1}{18N_c}\frac{3|R_V^{(0)}(0)|^2}{4\pi}\mathcal{E}_{00}, \tag{7d}$$

- c_F is the NRQCD (HQET) matching coefficient,
- $R_V^{(0)}(0)$ is the wave-function at the origin,
- $\mathcal{E}_{10;10}$, $\mathcal{B}_{00}$, and $\mathcal{E}_{00}$ are gluonic correlators of mass dimension 2.

Gluonic correlators

$$\mathcal{E}_{10;10} = \bigg| d^{dac} \int_0^\infty dt_1\, t_1 \int_{t_1}^\infty dt_2\, gE^{b,i}(t_2) \times \Phi_0^{bc}(t_1;t_2) gE^{a,i}(t_1) \Phi_0^{df}(0;t_1) \Phi_\ell^{ef} |\Omega\rangle \bigg|^2, \tag{8a}$$

$$\mathcal{B}_{00} = \bigg| \int_0^\infty dt\, gB^{a,i}(t) \Phi_0^{ac}(0;t) \Phi_\ell^{bc} |\Omega\rangle \bigg|^2, \tag{8b}$$

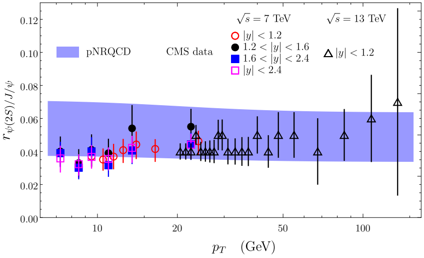
$$\mathcal{E}_{00} = \bigg| \int_0^\infty dt\, gE^{a,i}(t) \Phi_0^{ac}(0;t) \Phi_\ell^{bc} |\Omega\rangle \bigg|^2, \tag{8c}$$

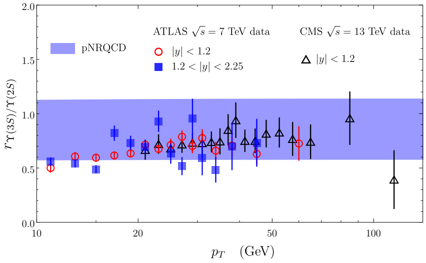
where $\Phi_0(t,t') = \mathcal{P}\exp[-ig \int_t^{t'} d\tau\, A_0^{\mathrm{adj}}(\tau, \mathbf{0})]$ is a Schwinger line.

- Gluonic correlators can be calculated using lattice QCD unlike CO LDMEs.
- By evolving the scale of $\mathcal{E}_{10;10}$, $\mathcal{B}_{00}$, and $\mathcal{E}_{00}$ from charm mass scale m_c to bottom mass scale m_b, we can related CO LDMEs between $\psi(nS)$ and $\Upsilon(nS)$.

pNRQCD predictive power

- Significantly reduces the number of independent CO LDMEs ($15 \to 3$).
- J/ψ and $\psi(2S)$ share the same $\mathcal{E}_{10;10}$, $\mathcal{B}_{00}$, and $\mathcal{E}_{00}$, their cross sections ratio equals the ratio of $|R^{(0)}_{J/\psi}(0)|^2$ and $|R^{(0)}_{\psi(2S)}(0)|^2$ (same for $\Upsilon(nS)$ states).





Figures from Brambilla, Chung, Vairo & **XPW**, JHEP 03 (2023) 242

- The prediction is based on NRQCD factorization and pNRQCD relations of the LDMEs without explicit perturbative calculations!

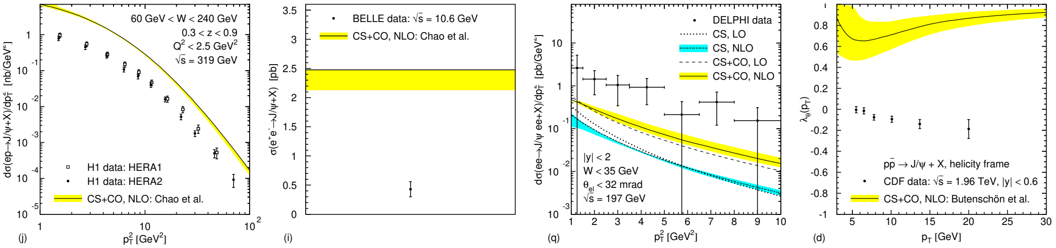
Score card of fittings - update 1

Table: Tests of the J/ψ LDMEs fits from high p_T pp, and low p_T γp, $e^+e^-, \gamma\gamma$ data.

Group	pp (p_T in fit)	pp (pol.)	pp (η_c)	$J/\psi + Z$	e^+e^-	γp	$\gamma\gamma$
Chao et al. set 1	✓ $(p_T > 7\,\mathrm{GeV})$	✓	✗	-	✗	✗	-
Chao et al. set 2	✓ $(p_T > 7\,\mathrm{GeV})$	✓	✓	-	✗	✗	-
Butenschön et al.	✓ $(p_T > 3\,\mathrm{GeV})$	✗	✗	✗	✓	✓	✗
Zhang et al. $+\eta_c$	✓ $(p_T > 7\,\mathrm{GeV})$	✓	✓	-	✗	✗	-
Bodwin et al.	✓ $(p_T > 10\,\mathrm{GeV})$	✓	✗	✗	✗	✗	-
Feng et al.	✓ $(p_T > 7\,\mathrm{GeV})$	✓	✗	-	✗	✗	-
pNRQCD	✓ $(p_T > 3 \times 2m_Q)$	✓	✗	✓(?)	✗	✗	-
pNRQCD	✓ $(p_T > 5 \times 2m_Q)$	✓	✓	✓(?)	✗	✗	-

- pNRQCD: fit the 3 gluonic correlators to high p_T hadroproduction data of $J/\psi, \psi(2S), \Upsilon(2S), \Upsilon(3S)$, constrains $\langle \mathcal{O}^{J/\psi}(^1S_0^{[8]}) \rangle$ to be small negative .
- Conflicts between descriptions for the low p_T and high p_T data still remain.

The remaining main conflicts/puzzles



Figures from Butenschön & Kniehl, Mod.Phys.Lett. A 28 (2013) 1350027.

- All high $p_T > 7$GeV fittings overshoot the low p_T $\gamma p, e^+e^-$ data by a factor of $\sim 5-10$, see left two figures (taking Chao et al. as an example).
- Global fit cannot describe the low p_T $\gamma\gamma$ data and the J/ψ polarization data, see right two figures.
- Conflict between low p_T and high p_T fittings and descriptions.

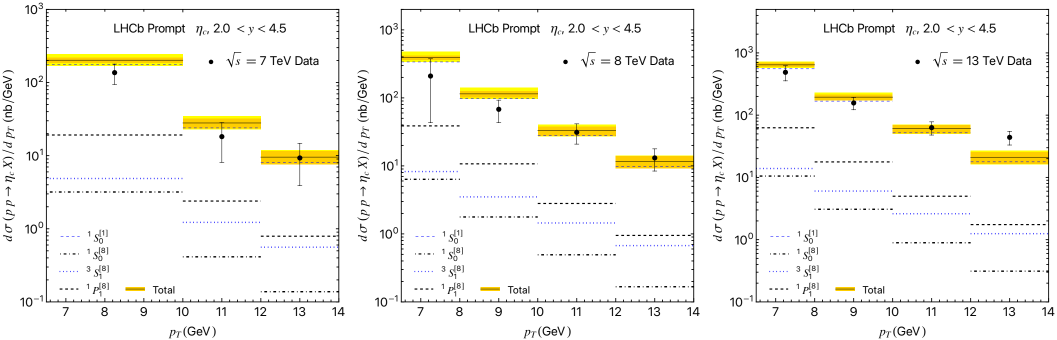
Our new fitting strategies and fitting results

- We combine LHC η_c and J/ψ data (42 data points) to fit 3 J/ψ CO LDMEs based on HQSS.
- We choose three different scale choices, $\mu_r = \mu_f = [\frac{1}{2}, 1, 2]m_T$, with the default scale choice $\mu_r = \mu_f = m_T$, where $m_T = \sqrt{4m_Q^2 + p_T^2}$;
- Systematically taking scale variations into account for the first time.
- $\psi(2S), \Upsilon(nS)$ CO LDMEs are related to those of J/ψ through pNRQCD relations. Feeddown from χ_{QJ} states are fitted from the measured data.

  We obtain (in units of 10^{-2} GeV^3),

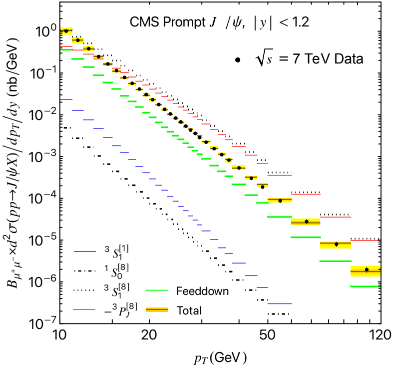
$\mu_r = \mu_f$	$\langle \mathcal{O}^{J/\psi}(^3S_1^{[8]}) \rangle$	$\langle \mathcal{O}^{J/\psi}(^1S_0^{[8]}) \rangle$	$\frac{\langle \mathcal{O}^{J/\psi}(^3P_0^{[8]}) \rangle}{m_c^2}$	$\frac{\chi^2_{\text{min}}}{\text{d.o.f}}$
$m_T/2$	0.592 ± 0.057	-0.205 ± 0.196	0.697 ± 0.089	0.34
m_T	1.050 ± 0.121	0.068 ± 0.2489	1.879 ± 0.261	0.22
$2m_T$	1.382 ± 0.189	0.358 ± 0.303	3.270 ± 0.533	0.21

Fitting results – LHCb η_c production



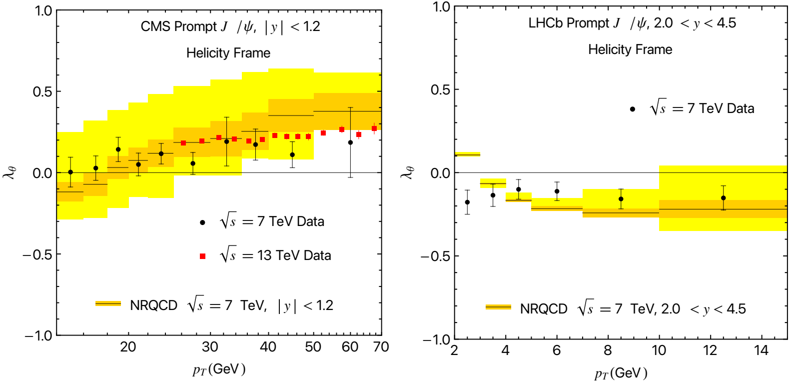
- Inner bands (orange) correspond to the default scale choice (both for SDCs and fitted LDMEs), the outer bands (yellow) encompass the uncertainties coming from the two other scale choices.
- CS channel saturates the cross sections and thus constrains $\langle \mathcal{O}^{J/\psi}(^1S_0^{[8]})\rangle$ to be small under HQSS.

Fitting results – CMS J/ψ production



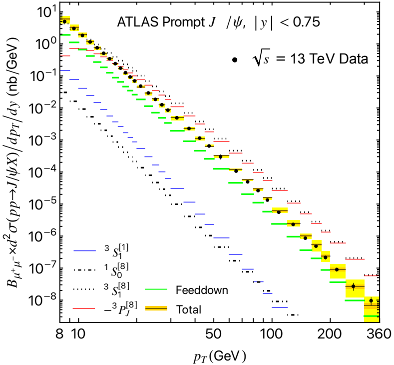
- Based on cancellation between large positive $^3S_1^{[8]}$ and large negative $^3P_J^{[8]}$ channels.
- This cancellation is not fine-tuning, because LDME mixing at NLO implies that only the sum of both contributions has physical significance.

Prediction – J/ψ polarization



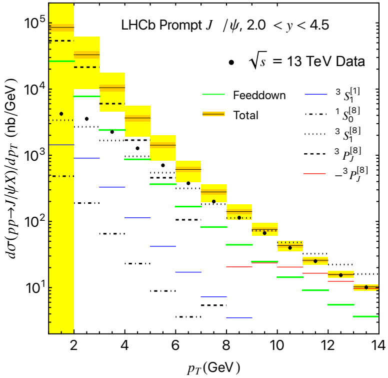
- In good agreement with the measurements and match the pattern that λ_θ turns from slightly negative at relatively low p_T to positive and converges to $\lambda_\theta \sim 0.3$ at high p_T.
- No polarization puzzle appears.

Prediction – ATLAS J/ψ production at very high p_T



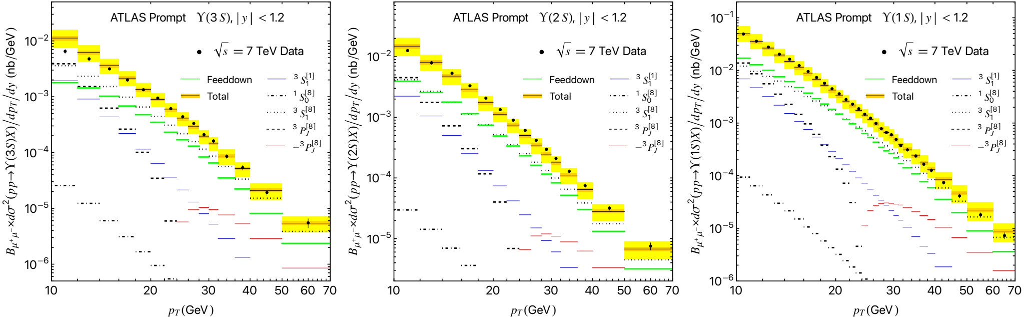
- Excellent description up to the highest measured p_T $(360\,\mathrm{GeV})$, suppising!
- A recent study on these very high p_T ATLAS data shows that the combined LP (large p_T) and threshold $(z \to 1)$ resummation has little effect on the fitting results. Chung, Kim & Lee, PRL 134, 071902 (2025)

Prediction – LHCb J/ψ production at low p_T



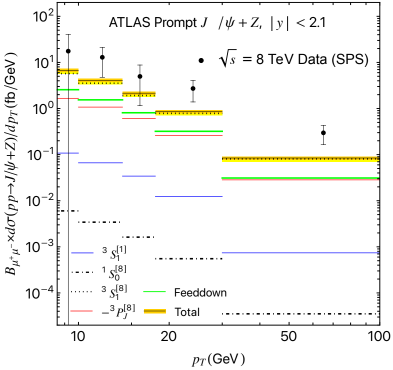
- Data with $p_T < 7$ GeV are not well described. Small-x resummation needed?

Prediction – ATLAS $\Upsilon(nS)$ production in pNRQCD



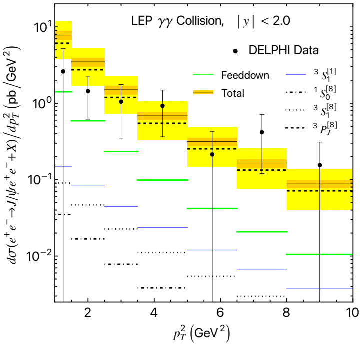
- $\Upsilon(1S)$, $\Upsilon(2S)$, $\Upsilon(3S)$ data well reproduced, highly nontrivial test of the pNRQCD relations.

Prediction – ATLAS $J/\psi + Z$, single parton scattering (SPS)



- For the two highest p_T bins, predictions lie $\sim 2\sigma$ deviations below data.
  Underestimated DPS contributions, unlikely? or?

Prediction – LEP $\gamma\gamma \to J/\psi + X$



- The cross section is exclusively dominated by single-resolved photon contributions. CS contribution is far below the data. $^3P_J^{[8]}$ channels dominate.

Prediction – HERA $\gamma p \to J/\psi + X$ $(0.1 < z < 0.6)$

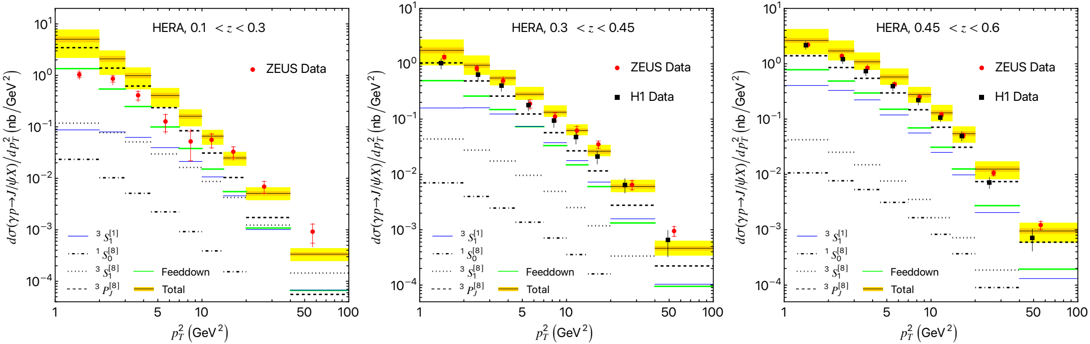
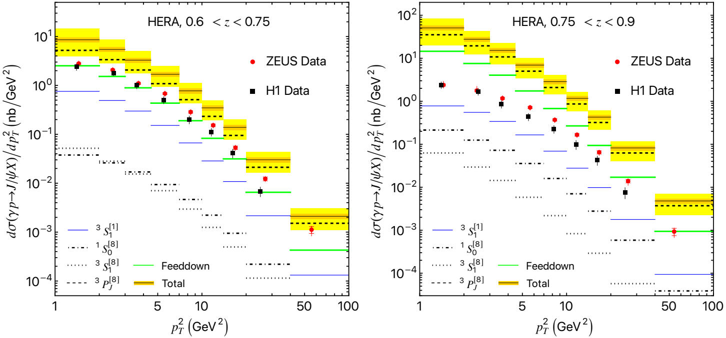


Figure: Prediction with divided z bins. Inelasticity $z = E_{J/\psi}/E_\gamma$ in the proton rest frame.

- For $0.1 < z < 0.6$, good description for all the data except for a few lowest p_T bins for $0.1 < z < 0.3$, where resolved photon ($gg \to J/\psi + X$, similar to hadroproduction) contribution dominates.

Prediction – HERA $\gamma p \to J/\psi + X$ $(0.6 < z < 0.9)$



- Obviously overshoot the data, regardless of p_T. For $0.75 < z < 0.9$, predictions overshoot the data by factors of 5.2 to 20.
- The region $z \to 1$ corresponds to the endpoint region, where the NRQCD factorization may not be valid, v^2 expansion becomes $v^2/(1-z)$ expansion. Beneke, Rothstein & Wise, PLB 408, 373 (1997).

Score card of fittings - update 2

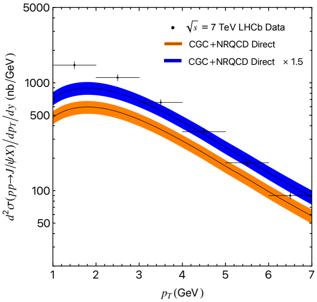
Table: Tests of the J/ψ LDMEs fits from high p_T pp, and low p_T γp, $e^+e^-, \gamma\gamma$ data.

Group	pp (p_T in fit)	pp (pol.)	pp (η_c)	$J/\psi + Z$	e^+e^-	γp	$\gamma\gamma$
Chao et al. set 1	✓ ($p_T > 7\,\mathrm{GeV}$)	✓	✗	-	✗	✗	-
Chao et al. set 2	✓ ($p_T > 7\,\mathrm{GeV}$)	✓	✓	-	✗	✗	-
Butenschön et al.	✓ ($p_T > 3\,\mathrm{GeV}$)	✗	✗	✗	✓	✓	✗
Zhang et al. $+\eta_c$	✓ ($p_T > 7\,\mathrm{GeV}$)	✓	✓	-	✗	✗	-
Bodwin et al.	✓ ($p_T > 10\,\mathrm{GeV}$)	✓	✗	✗	✗	✗	-
Feng et al.	✓ ($p_T > 7\,\mathrm{GeV}$)	✓	✗	-	✗	✗	-
pNRQCD	✓ ($p_T > 3 \times 2m_Q$)	✓	✗	✓(?)	✗	✗	-
pNRQCD	✓ ($p_T > 5 \times 2m_Q$)	✓	✓	✓(?)	✗	✗	-
Brambilla, Butenschön & **XPW**	✓ ($p_T > 7\,\mathrm{GeV}$)	✓	✓	✓(?)	✗	✓($z < 0.6$)	✓

- Less likely, the conflicts result from NRQCD factorization violations at relatively low p_T.

The remaining puzzles and possible solutions

- Observables still evade a consistent description: coincide with "extensions" of endpoint regions.
- Low p_T hadroproduction ✗ Possible solution: small-x resummation
- J/ψ photoproduction ($z > 0.6$), J/ψ from Belle ✗ Possible solution: quarkonium shape function



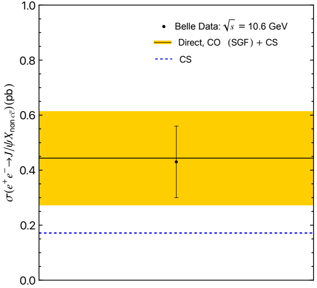


Figure: Plots unpublished. SDCs from Ma & Venugopalan, PRL 113, 192301 (2014); Chen, Jin, Ma & Meng, JHEP 03 (2022), 202

Summary and outlook

Summary:

- NRQCD factorization works pretty well except for end-point regions, where resummations are needed.
- Good descriptions for $\Upsilon(nS)$ production, highly nontrivial tests of the pNRQCD relations for LDMEs.

pp High p_T ($J/\psi, \eta_c, \Upsilon(nS)$, pol.)	$J/\psi + Z$	e^+e^-	γp	$\gamma\gamma$	pp (Low p_T, J/ψ)
✓	✓(?)	✓(SGF)	✓($z < 0.6$)	✓	✓ (small-x resum)

Outlook:

- Further study is needed to confirm or disprove: conflicts are not due to NRQCD factorization violations at relatively low p_T, but end-point regions.
- $z > 0.6$ for $\gamma p \to J/\psi X$, quarkonium shape functions, soft gluon factorization (SGF) ...
- Include inclusive quarkonium production and polarization in PDF fits.
- First lattice calculation of CO decay LDMEs in pNRQCD.