Family tree decomposition of cosmological correlators

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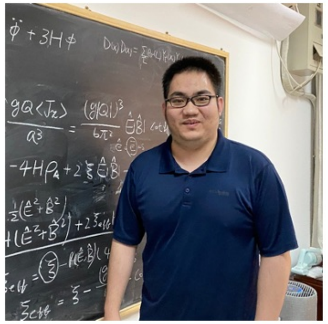
Workshop on Multi-front Exotic phenomena in Particle and Astrophysics

Yuxi | August 26, 2024

Based on:

w/ Zhehan Qin:

1. JHEP **10** (2022) 192 [2205.01692]
2. JHEP **04** (2023) 059 [2208.13790]
3. JHEP **07** (2023) 001 [2301.07047]
4. JHEP **09** (2023) 116 [2304.13295]
5. JHEP **01** (2024) 168 [2308.14802]
6. w/ Hongyu Zhang: JHEP **04** (2023) 103 [2211.03810]
7. w/ Jiaju Zang: JHEP **03** (2024) 070 [2309.10849]
8. w/ Bingchu Fan: 2403.07050
9. w/ Bingchu Fan & Jiaju Zang, to appear
10. w/ Haoyuan Liu & Zhehan Qin: 2407.12299
11. w/ Haoyuan Liu, Jiayi Wu, Hongyu Zhang, to appear



Bingchu Fan
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Zhehan Qin
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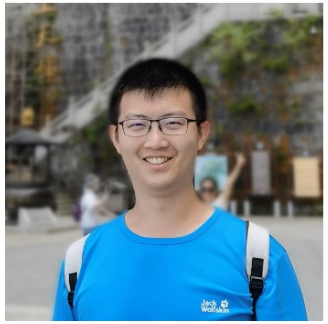

Jiayi Wu
吴家毅

Jiaju Zang
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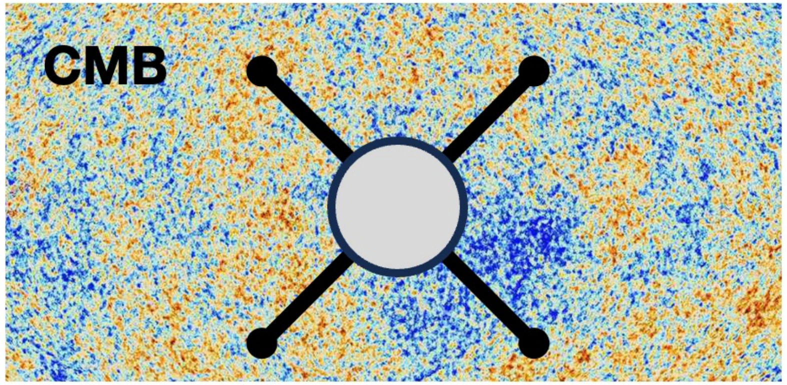
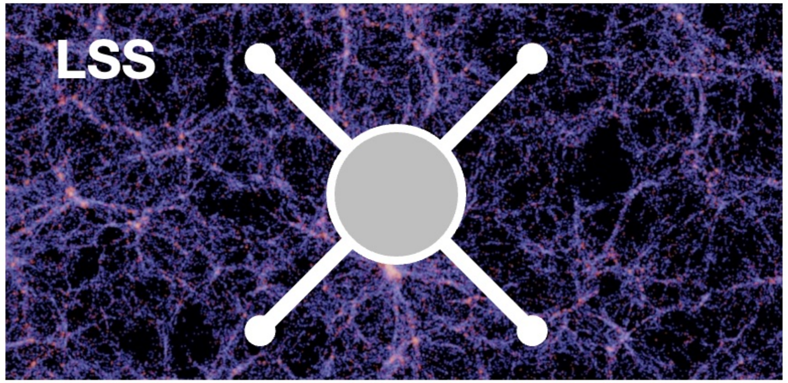
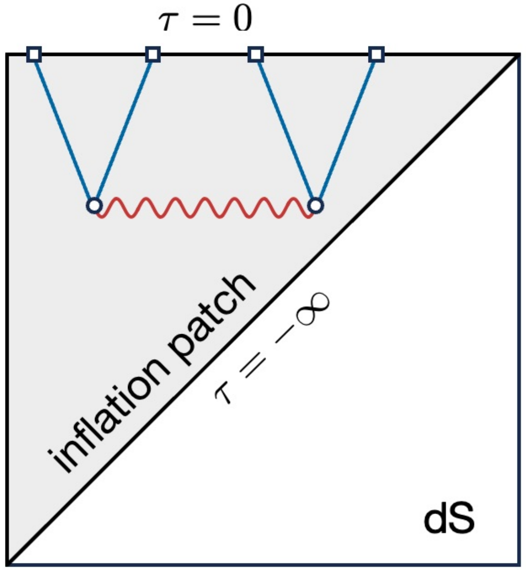
Hongyu Zhang
张洪语

A Cosmological collider program

[Chen, Wang, 0911.3380; Arkani-Hamed, Maldacena, 1503.08043 and many more]



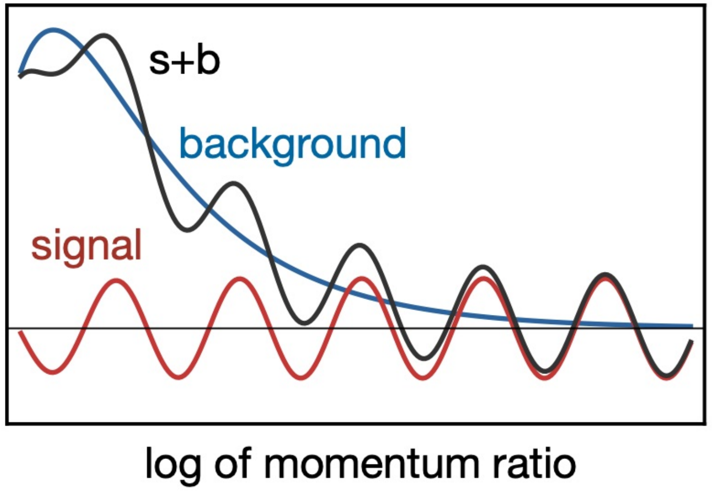




Inflation ~ dS

particle production

mass ~ 10^{14} GeV



superhorizon resonance

mass, spin, coupling, etc

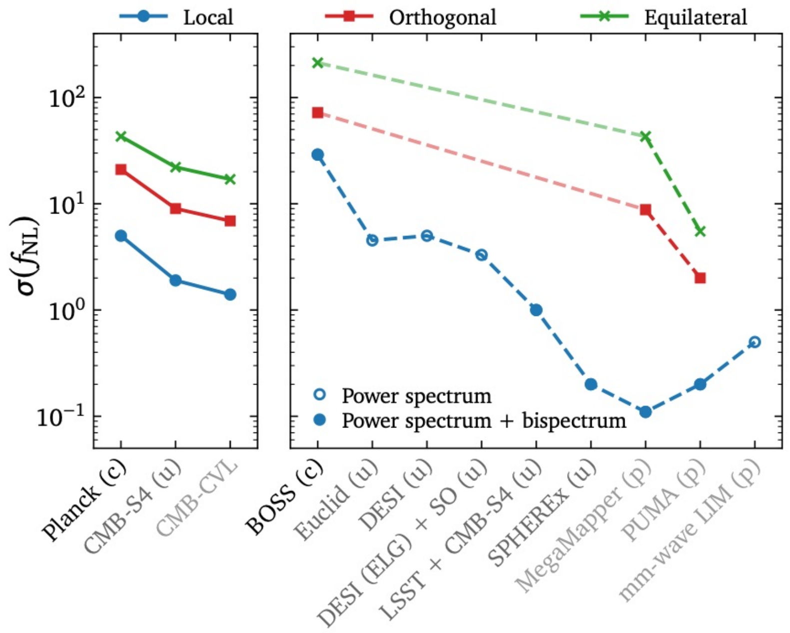
amplitude nonanalyticity

A Cosmological collider program

~ 2 orders in near future

~ 4 ultimately with 21cm

[Snowmass 2021: 2203.08128]



Constraints from CMB data [2404.07203]

Searching for Cosmological Collider in the Planck CMB Data

Wuhyun Sohn[1], Dong-Gang Wang[2], James R. Fergusson[2], and E. P. S. Shellard[2]

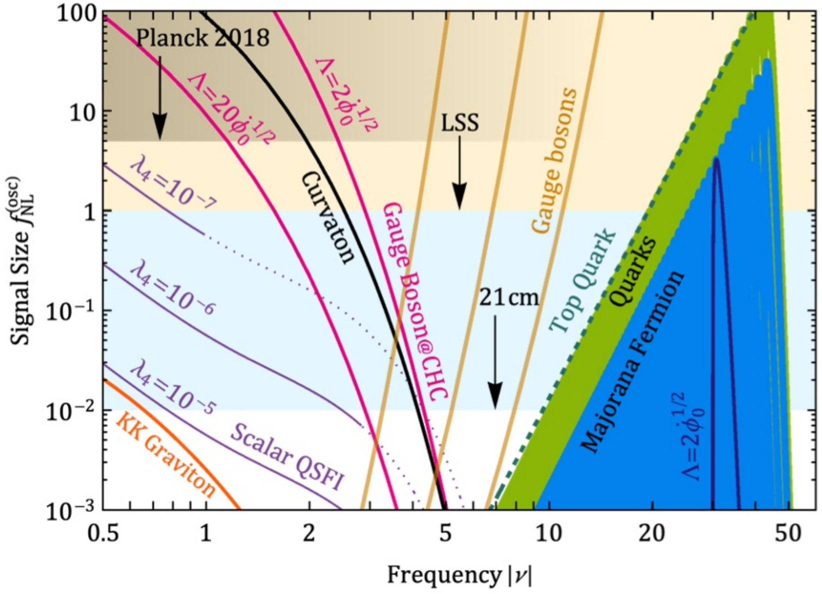
Constraints from LSS data [2404.01894]

BOSS Constraints on Massive Particles during Inflation: The Cosmological Collider in Action

Giovanni Cabass,[1,*] Oliver H.E. Philcox,[2,3,†] Mikhail M. Ivanov,[4,‡] Kazuyuki Akitsu,[5] Shi-Fan Chen,[6] Marko Simonović,[7,8] and Matias Zaldarriaga[6]

inflaton self-interactions. This is made possible through an improvement in Cosmological Bootstrap techniques and the combination of perturbation theory and halo occupation distribution models for galaxy clustering. Our work sets the standard for inflationary spectroscopy with cosmological observations, providing the ultimate link between physics on the largest and smallest scales.

A Cosmological collider program



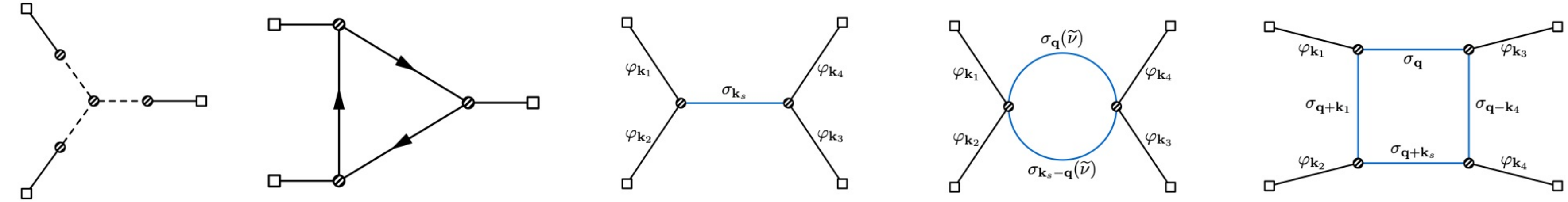
[Lian-Tao Wang, **ZX**, 1910.12876]

Over the years, many particle models identified in SM/BSM, with large signals

Many types of diagrams (tree + loop) involved

Understanding the amplitudes!

- efficient numerical implementation
- analytical structure



Cosmological correlators

[See Chen, Wang, **ZX**, 1703.10166 for a review]

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}\tau \int \mathrm{d}^d\boldsymbol{q} \times (-\tau)^p \times e^{\mathrm{i}E\tau} \times \mathrm{H}_{\mathrm{i}\widetilde{\nu}}\Big[-K(\boldsymbol{q},\boldsymbol{k})\tau\Big] \times \theta(\tau_i - \tau_j)$$

vertex int loop int ext line bulk line

Tree level: special functions and time orderings [difficulties increase with # vertices]

Special functions in propagators

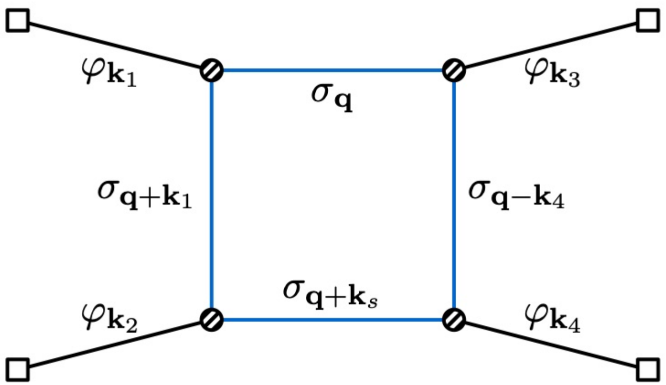
☐ partial Mellin-Barnes [Qin, **ZX**, 2205.01692, 2208.13790 etc]

Time ordered integrals

☐ family tree [**ZX**, Zang, 2309.10849]

Loop level techniques:

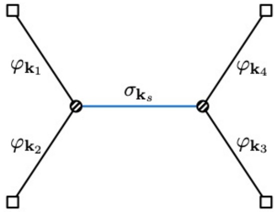
☐ spectral [**ZX**, Zhang, arXiv:2211.03810] ☐ dispersion [ongoing]

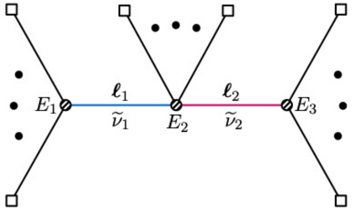


Cosmological correlators: status summary

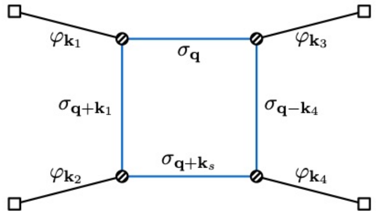
Progress from our group since 2022:

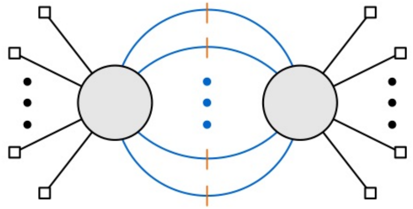
1. Arbitrary tree graphs solved [partial MB + family tree]
[arbitrary number of massive lines;
dS-boost broken dispersion, e.g. axion couplings]



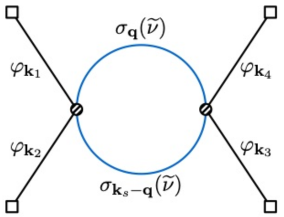


2. Leading nonlocal signals (oscillations) at all loop orders
[all-loop factorization theorems and cutting rules]





3. Full results for 1-loop bubble diagrams
[spectral + dispersion]



Analytical results enable fast numerical implementation:
1-loop: Brute-force numerical [O(105) CPU hours] vs. Analytical [O(10)s on a laptop]
[Lian-Tao Wang, **ZX**, Yi-Ming Zhong, 2109.14635] [**ZX**, Hongyu Zhang, 2211.03810]

Other methods: "cosmological bootstrap" [Arkani-Hamed et al. 1811.00024 etc]
AdS [Sleight et al 1907.01143 etc]

Partial Mellin-Barnes representation

[Qin, **ZX**, 2205.01692, 2208.13790]

Mellin transform & Mellin-Barnes rep:

$$F(s) = \int_0^\infty \mathrm{d}x\, x^{s-1} f(x) \qquad\qquad f(x) = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}}\, x^{-s} F(s)$$

Expanding in dilatation eigenmode [dS counterpart of Fourier transform in flat space]

Partial Mellin-Barnes rep: MB rep for all bulk lines; Special functions => powers

For example: Massive scalar propagator [Hankel function]

$$\mathrm{H}_\nu^{(1)}(-k\tau) = \frac{1}{\pi} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \left(\frac{k}{2}\right)^{-2s} (-\tau)^{-2s} e^{(2s-\nu-1)\pi\mathrm{i}/2} \Gamma\left[s-\frac{\nu}{2}, s+\frac{\nu}{2}\right]$$

↑ ↑

Time and momentum factorized

All time and momentum integrals factorized; We can deal with them separately:

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d\boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\right] \times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i - \tau_j)\right]$$

bulk lines — loop int — nested time int

bulk lines, loop int → Left poles

loop int, nested time int → Right poles

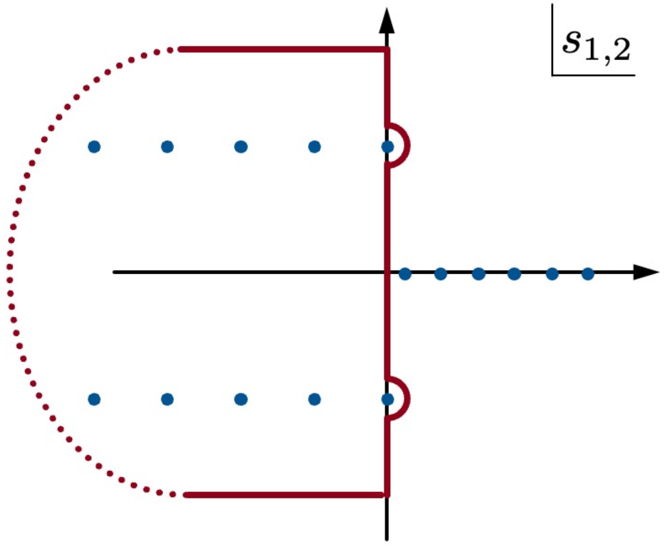
Loop integrals similar to flat space [simple loops doable]

Time integrals more challenging: arbitrary time orderings

Mellin integrands typically meromorphic [only poles]

Final results by residue theorem: pole collecting

Pole structure encodes rich physics!



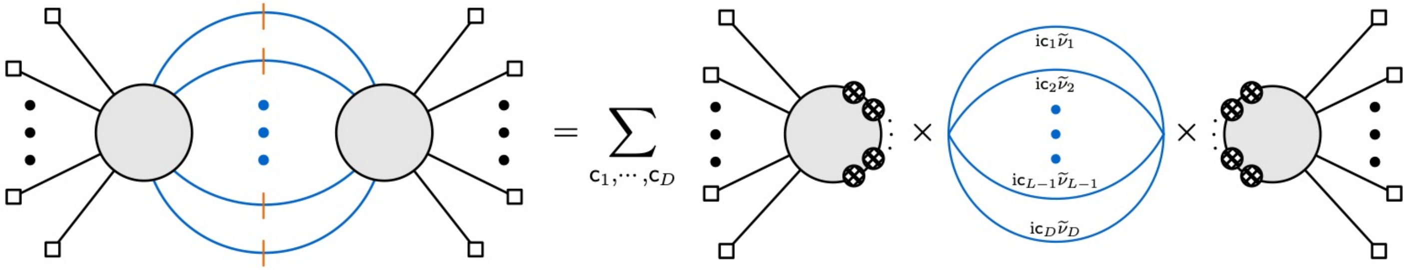
All-loop factorization theorem & cutting rule

[Zhehan Qin, **ZX**, 2304.13295; 2308.14802]

Thanks to PMB, nonanalyticities in K all come from the loop integral:

$$\mathcal{T}(\{\boldsymbol{k}\}) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \left[\int \mathrm{d}^d\boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\right] \times \left[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i-\tau_j)\right]$$

Nonlocal signals are factorized & cuttable to all loop orders by pole analysis:



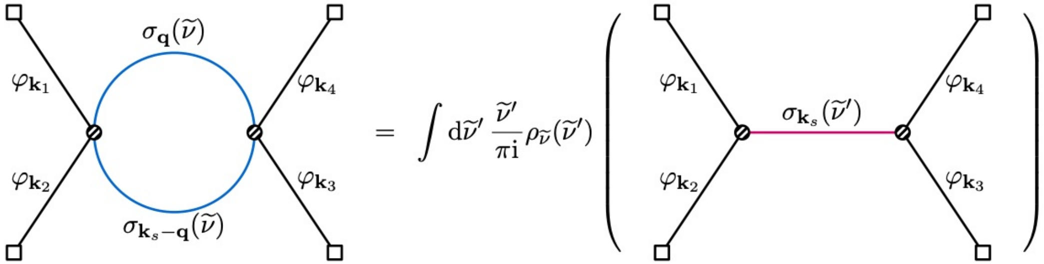
Moreover, all-loop signals are analytically calculable:

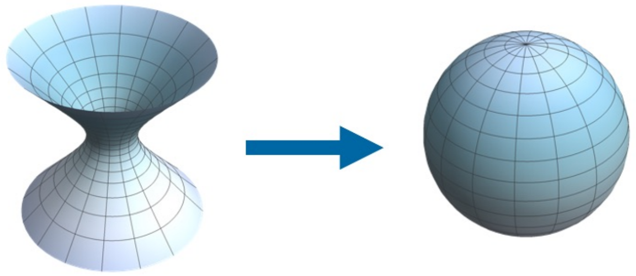
$$\mathfrak{M}_{\mathsf{c}_1\cdots\mathsf{c}_D}(P) \equiv \frac{P^{3(D-1)}}{(4\pi)^{(5D-3)/2}}\Gamma\left[\begin{matrix} -\sum_{i=1}^{D}\mathsf{c}_i\mathrm{i}\widetilde{\nu}_i - \frac{3}{2}(D-1) \\ \frac{3}{2}D + \sum_{i=1}^{D}\mathsf{c}_i\mathrm{i}\widetilde{\nu}_i \end{matrix}\right]\prod_{\ell=1}^{D}\left\{\Gamma\Big[\tfrac{3}{2}+\mathsf{c}_\ell\mathrm{i}\widetilde{\nu}_\ell, -\mathsf{c}_\ell\mathrm{i}\widetilde{\nu}_\ell\Big]\Big(\frac{P}{2}\Big)^{2\mathrm{i}\mathsf{c}_\ell\widetilde{\nu}_\ell}\right\}$$

Bootstrap with spectral decomposition & dispersion relations

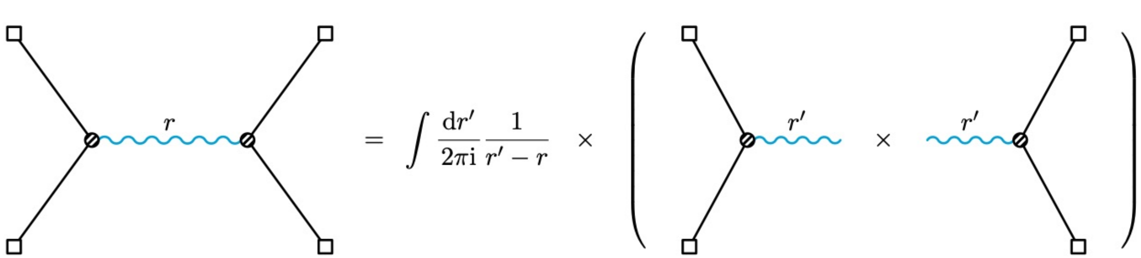
[**ZX**, Zhang, 2211.03810; Liu, Qin, **ZX**, 2407.12299; Liu, Wu, **ZX**, Zhang, to appear]

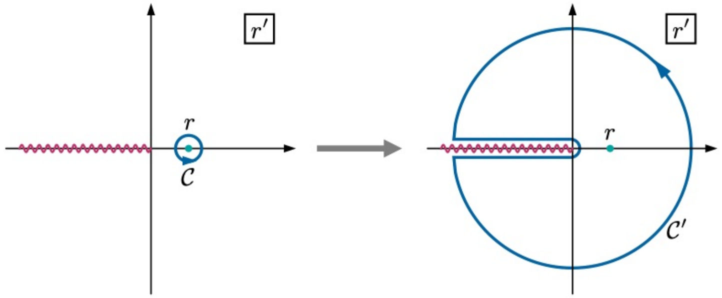
Full loop results obtained via spectral decomposition: loop => sum of trees





Furthermore, tree graphs reconstructable from its cut:





We obtained first and hitherto only known massive loop shapes with both methods

In particular, the dispersion methods is free of UV divergence: neat separation of renormalization-dep local terms with an "irreducible part" demanded by analyticity

Family tree decomposition

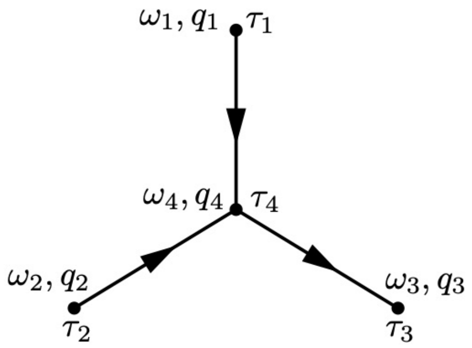
[**ZX**, Zang, 2309.10849]

$$\mathcal{T}\big(\{\boldsymbol{k}\}\big) \sim \int \mathrm{d}s \times \mathcal{G}(s) \times \bigg[\int \mathrm{d}^d \boldsymbol{q} K(\boldsymbol{q},\boldsymbol{k})^\alpha\bigg] \times \bigg[\int \mathrm{d}\tau e^{\mathrm{i}E\tau} \times (-\tau)^\beta \times \theta(\tau_i - \tau_j)\bigg]$$

bulk lines　　loop int　　nested time int

The most general time integral: $$(-\mathrm{i})^N \int_{-\infty}^{0} \prod_{\ell=1}^{N} \Big[\mathrm{d}\tau_\ell\, (-\tau_\ell)^{q_\ell - 1} e^{\mathrm{i}\omega_\ell \tau_\ell}\Big] \prod \theta(\tau_j - \tau_i)$$

It naturally acquires a graphic representation [NOT original Feynman diagrams]:



$$= (-\mathrm{i})^4 \int \prod_{\ell=1}^{4} \Big[\mathrm{d}\tau_\ell\, (-\tau_\ell)^{q_\ell - 1} e^{\mathrm{i}\omega_\ell \tau_\ell}\Big] \theta(\tau_4 - \tau_1)\theta(\tau_4 - \tau_2)\theta(\tau_3 - \tau_4)$$

Family tree decomposition

Complications all from theta functions

Irremovable, but can flip directions, at the expense of additional factorized graphs

$$\theta(\tau_1 - \tau_2) + \theta(\tau_2 - \tau_1) = 1$$

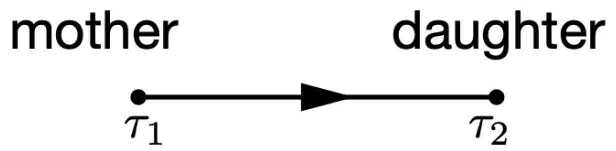
Family tree decomposition:

We always flip the directions such that all nested graphs are partially ordered

Partial order:

A mother can have any number of daughters

but a daughter must have only one mother



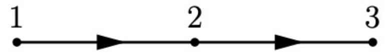
Every resulting nested graph can be interpreted as a maternal family tree

A useful notation for family trees: $[12(34\cdots)(5\cdots)]$

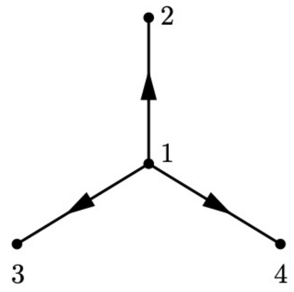
sisters

mother-daughter

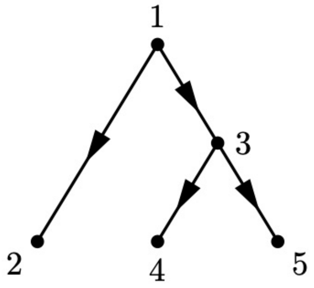
Examples:



$$[123] = (-\mathrm{i})^3 \int \prod_{i=1}^{3} \left[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \right] \theta_{32}\theta_{21}$$



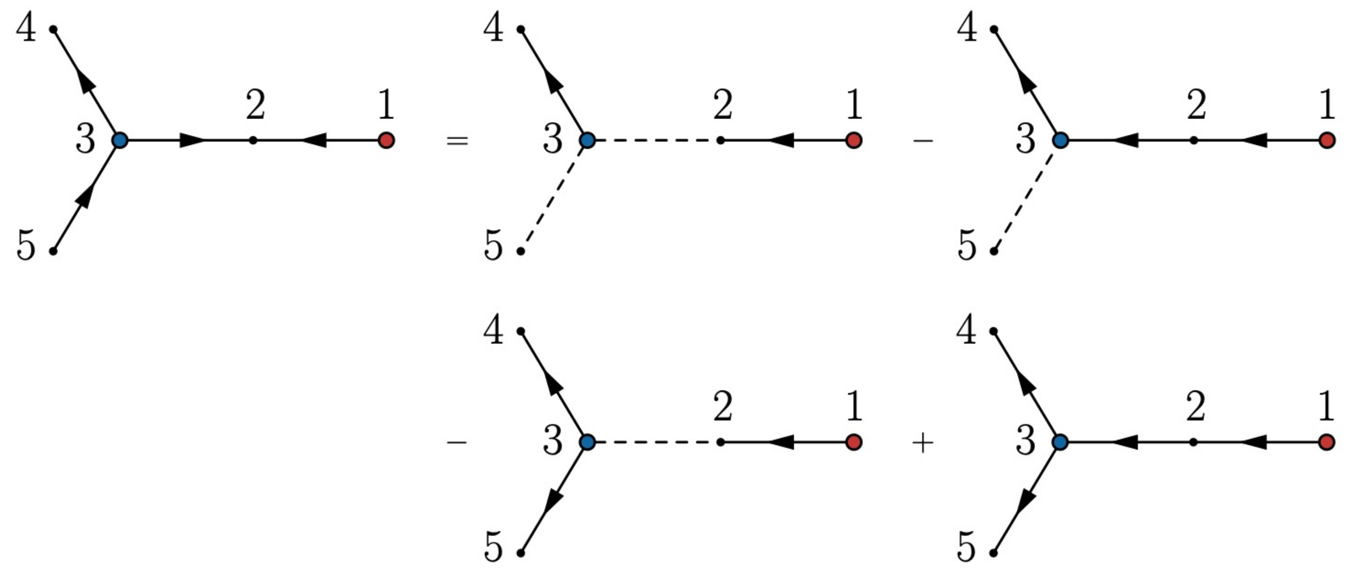
$$[1(2)(3)(4)] = (-\mathrm{i})^4 \int \prod_{i=1}^{4} \left[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \right] \theta_{41}\theta_{31}\theta_{21}$$



$$[1(2)\big(3(4)(5)\big)] = (-\mathrm{i})^5 \int \prod_{i=1}^{5} \left[\mathrm{d}\tau_i (-\tau_i)^{q_i - 1} e^{\mathrm{i}\omega_i \tau_i} \right] \theta_{43}\theta_{53}\theta_{31}\theta_{21}$$

$$\theta_{ij} \equiv \theta(\tau_i - \tau_j)$$

Example of family tree decomposition



Choose Site 1 as the earliest

1->2 good 2->3 flip

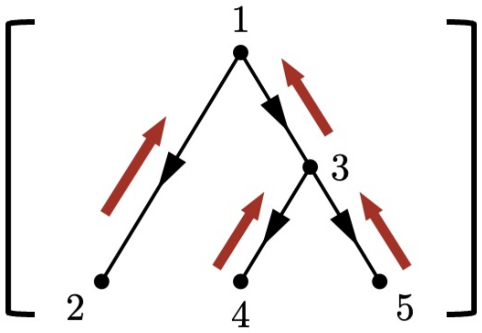
3->4 good 3->5 flip

$$\int \prod_{\ell=1}^{N} \left[\mathrm{d}\tau_\ell \, (-\tau_\ell)^{q_\ell - 1} e^{\mathrm{i}\omega_\ell \tau_\ell} \right] \theta(\tau_2 - \tau_1)\theta(\tau_2 - \tau_3)\theta(\tau_4 - \tau_3)(\tau_3 - \tau_5)$$

$$= [12]\,[34]\,[5] - [1234]\,[5] - [12]\,[3(4)(5)] + [123(4)(5)]$$

Computing the family tree

Expand-and-integrate strategy works, but more streamlined with MB reps

$\mathcal{R}$ 

$$\int_{-\infty}^{\tau_1} d\tau_2\, (-\tau_2)^{q_2-1} e^{i\omega_2\tau_2} = (-\tau_1)^{q_2} \mathrm{E}_{1-q_2}(-i\omega_2\tau_1)$$

(exp int)

$$\mathrm{E}_p(z) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(s) z^{-s}}{s+p-1}$$

MB rep

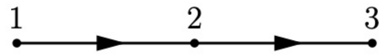
→ next layer: again powers and exp → go through all layers → finish MB int

Mellin integrals finished by the residue theorem, with a series expansion:

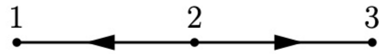
$$[\mathscr{P}(\widehat{1}2\cdots N)] = \frac{(-i)^N}{(i\omega_1)^{q_{1\cdots N}}} \sum_{n_2,\cdots,n_N=0}^{\infty} \Gamma(q_{1\cdots N} + n_{2\cdots N}) \prod_{j=2}^{N} \frac{(-\omega_j/\omega_1)^{n_j}}{(\widetilde{q}_j + \widetilde{n}_j) n_j!}$$

earliest site sum of all q's on Site j and her descendants $(q_{12\cdots} \equiv q_1 + q_2 + \cdots)$

Examples:



$$[123] = \frac{\mathrm{i}}{(\mathrm{i}\omega_1)^{q_{123}}} \sum_{n_2,n_3=0}^{\infty} \frac{(-1)^{n_{23}}\Gamma[n_{23}+q_{123}]}{n_2!n_3!(n_{23}+q_{23})(n_3+q_3)} \left(\frac{\omega_2}{\omega_1}\right)^{n_2} \left(\frac{\omega_3}{\omega_1}\right)^{n_3}$$



$$[2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{q_{123}}} \sum_{n_1,n_3=0}^{\infty} \frac{(-1)^{n_{13}}\Gamma[n_{13}+q_{123}]}{n_1!n_3!(n_1+q_1)(n_3+q_3)} \left(\frac{\omega_1}{\omega_2}\right)^{n_1} \left(\frac{\omega_3}{\omega_2}\right)^{n_3}$$

All family trees are multivariate hypergeometric series

Always expanded in reciprocal of earliest energy, prefactor gives the monodromy

When do FTD, always ask the maximal energy to sits at the earliest site

For simple family trees, the series sum to named hypergeometric functions [all dressed]

$$[1] = \frac{-\mathrm{i}}{(\mathrm{i}\omega_1)^{q_1}}\Gamma[q_1]$$ Euler Gamma function

$$[12] = \frac{-1}{(\mathrm{i}\omega_1)^{q_{12}}}\ {}_2\mathcal{F}_1\left[\begin{matrix} q_2, q_{12} \\ q_2+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}\right]$$ Gauss hypergeometric function

$$[2(1)(3)] = \frac{\mathrm{i}}{(\mathrm{i}\omega_2)^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle|\begin{matrix} q_1, q_3 \\ q_1+1, q_3+1 \end{matrix}\middle| -\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right]$$ Appell function

$$[123] = \frac{\mathrm{i}}{(\mathrm{i}\omega_1)^{q_{123}}}\ {}^{2+1}\mathcal{F}_{1+1}\left[\begin{matrix} q_{123}, q_{23} \\ q_{23}+1 \end{matrix}\middle|\begin{matrix} \text{-}\ , q_3 \\ \text{-}\ , q_3+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right]$$ Kampé de Fériet function

$$[1(2)\cdots(N)] = \frac{(-\mathrm{i})^N}{(\mathrm{i}\omega_1)^{q_{1\cdots N}}}\mathcal{F}_A\left[q_{1\cdots N}\middle|\begin{matrix} q_2, \cdots, q_N \\ q_2+1, \cdots, q_N+1 \end{matrix}\middle| -\frac{\omega_2}{\omega_1}, \cdots, -\frac{\omega_N}{\omega_1}\right]$$ Lauricella function

… while more complicated family trees are not yet named

Flexibility, functional identities, analytical continuation

[Bingchu Fan, **ZX**, Jiaju Zang, to appear]

The flexibility of MB rep leads to many distinct expansions of family trees in terms of large / small single energy, partial energy, total energy, or energy differences.

The many expansions of the same function yield many functional identities when the family tree sums to known functions:

$$[12] = [12] \qquad \frac{1}{\omega_1^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}q_2, q_{12}\\ q_2+1\end{matrix}\middle|-\frac{\omega_2}{\omega_1}\right] = \frac{\Gamma[q_2]}{\omega_{12}^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}1, q_{12}\\ q_2+1\end{matrix}\middle|\frac{\omega_2}{\omega_{12}}\right]$$

$$[12]+[21] = [1]\,[2] \qquad \frac{1}{\omega_1^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}q_2, q_{12}\\ q_2+1\end{matrix}\middle|-\frac{\omega_2}{\omega_1}\right] + \frac{1}{\omega_2^{q_{12}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}q_1, q_{12}\\ q_1+1\end{matrix}\middle|-\frac{\omega_1}{\omega_2}\right] = \frac{\Gamma[q_1, q_2]}{\omega_1^{q_1}\omega_2^{q_2}}$$

$$[123]+[2(1)(3)] = [1]\,[23] \qquad \frac{1}{\omega_1^{q_{123}}}\,{}_{2+1}\mathcal{F}_{1+1}\left[\begin{matrix}q_{123}, q_{23}\\ q_{23}+1\end{matrix}\middle|\begin{matrix}\text{-}\ , q_3\\ \text{-}\ , q_3+1\end{matrix}\middle|-\frac{\omega_2}{\omega_1}, -\frac{\omega_3}{\omega_1}\right] + \frac{1}{\omega_2^{q_{123}}}\mathcal{F}_2\left[q_{123}\middle|\begin{matrix}q_1, q_3\\ q_1+1, q_3+1\end{matrix}\middle|-\frac{\omega_1}{\omega_2}, -\frac{\omega_3}{\omega_2}\right]$$
$$= \frac{\Gamma[q_1]}{\omega_1^{q_1}\omega_2^{q_{23}}}\,{}_2\mathcal{F}_1\left[\begin{matrix}q_3, q_{32}\\ q_3+1\end{matrix}\middle|-\frac{\omega_3}{\omega_2}\right]$$

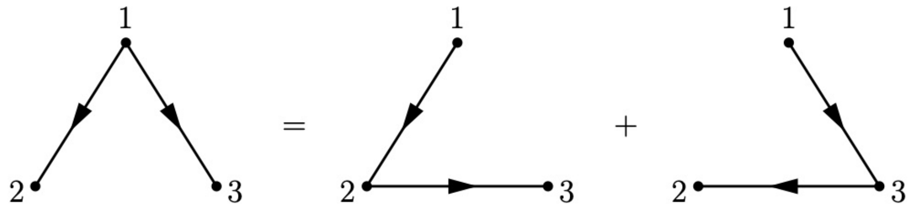
More importantly, when series do not close, the identities amount to analytical continuation beyond the region of convergence

Minimal set of functions: family chains

birthday rule: Compare the birthdays of all family members and sum over all possibilities

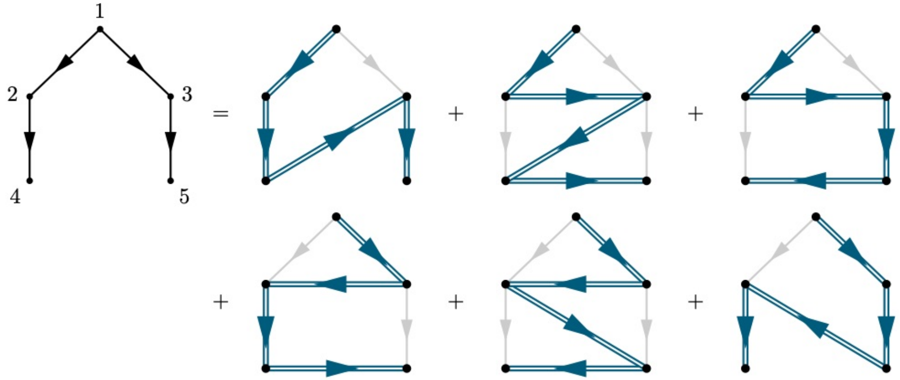
Formally, take shuffle products recursively among all subfamilies

$$\theta_{21}\theta_{31}(\theta_{32}+\theta_{23}) = \theta_{32}\theta_{21} + \theta_{23}\theta_{31}$$



Example:

$$\begin{aligned}\big[1(24)(35)\big] &= \big\{1(24) \sqcup\!\sqcup (35)\big\} \\ &= \{12435\} + \{12345\} + \{12354\} \\ &+ \{13245\} + \{13254\} + \{13524\}\end{aligned}$$



Family trees over-complete: further decomposable to chains; tree topology erased

Family chains: iterated integrals; Hopf algebra; transcendental weight; symbology?

Applications: Conformal-scalar amplitudes in FRW

[Fan, **ZX**, 2403.07050]

A nice toy model: conformal scalar with non-conformal self-interactions

$$S[\phi_c] = -\int \mathrm{d}^{d+1}x\sqrt{-g}\left[\frac{1}{2}(\partial_\mu\phi_c)^2 + \frac{1}{2}\xi R\phi_c^2 + \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi_c^n\right] \qquad \xi \equiv (d-1)/(4d)$$

With spatially-flat power-law FRW bg

=> massless scalar with time-dep couplings in Mink:

$$S[\varphi] = -\int \mathrm{d}\tau\mathrm{d}^d\boldsymbol{x}\left[\frac{1}{2}(\partial_\mu\varphi)^2 + \sum_{n\geq 3}\frac{\lambda_n(-\tau)^{P_n}}{n!}\varphi^n\right]$$

Mode function: $\varphi(k,\tau) = \frac{1}{\sqrt{2k}}e^{-\mathrm{i}k\tau}$

Many activities in recent years

- ☐ Cosmo polytopes [integrand for all] [Arkani-Hamed et al. 1709.02813 & many follow-ups]
- ☐ Symbol recursion [dS limit] [Hillman 1912.09450]
- ☐ Twisted cohomology [diff eq in FRW up to 3-site] [De & Pokraka 2308.03753]
- ☐ “Kinematic flow” [diff eq for all] [Arkani-Hamed et al. 2312.05300, 2312.05303]

With family trees, we found full analytical answers

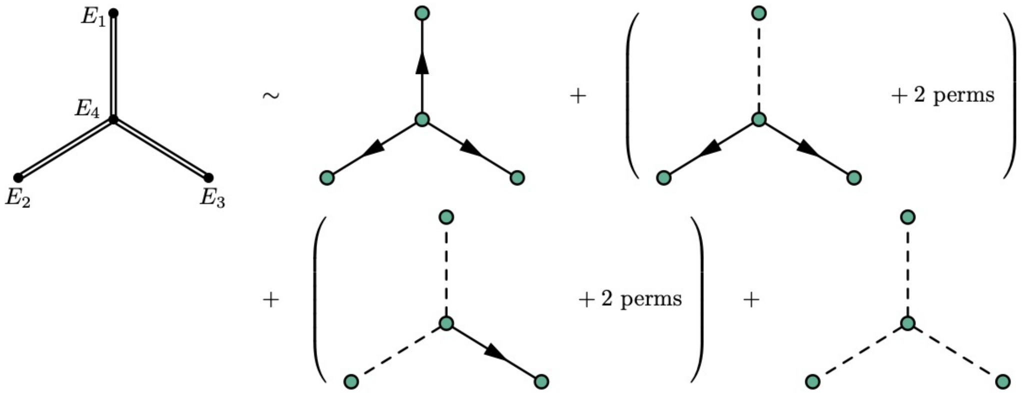
Applications: Conformal-scalar amplitudes in FRW

[Fan, **ZX**, 2403.07050]

Rule for conformal amplitudes: 1. Fix a partial order; 2. Write the uncut tree; 3. Cut!

Example: 4-site star:

Full analytical expressions in terms of family tress in two lines



$$\widetilde{\psi}_{\text{4-star}} = \sum_{\mathsf{a,b,c}=\pm} \mathsf{abc}\Big\{ \big[4_{1^{\mathsf{a}}2^{\mathsf{b}}3^{\mathsf{c}}}(1_{\bar{1}^{\mathsf{a}}})(2_{\bar{2}^{\mathsf{b}}})(3_{\bar{3}^{\mathsf{c}}})\big] + \Big(\big[4_{\bar{1}^{\mathsf{a}}2^{\mathsf{b}}3^{\mathsf{c}}}(2_{\bar{2}^{\mathsf{b}}})(3_{\bar{3}^{\mathsf{c}}})\big]\big[1\big] + 2 \text{ perms}\Big)$$
$$+ \Big(\big[4_{\bar{1}^{\mathsf{a}}\bar{2}^{\mathsf{b}}3^{\mathsf{c}}}3_{\bar{3}^{\mathsf{c}}}\big]\big[1_1\big]\big[2_2\big] + 2 \text{ perms}\Big) + \big[4_{\bar{1}^{\mathsf{a}}\bar{2}^{\mathsf{b}}\bar{3}^{\mathsf{c}}}\big]\big[1_1\big]\big[2_2\big]\big[3_3\big]\Big\}$$

Compared with kinematic flow: 64 coupled diff eqs!
Actually, much easier to derive diff eqs from family trees [Song He et al., 2407.17715]

Inflationary limit

Interesting to consider the special case of ϕ^3 theory in dS limit (all q = 0)

Boundary of IR safe region: A family tree of V sites contains q = 0 poles up to deg V
All poles cancel out in amplitudes, finite terms being polylogs

Example 2-site wavefunction: $\widetilde{\psi}_{\text{2-chain}} = [1_1 2_{\bar{1}}] - [1_{\bar{1}} 2_1] + [1_{\bar{1}}][2_1] - [1_1][2_1]$

$$[1_1 2_{\bar{1}}]_{q_1=q_2=q} = \frac{-1}{[\mathrm{i}(E_1+K)]^{2q}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma[n+2q]}{n+q} \left(\frac{E_2-K}{E_1+K}\right)^n$$

$$= \underbrace{-\frac{1}{2q^2} + \frac{\gamma_E + \log[\mathrm{i}(E_1+K)]}{q}}_{\text{divergent terms}} \underbrace{- \mathrm{Li}_2 \frac{K-E_2}{K+E_1} - \Big(\log[\mathrm{i}(E_1+K)] + \gamma_E\Big)^2 - \frac{\pi^2}{6}}_{\text{finite terms}} + \mathcal{O}(q)$$

Final answer: $\widetilde{\psi}_{\text{2-chain}} = \mathrm{Li}_2 \frac{E_2-K}{E_{12}} + \mathrm{Li}_2 \frac{E_1-K}{E_{12}} + \log\frac{E_1+K}{E_{12}} \log\frac{E_2+K}{E_{12}} - \frac{\pi^2}{6}$

More sites: integrated polylogs could be tedious, but easy to get the symbol

[See also Hillman 1912.09450]

Concluding remarks

✓ Arbitrary massive trees essentially solved [PMB + family tree];

✓ Nonlocal signals in arbitrary graphs at all loop orders obtained [PMB + factorization]

✓ A new class of special functions identified (family trees), many math structures

✓ FRW conformal amplitudes obtained: many lessons learnt from a good toy model

Challenges and progresses:

- Loops: bubble done; including spin? Beyond bubble?
- Degenerate kinematics: Understanding family trees!
- Reduced symmetry: boostless loops? (axion-type signals) non-inflation bg; slow roll
- Confronting data: analytical-result-inspired template design and constraining models!

Thank you!