An Improved Formula for Wigner Function and Spin Polarization at Local Thermodynamic Equilibirum

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Spin 2025

Sep. 22 - 26, 2025

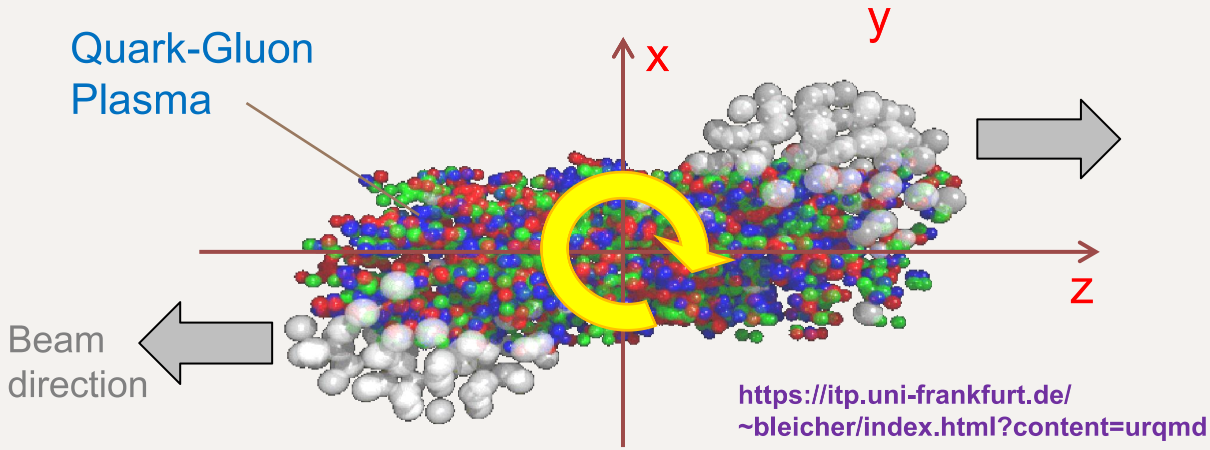
Contents

- Introduction
- New method for hypersurface integral
- Application in hyperon polarization
- Conclusions

XLS, F. Becattini, D. Roselli, arXiv: 2509.14301

Heavy-ion collisions

Strongly interacting matter with vorticity and magnetic fields



Initial orbital angular momentum → Vorticity field, Magnetic field, Strong field → Polarized quark/gluon → Spin polarization for baryons, Λ, Σ^0, Δ^{++}, Ω^-, ...

Spin alignment for vector mesons, ϕ, K^{*0}, ρ^0, ...

S. A. Voloshi, nucl-th/0410089

Z.-T. Liang, X.-N. Wang, PRL 94, 102301 (2005) ; PLB 629, 20 (2005)

F. Becattini, F. Piccinini, J. Rizzo, PRC 76, 044901 (2007)

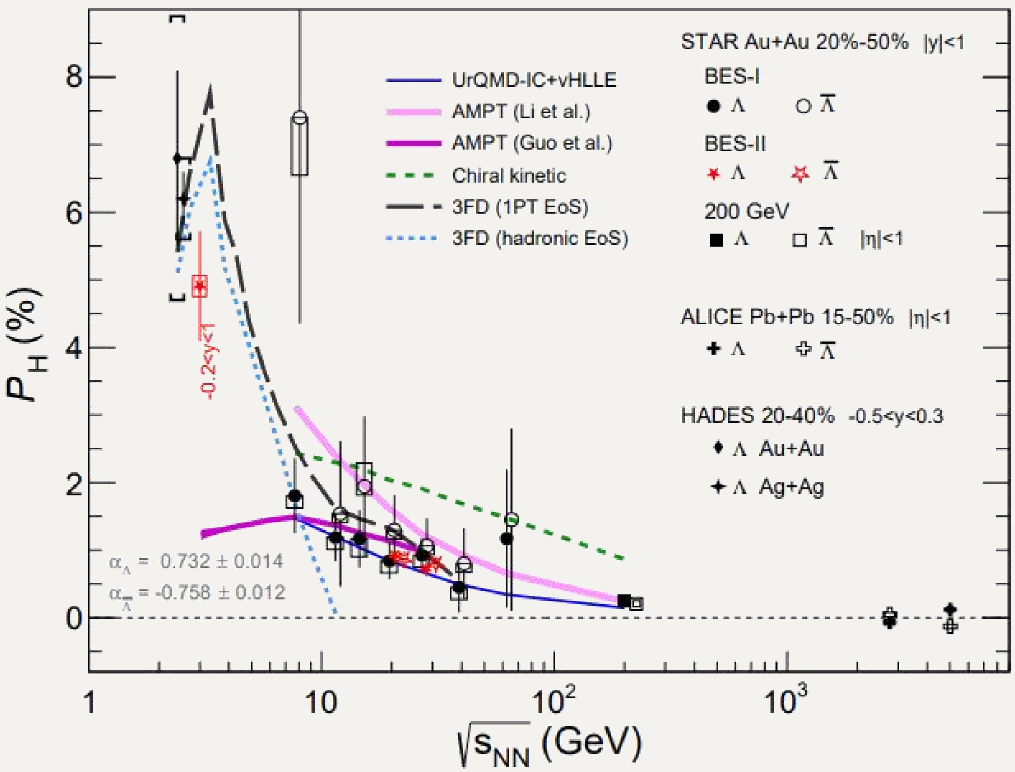
Plenary talks on Sep. 25:

T. Niida, 10:10 a.m. (Spin experiments)

F. Becattini, 10:50 a.m. (Spin theories)

J. Liao, 11:30 a.m. (Chiral magnetic eff.)

Global spin polarization



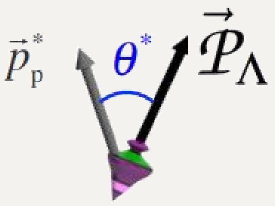
STAR, Nature 548, 62 (2017); Phys. Rev. C 104, L061901 (2021); Phys. Rev. C 98, 014910 (2018);Phys. Rev. C 108, 014910 (2023)

F. Becattini, M. Buzzegoli, T. Niida, S. Pu, A.-H. Tang, Int. J. Mod. Phys. E 33, 2430006 (2024).

Polarization along direction of global OAM

Evidence of vorticity field !

Measured by pairty-violating weak decay



$$\Lambda \rightarrow p + \pi^-$$

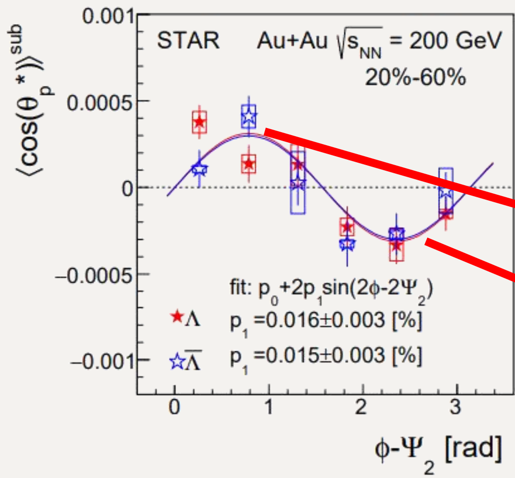
$$\frac{dN}{d\cos\theta^*} = \frac{1}{2}\left(1 + \alpha_H |\vec{\mathcal{P}}_H| \cos\theta^*\right)$$

α_H: decay constant
$\vec{P}_H$: Λ polarization
$\vec{p}_p^{\,*}$: proton momentum

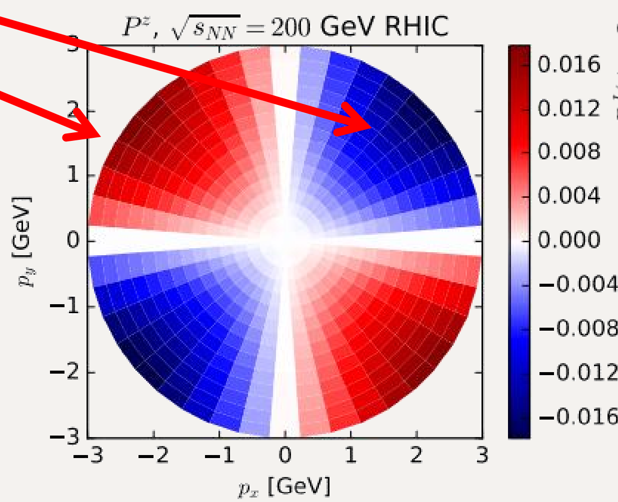
Local spin polarization

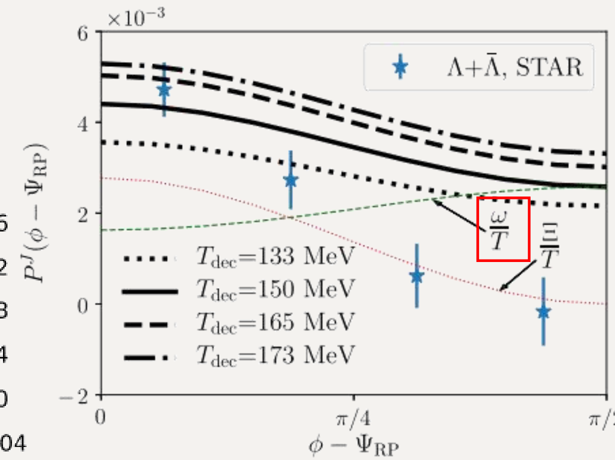
- Vorticity-induced polarization

$$S^\mu(p) = -\frac{1}{8m}\epsilon^{\mu\rho\sigma\tau}p_\tau \frac{\int_\Sigma d\Sigma_\lambda p^\lambda \varpi_{\rho\sigma} n_F(1-n_F)}{\int_\Sigma d\Sigma_\lambda p^\lambda n_F} \qquad \varpi_{\mu\nu} \equiv \frac{1}{2}(\partial_\nu\beta_\mu - \partial_\mu\beta_\nu)$$



Spin sign puzzle!





Polarization in longitudinal direction: experiments vs hydro simulations

Azimuthal angle dep. of polarization in global OAM direction

STAR, Nucl. Phys. A 982, 511 (2019); Phys.Rev.Lett. 123 (2019) 13, 132301.
F. Becattini, I. Karpenko, Phys.Rev.Lett. 120 (2018) 1, 012302
F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, and A. Palermo, Phys. Rev. Lett. 127, 272302 (2021).
F. Becattini, M. Buzzegoli, T. Niida, S. Pu, A.-H. Tang, Int. J. Mod. Phys. E 33, 2430006 (2024)

Local spin polarization

- Shear-induced polarization $\xi_{\mu\nu} = \frac{1}{2}\left(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu\right)$

$$S^\mu_{\text{shear}}(p) \begin{cases} -\frac{1}{4mN_p}\epsilon^{\mu\rho\sigma\tau}p_\tau \int_\Sigma \mathrm{d}\Sigma \cdot p\, n_F(1-n_F)\frac{\hat{t}_\rho}{\hat{t}\cdot p}\xi_{\sigma\lambda}p^\lambda \\ -\frac{1}{4mN_p}\epsilon^{\mu\rho\sigma\tau}p_\tau \int_\Sigma \mathrm{d}\Sigma \cdot p\, n_F(1-n_F)\frac{u_\rho}{u\cdot p}\xi_{\sigma\lambda}p^\lambda \\ -\frac{1}{4mN_p}\epsilon^{\mu\rho\sigma\tau}p_\tau \int_\Sigma \mathrm{d}\Sigma \cdot p\, n_F(1-n_F)\frac{n_\rho}{n\cdot p}\xi_{\sigma\lambda}p^\lambda \end{cases}$$

F. Becattini, M. Buzzegoli, A. Palermo, Phys.Lett.B 820 (2021) 136519

S. Liu, Y. Yin, JHEP 07 (2021) 188

Y.-C. Liu, X.-G. Huang, Sci.China Phys.Mech.Astron. 65 (2022) 7, 272011;

- Unit vector in time direction, fluid velocity, or arbitrary timelike vector?

- Additional assumptions:
  Isothermal local equilibrium - neglecting *T*-gradient
  “Strange memory” scenario - using quark mass instead of hyperon mass

- Non-dissipative or dissipative?
  J.-R. Wang, S. Fang, D.-L. Yang, S. Pu, arXiv: 2507.15238

- In this talk:
  Normal vector of hypersurface, automatical vanishing *T*-gradient

Related talks on Sep. 23 & 24, Parallel session - Spin in HIC:

Q. Wang, K. Sun, S. Pu, C. Yi, B. Fu, (Numerical cals.)

P. Das, T. Fu, C. Li (Experiments)

X.-G. Huang, J.-R. Wang (Theories)

Zubarev approach

- Density operator (invariant in Heisenberg picture)

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\int_{\Sigma_0} \mathrm{d}\Sigma_\mu(y)\ \widehat{T}^{\mu\nu}(y)\beta_\nu(y)\right]$$

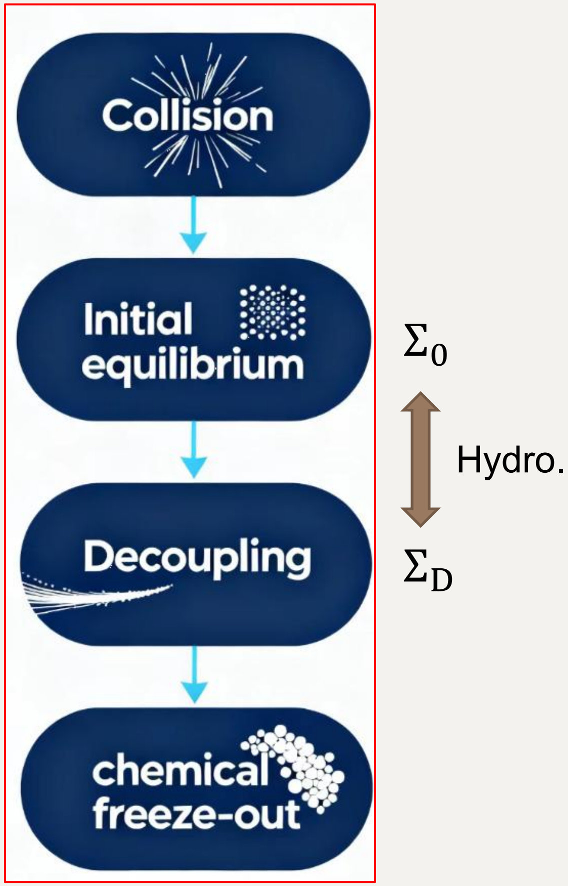
Gauss' theorem

$$\int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma_\mu(y)\ \widehat{T}^{\mu\nu}(y)\beta_\nu(y) + \int_\Omega \mathrm{d}^4 y\ \widehat{T}^{\mu\nu}(y)\partial_\mu\beta_\nu(y)$$

LTE at decoupling — Dissipation

- Mean value at LTE

$$O_{\mathrm{LE}}(x) \equiv \frac{1}{Z}\mathrm{Tr}\left(\exp\left[-\int_{\Sigma_\mathrm{D}} \mathrm{d}\Sigma_\mu(y)\widehat{T}^{\mu\nu}(y)\beta_\nu(y)\right]\widehat{O}(x)\right)$$



D. N. Zubarev, Soviet Physics Doklady 10, 850 (1966).
D. N. Zubarev, A. V. Prozorkevich, and S. A. Smolyanskii, Theor. Math. Phys. 40, 821 (1979).
D. N. Zubarev and M. V. Tokarchuk, Teor. Mat. Fiz. 88N2, 286 (1991).
C. van Weert, Annals of Physics 140, 133 (1982).
F. Becattini, M. Buzzegoli, and E. Grossi, Particles 2, 197 (2019).

Zubarev approach

- Expansion around global equilibrium

$$\beta_\nu(y) = \beta_\nu(x) + \Delta\beta_\nu(y,x)$$

$$O_{\rm LE}(x) = \frac{1}{Z}{\rm Tr}\left(\exp\left[-\beta(x)\cdot\widehat{P} - \int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu(y)\,\widehat{T}^{\mu\nu}(y)\Delta\beta_\nu(y)\right]\widehat{O}(x)\right)$$

$$O_{\rm GE}(x) \equiv \frac{{\rm Tr}[e^{-\beta(x)\cdot\widehat{P}}\widehat{O}(x)]}{{\rm Tr}[e^{-\beta(x)\cdot\widehat{P}}]}$$

GTE mean value

$$\Delta O_{\rm LE}(x) = -\int_{\Sigma_{\rm D}} {\rm d}\Sigma_\mu(y)\int_0^1 {\rm d}z\,\Delta\beta_\nu(y,x) \times\left\langle \widehat{O}(x),\, {\rm e}^{-z\beta(x)\cdot\widehat{P}}\widehat{T}^{\mu\nu}(y){\rm e}^{z\beta(x)\cdot\widehat{P}}\right\rangle_{c,{\rm GE}}$$

Linear response to $\Delta\beta$

- Hydrodynamic limit:
  β is slowly varing in spacetime

$$\Delta O_{\rm LE} \ll O_{\rm GE}$$

Keeps all orders in gradients

$$\Delta\beta_\nu \approx (y-x)^\lambda\partial_\lambda\beta_\nu(x) + \cdots$$

F. Becattini, M. Buzzegoli, A. Palermo, Phys.Lett.B 820 (2021) 136519
XLS, F. Becattini, D. Roselli, arXiv: 2509.14301

Wigner function

- Wigner operator for spin-1/2 fermions

$$\widehat{W}^{+}_{ab}(x,p) \equiv \theta(p^0)\int \frac{\mathrm{d}^4 y}{(2\pi)^4}\mathrm{e}^{-ip\cdot y}\overline{\Psi}_b\left(x+\frac{y}{2}\right)\Psi_a\left(x-\frac{y}{2}\right)$$

- Plane-wave quantization

$$\widehat{W}^{+}(x,p) = \frac{1}{2(2\pi)^3}\sum_{r,s}\int \mathrm{d}^4 q\, e^{iq\cdot x}\theta(p^0_+)\theta(p^0_-)\delta(p\cdot q) \times\delta\left(p^2+\frac{q^2}{4}-m^2\right)\widehat{a}^{\dagger}_r(p_+)\widehat{a}_s(p_-)u_s(p_-)\overline{u}_r(p_+)$$

$$p^\mu_\pm \equiv p^\mu \pm \frac{q^\mu}{2}$$

On-shell

- Linear response to $\Delta\beta$

$$\Delta W^{+}_{\mathrm{LE}}(x,p) = \frac{1}{2(2\pi)^3}\sum_{r,s}\int_{\Sigma_{\mathrm{D}}}\mathrm{d}\Sigma_\mu(y)\int \mathrm{d}^4 q\; e^{iq\cdot(x-y)}\theta(p^0_+)\theta(p^0_-) \times\frac{1-e^{\beta(x)\cdot q}}{\beta(x)\cdot q}\delta(p\cdot q)\delta\left(p^2+\frac{q^2}{4}-m^2\right)u_s(p_-)\overline{u}_r(p_+) \times\left\langle \widehat{a}^{\dagger}_r(p_+)\widehat{a}_s(p_-),\ \widehat{T}^{\mu\nu}(0)\right\rangle_{c,\mathrm{GE}}\Delta\beta_\nu(y,x)$$

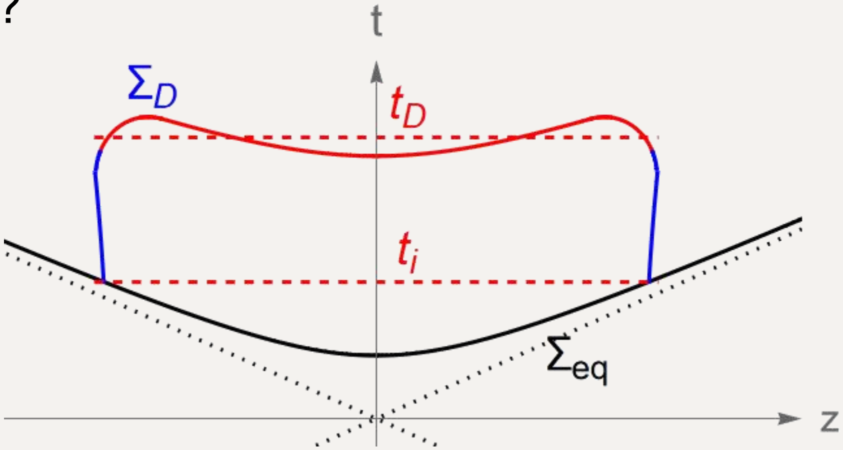
Depend on y

- How to deal with the hypersurface integral?

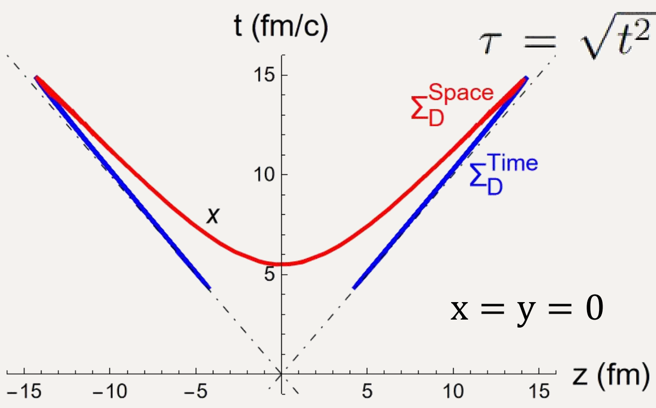
$$F_{\mu\nu}(q) = \int_{\Sigma_D} d\Sigma_\mu(y)\, e^{-iq\cdot(y-x)} \Delta\beta_\nu(y,x)$$

Nearly constant-t on spacelike part of Σ_D
Fluid's spatial region is sufficiently large

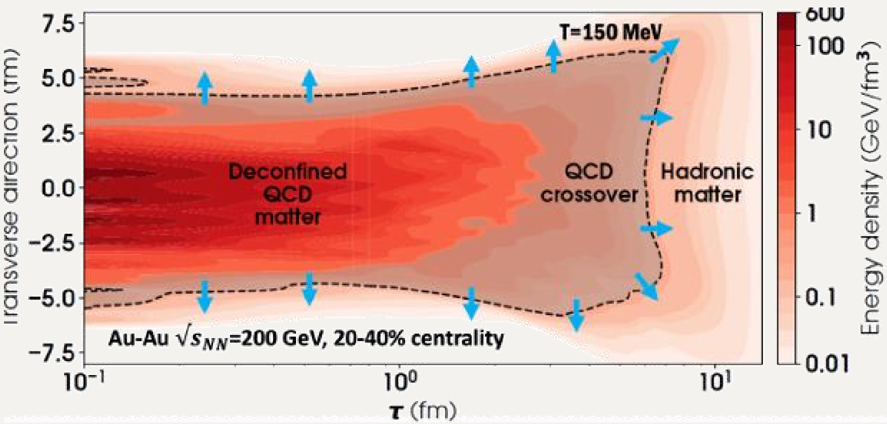
$$\int_{\Sigma_D} d\Sigma_\mu(y) \approx \hat{t}_\mu \int d^3\mathbf{y}\Big|_{y^0=t_i} + \int_{t_i}^{t_D} dy^0 \int d^3\mathbf{y}\, \partial_\mu^y$$



- Hydro-simulation: nearly contant-τ



CLVisc simulation @ Au+Au 27GeV



C. Gale, J.-F. Paquet, B. Schenke, S. Chun, PoS HardProbes2020 (2021) 039

H. Elfner, B. Muller, J.Phys.G 50 (2023) 10, 103001

Small-q expansion

- Expansion in hydrodynamic limit

$$\Delta W^{+}_{\rm LE}(x,p) = \frac{1}{(2\pi)^3}\int \mathrm{d}^4q\ \delta(p\cdot q)G^{\mu\nu}(q)F_{\mu\nu}(q)$$

$$F_{\mu\nu}(q) = \int_{\Sigma_{\rm D}} \mathrm{d}\Sigma_\mu(y)\ \mathrm{e}^{-iq\cdot(y-x)}\Delta\beta_\nu(y,x)$$

In hydrodynamic limit:
$\Delta\beta$ is slowly varing,
$F_{\mu\nu}$ is peaked at $q=0$

$$G^{\mu\nu}(q) = \frac{1}{2}\delta\left(p^2+\frac{q^2}{4}-m^2\right)\frac{1-\mathrm{e}^{\beta(x)\cdot q}}{\beta(x)\cdot q}\theta(p^0_+)\theta(p^0_-)$$
$$\times\sum_{r,s}u_s(p_-)\bar{u}_r(p_+)\left\langle\widehat{a}^\dagger_r(p_+)\widehat{a}_s(p_-),\widehat{T}^{\mu\nu}(0)\right\rangle_{c,\rm GE}$$

Non-divergent at $q=0$

$$=\sum_{N=0}^{\infty}\frac{1}{N!}\left[\partial^q_{\nu_1}\partial^q_{\nu_2}\cdots\partial^q_{\alpha_N}G^{\mu\nu}(q)\right]\Big|_{q^\mu=0}q^{\nu_1}q^{\nu_2}\cdots q^{\nu_N}$$

Taylor expansion in q

XLS, F. Becattini, D. Roselli, arXiv: 2509.14301

Small-q expansion

- Exchanging the order of integrals

$$\Delta W^{+}_{\rm LE}(x,p) = \frac{1}{(2\pi)^3}\sum_{N=0}^{\infty}\frac{1}{N!}\left[\partial^q_{\nu_1}\partial^q_{\nu_2}\cdots\partial^q_{\nu_N}G^{\mu\nu}(q)\right]\Big|_{q=0} \times\int_{\Sigma_{\rm D}}{\rm d}\Sigma_\mu(y)\, I_N^{\nu_1\nu_2\cdots\nu_N}(y-x)\Delta\beta_\nu(y,x)$$

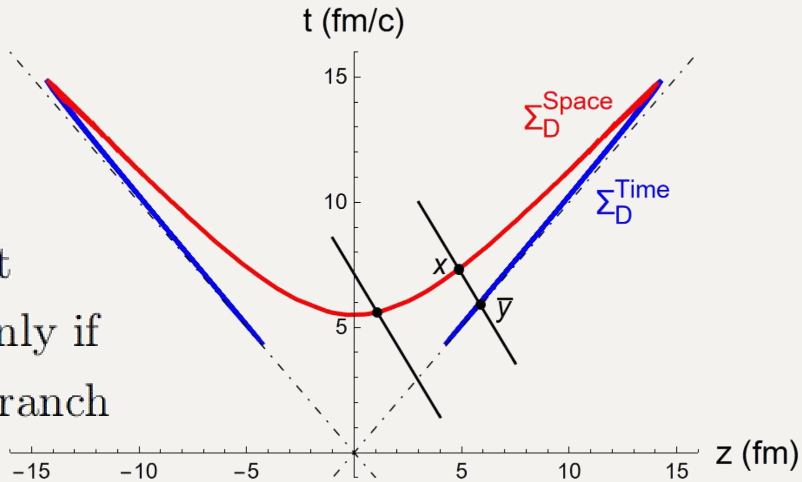
- Momentum integral

$$I_N^{\nu_1\nu_2\cdots\nu_N}(y-x) \equiv \int {\rm d}^4q\,\delta(p\cdot q){\rm e}^{-iq\cdot(y-x)}q^{\nu_1}\cdots q^{\nu_N} = (2\pi)^3\frac{(-i)^N}{|p^0|}\partial_x^{\nu_1}\cdots\partial_x^{\nu_N}\delta^3\left(\mathbf{y}-\mathbf{x}-\frac{\mathbf{p}}{p^0}(y^0-x^0)\right)$$

Singular at intersections of $\Sigma_{\rm D}$ and classical trajectory

$$\mathbf{y} = \mathbf{x} + \frac{\mathbf{p}}{p^0}(y^0-x^0)$$

$$\Rightarrow\; y^\mu = \bar{y}^\mu(x,p)\begin{cases} = x & \text{trivial, always exist} \\ \neq x & \text{non-trivial, exist only if } \Sigma_{\rm D} \text{ has time-like branch}\end{cases}$$



- Linear response of LTE Wigner function

$$\Delta W^{+}_{\rm LE}(x,p) = \sum_{N=0}^{\infty} \sum_{\bar{y}(x,p)} \frac{1}{N!} D_y^N(\bar{y}) \left[G^{\mu\nu}(q) \frac{n_\mu(y)}{|p\cdot n(y)|} \Delta\beta_\nu(y,x) \right] \Bigg|_{q=0,y=\bar{y}(x,p)}$$

order in gradient

② ①

$$D_y(\bar{y}) \equiv -i\Delta^{\nu\rho}(\bar{y})\partial^y_\rho \partial^q_\nu$$

$$\Delta^{\nu\rho}(\bar{y}) = g^{\nu\rho} - \frac{n^\nu(\bar{y})p^\rho}{p\cdot n(\bar{y})}$$

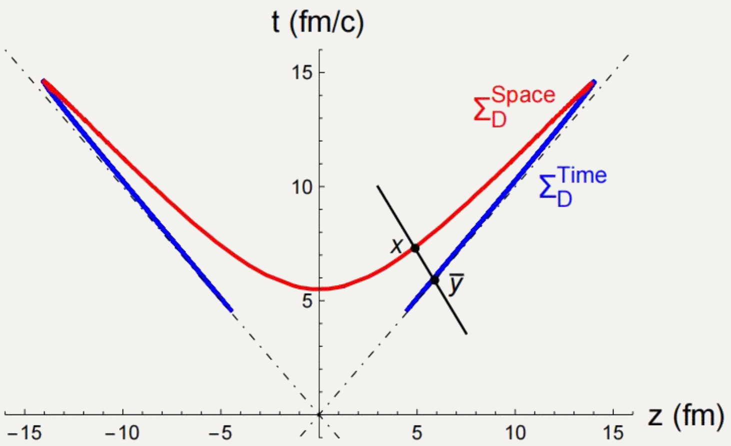
$$\boxed{\Delta^{\nu\rho}(\bar{y})n_\rho(\bar{y}) = 0}$$

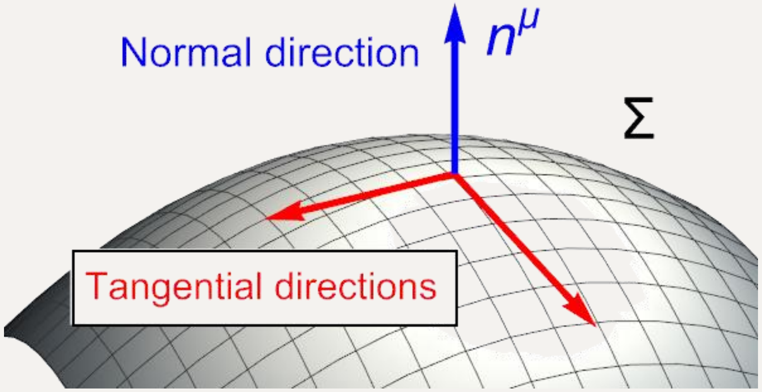
No spacetime derivative in normal direction

- Convergence conditions

$$l_c/L_H \ll 1, \quad l_c/L_G \ll 1\,.$$

- Sum over all intersections





Spin polarization

- Average spin polarization

$$S^\mu(p) = \frac{\int_{\Sigma_D} d\Sigma \cdot p\, \mathrm{tr}\left[\gamma^\mu \gamma^5 W^+(x,p)\right]}{2\int_{\Sigma_D} d\Sigma \cdot p\, \mathrm{tr}\left[W^+(x,p)\right]}$$

$$\Delta W^+_{\mathrm{LE}}(x,p) = \sum_{N=0}^{\infty} \sum_{\bar{y}(x,p)} \frac{1}{N!} D_y^N(\bar{y}) \times \left[G^{\mu\nu}(q) \frac{n_\mu(y)}{|p \cdot n(y)|} \Delta\beta_\nu(y,x) \right] \Bigg|_{q=0,\, y=\bar{y}(x,p)}$$

$$W^+_{\mathrm{GE}}(x,p) = \frac{1}{(2\pi)^3} \delta(p^2 - m^2)\theta(p^0)(\not{p} + m) n_F(x,p)$$

- Function $G^{\mu\nu}$ obtained by free-streaming operators

$\mathrm{tr}[\gamma^\mu \gamma^5 G^{\alpha\beta}(q)]$ is odd in *q* → odd-*N* terms contribute

$$\underbrace{\partial\beta,\ \partial n,}_{N=1} \quad \underbrace{\partial\partial\partial\beta,\quad (\partial\partial\beta)(\partial n),}_{N=3} \cdots$$

Spin polarization

- Average spin polarization at leading order in gradient

$$S^\mu(p) \simeq -\frac{1}{8mN_p}\epsilon^{\mu\nu\rho\lambda}p_\nu \int_{\Sigma_D} \mathrm{d}\Sigma(x)\cdot p\, n_F(x,p)[1-n_F(x,p)]$$

$$\times \sum_{\bar{y}(x,p)} \mathrm{sgn}[p\cdot n(\bar{y})] \left[\varpi_{\rho\lambda}(\bar{y}) + 2\frac{n_\rho(\bar{y})}{p\cdot n(\bar{y})}\xi_{\lambda\alpha}(\bar{y})p^\alpha\right]$$

① ② thermal vorticity ③ thermal shear

- Differences with previous results in literature

① Long-distance correlation

$$\Delta W^+_{\mathrm{LE}}(x,p) = \frac{1}{(2\pi)^3}\int_{\Sigma_D} \mathrm{d}\Sigma_\mu(y)\Delta\beta_\nu(y,x)\left[\int \mathrm{d}^4q\, \delta(p\cdot q)\mathrm{e}^{iq\cdot(x-y)}G^{\mu\nu}(q)\right]$$

$$\int_0^1 \mathrm{d}z\, \langle \widehat{W}^+(x,k), \mathrm{e}^{z\widehat{A}}\widehat{T}^{\mu\nu}(y)\mathrm{e}^{-z\widehat{A}}\rangle_{c,\mathrm{GE}}$$

Large correlation between *x* and any point on $\mathbf{y} = \mathbf{x} - \frac{\mathbf{p}}{p^0}(y^0 - x^0)$

② Additional sign factor $\mathrm{sgn}[p\cdot n(\bar{y})]$

No difference if hypersurface is purely spacelike

XLS, F. Becattini, D. Roselli, arXiv: 2509.14301

Spin polarization

- Average spin polarization at leading order in gradient

$$S^\mu(p) \simeq -\frac{1}{8mN_p}\epsilon^{\mu\nu\rho\lambda}p_\nu \int_{\Sigma_D} d\Sigma(x)\cdot p\, n_F(x,p)[1-n_F(x,p)]$$

$$\times \sum_{\bar{y}(x,p)} \mathrm{sgn}[p\cdot n(\bar{y})] \left[\varpi_{\rho\lambda}(\bar{y}) + 2\frac{n_\rho(\bar{y})}{p\cdot n(\bar{y})}\xi_{\lambda\alpha}(\bar{y})p^\alpha\right]$$

① ② thermal vorticity ③ thermal shear

- Differences with previous results in literature

③ Normal vector of hypersurface n

XLS, F. Becattini, D. Roselli, arXiv: 2509.14301 (this talk)

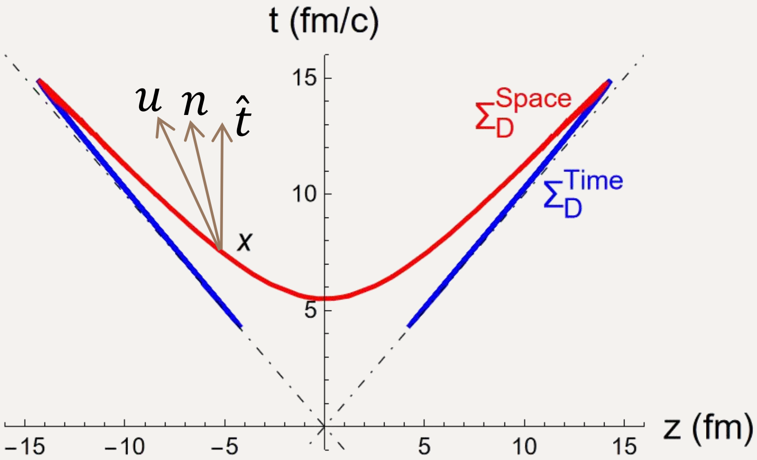
Unit vector in time direction $\hat{t}$

F. Becattini, M. Buzzegoli, A. Palermo, Phys.Lett.B 820 (2021) 136519

Fluid velocity u

S. Liu, Y. Yin, JHEP 07 (2021) 188

$n \sim \hat{t} \sim u$ at center region



Temperature gradient

- T-gradient in thermal vorticity/shear tensor

$$\varpi^{\mu\nu} = -\frac{1}{2T}(\partial^\mu u^\nu - \partial^\nu u^\mu) + \frac{1}{2T^2}(u^\nu \partial^\mu T - u^\mu \partial^\nu T)$$

$$\xi^{\mu\nu} = \frac{1}{2T}(\partial^\mu u^\nu + \partial^\nu u^\mu) - \frac{1}{2T^2}(u^\nu \partial^\mu T + u^\mu \partial^\nu T).$$

- Spin polarization induced by T-gradient

$$S^\mu_{\text{T-grad}}(p) = -\frac{1}{8mN_p}\int_{\Sigma_D} d\Sigma(x)\cdot p\, n_F(x,p)[1-n_F(x,p)]\epsilon^{\mu\nu\rho\lambda}p_\nu \times \sum_{\bar{y}(x,p)} \text{sgn}[p\cdot n]\frac{1}{T^2}\left\{u_\lambda\left[\partial_\rho T - n_\rho \frac{p\cdot\partial T}{p\cdot n}\right] - \frac{p\cdot u}{p\cdot n} n_\rho \partial_\lambda T]\right\}\Bigg|_{\bar{y}(x,p)}$$

- For hypersurface with T=const.

$$\partial_\mu T \propto n_\mu \Longrightarrow S^\mu_{\text{T-grad}}(p) = 0$$

★ A more general conclusion: Wigner function does not receive any contribution from T-gradient

"Iso-thermal approximation"

LTE density matrix does not contain any information beyond the hypersurface

F. Becattini, M. Buzzegoli, G. Inghirami, I. Karpenko, A. Palermo, Phys.Rev.Lett. 127 (2021) 27, 272302

Conclusions

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- An improved formula for Wigner function and spin polarization at local thermodynamic equilibrium

- Gradients of normal vector of decoupling hypersurface (curvature, gradient of curvature, ...)

  For spin polarization, such terms vanish at leading order

- Additional contribution from intersections between wordline and Σ_{D} (physical or non-physical?)

- For shear-induced polarization, vector n instead of $\hat{t}$ or u, automatically excludes T-gradient

  Needs numerical tests in hydro simulations

Thanks for your attention!