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Relaxation time approximation revisited

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e-Print: 2409.05131, Jin Hu

Outline

- Introduction
- Revisiting relaxation time approximation (RTA)
- A novel RTA
- Non-analytical structures in retarded correlators
- Summary and outlook

Introduction

Relativistic Boltzmann equation describes the non-equilibrium evolution of heavy-ion collision system, $p^\alpha \partial_\alpha f(x,p) = C[f]$

In general cases, we linearize it (we partly suppress x-dependence below)

$$f(x,p) = f_0(x,p)(1+\chi(x,p))$$

S.R.DeGroot et al, Relativistic kinetic theory

$$-f_0(p)\mathscr{L}_0[\chi] \equiv \frac{f_0(p)}{E_p}\int dP' dP_1 dP_2 f_0(p') W\big(\chi(p_1)+\chi(p_2)-\chi(p)-\chi(p')\big)$$

$\mathscr{L}_0$ is self-adjoint, positive semidefinite.

Though linearized, it is still intractable.

Anderson and Witting RTA $-\mathscr{L}_0[\chi] \sim -1/\tau_R\,\chi$ J.L.Anderson, H.R.Witting, Physica 74, 466 (1974)

Revisiting RTA

Anderson and Witting (AW) RTA $-\mathscr{L}_0[\chi] \sim -1/\tau_R\,\chi$

$$p^\alpha \partial_\alpha f(x,p) = -u\cdot p/\tau_R\,(f - f_0)$$

J.L.Anderson, H.R.Witting, Physica 74, 466 (1974)

RTA **widely** used in

deriving hydrodynamic equations, transport coefficients, exact solutions,

gradient expansion convergence, hydro attractors, numerical simulations $\cdots$

energy-dependent RTA (eRTA) $\tau_R = (\beta E_p)^\alpha t_0$

simulations of fluids, jets in relativistic heavy-ion collisions,

analytical study on retarded correlators,

studying the system of Weyl semimetal …

Revisiting RTA

What does this work do ?

Though RTA (also eRTA) has seen extremely widespread application in relativistic kinetic theory and hydrodynamics, a rigorous mathematical justification is still lacking !

I now try to close this gap !

Revisiting RTA

How to construct RTAs ? 🤔

$\mathscr{L}_0\,(\mathscr{L}_\alpha)$ is self-adjoint, positive semidefinite.

AW RTA : $-\mathscr{L}_0[\chi] \sim -\gamma_6\chi$

eRTA : $-E_p^{-\alpha}\mathscr{L}_\alpha[\chi] \equiv -E_p^{-\alpha}(E_p^{\alpha}\mathscr{L}_0[\chi])$
$\sim -E_p^{-\alpha}\gamma_6'\chi$

} $\gamma_6\,,\,\gamma_6'$ nonzero smallest eigenvalue

still self-adjoint and semi-positive,
but square-integrable function
space changes !

How to justify RTA ? 🤔

analyze the spectrum of $\mathscr{L}_0\,(\mathscr{L}_\alpha)$, to see if the truncation is allowed

Revisiting RTA

AW RTA : $\mathscr{L}_0$ must have an eigenvalue spectrum **gapped** from zero except collision invariants.

Does all the dynamics admit such a gapped spectrum ?

What dynamics admits a gapped spectrum ?

Hard interactions or hard collisions

e.g. 4-fermion interactions (UV-incomplete)

$\exists \gamma > -2, 0 \leq \beta < \gamma + 2, B > 0, c_0 > 0,$

so that $\sigma(g, \Theta) > B \dfrac{g^{\beta+1}}{c_0 + g} \sin^{\gamma} \Theta,$

the interactions are hard.

$\mathscr{L}_0$ featuring a gapped eigenspectrum of $[\nu_{min}, \infty], \nu_{min} > 0$

Mathematical aspects see: M. Dudyn´ski, Journal of Statistical Physics 153, 1084–1106 (2013)

M. Dudyn´ski and M. Ekiel-Jezewska, Communications in Mathematical Physics 115, 607 (1988)

Revisiting eRTA

J.Hu, 2409.05131

We still retain the smallest eigenvalue. That is what eRTA keeps !

$$\mathscr{L}_\alpha \to \gamma'_6 \qquad \tau_R = E_p^\alpha/\gamma'_6$$

We can always find a slower mode as long as

$$\alpha > 0, \qquad E_p^\alpha/\gamma'_6 < E_{p'}^\alpha/\gamma'_7 \;\; (\gamma'_7 > \gamma'_6, \text{ but } \; p \ll p')$$

In fact, the hierarchy of eigenvalues is irrelevant when $p, p' \to \infty$.

$\tau_{R,n} \equiv E_p^\alpha/\gamma'_n$ is unbounded. Retaining only one eigenvalue is not enough !

This bound exists for massive transport for $\alpha < 0$.

$E_p^\alpha/\gamma'_6 \to m^\alpha/\gamma'_6$ as $p \to 0$. But it is a redundant description to AW RTA.

Summary and outlook

1. RTA can be only justified in the cases of hard interactions .
2. eRTA can not be justified in any cases .
3. Summary-outlook duality : study non-analytical structures

   in retarded correlators within kinetic description.

(outlook for this talk, summary for the full presentation)

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Back Up

Revisiting RTA

$$Df(x,p) + E_p^{-1} p^{\langle\nu\rangle}\partial_\nu f(x,p) = -f_0(x,p)\mathscr{L}_0[\chi],$$

$$\partial_\mu \equiv u_\mu D + \Delta_\mu^\nu \partial_\nu\,, \quad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \quad p^{\langle\mu\rangle} = \Delta^\mu_\nu p^\nu$$

When we talk about RTA, it only concern with collision effects !

So the spatial-gradient term can be effectively neglected !

Equilibration typically involves {
- collision, momentum isotropization, fast
- drift, spatial gradients smoothing, slow

Local collisions are independent of x^μ .

Revisiting RTA

It is convenient to think about it in rest frame.

$$\nabla f_0(x,p) = \nabla \chi(x,p) = 0$$

Then $\partial_t \chi(t) = -\mathscr{L}_0[\chi], \qquad \chi(t) = e^{-\mathscr{L}_0 t}\chi(0)$

Except collision invariants, the mode with the smallest eigenvalue lives longest. That is what RTA keeps !

$\mathscr{L}_0 \to \gamma_6$

$\tau_R = 1/\gamma_6$

1. $\mathscr{L}_0$ has an eigenvalue spectrum gapped from zero .

2. The smallest eigenvalue is separated from other nonzero ones.

We may loose the 2nd condition using the infimum of the spectrum.

Revisiting RTA

$\mathscr{L}_0$ has an eigenvalue spectrum gapped from zero .

Does all the dynamics admit such a gapped spectrum ?

No !

What dynamics admits a gapped spectrum ?

Hard interactions or hard collisions !

Mathematical aspects see: M. Dudyn´ski, Journal of Statistical Physics 153, 1084–1106 (2013)

M. Dudyn´ski and M. Ekiel-Jezewska, Communications in Mathematical Physics 115, 607 (1988)

Revisiting RTA

Hard interactions:

✅ $-\mathscr{L}_0$ featuring an eigenspectrum of $[-\infty, -\nu_{min}]$

Soft interactions:

$-\mathscr{L}_0$ featuring an eigenspectrum of $[-\nu_{max}, 0]$

where $0 < \nu_{max}, \nu_{min} < \infty, \quad \nu_{max}|_{m\to 0} \to \infty$

M. Dudyn´ski, Journal of Statistical Physics 153, 1084–1106 (2013)

M. Dudyn´ski and M. Ekiel-Jezewska, Communications in Mathematical Physics 115, 607 (1988)

Revisiting RTA

Hard interactions, e.g. $g \equiv \sqrt{-(p-p')\cdot(p-p')}$

4-fermion interactions (UV-incomplete) $\sigma(g,\Theta) \sim s = g^2 + 4m^2 > \frac{g^3}{c_0+g} \geq \frac{g^3}{c_0+g}\sin\Theta$

Soft interactions:

Most cases in relativistic interactions: (LO) QED, QCD, $\lambda\phi^4 \cdots$

$\lambda\phi^4: \quad \sigma(g,\Theta) \sim \frac{1}{s} = \frac{1}{g^2+4m^2} < \frac{2}{g^2}$

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M. Dudyn´ski, Journal of Statistical Physics 153, 1084–1106 (2013)

M. Dudyn´ski and M. Ekiel-Jezewska, Communications in Mathematical Physics 115, 607 (1988)

In non-relativistic cases, interactions are **easier** to be hard.

R. Liboff, Kinetic Theory, Springer (2003).

Revisiting RTA

I. RTA can be only justified in the cases of hard interactions .

Hard interactions are rare in relativistic cases.

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Revisiting eRTA

$$\partial_t \chi(t) = -E_p^{-\alpha} \mathscr{L}_\alpha[\chi], \qquad \chi(t) = e^{-E_p^{-\alpha} \mathscr{L}_\alpha t} \chi(0)$$

We still retain the smallest eigenvalue. That is what eRTA keeps !

$$\mathscr{L}_\alpha \to \gamma'_6 \qquad \tau_R = E_p^\alpha / \gamma'_6$$

We can always find a slower mode as long as

$$\alpha > 0, \qquad E_p^\alpha / \gamma'_6 < E_{p'}^\alpha / \gamma'_7 \ \ (p \ll p')$$

In fact, the hierarchy of eigenvalues is irrelevant when $p, p' \to \infty$.

$\tau_{R,n} \equiv E_p^\alpha / \gamma'_n$ is unbounded. Retaining only one eigenvalue is not enough !

Revisiting eRTA

$$\partial_t \chi(t) = -E_p^{-\alpha} \mathscr{L}_\alpha[\chi], \qquad \chi(t) = e^{-E_p^{-\alpha} \mathscr{L}_\alpha t} \chi(0)$$

However this bound exists for massive transport for $\alpha < 0$.

$$E_p^\alpha / \gamma_6 \to m^\alpha / \gamma_6' \quad \text{as} \quad p \to 0$$

But this not energy-energy-dependent but mass-dependent.

Even though, it is still a redundant description to AW RTA.

To explain this, we introduce a useful argument below.

Revisiting eRTA

Disturb the eq system with a homogeneous perturbation.

$$\partial_t \chi(t) = -E_p^{-\alpha}\mathscr{L}_\alpha[\chi], \qquad \chi(t) = e^{-E_p^{-\alpha}\mathscr{L}_\alpha t}\chi(0)$$

The observer A sees the disturbance damping out at

$$\tau_R = E_p^\alpha/\gamma_6' \to m^\alpha/\gamma_6'$$

The observer B sees the disturbance damping out at $\tau_R = 1/\gamma_6$

$$\partial_t \chi(t) = -\mathscr{L}_0[\chi], \qquad \chi(t) = e^{-\mathscr{L}_0 t}\chi(0)$$

The same equation should give the same observation. $1/\gamma_6 \simeq m^\alpha/\gamma_6'$

Revisiting RTA

II. eRTA can not be well-justified in any cases.

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Non-analytical structures in retarded correlators

Introduction flashing back

$$G_R^{\alpha\beta} \equiv i\theta(x - x')\langle[A^{\alpha}(x), B^{\beta}(x')]\rangle$$

Retarded correlators encoding the full information of linear response.

They carry rich and fundamental physical meaning in non-equilibrium.

statistical physics.

Introduction flashing back

$$G_R^{\alpha\beta} \equiv i\theta(x - x')\langle[A^\alpha(x), B^\beta(x')]\rangle$$

Poles are typically indicative of hydrodynamic collective excitations.

Cuts correspond to non-hydrodynamic modes.

Introduction flashing back

Fourier transform

$$G_R^{\alpha\beta} \equiv i\theta(t - t')\langle[A^\alpha(x), B^\beta(x')]\rangle \quad \longrightarrow \quad G_R^{\alpha\beta}(\omega, k)$$

In strongly-coupled theory, e.g. large N thermal

$\mathcal{N} = 4$ Super Yang-Mills theory, they contain poles.

S. A. Hartnoll, S.P. Kumar, *JHEP* 12 (2005) 036

P. K.Kovtun, A.O. Starinets, *Phys.Rev.D* 72 (2005) 086009

Q: How about the behavior of weakly-coupled theory ?

More greedily, what is the **general** linear response behavior ?

Non-analytical structure in retarded correlators

For **weakly-coupled** theory, e.g. massless $\lambda\phi^4$ theory,

It is shown that stress-stress correlator contains a cut. $\Im\omega \in [-\infty,0]$

Variation method to solve $\mathscr{L}_0$ G.D.Moore, *JHEP* 05 (2018) 084

This poses a so-called **pole/cut dilemma** !

Cut confirmed using analytical spectrum of $\mathscr{L}_1$ G.S.Rocha etc, *Phys.Rev.D* 110 (2024) 7, 076003

Also confirmed numerically for $\mathscr{L}_0$ S.Ochsenfeld, S.Schlichting, *JHEP* 09 (2023) 186

Now you can understand it mathematically ! $\lambda\phi^4$ theory is soft !

$-\mathscr{L}_0$ featuring an eigenspectrum of $[-\nu_{max}, 0]$, $\nu_{max}|_{m\to 0} \to \infty$

Non-analytical structure in retarded correlators

How about hard cases ?

Note AW RTA is now well-defined. We can trust it.

Massless RTA (finte k):

Paul Romatschke, *Eur.Phys.J.C* 76 (2016) 6, 352

1. Onset transition behavior of hydro poles

exist for low k ➡ vanish for high k

More universal in non-relativistic cases because of the softness

Studies in the Statistical Mechanics (1970), J.De Boer and G.E.Uhlenbeck, Chap V

2. Gapped branch cut line $\mathfrak{R}\omega \in [-k, k], \quad \mathfrak{I}\omega = -i/\tau_R$

Hydro modes can exist as the long-lived low energy dofs !

Non-analytical structure in retarded correlators

Gapped branch cut line $\Re\omega \in [-k, k], \quad \Im\omega = -i/\tau_R$

$k \to 0, \quad \Re\omega = 0, \quad \Im\omega = -i/\tau_R$ Gapped branch point

Paul Romatschke, *Eur.Phys.J.C* 76 (2016) 6, 352

Romatschke's conclusion is questioned by an analytical calculation with eRTA

A.Kurkela, U.A.Wiedemann, *Eur.Phys.J.C* 79 (2019) 9, 776

1. Onset transition behavior is an artifact of analytical continuation.

2. Hydro poles embedded in s strip of non-hydro modes.

   Can hydro modes exist as the long-lived low energy dofs ?

   No way !

Non-analytical structure in retarded correlators

Romatschke's conclusion is questioned by an analytical calculation with eRTA

A.Kurkela, U.A.Wiedemann, *Eur.Phys.J.C* 79 (2019) 9, 776

2. Hydro poles embedded in s strip of non-hydro modes.

Can hydro modes exist as the long-lived low energy dofs ?

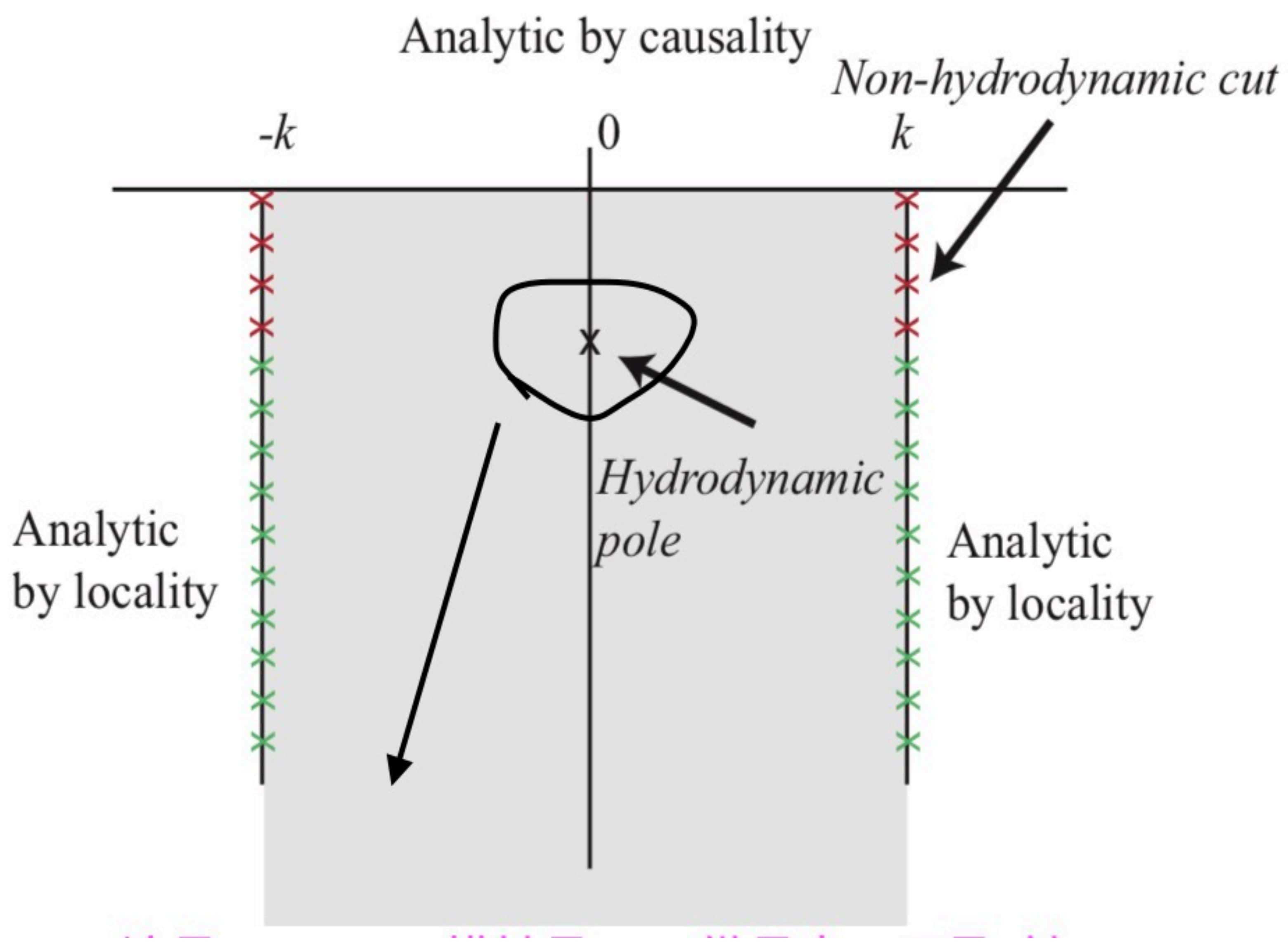
No way !

Comments on the debate :

I. Romatschke's analysis applies for hard cases.

II. eRTA is not well-justified. soft cases.

III. Both two give one side of the coin.



Non-analytical structure in retarded correlators

General statements on non-analytical structures within retarded correlators

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Why I call it general ?

Finite mass, finite k and softness all into account.

Hard interactions :

$m = 0$ or $k = 0,$ Romatschke's analysis applies.

$$\Im\omega = -\frac{1}{\tau_R}, \quad -k \leq \Re\,\omega \leq k,$$ two finite branch points

This and massive RTA can be tested with finite-element analysis.

Non-analytical structure in retarded correlators

Hard interactions :

$m \neq 0\,,\; k \neq 0 \qquad G_R(\omega,k) \sim \int dp \int_{-1}^{1} dx \frac{p^n e^{-\sqrt{m^2+p^2}/T}}{\frac{1}{\tau_R} - i\omega + \frac{ipkx}{\sqrt{m^2+p^2}}}$

$m \to 0 \qquad G_R(\omega,k) \sim \int dp p^n e^{-p/T} \int_{-1}^{1} dx \frac{1}{\frac{1}{\tau_R} - i\omega + ikx}$

$m \neq 0\,,\; k \neq 0 \qquad \omega = -\frac{i}{\tau_R} \mp \frac{pk}{\sqrt{m^2+p^2}},\; p \in [0,\infty]$

$\Im\omega = -\frac{1}{\tau_R}, \quad -k \leq \Re\,\omega \leq k,$ line formed by branch points !

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Non-analytical structure in retarded correlators

Soft interactions :

$$k = 0 \qquad \Re\omega = 0, \quad \Im\omega \in [-\nu_{max}, 0], \quad \nu_{max}|_{m\to 0} \to \infty$$

$$m \to 0 \qquad \Re\omega = 0, \quad \Im\omega \in [-\infty, 0]$$

G.D.Moore, *JHEP* 05 (2018) 084 **G.S.Rocha etc, *Phys.Rev.D* 110 (2024) 7, 076003**

$k \neq 0,\ m = 0$ eRTA is not well-justified !

$$[\cos\theta, \mathscr{L}_0] \neq 0, \quad \cos\theta \equiv \langle p, k\rangle$$

Hydro modes could exist as the long-lived low energy dofs !

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Non-analytical structure in retarded correlators

J. Hu, 2409.05131

Soft interactions :

$k \neq 0, \; m = 0$ eRTA is not well-justified !

To complete the analysis given in **A.Kurkela, U.A.Wiedemann, *Eur.Phys.J.C* 79 (2019) 9, 776**

It is easier to use $\mathscr{L}_0$ instead of $\mathscr{L}_1$, **S.Ochsenfeld, S.Schlichting, *JHEP* 09 (2023) 186**

Or you need to extend **G.S.Rocha etc, *Phys.Rev.D* 110 (2024) 7, 076003** to finite k.

$k \neq 0, \; m \neq 0$

Finite-element analysis can work well.

Massive $\lambda\phi^4$ and 4-fermions interaction **+ T.Z.Zhou, Q.Z.Sun, In progress.**

Summary and outlook

- I present a mathematical justification of RTA and eRTA.
- Propose a new way to restore collision invariance.
- General statements on non-analytical structure in retarded correlators within kinetic description.
- Nonlinear response and fluctuations.

Back up

self-adjoint and positive semidefinite :

$$\int \mathrm{dP} f_0(p) E_p \psi(p) \mathscr{L}_0 \phi(p) = \int \mathrm{dP} f_0(p) E_p \phi(p) \mathscr{L}_0 \psi(p)$$

$$\int \mathrm{dP} f_0(p) E_p \psi(p) \mathscr{L}_0 \psi(p) \geq 0$$

after redefinition, function space changes

$$\int \mathrm{dP} f_0(p) E_p^{1-\alpha} \psi(p) \mathscr{L}_\alpha \phi(p) = \int \mathrm{dP} f_0(p) E_p^{1-\alpha} \phi(p) \mathscr{L}_\alpha \psi(p)$$

$$\int \mathrm{dP} f_0(p) E_p^{1-\alpha} \psi(p) \mathscr{L}_\alpha \psi(p) \geq 0$$

Revisiting RTA

energy-dependent RTA (eRTA) $\tau_R = (\beta E_p)^\alpha t_0$

$$p^\alpha \partial_\alpha f(x,p) = -u \cdot p/\tau_R (f - f_0)$$

used in simulations of fluids in relativistic heavy-ion collisions

K.Dusling, G.D. Moore, D.Teaney, *Phys.Rev.C* 81 (2010) 034907

in simulations of jets in relativistic heavy-ion collisions,

R.Baier etc, *Nucl.Phys.B* 483 (1997) 291-320

in analytical study on retarded correlators,

A.Kurkela, U.A.Wiedemann, *Eur.Phys.J.C* 79 (2019) 9, 776

in the study of Weyl semimetal systems …

A.Amoretti etc, *JHEP* 02 (2024) 071

Back up

For massless particles:

$$\int dP f_0(p) E_p^{1-\alpha} \to \infty, \quad \alpha \geq 4$$

In this case, to avoid IR divergence :

$$\alpha < 4$$

Revisiting RTA

Theorem 1 • *Assume that* $\exists \gamma > -2$, $0 \leq \beta < \gamma + 2$, $B > 0$ *and* $c_0 > 0$, *so that* $\sigma(g, \Theta) > B \frac{g^{\beta+1}}{c_0+g} \sin^\gamma \Theta$, *then* $\nu(p) > \nu_0 (p_0/m)^{\beta/2}$ *where* ν_0 *is a constant, the interaction is hard.*

Theorem 2 • *Assume that* $\exists 0 < \alpha < 4$, $\gamma > -2$ *and* $B' > 0$, *so that* $\sigma(g, \Theta) < B' g^{-\alpha} \sin^\gamma \Theta$, *then* $\nu(p) < \nu_0 (p_0/m)^{-\epsilon/2} \leq \nu_0$, *the interaction is soft, where*

$$\epsilon = \begin{cases} \alpha, & \text{for} \quad 0 < \alpha < 3, \\ \alpha - 2, & \text{for} \quad 3 < \alpha < 4, \\ \delta + 1, & \text{for} \quad \alpha = 3, \quad \text{and} \quad 0 < \delta < 1, \end{cases} \tag{27}$$

and ν_0 *is a constant.*

softness of interactions: M. Dudyn´ski, Journal of Statistical Physics 153, 1084–1106 (2013)

M. Dudyn´ski and M. Ekiel-Jezewska, Communications in Mathematical Physics 115, 607 (1988)

Introduction (for national)

Also in rare cases, the eigen spectrum can be solved analytically

$$-\mathscr{L}[\chi] \equiv \int dP' dP_1 dP_2 f_{leq}(p) f_{leq}(p') W\left(\chi(p_1) + \chi(p_2) - \chi(p) - \chi(p')\right)$$

$\mathscr{L}$ is self-adjoint, positive semidefinite.

Non-relativistic monatomic gases

given by C.S.Wang and U.E.Uhlenbeck

Studies in the Statistical Mechanics (1970), J.De Boer and G.E.Uhlenbeck, Chap IV