Spin hydrodynamics and an extension of the Bargmann-Michel-Telegdi equation

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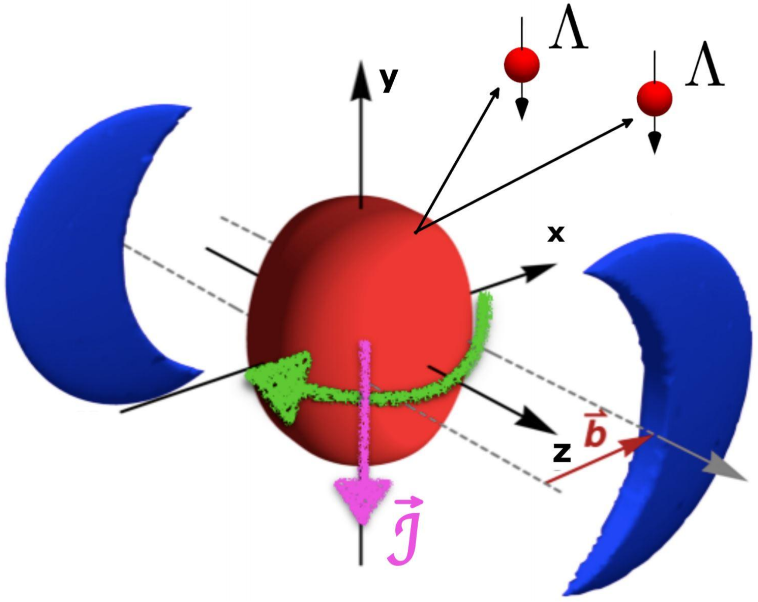
Outline

- Introduction
- Classical equations for spin
- Relativistic spin hydrodynamics
- Summary

Introduction

Rotation and polarization



Picture from Florkowski, Kumar and Ryblewski, PPNP (2019)

- **Large angular momentum in non-central heavy ion collisions**

$$J \sim \frac{A\sqrt{s}}{2} b \sim 10^5 \hbar$$

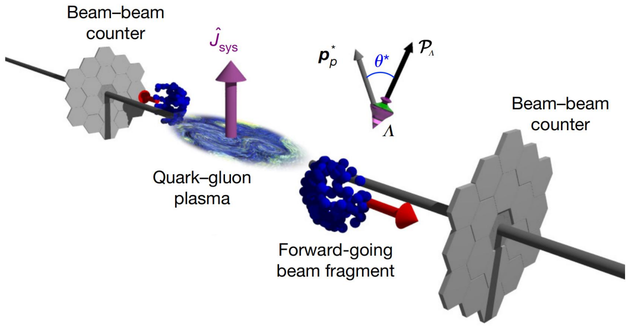
Vorticity of quark gluon plasma

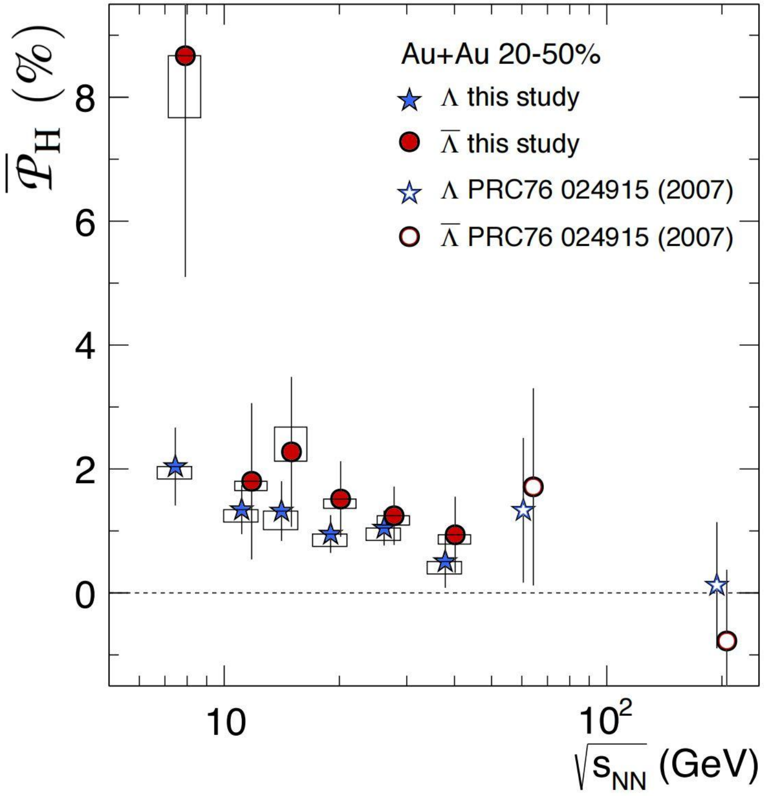
⇩ **(Total angular momentum conservation)**

Polarization of particles along vorticity

Liang and Wang, PRL (2005); PLB (2005)
Gao, Chen, Deng, Liang, Wang and Wang, PRC (2008)

Most vortical fluid and spin polarization





- **Weak decay:** $\Lambda \to p + \pi^-$
- **Proton distribution:**

$$\frac{dN}{d\cos\theta^*} = \frac{1}{2}(1 + \alpha_H |\vec{\mathcal{P}}_H| \cos\theta^*)$$

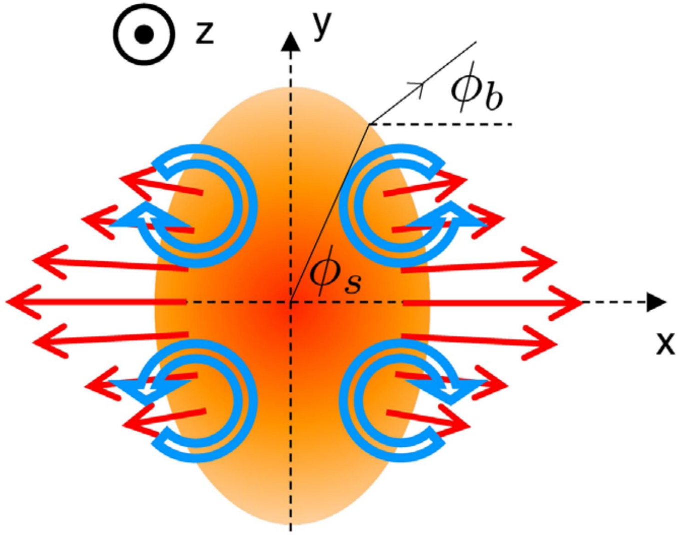
- **Large vorticity:**

$$\omega \approx (9 \pm 1) \times 10^{21} s^{-1}$$

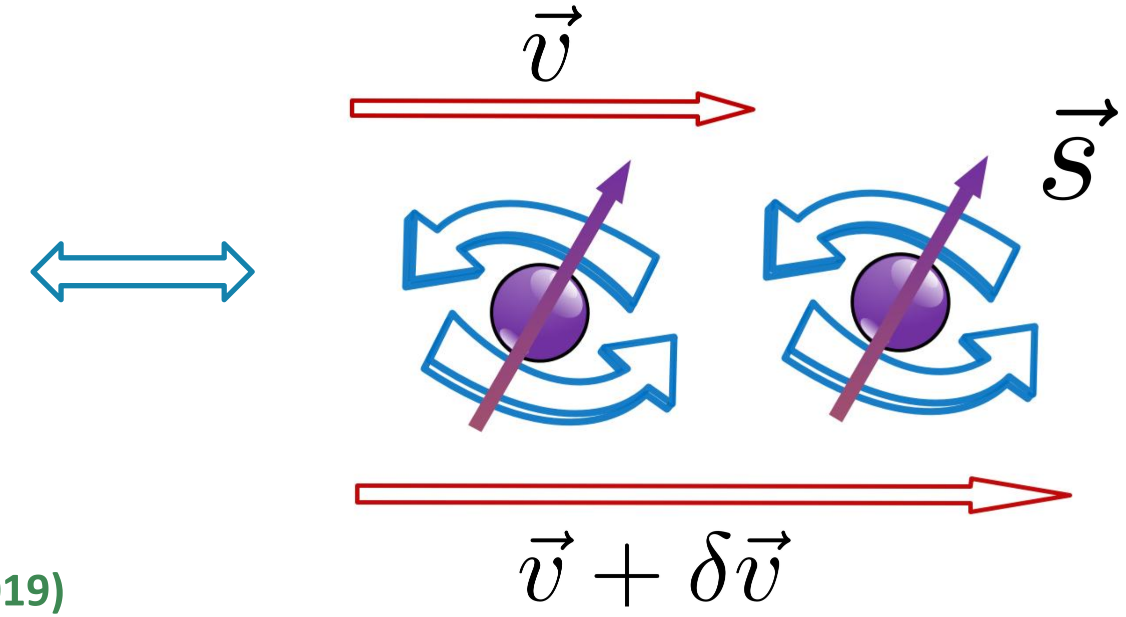
Most vortical fluid!

Adamczyk et al. (STAR), Nature 548, 62 (2017)

Spin-orbit angular momentum transfer



Adam et al. (STAR), PRL. 123, 132301 (2019)



How to describe the spin evolution?

Classical Equations for Spin

Bargmann-Michel-Telegdi (BMT) equation

- **The relativistic spin evolution of a charged particle in an electromagnetic field:**

$$s^\mu u_\mu = 0 \quad \dot{s}^\mu = \frac{ds^\mu}{d\tau}$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

Thomas precession
(caused by relativistic acceleration)

$$\dot{s}^\mu = \boxed{-u^\mu s^\nu \dot{u}_\nu} + \boxed{\gamma \Delta^{\mu\nu} F_{\nu\lambda} s^\lambda} + \text{damping}$$

Larmor precession
(caused by external electromagnetic fields)

Bargmann, Michel and Telegdi (1959)

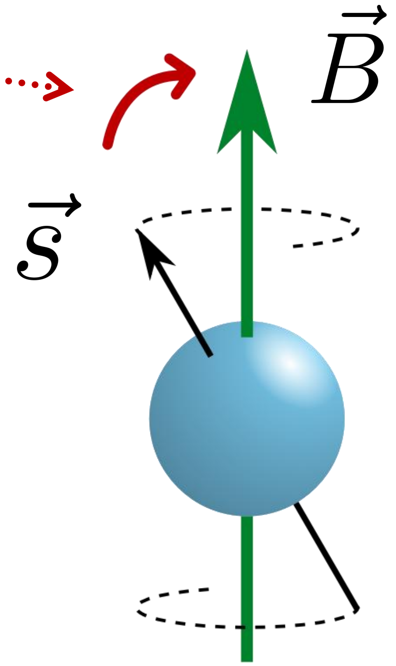
Landau-Lifshitz-Gilbert (LLG) equation

- In the non-relativistic limit, the Thomas precession term vanishes, and the spin evolution is governed by

Precession **Damping**

$$\frac{d}{dt}\vec{s} = -\gamma\vec{B}\times\vec{s}+\lambda\gamma\vec{s}\times(\vec{s}\times\vec{B})$$

$$\approx -\gamma\vec{B}\times\vec{s}+\lambda\vec{s}\times\frac{d}{dt}\vec{s}$$



- Finally, spin is parallel to the magnetic field.

Landau and Lifshitz (1935); Gilbert (1955)

What is the equation for spin in fluids?

- **We expect the corresponding equation to include terms for Thomas precession, spin-vorticity precession, ...**

$$\dot{s}^{\mu} = \text{Thomas precession} + \text{spin-vorticity} + \text{spin-dissipation} + \ldots$$

spin-vorticity ⇨ **similar to** $\vec{B} \times \vec{s}$

Let's see if this guess is correct...

Relativistic Spin Hydrodynamics

Canonical spin hydrodynamics

- Total angular momentum is decomposed into the orbital part and spin part

spin current

$$J^{\lambda\mu\nu} = x^{\mu}\Theta^{\lambda\nu} - x^{\nu}\Theta^{\lambda\mu} + \boxed{\Sigma^{\lambda\mu\nu}}$$

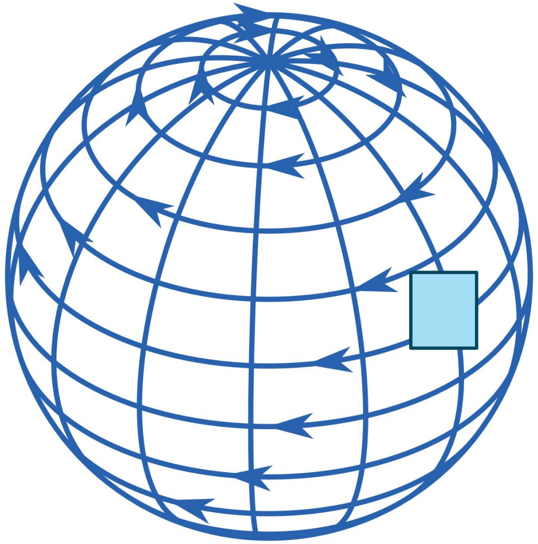
- Conserved laws give the spin evolution equation

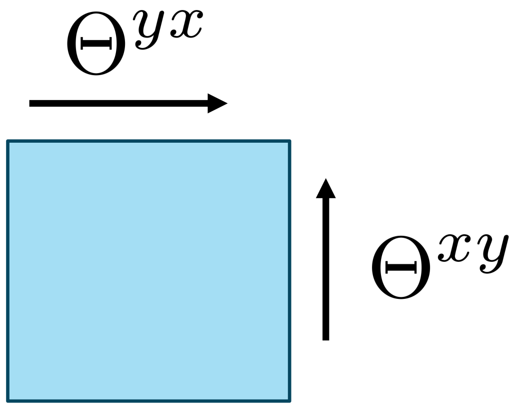
$$\partial_{\lambda}J^{\lambda\mu\nu} = \partial_{\mu}\Theta^{\mu\nu} = 0 \Longrightarrow \boxed{\partial_{\lambda}\Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}}$$

spin evolution equation

Becattini, PRL (2012)
Florkowski, Kumar and Ryblewski, PPNP (2019)
Hattori, Hongo, Huang, Matsuo and Taya, PLB (2019)

The role of the anti-symmetric part of EMT





$$\Theta^{xy} \neq \Theta^{yx}$$

There exists nonzero net torque acting on the fluid cell

$$\partial_\lambda \Sigma^{\lambda\mu\nu} = -2\Theta^{[\mu\nu]}$$

Two different choices for the spin current

- **Choice-1**

(Widely used in spin hydrodynamics)

$$\Sigma_1^{\lambda\mu\nu} = \frac{i}{4}\overline{\psi}\gamma^\lambda[\gamma^\mu,\gamma^\nu]\psi$$

1. **Non-Hermitian.**
2. **Anti-symmetric with respect to $\mu\nu$.**
3. **Spin tensor has 6 DoFs.**

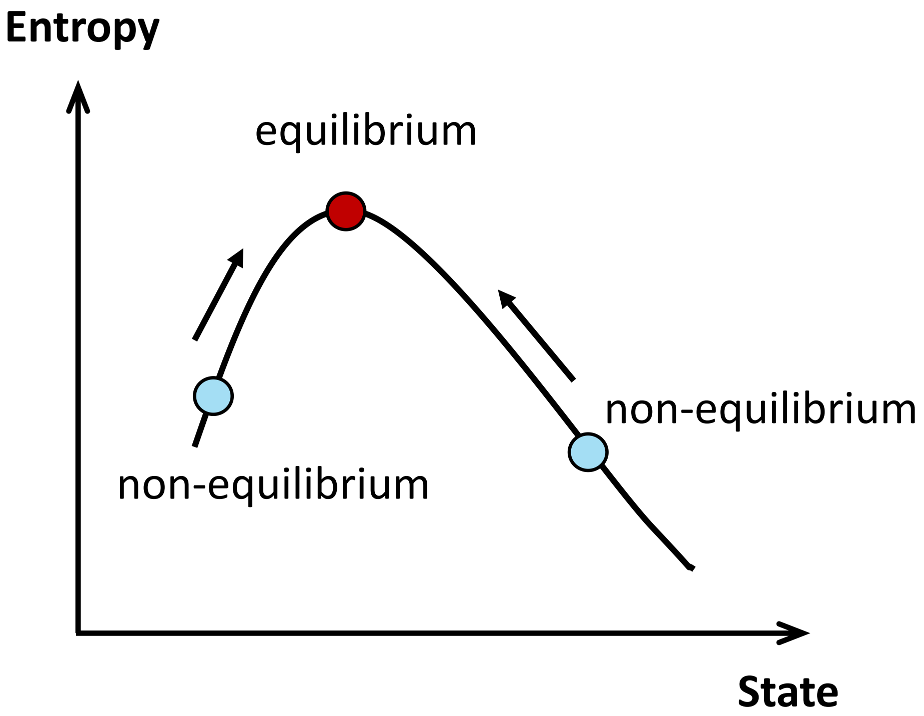
- **Choice-2**

$$\Sigma_2^{\lambda\mu\nu} = \frac{i}{8}\overline{\psi}\{\gamma^\lambda,[\gamma^\mu,\gamma^\nu]\}\psi$$
$$= \frac{1}{2}(\Sigma_1^{\lambda\mu\nu} + \Sigma_1^{\dagger\lambda\mu\nu})$$

1. **Hermitian.**
2. **Totally anti-symmetric.**
3. **Spin tensor has 3 DoFs.**

Becattini et al., Int.J.Mod.Phys.E 33 (2024)

Spin hydrodynamics based on Choice-1



$$\partial_\mu \mathcal{S}^\mu \geq 0$$

$$\Sigma_1^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + O(\partial)$$

$$\Theta^{[\mu\nu]} = 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}$$

$$\Downarrow$$

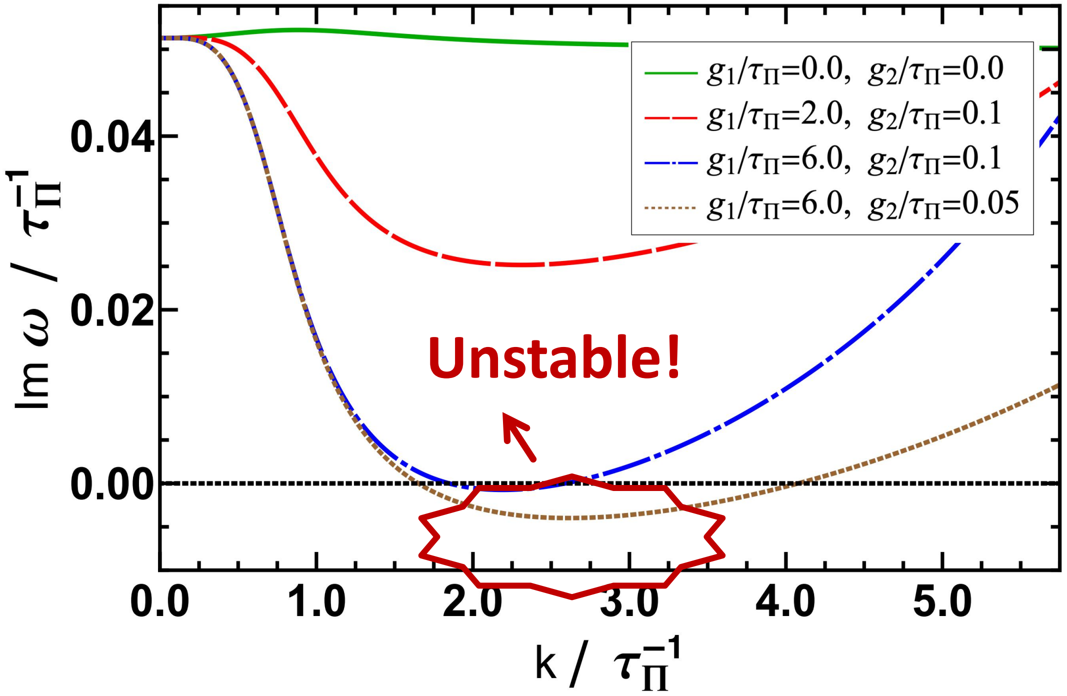
$$q^\mu = \lambda(\beta\dot{u}^\mu - \Delta^{\mu\nu}\partial_\nu\beta - 4\beta\omega^{\mu\nu}u_\nu)$$

$$\phi^{\mu\nu} = \gamma(\Delta^{\mu\alpha}\Delta^{\nu\beta}\partial_{[\alpha}u_{\beta]} + 2\omega^{\mu\nu})$$

Hattori, Hongo, Huang, Matsuo and Taya, PLB (2019)

Two possible issues of Choice-1

1. **Stability conditions impose strong constraints on the equation of state, limiting the scope of application.**



Xie, DLW, Yang and Pu, PRD (2023)
Ren, Yang, DLW and Pu, PRD (2024)

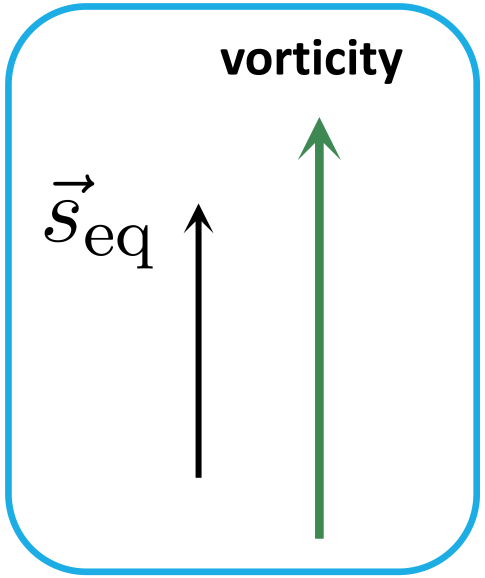
Also see:
Sarwar et al., PRD (2023)
Daher, Das and Ryblewski, PRD (2023)

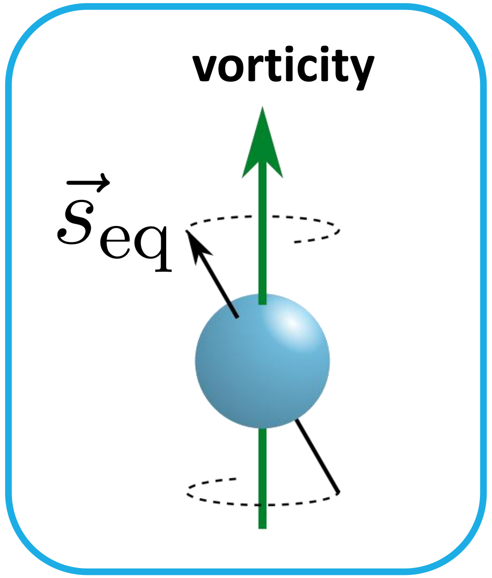
Two possible issues of Choice-1

2. **In the equilibrium state, the spin vector should "look the same" when observed along the flow generated by the killing vector.**

$$\mathcal{L}_\beta s^\mu_{\rm eq} = 0$$

(vanishing Lie derivative)





However, the spin hydrodynamics based on Choice-1 fails to satisfy the condition!

Choice-2: Totally anti-symmetric spin current

- **Adopting Choice-2, the spin current tensor is totally anti-symmetric and can be decomposed as:**

  Hongo, Huang, Kaminski, Stephanov and Yee, JHEP (2021)
  Cao, Hattori, Hongo, Huang and Taya, PRD (2022)

$$\Sigma_2^{\lambda\mu\nu} = u^\lambda S^{\mu\nu} + u^\mu S^{\nu\lambda} + u^\nu S^{\lambda\mu} + O(\partial)$$

- **Additional terms emerge in the entropy production rate:**

$$\partial_\mu \mathcal{S}^\mu = \underline{q^\mu(\partial_\mu \beta - \beta \dot{u}_\mu)} + \underline{2\beta\omega_{\mu\nu} S^{\lambda\mu}\partial_\lambda u^\nu} + \ldots$$

$$q^\mu = -\frac{1}{2} u_\nu \partial_\lambda \Sigma^{\lambda\mu\nu}$$

It seems impossible to ensure $\boxed{\partial_\mu \mathcal{S}^\mu \geq 0}$?

Constitutive relations with spin corrections

- **These additional terms can be absorbed into the terms related to heat flow h^μ, viscous tensor $\pi^{\mu\nu}$, and $\phi^{\mu\nu}$:**

$$
\begin{aligned}
\partial_\mu \mathcal{S}'^\mu \quad = \quad & (h^\mu - \mathcal{H}\nu^\mu + h_s^\mu)(\partial_\mu \beta + \beta \dot{u}_\mu) \\
& + \beta(\phi^{\mu\nu} + \phi_s^{\mu\nu})(\partial_{[\mu} u_{\nu]} + 2\omega_{\mu\nu}) \\
& + \beta(\pi^{\mu\nu} + \pi_s^{\mu\nu})\partial_{(\mu} u_{\nu)} + O(\partial^3)
\end{aligned}
$$

$$\mathcal{S}'^\mu = \mathcal{S}^\mu + S^{\mu\nu}\partial_\nu \beta$$

Fang, Fukushima, Pu and DLW, arXiv:2506.20698

Constitutive relations with spin corrections

- These additional terms can be absorbed into the terms related to heat flow h^μ, viscous tensor $\pi^{\mu\nu}$, and $\phi^{\mu\nu}$:

$$
\begin{aligned}
h^\mu - \mathcal{H}\nu^\mu + h_s^\mu &= -\sigma\Delta^{\mu\nu}(\partial_\nu\beta + \beta\dot{u}_\nu) \\
\pi^{\mu\nu} + \pi_s^{\mu\nu} &= \eta\partial^{<\mu}u^{\nu>} + \zeta(\partial\cdot u)\Delta^{\mu\nu} \\
\phi^{\mu\nu} + \phi_s^{\mu\nu} &= \gamma(\Delta^{\mu\alpha}\Delta^{\nu\beta}\partial_{[\alpha}u_{\beta]} + 2\omega^{\mu\nu})
\end{aligned}
$$

spin corrections

Fang, Fukushima, Pu and DLW, arXiv:2506.20698

Extended BMT equation

- **Spin corrections modify the spin evolution equation:**

$\phi_s^{\mu\nu}$

$$\Delta^\mu_\rho \Delta^\nu_\sigma \partial_\lambda \Sigma_2^{\lambda\rho\sigma} = 2[v_n^2(\partial\cdot u)S^{\mu\nu} - 2\omega_\alpha^{\ [\nu}S^{\mu]\alpha}] + \text{original terms}$$

- **Extended BMT equation from the spin evolution equation:**

$$\dot{s}^\mu = -u^\mu s^\nu \dot{u}_\nu + (\epsilon^{\mu\nu\rho\sigma} s_\nu u_\rho - 2\gamma g^{\mu\sigma})(2\omega_\sigma - \mathfrak{m}_\sigma) - s_\nu \partial^{<\mu} u^{\nu>} - (2v_n^2 + 1/3)s^\mu(\partial\cdot u)$$

Fang, Fukushima, Pu and DLW, arXiv:2506.20698

Extended BMT equation

- **Extended BMT equation**

Thomas precession

$$\dot{s}^{\mu} = \boxed{-u^{\mu}s^{\nu}\dot{u}_{\nu}} + (\epsilon^{\mu\nu\rho\sigma}s_{\nu}u_{\rho} - 2\gamma g^{\mu\sigma})(2\omega_{\sigma} - \mathfrak{m}_{\sigma})$$
$$- s_{\nu}\partial^{<\mu}u^{\nu>} - (2v_n^2 + 1/3)s^{\mu}(\partial \cdot u)$$

- **Classical BMT equation:**

$$\dot{s}^{\mu} = \boxed{-u^{\mu}s^{\nu}\dot{u}_{\nu}} + \gamma\Delta^{\mu\nu}F_{\nu\lambda}s^{\lambda} + \text{damping}$$

Thomas precession

Fang, Fukushima, Pu and DLW, arXiv:2506.20698

Extended BMT equation

➢ **Extended BMT equation**

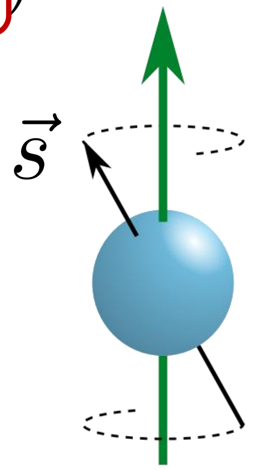
spin precession in hydro.

$$\dot{s}^{\mu} = -u^{\mu} s^{\nu} \dot{u}_{\nu} + \left(\epsilon^{\mu\nu\rho\sigma} s_{\nu} u_{\rho} - 2\gamma g^{\mu\sigma}\right)\left(2\omega_{\sigma} - \mathfrak{m}_{\sigma}\right) - s_{\nu}\partial^{<\mu} u^{\nu>} - (2v_n^2 + 1/3) s^{\mu} (\partial \cdot u)$$

➢ **Classical BMT equation:**

$$\dot{s}^{\mu} = -u^{\mu} s^{\nu} \dot{u}_{\nu} + \gamma \Delta^{\mu\nu} F_{\nu\lambda} s^{\lambda} + \text{damping}$$

spin precession in EM fields



precession

Fang, Fukushima, Pu and DLW, arXiv:2506.20698

Extended BMT equation

- **Extended BMT equation**

$$\dot{s}^{\mu} = -u^{\mu} s^{\nu} \dot{u}_{\nu} + (\epsilon^{\mu\nu\rho\sigma} s_{\nu} u_{\rho} - \boxed{2\gamma g^{\mu\sigma})(2\omega_{\sigma} - \mathfrak{m}_{\sigma})} - \boxed{s_{\nu}\partial^{<\mu} u^{\nu>}} - \boxed{(2v_n^2 + 1/3) s^{\mu} (\partial \cdot u)}$$

shear effect **volumetric expansion**

- **Classical BMT equation:**

$$\dot{s}^{\mu} = -u^{\mu} s^{\nu} \dot{u}_{\nu} + \gamma \Delta^{\mu\nu} F_{\nu\lambda} s^{\lambda} + \boxed{\text{damping}}$$

Fang, Fukushima, Pu and DLW, arXiv:2506.20698

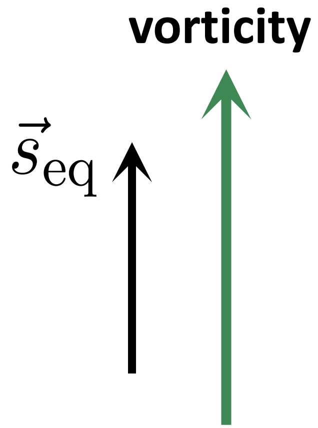
Extended BMT equation

➢ **Extended BMT equation in the equilibrium state**

$$\dot{s}^\mu = -u^\mu s^\nu \dot{u}_\nu + (\epsilon^{\mu\nu\rho\sigma} s_\nu u_\rho - 2\gamma g^{\mu\sigma})(2\omega_\sigma - \mathfrak{m}_\sigma)$$
$$- s_\nu \partial^{<\mu} u^{\nu>} - (2v_n^2 + 1/3) s^\mu (\partial \cdot u)$$

⇩

$$\dot{s}^\mu_{\rm eq} = -u^\mu s^\nu_{\rm eq} \dot{u}_\nu$$



Fang, Fukushima, Pu and DLW, arXiv:2506.20698

Issues disappear

1. **Stable modes in linear mode analysis**

$$\chi_s = \delta\omega^{\mu\nu}/\delta S^{\mu\nu}$$

$$\omega = \pm 4i\gamma\chi_s$$

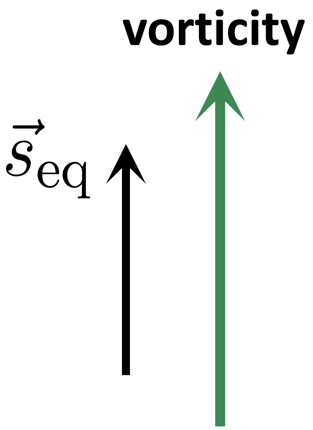
Choice-1

$$\omega = -4i\gamma\chi_s$$

Choice-2

2. **Symmetry in the equilibrium state**

$$\mathcal{L}_\beta s^\mu_{\text{eq}} = 0$$



Fang, Fukushima, Pu and DLW, arXiv:2506.20698

Summary

Summary

- Spin hydrodynamics provides an effective tool for describing the spin evolution in quark-gluon-plasma.
- There are two different choices for the canonical spin current. Our results suggest that the totally anti-symmetric one may be better.
- The extended BMT equation includes terms for Thomas precession and spin-hydrodynamic field couplings.