Covariant L-S/helicity Coupling Decomposition by spinors

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Outline

1. Particle states
2. Spinor-helicity formalism
3. (Covariant) L-S/Helicity coupling
4. Spin Operator, Form Factor

Ordinary QFT

Field to particle:

$$\boxed{\psi(x)} = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[\boxed{a_s(\vec{p})} u_s(\vec{p}) e^{-ip\cdot x} + \boxed{b_s^\dagger(\vec{p})} v_s(\vec{p}) e^{ip\cdot x}\right]$$

Irrep of SO(3,1)

Canonical Quantization

Excitation of field

$$\psi(x)|0\rangle = e^{ip\cdot x} v_s(p)|p,\sigma\rangle$$

$$U_0(\Lambda,a)\psi_\ell^+(x)U_0^{-1}(\Lambda,a) = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}\left(\Lambda^{-1}\right)\psi_{\bar{\ell}}^+(\Lambda x + a)$$

SO(3,1)=$\mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R}: (j_1, j_2)$

$$[\mathscr{J}_i, \mathscr{J}_j] = \mathrm{i}\epsilon_{ijk}\mathscr{J}_k$$
$$[\mathscr{J}_i, \mathscr{K}_j] = \mathrm{i}\epsilon_{ijk}\mathscr{K}_k\,,$$
$$[\mathscr{K}_i, \mathscr{K}_j] = -\mathrm{i}\epsilon_{ijk}\mathscr{J}_k$$

Weinberg's construction

Particle to field:

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_s(\vec{p}) u_s(\vec{p}) e^{-ip\cdot x} + b_s^\dagger(\vec{p}) v_s(\vec{p}) e^{ip\cdot x} \right]$$

Irrep of SO(3,1)

Unitary Irrep of ISO(3,1)

$$U_0(\Lambda, a)\psi_\ell^+(x)U_0^{-1}(\Lambda, a) = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}\left(\Lambda^{-1}\right)\psi_{\bar{\ell}}^+(\Lambda x + a)$$

(m,s) given by Casimir P^2, W^2.

1. define: $|k, \sigma\rangle$, L_p

2. $|p, \sigma\rangle \equiv U(L_p)|k, \sigma\rangle$,

$$U(\Lambda)|p, \sigma\rangle = |\Lambda p, \sigma'\rangle W_{\sigma'}{}^{\sigma}.$$

$$W(\Lambda, p) = L_{\Lambda p}^{-1} \Lambda L_p$$

$$[J_k, J_l] = i\varepsilon_{kln}J_n, [J_k, P_l] = i\varepsilon_{kln}P_n, [J_k, B_l] = i\varepsilon_{kln}B_n, [J_k, H] = 0,$$

$$[P_k, P_l] = 0, [B_k, B_l] = 0, [B_k, P_l] = iM\delta_{kl}, [P_k, H] = 0, [B_k, H] = iP_k,$$

$$[J_k, M] = [P_k, M] = [B_k, M] = [H, M] = 0.$$

Standard boost L_p : change p,
Little group: change spin

Particle states

$$k^\mu = (m, 0, 0, 0)$$

$$L_p^{\mathbf{c}} \equiv R(\theta,\varphi)e^{i\eta K_3}R(\theta,\varphi)^{-1},$$

Canonical state

2-particle

Orbit part   L-S couple

$$\langle \vec{0}, s_{3z}|\vec{q}, s_{1z}\rangle| -\vec{q}, s_{2z}\rangle \propto Y^*_{Lm_z}(\hat{q}) C^{s1,s_{1z};s2,s_{2z}}_{s,s_{3z}} C^{L,m_z;s,s_z}_{s3,s_{3z}}$$

$$\downarrow (\text{boost})$$

$$|\vec{P}, J_z, J\rangle \equiv U(L_P)|\vec{0}, J_z, J\rangle$$

$$= C^{J,J_z}_{L,L_z;s,s_z} C^{s,s_z}_{s_1,s_{1z};s_2,s_{2z}} \sqrt{\frac{4E_{cm}}{|\vec{p}_{1cm}|}} \int d\Phi Y_{L,L_z}(\Omega)|\vec{p}_1, s'_{1z}\rangle|\vec{p}_2, s'_{2z}\rangle$$

$$\times W(L_P^{-1}, p_1)^{s_{1z}}_{s'_{1z}} W(L_P^{-1}, p_2)^{s_{2z}}_{s'_{2z}}.$$

Non-covariant little group trans

$$L_p^{\mathbf{h}} \equiv R(\theta,\varphi)e^{i\eta K_3}$$

h: polarization along p-axis

Helicity state

2-particle

Helicity couple

$$\langle \vec{0}, h_3; h_1^0, h_2^0|\vec{q}, h_1\rangle| -\vec{q}, h_2\rangle \propto \delta^{h_1^0}_{h_1}\delta^{h_2^0}_{h_2} D^{(s)}_{h_3,h_1-h_2}(\hat{q})$$

$$\downarrow (\text{boost})$$

$$|\vec{P}, h, J; h_1, h_2\rangle = U(L_P)\int \mathrm{d}\Omega D^{J*}_{h,h_1-h_2}(\Omega)|\vec{0}, \Omega(\vec{p}_{1cm}); h_1, h_2\rangle$$

$$= \int \mathrm{d}\Omega D^{J*}_{h,h_1-h_2}(\Omega)|\vec{p}_1, h'_1\rangle|\vec{p}_2, h'_2\rangle W(L_P, p_{1cm})^{h_1}_{h'_1} W(L_P, p_{2cm})^{h_2}_{h'_2}$$

Non-covariant little group trans

3-pt Amplitude(CGC)

There are already several ways to construct, all non-covariant!

$$\langle \vec{p_1}, s_{1z}|\langle \vec{p_2}, s_{2z}|\vec{P}, J_z, J\rangle = \delta^4(P-p_1-p_2) C^{J,J_z}_{L,L_z;s,s_z} C^{s,s_z}_{s_1,s'_{1z};s_2,s'_{2z}} \sqrt{\frac{4E_{cm}}{|\vec{p}_{1cm}|}} Y_{L,L_z}(\Omega) W(L_P^{-1}, p_1)^{s'_{1z}}_{s_{1z}} W(L_P^{-1}, p_2)^{s'_{2z}}_{s_{2z}}$$

L-S coupling

$$\langle \vec{p}_1, h_1|\langle \vec{p}_2, h_2|\vec{P}, h, J; h_1^0, h_2^0\rangle = \delta^4(P - p_1 - p_2) D^{J*}_{h,h_1^0-h_2^0} W^{h_1^0}_{h_1} W^{h_2^0}_{h_2}$$

Helicity coupling

$$M^{\lambda_3}_{\lambda_1\lambda_2}(k_3, p_1^*, p_2^*, L, S) = H^{\lambda_3}_{\lambda_1,\lambda_2}(k_3, p_1^*, p_2^*, L, S) D^{s3*}_{\lambda_3,\lambda_1-\lambda_2}(\phi, \theta, 0)$$

$$H^{\lambda_3}_{\lambda_1,\lambda_2}(k_3, p_1^*, p_2^*, L, S) = \sqrt{\frac{2L+1}{2s_3+1}} (C^{s3}_{L,S})^{\lambda_3}_{0,\lambda_1-\lambda_2} (C^{S}_{s_1,s_2})^{\lambda_1-\lambda_2}_{\lambda_1,-\lambda_2} q^L$$

S.U.Chung, SPIN FROMALISMS

$$\mathcal{A}^{\sigma_2\sigma_3}_{\sigma_1}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*) = \underbrace{\Gamma^{\alpha_2\alpha_3}_{\alpha_1}(\mathbf{k}_1, \mathbf{p}_2^*, \mathbf{p}_3^*)}_{\text{pure-orbital part}} \times \underbrace{\bar{u}^{\alpha_1}_{\sigma_1}(s_1)\, u^{\sigma_2}_{\alpha_2}(\mathbf{p}_2^*, s_2)\, u^{\sigma_3}_{\alpha_3}(\mathbf{p}_3^*, s_3)}_{\text{pure-spin part}}$$

$$\Gamma^{\alpha_1\alpha_2}_{\alpha_3}(k_3, p_1^*, p_2^*; L, S) = P^{\alpha_L\alpha_S}_{\alpha_3}(k_3; s_3, L, S) P^{\alpha_1\alpha_2}_{\alpha_S}(k_3; S, s_1, s_2) \tilde{t}^{(L)}_{\alpha_L}(k_3; p_1^* - p_2^*)$$

$$p_i^* = L_{p_1}^{-1} p_i \text{ for } i = 2, 3$$

Covariant Tensor Method, H.J. Jing, et, JHEP 06 (2023) 039

Spinor Variable

$\alpha\beta\gamma\ ...: SU(2)_L,\ \dot\alpha\dot\beta\dot\gamma\ ...: SU(2)_R\ ,\ IJK: SU(2)$-little group

$$p^\mu \sigma_{\mu\alpha\dot\alpha} = p_{\alpha\dot\alpha}, p^{\dot\alpha\alpha} = p^\mu \bar\sigma_\mu^{\dot\alpha\alpha}.$$

$$u(p) \sim \begin{pmatrix} |p^I] \\ |p^I\rangle \end{pmatrix}, v(p) \sim \begin{pmatrix} |p_I] \\ -|p_I\rangle \end{pmatrix}, \bar u(p) \sim ([p_I|, -\langle p_I|), \bar v \sim ([p^I|, \langle p^I|).$$

Massless: $p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha} = |p\rangle_\alpha [p|_{\dot\alpha}$

$$\lambda_\alpha \equiv \begin{pmatrix} \sqrt{2E}c \\ \sqrt{2E}s \end{pmatrix}$$

$$\tilde\lambda_{\dot\alpha} \equiv \begin{pmatrix} \sqrt{2E}c \\ \sqrt{2E}s^* \end{pmatrix}$$

Massive: $p_{\alpha\dot\alpha} = \lambda_\alpha^I\ \tilde\lambda_{\dot\alpha I} = |p^I\rangle_\alpha [p_I|_{\dot\alpha}$

$$\lambda_\alpha^I = \begin{pmatrix} \sqrt{E+p}c & \sqrt{E-p}(-s^*) \\ \sqrt{E+p}s & \sqrt{E-p}c \end{pmatrix}_{\alpha,I}, \tilde\lambda_{\dot\alpha I} = \begin{pmatrix} \sqrt{E+p}c & \sqrt{E-p}(-s) \\ \sqrt{E+p}s^* & \sqrt{E-p}c \end{pmatrix}_{\dot\alpha,I}.$$

Helicity state

$$\lambda_\alpha^{\mathbf{c}I} = \begin{pmatrix} \sqrt{E+p}c^2 + \sqrt{E-p}|s|^2 & (\sqrt{E+p} - \sqrt{E-p})s^*c \\ (\sqrt{E+p} - \sqrt{E-p})sc & \sqrt{E+p}|s|^2 + \sqrt{E-p}c^2 \end{pmatrix}_{\alpha,I},$$

$$\tilde\lambda_{\dot\alpha I}^{\mathbf{c}} = \begin{pmatrix} \sqrt{E+p}c^2 + \sqrt{E-p}|s|^2 & (\sqrt{E+p} - \sqrt{E-p})sc \\ (\sqrt{E+p} - \sqrt{E-p})s^*c & \sqrt{E+p}|s|^2 + \sqrt{E-p}c^2 \end{pmatrix}_{\dot\alpha,I}.$$

Canonical state

e.g.

$$\langle 12\rangle = \lambda_1^\alpha \lambda_{2\alpha} = \epsilon^{\alpha\beta}\lambda_{1\beta}\lambda_{2\alpha}$$

$$[2|3|\mathbf{1}\rangle = \tilde\lambda_{2\dot\beta} p_3^{\dot\beta\alpha} \lambda_{1\alpha}^I \qquad [\mathbf{1}|3|\mathbf{1}\rangle = \tilde\lambda_{1\dot\beta}^{(I} p_3^{\dot\beta\alpha} \lambda_{1\alpha}^{J)}$$

Bold = Total Symmetric = Spin-s:

$$|\mathbf{1}\rangle, |\mathbf{1}] \sim spin\ \frac{1}{2}$$

$$|\mathbf{1}\rangle^2, |\mathbf{1}\rangle|\mathbf{1}], |\mathbf{1}]^2 \sim spin\ 1$$

$$|\mathbf{1}\rangle^3, |\mathbf{1}\rangle^2|\mathbf{1}], |\mathbf{1}\rangle|\mathbf{1}]^2, |\mathbf{1}]^3 \sim spin\ \frac{3}{2}$$

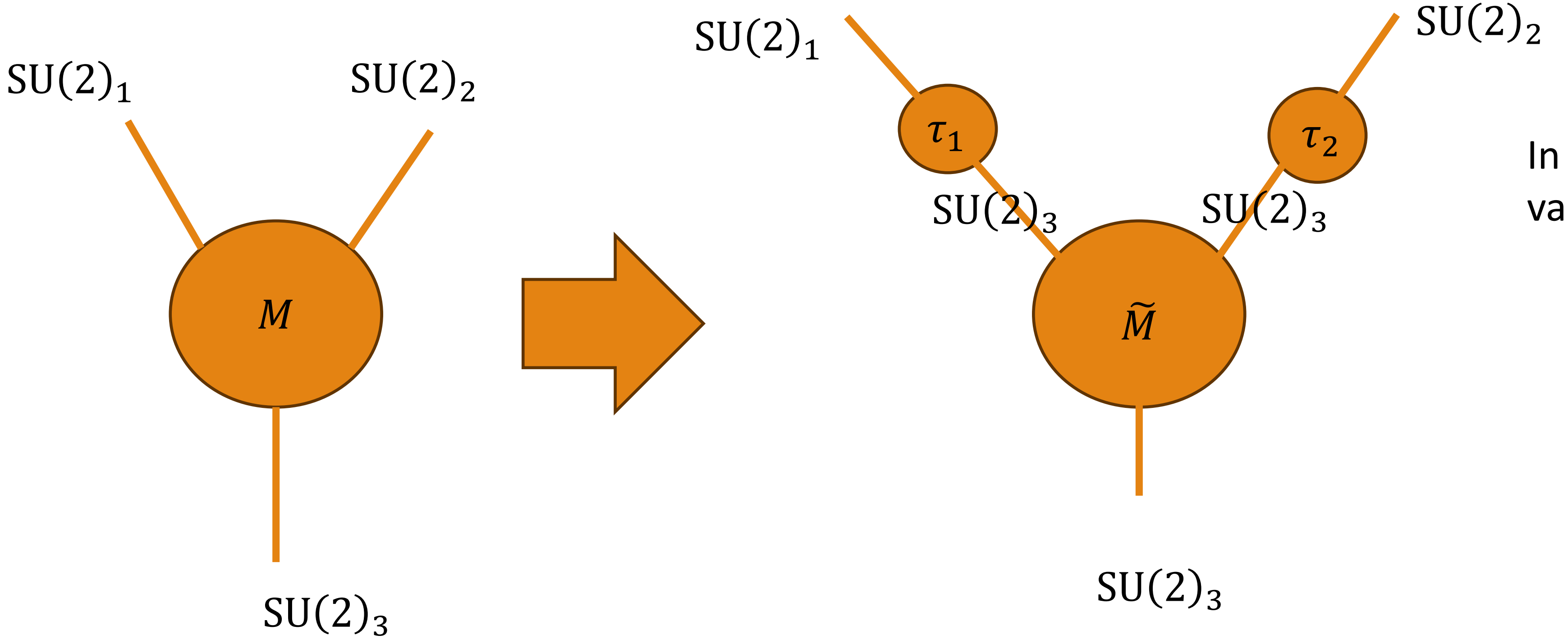
... ...

$$j_L + j_R = s \sim |\mathbf{p}]^{2j_R}\langle \mathbf{p}|^{2j_L}$$

Spin coupling

$\mathbf{1}^{s_1} + \mathbf{2}^{s_2} \to \mathbf{3}^{s_3}$:

- Since $SU(2)_1 \neq SU(2)_2$, they don't couple directly.
- In CoM frame, $SU(2)_i$ are mixed.



In CoM frame (spinor-canonical variable):

$$\tau_1 \propto \tau_2 \propto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Spin coupling

$\mathbf{1}^{s_1} + \mathbf{2}^{s_2} \to \mathbf{3}^{s_3}$:

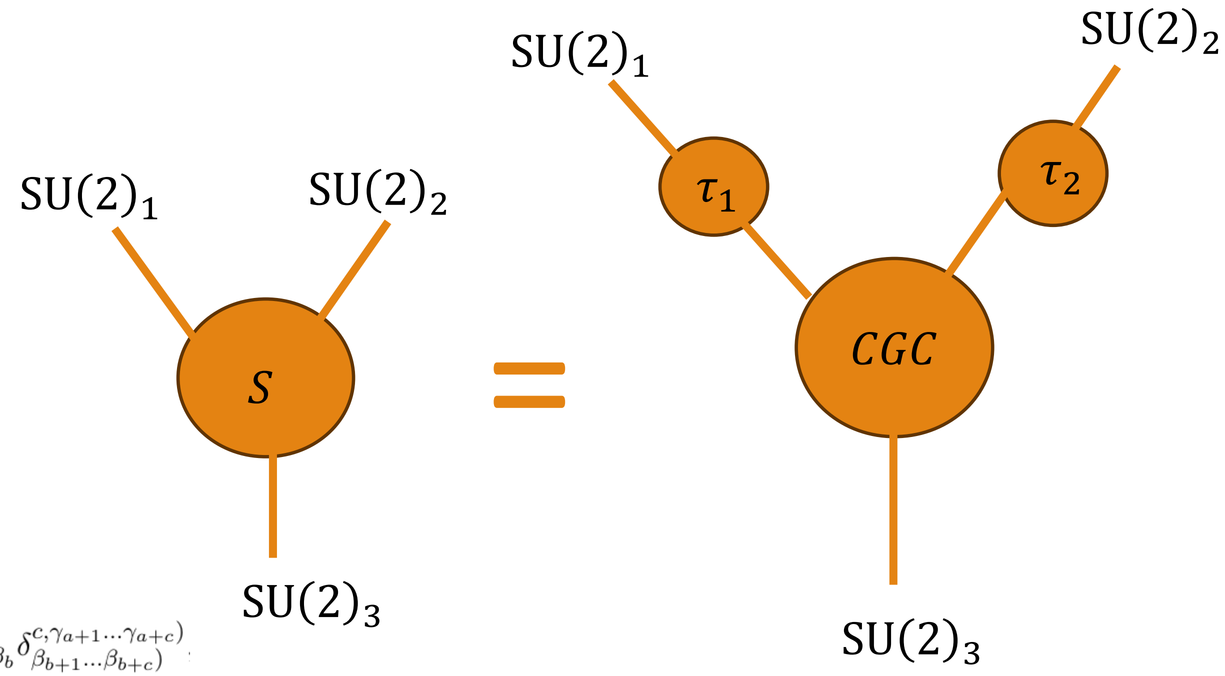
- Since $\mathrm{SU}(2)_1 \neq \mathrm{SU}(2)_2$, they can't couple directly.
- In CoM frame, $\mathrm{SU}(2)_i$ are mixed.

Spin-Coupling:

$$\mathcal{S}^{*\{K\}}_{\{I\},\{J\}} \equiv (\tau_1^{2s_1})^{\{A\}}_{\{I\}} (\tau_2^{2s_2})^{\{B\}}_{\{J\}} C^{S,\{K\}}_{s_1,\{A\};s_2,\{B\}}$$

$$a = s_1 + s - s_2,$$
$$b = s_1 + s_2 - s,$$
$$c = s_2 + s - s_1.$$

$$C^{s,\{\gamma\}}_{s_1,\{\alpha\};s_2,\{\beta\}} \equiv \sqrt{\frac{(a+b)!(b+c)!(a+c+1)!}{b!(a+b+c+1)!a!c!}} \delta^{a,(\gamma_1\ldots\gamma_a}_{(\alpha_1\ldots\alpha_a} \varepsilon^{b}_{\alpha_{a+1}\ldots\alpha_{a+b}),(\beta_1\ldots\beta_b} \delta^{c,\gamma_{a+1}\ldots\gamma_{a+c})}_{\beta_{b+1}\ldots\beta_{b+c})}$$



L-S coupling

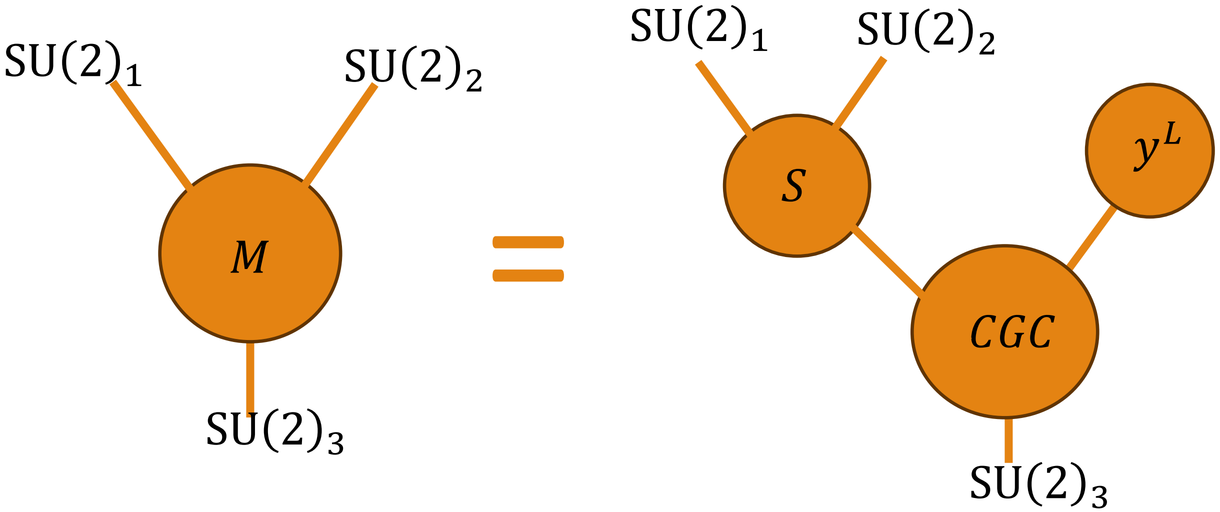
$\mathbf{1}^{s_1} + \mathbf{2}^{s_2} \to \mathbf{3}^{s_3}$:

$$\mathcal{S}^{*\{K\}}_{\{I\},\{J\}} \equiv (\tau_1^{2s_1})^{\{A\}}_{\{I\}}(\tau_2^{2s_2})^{\{B\}}_{\{J\}} C^{S,\{K\}}_{s_1,\{A\};s_2,\{B\}}$$ Spin part

For orbit, we construct $\mathcal{Y}^{*L} \sim Y^*_{Lm}(\Omega)$ in COM as a spin-L rep of $SO(3)$.

orbit part

$\vec{p}_1^{\,L}$ ~: covariant spherical harmonic.



LS coupling

$$M^{*J^S_L\{K\}}_{\{I\},\{J\}} = \mathcal{S}^{*\{A\}}_{\{I\},\{J\}} \mathcal{Y}^{*L,\{B\}} C^{J,\{K\}}_{S,\{A\};L,\{B\}}.$$

$$M \in span\{\tau_1^{2s_1} \otimes \tau_2^{2s_2} \otimes \mathcal{Y}^{*L}\}$$

L-S coupling

$\mathbf{1}^{\frac{1}{2}} + \mathbf{2}^{0} \rightarrow \mathbf{3}^{\frac{1}{2}}$:

$S = \frac{1}{2}, L = 1$: $([\mathbf{31}] + \langle\mathbf{31}\rangle)$

$\mathbf{1}^{\frac{1}{2}} + \mathbf{2}^{\frac{1}{2}} \rightarrow \mathbf{3}^{1}$:

$S = 0, L = 1$: $([\mathbf{12}] - \langle\mathbf{12}\rangle)[\mathbf{3}|\mathbf{1}|\mathbf{3}\rangle$

$S = 1, L = 0$: $([\mathbf{31}] - \langle\mathbf{31}\rangle)([\mathbf{32}] - \langle\mathbf{32}\rangle)$

$S = 1, L = 1$:

$$\frac{1}{2}\left((p_2 \cdot p_3 + m_2 m_3)([\mathbf{31}] - \langle\mathbf{31}\rangle)([\mathbf{32}] + \langle\mathbf{32}\rangle) - (p_1 \cdot p_3 + m_1 m_3)([\mathbf{32}] - \langle\mathbf{32}\rangle)([\mathbf{31}] + \langle\mathbf{31}\rangle)\right)$$

$\mathbf{1}^{1} + \mathbf{2}^{\frac{1}{2}} \rightarrow \mathbf{3}^{\frac{1}{2}}$:

$S = \frac{1}{2}, L = 0$: $([\mathbf{31}] - \langle\mathbf{31}\rangle)([\mathbf{12}] - \langle\mathbf{12}\rangle)\sqrt{\frac{2}{3}}$

$S = \frac{1}{2}, L = 1$: $([\mathbf{31}] + \langle\mathbf{31}\rangle)([\mathbf{12}] - \langle\mathbf{12}\rangle)$

$S = \frac{3}{2}, L = 1$:

$$((m_3 + m_2 - m_1)([\mathbf{12}] + \langle\mathbf{12}\rangle)([\mathbf{31}] - \langle\mathbf{31}\rangle) + [\mathbf{1}|\mathbf{2}|\mathbf{1}\rangle([\mathbf{32}] - \langle\mathbf{32}\rangle))$$

helicity coupling

$\mathbf{1}^{s_1} + \mathbf{2}^{s_2} \to \mathbf{3}^{s_3}$:

Select certain $h_1^0 h_2^0$: $\delta_{h_1 h_1^0}\delta_{h_2 h_2^0}$

$s_3 = \left|h_1^0 - h_2^0\right| + Z,$

$h_1^0 = -s_1, -s_1+1, \dots, s_1, h_2^0 = -s_2, \dots, s_2.$

$$\tau_{1+K}^{\;\;I} \equiv m_3|\vec{p}_{1cm}|\tau_{1K}^{I} - \mathcal{Y}_K^A \tau_{1A}^I \propto \begin{pmatrix} c & 0 \\ s & 0 \end{pmatrix}$$

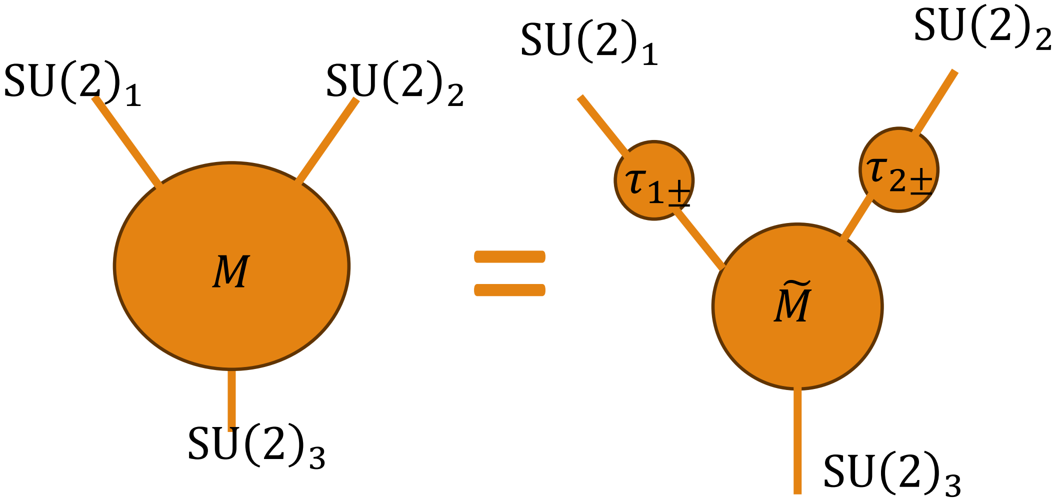
$$\tau_{1-K}^{\;\;I} \equiv m_3|\vec{p}_{1cm}|\tau_{1K}^{I} + \mathcal{Y}_K^A \tau_{1A}^I \propto \begin{pmatrix} 0 & -s^* \\ 0 & c \end{pmatrix}$$

$$M^{h_1^0,h_2^0} \sim \tau_{1+(\{J_1\}}^{s_1+h_1^0}\tau_{2-\{J_4\})}^{s_2-h_2^0}\tau_{1-(\{J_2\}}^{s_1-h_1^0}\tau_{2+\{J_3\})}^{s_2+h_2^0} \sim \delta_{h_1}^{h_1^0}\delta_{h_2}^{h_2^0}$$

$$M_{\{K\}}^{h_1^0 h_2^0\{I\}\{J\}} \overset{\text{CoM}}{\propto} M_{\{K\}}^{h_1^0 h_2^0\{++\cdots--\}\{+++\cdots---\}},$$

$h_1 = h_1^0, h_2 = h_2^0$

or $M_{\{K\}}^{h_1^0 h_2^0\{I\}\{J\}} = 0.$

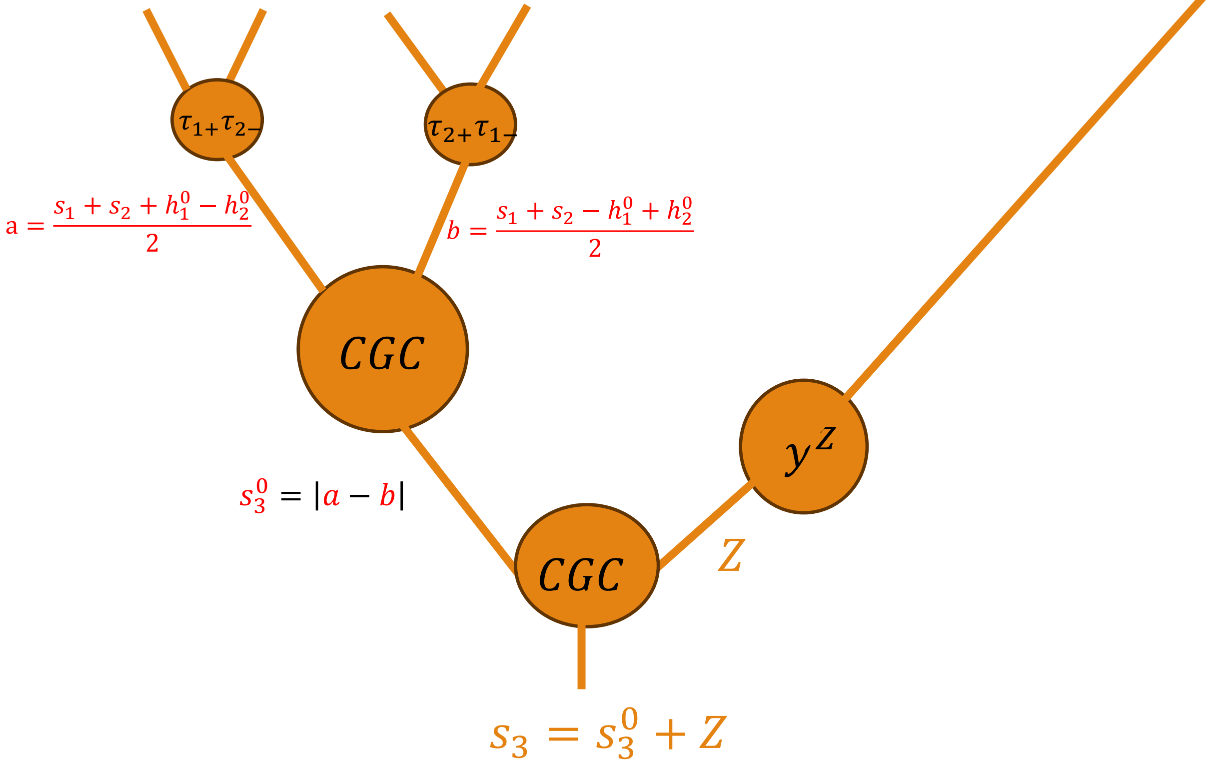


helicity coupling

$\mathbf{1}^{s_1} + \mathbf{2}^{s_2} \to \mathbf{3}^{s_3}$: $s_3 = \underbrace{\left|h_1^0 - h_2^0\right|}_{s_3^0} + Z,$

h_1^0, h_2^0, Z determine the unique amplitude with
$s_3 = \left|h_1^0 - h_2^0\right| + Z$:

$$\tau^{s_1+h_1^0}_{1+(\{J_1\}}\tau^{s_2-h_2^0}_{2-\{J_4\})}\tau^{s_1-h_1^0}_{1-(\{J_2\}}\tau^{s_2+h_2^0}_{2+\{J_3\})} C^{\frac{s_1+s_2+h_1^0-h_2^0}{2},\{J_1,J_4\};\frac{s_1+s_2-h_1^0+h_2^0}{2},\{J_2,J_3\}}_{|h_1^0-h_2^0|,(\{K_1\}} \mathcal{Y}^Z_{\{K_2\})}$$



Casimirs

A lot of relativistic spin operators
(table form 1303.3862)

TABLE 1: Definitions and commutation properties of various relativistic spin operators.

Operator name	Definition	$[\hat{H}_0, \hat{S}] = 0$?	$[\hat{S}_i, \hat{S}_j] = i\varepsilon_{i,j,k}\hat{S}_k$?	Eigenvalues $= \pm 1/2$?
Pauli [30–32]	$\hat{S}_{P} = \frac{1}{2}\hat{\Sigma}$	no	yes	yes
Foldy-Wouthuysen [33–37]	$\hat{S}_{FW} = \frac{1}{2}\hat{\Sigma} + \frac{i\beta}{2\hat{p}_0}\hat{p}\times\alpha - \frac{\hat{p}\times(\hat{\Sigma}\times\hat{p})}{2\hat{p}_0(\hat{p}_0 + m_0 c)}$	yes	yes	yes
Czachor [38]	$\hat{S}_{Cz} = \frac{m_0^2c^2}{2\hat{p}_0^2}\hat{\Sigma} + \frac{i m_0 c\beta}{2\hat{p}_0^2}\hat{p}\times\alpha + \frac{\hat{p}\cdot\hat{\Sigma}}{2\hat{p}_0^2}\hat{p}$	yes	no	no
Frenkel [39–41]	$\hat{S}_{F} = \frac{1}{2}\hat{\Sigma} + \frac{i\beta}{2m_0 c}\hat{p}\times\alpha$	yes	no	no
Chakrabarti [42–48]	$\hat{S}_{Ch} = \frac{1}{2}\hat{\Sigma} + \frac{i}{2m_0 c}\alpha\times\hat{p} + \frac{\hat{p}\times(\hat{\Sigma}\times\hat{p})}{2m_0 c(m_0 c + \hat{p}_0)}$	no	yes	yes
Pryce [47, 49–52]	$\hat{S}_{Pr} = \frac{1}{2}\beta\hat{\Sigma} + \frac{1}{2}\hat{\Sigma}\cdot\hat{p}(1-\beta)\frac{\hat{p}}{\hat{p}^2}$	yes	yes	yes
Fradkin-Good [46, 53]	$\hat{S}_{FG} = \frac{1}{2}\beta\hat{\Sigma} + \frac{1}{2}\hat{\Sigma}\cdot\hat{p}\left(\frac{\hat{H}_0}{c\hat{p}_0} - \beta\right)\frac{\hat{p}}{\hat{p}^2}$	yes	no	yes

What is the relativistic spin operator?

Bogolubov et al.

$$\mathbf{S}_{BG} = \frac{1}{m}\left(\mathbf{W} - \frac{W^0\mathbf{P}}{m + P^0}\right)$$

$$\boldsymbol{L} = \boldsymbol{R}\times\boldsymbol{P}, \boldsymbol{S} = \boldsymbol{J} - \boldsymbol{L}$$

$$\mathbf{R}_{CM} = -\frac{1}{2}\left(\frac{1}{P^0}\mathbf{K} + \mathbf{K}\frac{1}{P^0}\right),$$

$$\mathbf{R}_{NW} = \mathbf{R}_{CM} - \frac{\mathbf{P}\times\mathbf{W}}{mP^0(m + P^0)}$$

Spin Operators

CSCO

L-S coupling: $\{P^2, \vec{P}, S^2, L^2, J^2, J_z\}$

$$\hat{W}_{3IJ} = \lambda^{\alpha}_{3I}\tilde{\lambda}^{\dot{\alpha}}_{3J}\hat{W}_{3\alpha\dot{\alpha}}/m_3$$

Pauli-Lubanski

$$\hat{S}_{iIJ} \equiv \frac{1}{2(m_i m_3 + p_i \cdot p_3)}\hat{W}_{iI'J'}\tau^{I'}_{iI}\tau^{J'}_{iJ}$$

Rotate $SU(2)_i$ indices

Rotate $SU(2)_3$ indices of τ_i

$$\hat{S}_{IJ} \equiv \hat{S}_{1IJ} + \hat{S}_{2IJ}$$ Spin operator

$$\hat{L} \equiv \hat{W} - \hat{S}$$ Orbit-anlugr-momentum operator

$$\hat{S}^2 M = S(S+1)M$$
$$\hat{L}^2 M = L(L+1)M$$

CSCO

helicity coupling: $\{P^2, \vec{P}, H_1, H_2, J^2, J_z\}$

$$\hat{\mathcal{H}}_1 = \frac{1}{2im_3|\vec{p}_{1cm}|}\mathcal{Y}^{IJ}\hat{S}_{1IJ}, \hat{\mathcal{H}}_2 = \frac{-1}{2im_3|\vec{p}_{1cm}|}\mathcal{Y}^{IJ}\hat{S}_{2IJ}.$$

$$\hat{\mathcal{H}}_1 M_J^{h_1^0,h_2^0} = h_1^0 M_J^{h_1^0,h_2^0}, \hat{\mathcal{H}}_2 M_J^{h_1^0,h_2^0} = h_2^0 M_J^{h_1^0,h_2^0}.$$

In CoM frame:

$$S_i = (\frac{p^0 W^0 - \vec{p}_i \cdot \vec{W}_i}{m_i}, \vec{W}_i - \frac{W_i^0}{p_i^0 + m_i}\vec{p}_i) = (0, \vec{W}_i - \frac{W_i^0}{p_i^0 + m_i}\vec{p}_i)$$

Same as $\hat{S}_{BG}$

Form factor

Analyse L, S of form factors:

$$\mathcal{O}_1 = \bar{\psi}\psi \sim [\mathbf{12}] + \langle \mathbf{12} \rangle$$

$$\mathcal{O}_2 = i\bar{\psi}\gamma_5\psi \sim [\mathbf{12}] - \langle \mathbf{12} \rangle$$

$$\mathcal{O}_3 = \bar{\psi}\gamma_0\psi \sim [\mathbf{2}|3|\mathbf{1}\rangle + \langle \mathbf{2}|3|\mathbf{1}] = 0.$$

$$\mathcal{O}_4 = \bar{\psi}\gamma_i\psi \sim [\mathbf{23}]\langle \mathbf{31} \rangle + \langle \mathbf{23} \rangle[\mathbf{31}]$$

Assume that p_3 is rest, get the same results in Jason Kumar, Danny Marfatia, 1305.1611.

$$p_3 = p_1 + p_2$$

bilinear	Amplitude	r=(L,S,J)	CP
$\bar{\psi}\psi$	$\bar{v}_1 u_2$	(1,1,0))	++
$i\bar{\psi}\gamma_5\psi$	$\bar{v}_1\gamma_5 u_2$	(0,0,0)	+-
$\bar{\psi}\gamma_0\psi$	$p_3^\mu \bar{v}_1\gamma_\mu u_2$	none	-+
$\bar{\psi}\gamma_i\psi$	$\epsilon_3^\mu \bar{v}_1\gamma_\mu u_2$	(0,1,1)+(2,1,1)	–
$\bar{\psi}\gamma_5\gamma_0\psi$	$p_3^\mu \bar{v}_1\gamma_5\gamma_\mu u_2$	(0,0,0)	+-
$\bar{\psi}\gamma_5\gamma_i\psi$	$\epsilon_3^\mu \bar{v}_1\gamma_5\gamma_\mu u_2$	(1,1,1)	++
$\bar{\psi}\sigma^{0i}\psi$	$p_3^\mu \epsilon_3^\nu \bar{v}_1\sigma_{\mu\nu} u_2$	(0,1,1)+(2,1,1)	–
$\bar{\psi}\sigma^{ij}\psi$	$\epsilon_3^\mu \epsilon_3^\nu \bar{v}_1\sigma_{\mu\nu} u_2$	(1,0,1)	-+
$\phi^\dagger\phi$	1	(0,0,0)	++
$iIm(\phi^\dagger\partial_0\phi)$	$p_3^\mu(p_{1\mu} - p_{2\mu})$	none	-+
$iIm(\phi^\dagger\partial_i\phi)$	$\epsilon_3^\mu(p_{1\mu} - p_{2\mu})$	(1,0,1)	–
$B_\mu^\dagger B^\mu$	$\epsilon_1^\mu \epsilon_{2\mu}$	(0,0,0)+(2,2,0)	++
$iIm(B_\mu^\dagger\partial_0 B^\mu)$	$\epsilon_1^\mu \epsilon_{2\mu} p_3^\mu (p_{1\mu} - p_{2\mu})$	none	-+
$iIm(B_\mu^\dagger\partial_i B^\mu)$	$\epsilon_1^\nu \epsilon_{2\nu} \epsilon_3^\mu (p_{1\mu} - p_{2\mu})$	(1,0,1)+(1,2,1)+(3,2,1)	–
$i(B_i^\dagger B_j - B_j^\dagger B_i)$	$\epsilon_3^\mu \epsilon_3^\nu (\epsilon_{1\mu}\epsilon_{2\nu} - \epsilon_{1\nu}\epsilon_{2\mu})$	(0,1,1)+(2,1,1)	-+
$i(B_i^\dagger B_0 - B_0^\dagger B_i)$	$\epsilon_3^\mu p_3^\nu (\epsilon_{1\mu}\epsilon_{2\nu} - \epsilon_{1\nu}\epsilon_{2\mu})$	(1,0,1)+(1,2,1)+(3,2,1)	–
$\epsilon^{0ijk} B_i \partial_j B_k$	$p_{3\mu}\epsilon^{\mu\nu\kappa\rho}\epsilon_{1\nu} p_{2\kappa}\epsilon_{2\rho}$	(1,1,0)	+-
$\epsilon^{0ijk} B_0 \partial_j B_k$	$\epsilon_{3\mu}\epsilon^{\mu\nu\kappa\rho}\epsilon_{1\nu} p_{2\kappa}\epsilon_{2\rho}$	(2,2,1)	++
$B^\nu \partial_\nu B_0$	$\epsilon_{1\mu} p_2^\mu p_{3\nu} \epsilon_2^\nu$	(0,0,0)+(2,2,0)	++
$B^\nu \partial_\nu B_i$	$\epsilon_{1\mu} p_2^\mu \epsilon_{3\nu} \epsilon_2^\nu$	(1,1,1)	+-

Form factor

Construct form factors by L, S:

1,2 are spinning or spinless:

$$\int dx e^{ip_3\cdot x}\langle 0|O^{(\alpha_1\ldots\dot{\alpha}_1\ldots)}(x)|p_1p_2\rangle \in span\,\{\tau_1^{2s_1}\otimes\tau_2^{2s_2}\otimes\mathcal{Y}^{*L}\otimes|3\rangle^{2j_L}\otimes|3]^{2j_R}\}$$

As a comparison, amplitude M:

$$M\in span\{\tau_1^{2s_1}\otimes\tau_2^{2s_2}\otimes\mathcal{Y}^{*L}\}$$

E.g.

$$\int e^{ip_3x}\langle 0|\bar{\psi}\gamma^\mu\psi|p_1p_2\rangle = aM_{s_3=0}p_3^\mu + b|3^{I_1}]\langle 3^{I_2}|M_{s_3=1,\{I_1I_2\}}$$

$$p_3 = p_1 + p_2$$

Thanks