量子色动力学

中国科学院理论物理研究所

馬建平

QCD Lectures

Perturbative Theory and Factorization

Versions:

2006: Notes, hand-writting on papers.

2007:

2008:

2017: Remade on Ipad.

2018: Improved

2022 "

2025. Add TMD + Resunat. ([illegible])

Contents:

1. QCD Lagrangian:

SU(3) gauge group, or $SU(N_c)$

$$\mathcal{L}_{QCD} = -\frac{1}{2}\,\mathrm{Tr}\, G^{\mu\nu} G_{\mu\nu} + \sum_{i=1}^{N_f} \bar{q}_i \left(i\gamma\cdot D - m_i \right) q_i ,$$

$q_i(x)$: quark field,

$G^\mu(x)$: gauge field, $N_c \times N_c$ matrix,

$$G^\mu = G^{a,\mu} T^a, \quad a = 1, \cdots N_c^2 - 1, \quad \mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}.$$

$D^\mu = \partial^\mu + i g_s G^\mu$, covariant derivative

$$G^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + i g_s [G^\mu, G^\nu] = G^{a,\mu\nu} T^a,$$

$$G^{a,\mu\nu} = \partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu} - g_s f^{abc} G^{b,\mu} G^{c,\nu}$$

Gauge transformation: $U(x)$, element of $SU(N_c)$ #

$$q(x) \to U(x)\, q(x),$$

$$G^\mu(x) \to U(x)\, G^\mu(x)\, U^\dagger(x) - \frac{i}{g_s} U(x)\, \partial^\mu U^\dagger(x),$$

$\mathcal{L}_{QCD}$ is invariant under the transformation.

1.2.

Quantization:

$\mathcal{L}_{eff} = \mathcal{L}_{QCD} + \mathcal{L}_{GF} + \mathcal{L}_{FP}$,

$\mathcal{L}_{GF}$: gauge fixing term.

covariant gauge: $\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu G^{a,\mu})^2$.

$\mathcal{L}_{FP}$: Faddev-Popov term, ghost field.

$\Longrightarrow$ Feynman rule, Feynman diagrams

$\xi = 1$: Feynman gauge

other useful gauge: light cone gauge

$n \cdot G = 0$, $n^2 = 0$ or $(\neq 0)$

Physical gauge, no ghost needed.

1.3

Parameters in QCD:

m_i: mass of quark.

$m_u \sim m_d \sim O(1)$ MeV, $m_s \sim 100$ MeV

$m_c \sim 1.4$ GeV, $m_b \sim 5$ GeV, $m_t = 175$ GeV.

g_s: coupling constant

Def:
$$\alpha_s = \frac{g_s^2}{4\pi} = \alpha_s(\mu),$$

μ: renormalization scale

$$\mu^2 \frac{\partial \alpha_s(\mu)}{\partial \mu^2} = \beta(\alpha_s) = -\left(b_0 \alpha_s^2 + b_1 \alpha_s^3 + b_2 \alpha_s^4 + \cdots\right),$$

$$b_0 = \frac{1}{4\pi}\left(11 - \frac{2}{3} N_f\right), \quad b_1 = \frac{1}{8\pi^2}\left(51 - \frac{19}{3} N_f\right),$$

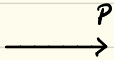
$$b_2 = \frac{1}{128\pi^3}\left(2857 - \frac{5033}{9} N_f + \frac{325}{27} N_f^2\right),$$

$$\Rightarrow \quad \beta < 0, \qquad \mu^2 \to \infty, \quad \alpha_s(\mu) \to 0$$

Asymptotic freedom. (2014 诺贝奖)

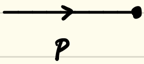
1.4

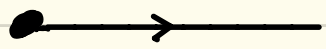
Feynman rule:

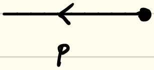


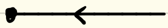
momentum flow

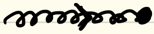
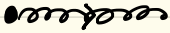
External lines:

q 

$u(p)$,  $\bar{u}(p)$

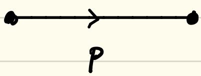
$\bar{q}$ 

$\bar{v}(p)$,  $v(p)$

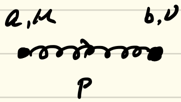
a, μ  ε^{μ},  $\varepsilon^{*\mu}$

g

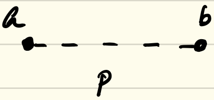
Propagators:



$\dfrac{i}{\gamma \cdot p - m + i\varepsilon}$,



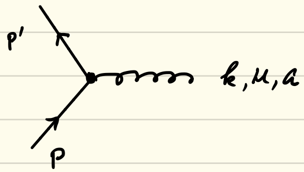
$\dfrac{-i g^{\mu\nu} \delta^{ab}}{p^2 + i\varepsilon}$,



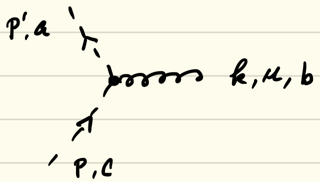
$\dfrac{i \delta^{ab}}{p^2 + i\varepsilon}$,

1.5.

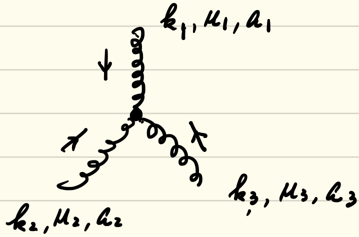
Vertex:



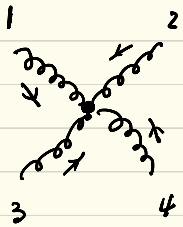
$$-ig_s \gamma^\mu T^a$$



$$g_s f^{abc} p'^\mu$$



$$-g_s f^{a_1 a_2 a_3}\Big[(k_1-k_2)_{\mu_3} g_{\mu_1\mu_2} + (k_2-k_3)_{\mu_1} g_{\mu_2\mu_3} + (k_3-k_1)_{\mu_2} g_{\mu_1\mu_3}\Big]$$



$$ig_s^2\Big[f^{a_1a_2b}f^{a_3a_4b}\left(g_{\mu_2\mu_3}g_{\mu_1\mu_4} - g_{\mu_1\mu_3}g_{\mu_2\mu_4}\right) + \text{"cylinder circular permutation"}\Big],$$

(1, 2, 3, 4), (2, 3, 4, 1)

(4, 1, 2, 3), $\underline{(3,4,1,2)}$ ×

1.6.

Perturbative theory :

U.V. divergences, d-dim. regularization,

$$\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right),\qquad \text{(Buras, Bardeen)}.$$

$\overline{MS}$ schema of renormalization :

subtration of these pole combinations.

θ - term.

Massless QCD: high energy scattering + light hadrons

⇒ We can neglect mass of light quarks and heavy quarks.

$$\Rightarrow \mathcal{L}_{QCD} = -\frac{1}{2} \mathrm{Tr}\, G^{\mu\nu} G_{\mu\nu} + \sum \bar{q}\, i\gamma\cdot D\, q$$

$$q = u, d, s$$

Only one dimensionless parameter: g_s or α_s

energy scale. $\Lambda_{QCD} \sim 100\,\text{MeV}$.

2. Divergences in QCD and $e^+e^- \to$ hadrons.

* Light-cone coordinate system:

A momentum P (or vector) in Cartesian coordinate system: $p^\mu = (p^0, p^1, p^2, p^3)$, metric $g_{\mu\nu}$

In light-cone coordinate system:

$$p^\mu = (p^+, p^-, p^1, p^2), \qquad p_\perp^\mu = (0, 0, p^1, p^2),$$

$$p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3), \qquad p^- = \frac{1}{\sqrt{2}}(p^0 - p^3).$$

Dot-product of two vectors:

$$A \cdot B = \underbrace{A^+ B^- + A^- B^+ - A^1 B^1 - A^2 B^2}_{\text{metric } g_{\mu\nu}} = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3.$$

Lorentz boost along the z-direction:

$$p^\mu \to p'^\mu, \quad p'^+ = \alpha p^+, \quad p'^- = \frac{1}{\alpha} p^-, \quad p_\perp'^\mu = p_\perp^\mu \ \#$$

2.2.

The advantage: large momentum in the z-direction:

$$p^3 \to \infty,$$

$$P^\mu = (P^+, P^-, P_\perp^1, P_\perp^2) \approx (P^+, 0, 0, 0)$$

Two light-cone vectors:

$$l^\mu = (1,0,0,0), \quad n^\mu = (0,1,0,0), \quad l^2 = n^2 = 0$$

$$l \cdot n = 1, \qquad g_\perp^{\mu\nu} = g^{\mu\nu} - n^\mu l^\nu - l^\mu n^\nu,$$

$$A_\perp^\mu = g_{\perp\mu\nu} A^\nu.$$

rapidity: $\quad y = \frac{1}{2} \ln \frac{P^+}{P^-}$ *

$$n \cdot A = A^+, \quad l \cdot A = A^-,$$

2.3.

Divergences in QCD:

U.V. divergences — Regularization + renormalization

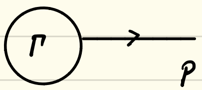
Collinear divergences

I.R. divergences

Glauber divergences

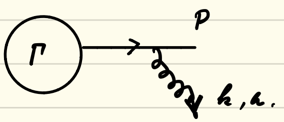
??

Consider a massless quark in final state:



: $\bar{u}(p)\Gamma(p) \qquad p^\mu = (p^+, 0, 0, 0)$

The quark can emit gluons.



$$\bar{u}(p)\left(-ig_s \gamma\cdot\varepsilon^* T^a\right) \frac{i\gamma\cdot(p+k)}{(p+k)^2+i\varepsilon}\,\Gamma(p+k).$$

Collinear divergence: $k^\mu \sim (1, \lambda^2, \lambda, \lambda)$, $\lambda \to 0$

$$\frac{i\gamma\cdot(p+k)}{(p+k)^2+i\varepsilon} = \frac{i\gamma^-(p+k)^+}{2p^+k^-+i\varepsilon}\left(1+O(\lambda)\right) \sim \frac{1}{\lambda^2}, \quad \text{divergent!}$$

2.4.

I.R. divergence: $k^\mu \sim (\lambda, \lambda, \lambda, \lambda)$, soft gluon

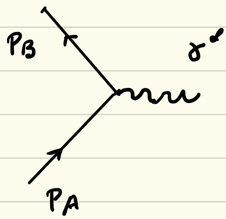
$$\frac{i\gamma\cdot(p+k)}{(p+k)^2+i\varepsilon} = \frac{i\gamma^- p^+}{2p^+k^- + i\varepsilon}\Big(1+O(\lambda)\Big) \sim \frac{1}{\lambda}, \quad \text{divergent !}$$

In QED: such divergences are "eliminated" with physical requirements.

in QCD: We have no quarks and gluons as observable states. ??

A loop-example: quark form factor

Tree-level:

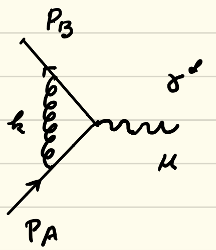


$$\bar{v}(P_B)\,\gamma^\mu\, u(P_A), \qquad P_A^\mu = (P_A^+, 0, 0, 0), \quad P_B^\mu = (0, P_B^-, 0, 0).$$

Corrections from high orders of α_s.

$$\gamma\cdot P_A\, u(P_A) = \gamma^- P_A^+ u(P_A) = 0, \quad \bar{v}(P_B)\,\gamma\cdot P_B = \bar{v}(P_B)\,\gamma^+ P_B^- = 0$$

2.5



$$\Gamma = \bar{v}(P_B)\int \frac{d^4k}{(2\pi)^4}(-ig_s\gamma_\rho T^a)\frac{i\gamma\cdot(-P_B-k)}{(P_B+k)^2+i\varepsilon}$$

$$\cdot\, \gamma^\mu \frac{i\gamma\cdot(P_A-k)}{(P_A-k)^2+i\varepsilon}(-ig_s\gamma^\rho T^a)\, u(P_A)$$

$$\cdot\, \frac{-i}{k^2+i\varepsilon}\,,$$

$$\gamma_\rho\otimes\gamma^\rho \sim \gamma^-\otimes\gamma^+ + \gamma_\perp\otimes\gamma_\perp$$

Collinear to A; $\quad k^\mu \sim (1,\lambda^2,\lambda,\lambda)$, $\quad \lambda \ll 1$

Expand the integrand in λ, the leading order:

$$\Gamma_A = \int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\varepsilon}\bar{v}(P_B)\left(\frac{-g_s n_\rho T^a}{k^+ + i\varepsilon}\right)\gamma^\mu\frac{i\gamma\cdot(P_A-k)}{(P_A-k)^2+i\varepsilon}$$

$$\cdot\left(-ig_s\gamma^\rho T^a\right)u(P_A)$$

$$\sim \int\frac{d^4k}{(2\pi)^4}\cdot\mathcal{O}\left(\frac{1}{\lambda^4}\right), \qquad k^2\sim\lambda^2, \quad (P_A-k)^2\sim\lambda^2$$

divergent !! $\qquad d^4k \sim \lambda^4.$

Power counting

2.6.

Collinera to B: $k^\mu \sim (\lambda^2, 1, \lambda, \lambda)$

$\Rightarrow$

$$\Gamma_B = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\varepsilon} \bar{v}(p_B)\left(-ig_s\gamma_\rho T^a\right) \frac{i\gamma\cdot(-p_B-k)}{(p_B+k)^2+i\varepsilon} \gamma^\mu \left(\frac{-g_s \ell^\rho T^a}{k^- - i\varepsilon}\right) u(p_A),$$

divergent !

$\lambda_s, \lambda.$

I.R. divergence: $k^\mu \sim (\lambda_s, \lambda_s, \lambda_s, \lambda_s)$

$$\Gamma_s = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\varepsilon} \frac{1}{k^+ + i\varepsilon} \frac{1}{k^- - i\varepsilon} \bar{v}(p_B)\left(-g_s n_\rho T^a\right)\gamma^\mu\left(-g_s \ell^\rho T^a\right) u(p_A),$$

divergent.

Subtraction:

$$\Gamma = \underbrace{\left(\Gamma - \Gamma_A - \Gamma_B + \Gamma_s\right)}_{} + \left(\Gamma_A + \Gamma_B - \Gamma_s\right)$$

$\Downarrow$ free from collinear- and I.R. divergences !!

* Γ_A, Γ_B also have I.R. !! (-)

But: light-cone singularity ....

2.7

Glauber-gluon: $k^\mu \sim (\lambda^2, \lambda^2, \lambda, \lambda)$

$$\Gamma_G = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{-k_\perp^2 + i\varepsilon} \frac{1}{k^+ + i\varepsilon} \frac{1}{k^- - i\varepsilon}$$
$$\bar{v}(p_B) \left(-g_s n_\rho T^a\right) \gamma^\mu \left(-g_s \ell^\rho T^a\right) u(p_A),$$

it gives a divergent absorptive part, it is similar to Coulomb singularity.

$\Rightarrow$ Perturbation theory of QCD contains I.R. singularity and collinear singularity !!

Perturbation theory of QCD is meaningless ??

There are no S-matrix elements with quarks or gluons as physical states. Unlike QED !

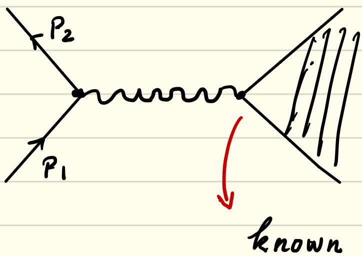
But

2.8.

Consider:

$$e^+ e^- \longrightarrow \text{hadrons}, \quad \text{or} \quad e^+ e^- \longrightarrow X$$

Leading order of QED



$$s = (P_1 + P_2)^2$$

$$\hat{O}(x) = e^{i\hat{P}\cdot x}\, \hat{O}(0)\, e^{-i\hat{P}\cdot x}$$

$$\Rightarrow \quad \sigma = \frac{1}{2s} L^{\mu\nu} \frac{1}{s^2} W^{\mu\nu},$$

$$L^{\mu\nu} = \sum{}' \bar{v}(P_2)\, \gamma^\mu\, u(P_1)\, \bar{u}(P_1)\, \gamma^\nu\, v(P_2)$$

$$= \left(P_1^\mu P_2^\nu + P_1^\nu P_2^\mu - P_1\cdot P_2\, g^{\mu\nu}\right),$$

Hadronic tensor:

$$W^{\mu\nu} = \int d^4x\, e^{iq\cdot x} \sum_X \langle 0|\, J^\nu(x)\, |X\rangle\langle X|\, J^\mu(0)\, |0\rangle,$$

T-order product:

$$T\left(J^\nu(x) J^\mu(0)\right) = \theta(x_0)\, J^\nu(x) J^\mu(0) + \theta(-x_0)\, J^\mu(x) J^\nu(0),$$

2. f

Def:

$$T^{\mu\nu}(q) = \int d^4x\, e^{iq\cdot x} \langle 0 | T\left(J^\nu(x) J^\mu(0)\right) | 0 \rangle .$$

Using
$$\Theta(t) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \frac{e^{-i\omega t}}{\omega + i\epsilon} = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

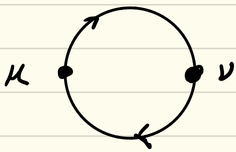
$$\Rightarrow \quad \boxed{W^{\mu\nu}(q) = 2\,\mathrm{Im}\, T^{\mu\nu}(q)}$$

"Cutkosky rule, cutting diagrams"

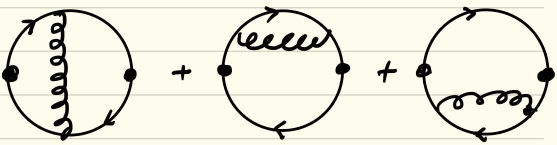
Now: we know J^μ, $J^\mu(x) = \sum_q \bar{q}(x)\gamma^\mu q(x)\, e Q_q$,

we can calculate $T^{\mu\nu}$ with perturbative theory of QCD.

Tree-level:



One-loop:



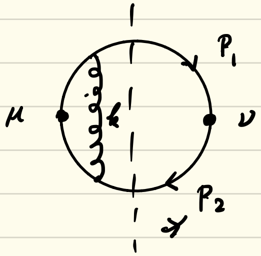
$$R = \frac{\sigma(e^+e^- \to X)}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q Q_q^2 \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} + O(\alpha_s^2) \right\} \Theta(s - 4m_q^2)$$

$N_c = 3$.

It is finite, it contains no collinear- or I.R. divergences! Why??

2.10.

Why? consider: $W^{\mu\nu}$ or $\mathrm{Im}\, T^{\mu\nu}$, cut diagrams



(a).

$$W_a^{\mu\nu} = \frac{1}{2}\int \frac{d^4p_1\, d^4p_2}{(2\pi)^4(2\pi)^4}\, 2\pi\delta(p_1^2)\, 2\pi\delta(p_2^2)$$

$$\cdot (2\pi)^4 \delta^4(p_1+p_2-q) \int \frac{d^4k}{(2\pi)^4}\,(-1)$$

$$\frac{-i}{k^2+i\varepsilon}\;\frac{i}{(p_1-k)^2+i\varepsilon}$$

$$\frac{i}{(p_2+k)^2+i\varepsilon}\cdot \mathrm{Tr}\Big[\gamma\cdot p_1\,(-ig_s\gamma^\rho T^a)$$

$$\gamma\cdot(p_1-k)\,\gamma^\mu\,\gamma\cdot(-p_2-k)\,(-ig_s\gamma_\rho T^a)\,\gamma\cdot p_2\,\gamma^\nu\Big],$$

Consider: the gluon is soft, $k^\mu \sim (\lambda,\lambda,\lambda,\lambda)$, 2.10.1

Expanding in λ, leading order: $\quad (p_1^2 = p_2^2 = 0 \,!)$

$$W^{\mu\nu}_{a,IR} = \frac{1}{2} \int \frac{d^4 p_1\, d^4 p_2}{(2\pi)^4 (2\pi)^4}\, 2\pi\delta(p_1^2)\, 2\pi\delta(p_2^2)$$

$$(2\pi)^4 \delta^4(p_1 + p_2 - q)\; \Pi^{\mu\nu}_{a,I.R.}\,,$$

$$\Pi^{\mu\nu}_{a,I.R.} = i g_s^2 \int \frac{d^4 k}{(2\pi)^4} \mathrm{Tr}\left[\gamma\cdot p_1\, \gamma^\rho\, \gamma\cdot p_1\, \gamma^\mu\, \gamma\cdot(-p_2)\, \gamma_\rho\, \gamma\cdot p_2\, \gamma^\nu\right]$$

$$\cdot \frac{1}{(-2p_1\cdot k + i\varepsilon)(2p_2\cdot k + i\varepsilon)(k^2 + i\varepsilon)} (N_c C_F).$$

If we take $\quad p_1^\mu = (p_1^+, 0, 0, 0),\quad p_2^\mu = (0, p_2^-, 0, 0)$

The k^+ - integration can be done with Cauchy theorem:

$$\Rightarrow \Pi^{\mu\nu}_{a,I.R} = -g_s^2 \int_0^\infty \frac{dk^-}{2\pi} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{(2p_1^+ k^- + i\varepsilon)(2p_2^- k_\perp^2 - i\varepsilon)}$$

$$(k^- \to -k^-),\quad \mathrm{Tr}\left[\gamma\cdot p_1\, \gamma^\rho\, \gamma\cdot p_1\, \gamma^\mu\, \gamma\cdot(-p_2)\, \gamma_\rho\, \gamma\cdot p_2\, \gamma^\nu\right] (N_c C_F),$$

2.11

The contribution comes from the pole $k^2 + i\varepsilon = 0$.

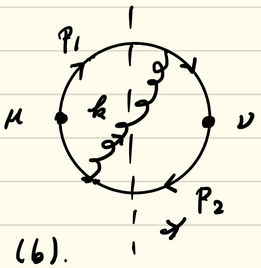
We can re-write the results as:

$$\Pi^{\mu\nu}_{a,I.R} = -g_s^2 \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2) \frac{1}{(2P_1\cdot k + i\varepsilon)(2P_2\cdot k - i\varepsilon)}$$

$$\mathrm{Tr}\left[\gamma\cdot P_1 \gamma^\rho \gamma\cdot P_1 \gamma^\mu \gamma\cdot(-P_2)\gamma_\rho \gamma\cdot P_2 \gamma^\nu\right](N_c C_F),$$

Without specificat of P_1^μ and P_2^μ, covariant form.

Now consider I.R. contribut from the diagram:



(6).

$k^\mu \sim (\lambda, \lambda, \lambda, \lambda)$. $\lambda \to 0$.

Overall factor:

$$\delta^4(q - P_1 - P_2 - k) \approx \delta^4(q - P_1 - P_2)$$

$$W^{\mu\nu}_{b,IR} = \frac{1}{2}\int \frac{d^4P_1\, d^4P_2}{(2\pi)^4(2\pi)^4} 2\pi\delta(P_1^2)\, 2\pi\delta(P_2^2)$$

$$(2\pi)^4\delta^4(P_1 + P_2 - q)\ \Pi^{\mu\nu}_{b,I.R.},$$

2.11.1

Following the same steps :

$\Rightarrow$

$$\Pi^{\mu\nu}_{b,I.R} = + g_s^2 \int \frac{d^4k}{(2\pi)^4}\, 2\pi\,\delta(k^2)\, \frac{1}{(2p_1\cdot k - i\varepsilon)(2p_2\cdot k + i\varepsilon)}$$

$$\mathrm{Tr}\left[\gamma\cdot p_1\,\gamma^\rho\,\gamma\cdot p_1\,\gamma^\mu\,\gamma\cdot(-p_2)\,\gamma_\rho\,\gamma\cdot p_2\,\gamma^\nu\right](N_c C_F),$$

Neglecting the differences in "$i\varepsilon$", one has already:

$$\Pi^{\mu\nu}_{a,I.R} + \Pi^{\mu\nu}_{b,I.R} = 0.$$

or :

$$W^{\mu\nu}_{a,I.R} + W^{\mu\nu}_{b,I.R} = 0$$

With the difference: the I.R. contribut to σ:

$$\sigma \sim \Pi^{\mu\nu}_{a,I.R} + \Pi^{\mu\nu}_{b,I.R} + \left(\Pi^{\mu\nu}_{a,I.R} + \Pi^{\mu\nu}_{b,I.R}\right)^* = 0.$$

$\Rightarrow$ σ has no I.R. divergences at one-loop.

One can also show in a similar way that σ has no collinear divergences at one-loop.

Common statements: The divergence from virtual part is cancelled by that from real part.

General statements:

KLN theorem !!

2.13.

KLN theorem: (Kinoshita, Lee, N

state a, b; $a \to b$

the probability: $|S_{ba}|^2 = |\langle b|S|a\rangle|^2$.

In general, it contains divergences from degenerate states of $|a\rangle$ and $|b\rangle$, like I.R.- and collinear divergences. Suppose: these divergences are regularized by a set of parameters $[\mu]$, e.g., quark mass ...

$\mu \to 0$, divergences appear.

If we sum those energy degenerate states of $|a\rangle$ an $|b\rangle$, then the sum:

$$\sum_{D[a]} \sum_{D[b]} |S_{ba}|^2$$ is free from these divergences!!

Note: a and b do not to have the same energy, or in the same state.

2.14.

For $e^+e^- \to$ hadrons, special case of KLN,

because of that hadrons or QCD states only appear

in final state. Block - Nordsieck theorm.

OPE:

$$J^{\mu}(x) J^{\nu}(0) = C_0^{\mu\nu}(x)\, I + C_1^{\mu\nu}(x) :\bar{q} q: + \cdots$$

We have only take the leading order here.

The remaining terms are power-suppressed $\sim \Lambda^2/s$.

SVZ - sum-rule.

R-ratio

Renormalon $\Rightarrow$ power correction at Λ/s.

3.1.

3. DIS and QCD Factorization

!!

DIS: Deeply Inelastic Scattering

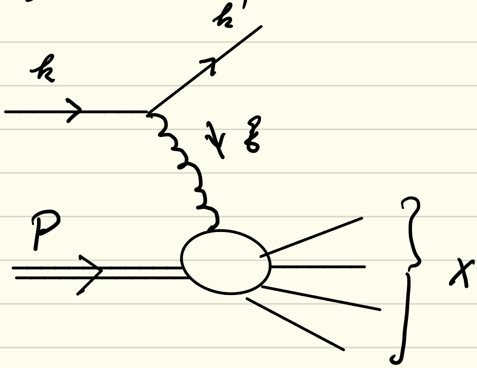
A classical example of QCD applications

DIS: (unpolarized case)

$$e(k) + h(P) \to e(k') + X,$$

h: hadron, usually it is a proton.

At leading order of QED:



$q = k - k'$,

$q^2 = (k-k')^2 = -Q^2 < 0$,

$$y = \frac{q\cdot P}{k\cdot P},$$

$$x_B = \frac{Q^2}{2q\cdot P}, \quad \text{Bjorken variable}$$

3.2.

The cross-section:

$$k'^0 \frac{d\sigma}{d^3k'} = \frac{2}{k\cdot p}\left(\frac{\alpha^2}{q^2}\right)^2 L_{\mu\nu} W^{\mu\nu},$$

$$L^{\mu\nu} = k'^\mu k^\nu + k^\mu k'^\nu - k\cdot k' g^{\mu\nu},$$ the leptonic tensor

The hadronic tensor:

$$W^{\mu\nu} = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} \sum_X \langle h(P)| J^\mu(x) | X\rangle \langle X| J^\nu(0) | h(P)\rangle,$$

$$J^\mu = Q_q \bar{q}\gamma^\mu q,$$ (take $Q_q = 1$ for brevity).

The decomposition:

$$W^{\mu\nu}(P,q) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x,Q^2) + \frac{\hat{p}^\mu \hat{p}^\nu}{P\cdot q} F_2(x,Q^2),$$

$$\hat{p}^\mu = p^\mu - \frac{q\cdot p}{q^2} q^\mu,$$

$$q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$$

$x = x_B.$

em-gauge invariance.

Need to know F_1, F_2 !!

3.3.

Kinematical region of DIS: Bjorken limit

$Q^2 \to \infty, \quad 2q\cdot P \to \infty, \quad x_B = \frac{Q^2}{2q\cdot P}$ fixed

$0 < x_B < 1.$

Bjorken scaling: $F_{1,2}(x, Q^2) \approx F_{1,2}(x), \quad Q^2 \to \infty,$

Pre-QCD:

(Naive) parton model:

$F_2(x, Q^2) = x\, f_p(x), \qquad 2x F_1(x, Q^2) = F_2(x, Q^2)$

Automatical scaling. "p" for parton,

The initial hadron $h(P)$ consists of many partons.

$f_p(x)$: the probability to find a parton with the momentum xP, x %.

DIS:

$\gamma^*(q) + P(xP) \to P + X.$

3.4

With QCD: Improved parton model

QCD factorization theorem for DIS:

$$F_2(x,Q^2) = x \sum_a \int_x^1 \frac{d\xi}{\xi}\, C_a\left(\frac{x}{\xi}, Q^2, \mu^2\right) f_{a/h}(\xi,\mu^2) + \cdots$$

$$= x \sum_a C_a \otimes f_{a/h} \left(1 + O\left(\frac{\Lambda^2}{Q^2}\right)\right),$$

$a = q, \bar{q}, G,$

$f_{a/h}(\xi,\mu^2)$: parton distribution function (PDF).

defined with QCD operators, ξ: %.

It is a distribution, not a probability. (!)

$C_a\left(\frac{x}{\xi}, Q^2, \mu^2\right)$: perturbative coefficient function,

free from collinear- and I.R. divergences.

At leading order: $C_q(z, Q^2, \mu^2) = \delta(1-z) + O(\alpha_s)$,

it reproduces the (naive) parton model,

and "Partons" = quarks, $\bar{q}$.

3.5.

Question: We don't know the inner-structure of hadrons, how we make predictions?

Traditional way: Operator Product Expansion (OPE).

Modern way to discuss DIS:

Breit frame for Bjorken limit:

h: moving in the z-direction: $P^\mu = (P^+, \frac{m_h^2}{2P^+}, 0, 0)$,

γ^*: moving in the $-z$-direction: $q^\mu = (q^+, q^-, 0, 0)$

$q^+ < 0$.

Bjorken limit is realized $\quad q^- > 0$

by $q^- \to \infty$, $\Rightarrow Q^2 = -2q^+q^- \to \infty$,

$2P\cdot q = 2P^+q^- + 2P^-q^+ \to \infty$

$$x = \frac{Q^2}{2P\cdot q} = -\frac{q^+}{P^+}, \text{ fixed.}$$

If P^+ is large, $P^\mu \approx (P^+, 0, 0, 0)$.

3.6

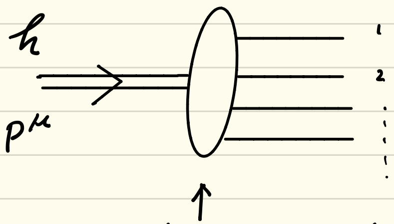
To derive the factorization:

Def: $$T^{\mu\nu}(q,P) = \frac{1}{4\pi}\int d^4x\, e^{iq\cdot x} \langle P|T(J^\mu(x)J^\nu(0))|P\rangle,$$

$$\Longrightarrow \quad W^{\mu\nu}(q,P) = 2\,\mathrm{Im}\,T^{\mu\nu}(q,P).$$

We know: A hadron consists of partons,

partons: q, $\bar{q}$, G.



n-parton component $(q, \bar{q}, G)$

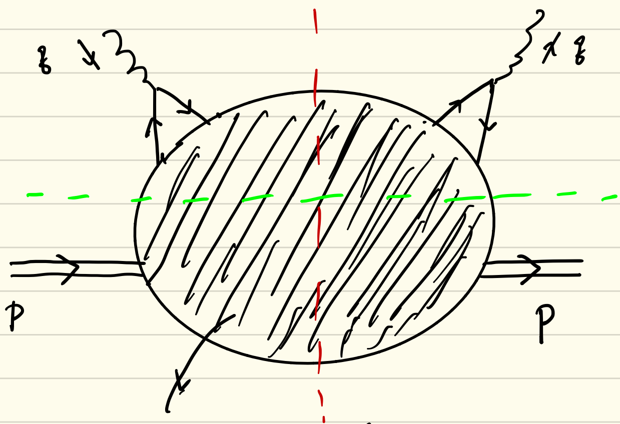
like a vertex

Suppose: We know these vertices, and use perturbative theory of QCD to calculate $T^{\mu\nu}$,

$T^{\mu\nu}$ is a Green's funct,

3.7.

A 4-point Green's functn:



complicated Feyman diagrams, $q, \bar{q}, G$.

Classification of diagrams:

Dividing a given diagram into two parts, (green line)

Upper part: containing the two photon external lines.

lower-part: " the two hadron external lines.

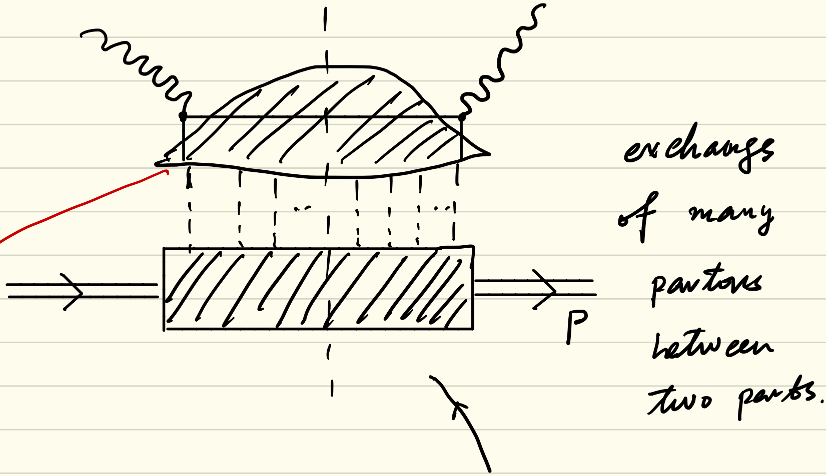
two parts are connected with interal lines of quarks and gluons

3.8.

The diagrams are classfied as 2-particle reducable (2PR),

3PR- ... nPR-diagrams, (one-PR?)

Structure of diagrams: (nPR)



- - - - : lines for quarks or gluons.

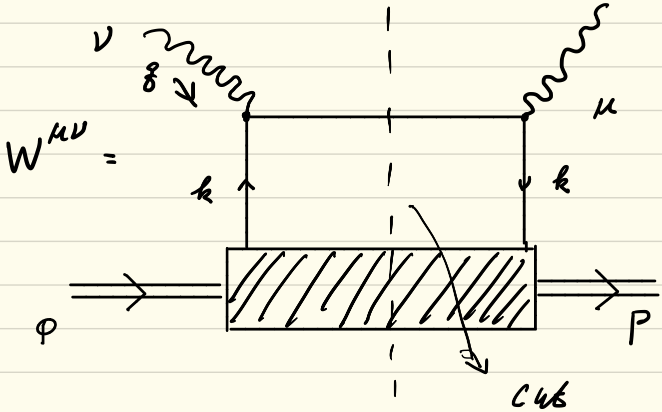
The lower-part is the matrix elements of hadron sandwitched with operators represented by those lines.

The upper-parts can be classified into tree-level-, one-loop- ... n-loop diagrams.

3.P.

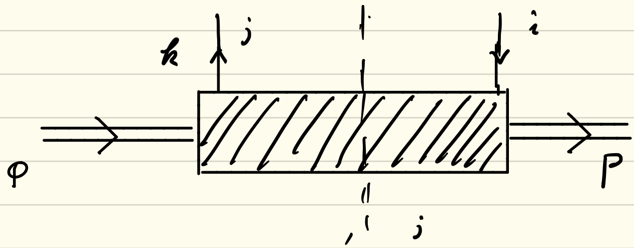
Tree-level of upper-part:

· 2-PR: only one diagram,



"hand-bag diagram"

The black box



"jet function"

$$= \boxed{\Gamma_{ji}(k,P) = \int d^4x\, e^{-ik\cdot x} \langle h(P) |\, \bar{q}_i(x)\, q_j(0)\, | h(P) \rangle}$$

i, j: Indices of spinor and color.

3.10.

Expanding around $k^\mu = \hat{k}^\mu (1+O(\lambda))$, $\hat{k}^\mu = (k^+, 0,0,0)$

The hand-bag diagram:

$$W^{\mu\nu} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \left(\gamma^\mu \gamma\cdot(k+q)\gamma^\nu \delta((k+q)^2)\right)_{ij} \Gamma_{ji}(k,P),$$

For $\Gamma(k,P)$: In the case of $P^+ \to \infty$, $P^\mu \approx (P^+,0,0,0)$,

the dominant contribution comes from the region:

$$\boxed{k^\mu \sim Q(1,\lambda^2,\lambda,\lambda), \quad \lambda = \frac{\Lambda}{Q} \ll 1.}$$

Power counting for parton momentum. $\Lambda \sim \Lambda_{QCD}, m_q, \cdots$

$$W^{\mu\nu} = \frac{1}{2}\int dk^+ \left(\gamma^\mu \gamma\cdot(\hat{k}+q)\gamma^\nu\right)_{ij} \delta((\hat{k}+q)^2)$$

$$\cdot \int \frac{dx^-}{2\pi} e^{-ik^+x^-} \langle h(P)|\bar{q}_i(x^-n)\, q_j(0)|h(P)\rangle$$

$$+ O(\lambda) + \cdots$$

The quark field: $i\gamma\cdot D\, q = \left(iD^+\gamma^- + iD^-\gamma^+ - i\vec{\gamma}_\perp\cdot\vec{D}_\perp\right) q(x) = 0$

in the hadron $\qquad O(P^+),\ O(\Lambda^2),\ O(\Lambda),$

↓

3.11

Not all components of $\xi(x)$ are important.

$$\xi(x) = \xi^{(+)}(x) + \xi^{(-)}(x), \qquad \xi^{(+)}(x) = \frac{1}{2}\gamma^-\gamma^+\xi(x), \quad \xi^{(-)}(x) = \frac{1}{2}\gamma^+\gamma^-\xi(x).$$

$$\gamma^-\gamma^+ + \gamma^+\gamma^- = 2, \qquad \gamma^-\gamma^- = \gamma^+\gamma^+ = 0.$$

Using EOM: $\quad \xi^{(-)} \sim O\left(\frac{\Lambda}{P^+}\right)\xi^{(+)}$,

Power-counting for ξ.

$\Rightarrow$ $\xi^{(+)}(x)$ is the large component.

$$\int \frac{dx^-}{2\pi} e^{-ik^+x^-} \langle h(P) | \bar{\xi}_i(x^- n) \, \xi_j(0) | h(P) \rangle$$

$$= \frac{1}{2N_c} (\gamma^-)_{ji} \, f_{\xi/P}(z) + O(\Lambda),$$

color diagonal.

PDF: $$f_{\xi/P}(z) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{-ik^+\lambda} \langle h(P) | \bar{\xi}(\lambda n) \gamma^+ \xi(0) | h(P) \rangle,$$

↑

dimensionless, depends only on $k^+/P^+ = z$.

momentum conservation: $-P^+ < k^+ < P^+$,

$k^+ < 0$: antiquark. we take $k^+ > 0$

$$(\hat{k}+q)^2 = 2(\hat{k}+q)^+ q^- = 0$$

$$\Rightarrow \quad k^+ + q^+ = 0, \quad (\hat{k}+q)^\mu = (0, q^-, 0, 0) \;!!$$

3.12

$$W^{\mu\nu} = \frac{1}{4N_c}\int_0^{P^+} dk^+ f_{q/p}(z)\, \delta((\hat{k}+q)^2)\, \mathrm{Tr}\left[\gamma^\mu \gamma\cdot(\hat{k}+q)\gamma^\nu\gamma^-\right],$$

$$\Rightarrow \quad F_1(x,Q^2) = \frac{1}{2}\int_0^1 dz\, \delta(x-z)\, f_{q/p}(z) = \frac{1}{2} f_{q/p}(x),$$

$$F_2(x,Q^2) = 2x F_1(x,Q^2) = x f_{q/p}(x) \quad \#$$

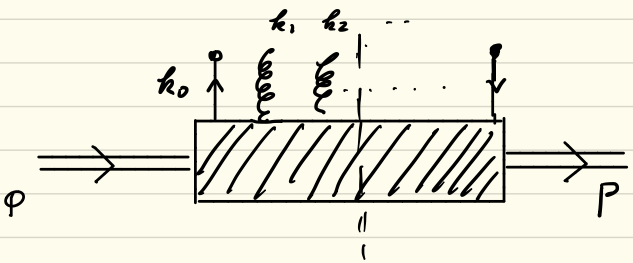
But:

$$f_{q/p}(z) = \frac{1}{2}\int\frac{d\lambda}{2\pi} e^{-ik^+\lambda}\langle h(P)|\bar{q}(\lambda n)\gamma^+ q(0)|h(P)\rangle$$

It is not gauge invariant!

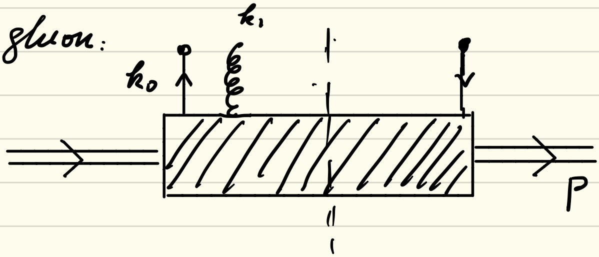
In fact, at tree-level there are many diagrams at the leading order of Λ or $\lambda = \frac{\Lambda}{P^+}$!!

Back to the black box, it can have gluons.



3.13

The case of one gluon:



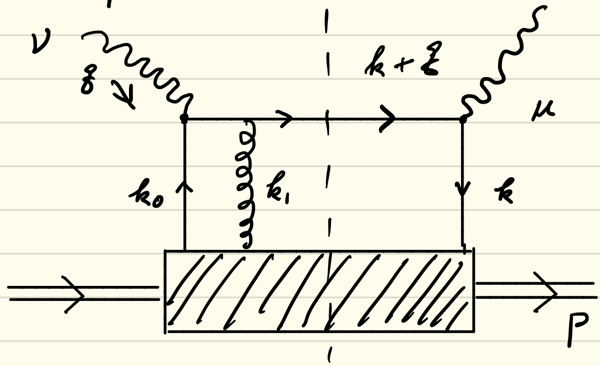
FT: $\langle h(P) | \bar{\xi}(x_0)\, G^{\mu}(x_1)\, \xi(0) | h(P) \rangle$

$$\sim P^+ \left(1, \frac{\Lambda^2}{(P^+)^2}, \frac{\Lambda}{P^+}, \frac{\Lambda}{P^+}\right) \sim k_{0,1}^{\mu}$$

$\mu = +$: The largest component.

The contribution to $W^{\mu\nu}$:

After expansion in Λ:



$$W_{1g}^{\mu\nu} = \frac{1}{2}\int dk^+ dk_1^+ \left[\gamma^\mu\, \gamma\cdot(\hat{k}+q)\, (-ig_s\gamma^-)\, \frac{i\gamma\cdot(\hat{k}+q-\hat{k}_1)}{(\hat{k}+q-\hat{k}_1)^2+i\varepsilon} \right.$$

$$\left. \cdot\, \gamma^\nu\, \delta((\hat{k}+q)^2) \right]_{ij} \frac{1}{2N_c} (\gamma^-)_{ji}$$

$$\cdot \int \frac{d\lambda\, d\lambda_1}{(2\pi)^2} e^{-i\lambda k^+ + i\lambda_1 k_1^+} \langle h | \bar{\xi}(\lambda n)\, \gamma^+ G^+(\lambda_1 n)\, \xi(0) | h \rangle,$$

3.14

$$\hat{k} = (k^+, 0, 0, 0), \quad \hat{k}_1 = (k_1^+, 0, 0, 0), \quad \rho = -$$

$$\gamma^\rho \frac{\gamma\cdot(\hat{k}+q-\hat{k}_1)}{(\hat{k}+q-\hat{k}_1)^2+i\varepsilon}\gamma^\nu\gamma^- = \gamma^\nu\gamma^- \frac{1}{-k_1^+ + i\varepsilon} n^\rho$$

Note : $\nu = \perp$ and $k^+ + q^+ = 0$, on-shell.

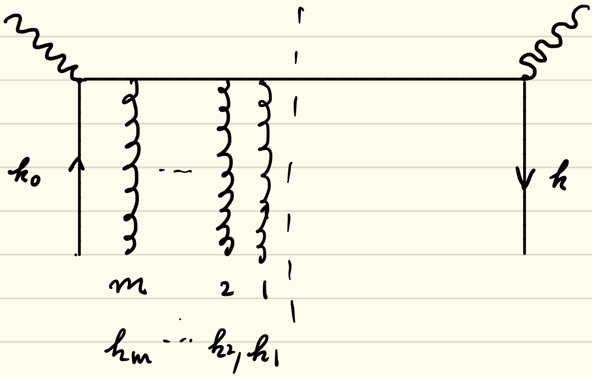
$$\Rightarrow \quad W_{1g}^{\mu\nu} = \frac{1}{4N_c}\int dk^+ \left[\mathrm{Tr}\left(\gamma^\mu \gamma\cdot(\hat{k}+q)\gamma^\nu\gamma^-\right) \delta\left((\hat{k}+q)^2\right)\right]$$

$$\cdot \frac{1}{2}\int dk_1^+ \frac{i}{-k_1^+ + i\varepsilon}(-ig_s n_\rho)\int \frac{d\lambda d\lambda_1}{(2\pi)^2} e^{-i\lambda k^+ + i\lambda_1 k_1^+}$$

$$\langle h | \bar{q}(\lambda n)\gamma^+ G^\rho(\lambda_1 n) q(0) | h \rangle,$$

$n_\rho G^\rho = G^+$, the factor in $[\cdots]$ is the same as before.

One can work out the result for exchanges of m-gluons.



3.15

The contribution with m-gluons:

$$W^{\mu\nu}_{mg} = \frac{1}{4N_c}\int dk^+ \left[\mathrm{Tr}\left(\gamma^\mu \gamma\cdot(\hat{k}+q)\gamma^\nu \gamma^-\right) \delta\left((\hat{k}+q)^2\right)\right]$$

$$\frac{1}{2}\int \prod_{i=1}^{m} dk_i \frac{g_s \qquad g_s \quad \cdots \quad g_s}{(-k_1^+ + i\varepsilon)(-\tilde{k}_2^+ + i\varepsilon)\cdots(-\tilde{k}_m^+ + i\varepsilon)}$$

$$\cdot \int \frac{d\lambda}{2\pi} e^{-i\lambda k^+} \prod_{i=1}^{m} \frac{d\lambda_i}{2\pi} e^{+i\lambda_i k_i^+}$$

$$\cdot \langle h(P) | \bar{q}(\lambda n)\gamma^+ G^+(\lambda_1 n) G^+(\lambda_2 n)\cdots G^+(\lambda_m n) q(0) | h(P)\rangle,$$

$\hookrightarrow$ ordered "statistical factor"

$\tilde{k}_i^+ = k_1^+ + k_2^+ + \cdots k_i^+$.

The contributions are at the same order of Λ !!

need to be summed.

Gauge link: path-ordered along n-direction

$$V(x,\infty) = P \exp\left\{-ig_s\int_0^\infty d\lambda\, G^+(\lambda n + x)\right\}$$

$$= 1 + \sum_{i=1} (-ig_s)^i \int \prod_{j=1}^{i} d\lambda_j\, G^+(\lambda_1 n + x) G^+(\lambda_2 n + x)$$

$$G^+(\lambda_3 n + x) \cdots G^+(\lambda_j n + x) \cdot \theta(\lambda_1 - \lambda_2)\theta(\lambda_2 - \lambda_3)$$

$$\theta(\lambda_3 - \lambda_4) \cdots \theta(\lambda_{j-1} - \lambda_j)\,\theta(\lambda_j),$$

3.16

Using $\theta(\lambda) = i\int \frac{d\omega}{2\pi} \frac{e^{-i\omega\lambda}}{\omega + i\varepsilon}$

$$V(x,\infty) = 1 + \sum_{i=1}^{\infty} (-ig_s)^i \int \prod_{j=1}^{i} dk_i^+$$

$$\cdot \frac{i}{(-k_1^+ + i\varepsilon)} \frac{i}{(-\tilde{k}_2^+ + i\varepsilon)} \frac{i}{(-\tilde{k}_3^+ + i\varepsilon)} \cdots \frac{i}{(-\tilde{k}_i^+ + i\varepsilon)}$$

$$\cdot \int \prod_{j=1}^{i} \frac{d\lambda_j}{2\pi} e^{ik_j^+\lambda_j} \quad G^+(\lambda_1 n + x)\, G^+(\lambda_2 n + x) \cdots G^+(\lambda_i n + x),$$

$\Rightarrow \sum_{m=0}^{\infty} W_{mg}^{\mu\nu}$ can be summed with V. the left part.

Doing the same for the right part ..

After the sum: the leading order of Λ at tree-level:

$$\boxed{\begin{aligned} W^{\mu\nu} &= \frac{1}{4N_c}\int dk^+ \, \mathrm{Tr}\left[\gamma^\mu \gamma\cdot(\hat{k}+q)\gamma^\nu\gamma^-\right] \delta\left((\hat{k}+q)^2\right) \\ &\cdot \frac{1}{2}\int \frac{d\lambda}{2\pi} e^{-i\lambda k^+} \langle h|\, \bar{q}(\lambda n)\, V^+(\lambda n,\infty)\gamma^+ V(0,\infty)\, q(0)\,|h\rangle \end{aligned}}$$

The gauge invariant definition of PDF:

$$\boxed{\begin{aligned} f_{q/p}(z,\mu) &= \\ &\cdot \frac{1}{2}\int \frac{d\lambda}{2\pi} e^{-i\lambda k^+} \langle h|\, \bar{q}(\lambda n)\, V^+(\lambda n,\infty)\gamma^+ V(0,\infty)\, q(0)\,|h\rangle \end{aligned}}$$

$k^+ = zP^+$,

3.17.

μ: renormalization scale, introduced by U.V. subtraction. Because of the subtraction, one can not show that PDF is positive as a probability. !

Gauge transformation:

$$q(x) \to \mathcal{U}(x)\, q(x), \quad V(x,\infty) \to \mathcal{U}(\infty)\, V(x,\infty)\, \mathcal{U}^{+}(x).$$

The defined PDF is gauge invariant!

Is the obtained $W^{\mu\nu}$ gauge invariant??

We need to check the factor $\mathrm{Tr}\left[\gamma^{\mu}\gamma\cdot(\hat{k}+q)\gamma^{\nu}\gamma^{-}\right]$ with $(\hat{k}+q)^2=0$.

$$\frac{1}{2N_c}\mathrm{Tr}\left[\gamma^{\mu}\gamma\cdot(\hat{k}+q)\gamma^{\nu}\gamma^{-}\right] = \frac{1}{k^{+}}\sum{}' \,\bar{u}(\hat{k})\gamma^{\mu}u(\hat{k}+q)\, u(\hat{k}+q)\gamma^{\nu}u(\hat{k})$$

$\hat{k}^2=0$

$u(\hat{k}+q)\gamma^{\nu}u(\hat{k})$: the scattering amplitude for

$$\gamma^{*} + q(\hat{k}) \longrightarrow q(\hat{k}+q)$$

Therefore, the obtained $W^{\mu\nu}$ is gauge invariant.

in fact: $\gamma^{*} + \left[q(\hat{k}) + (+\text{gluons})\right] \to q(\hat{k}+q)$

3.18

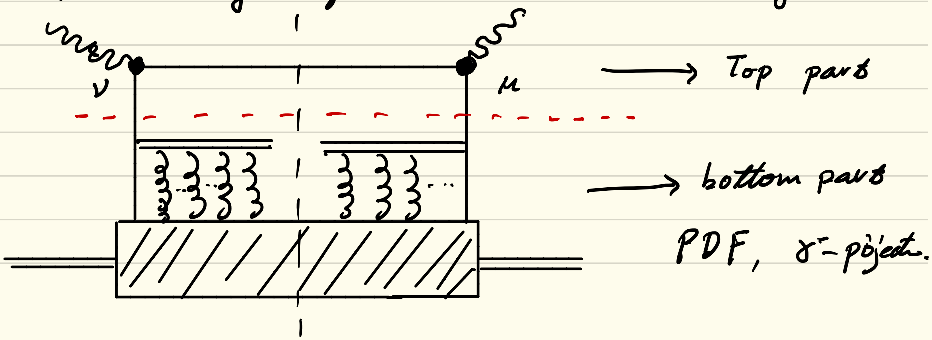
- For gauge invariance it is crucial that the initial quark is on-shell! It is also important for summing of collinear gluons by Ward identity beyond tree-level to prove the factorization.

To include antiquark contributions: $z<0$.

$$f_{\bar{q}/p}(z) = -f_{q/p}(-z).$$

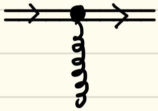
$$F_2(x,Q^2) = x\left(f_{q/p}(x) + f_{\bar{q}/p}(x)\right) \#$$

To discuss the factorization beyond tree-level, we modify the notation of the "black box".
After summing all gluons, our result can be given as:



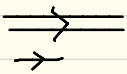
3.18

Feynman rule for $V(x,\infty)$:

Vertex:  $(-i g_s T^a n^\mu)$,

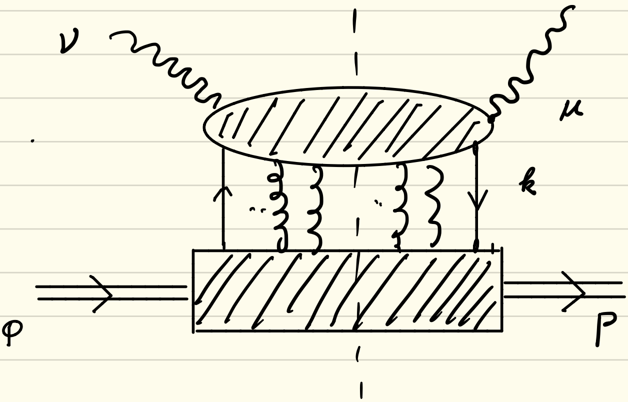
μ, a,

Propagator:

 $\dfrac{i}{k_1^+ + i\varepsilon}$

k

Beyond tree-level:



(loops)

The top bubble can be at any order of α_s.

We have studied the case at order α_s^0.

3.20.

We need to sum all contributions of gluon exchanges.

After the collinear expansion: All parton lines carries "+" momentum $\hat{k} = (\hat{k}^+, 0, 0, 0)$,

the quark lines are for on-shell quarks.

The gluon lines are for G^+ gluons!

In covariant gauge, one can derive

Ward identity: (BRST)

$$\boxed{\langle f | \partial^{\mu_1} G_{\mu_1}(x_1) \partial^{\mu_2} G_{\mu_2}(x_2) \cdots \partial^{\mu_n} G_{\mu_n}(x_n) | i \rangle = 0}$$

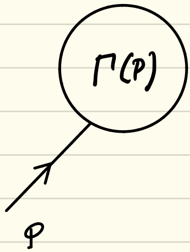
$n = 1, 2, 3 \cdots$, matrix relation!

$|i\rangle$, $|f\rangle$, physical states, on-shell !!

3.21

Illustration: one gluon attachment, one quark in the initial state

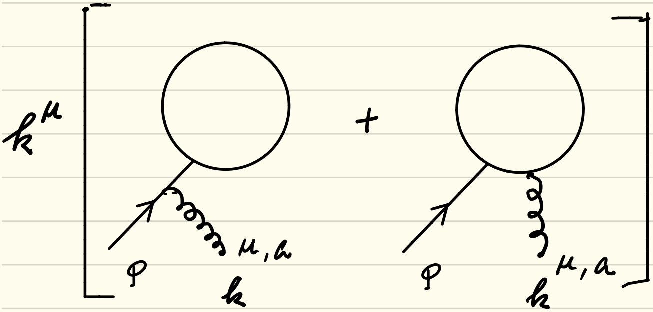
$$p^\mu = (p^+, 0, 0, 0)$$



$$\langle f \,|\, q(p) \rangle = \Gamma(p)\, u(p)$$

One-gluon insertion:

$$\langle f \,|\, \partial_\mu G^{a,\mu}(x) \,|\, i \rangle = 0 .$$



$= 0$!

(a) (b)

(b): All possible insertions except (a).

3.22

We decompose

$$G^{a,\mu}(k) = \int d^4x\, e^{ik\cdot x} G^{a,\mu}(x)$$

$$= \frac{k^\mu}{n\cdot k} n\cdot G^a(k) + \left(G^{a,\mu}(k) - \frac{k^\mu}{n\cdot k} n\cdot G^a(k) \right)$$

$\uparrow$

scalar gluon, longitudinal polarized.

All possible attachment of scalar gluon in (b) = Attachments to the external leg.

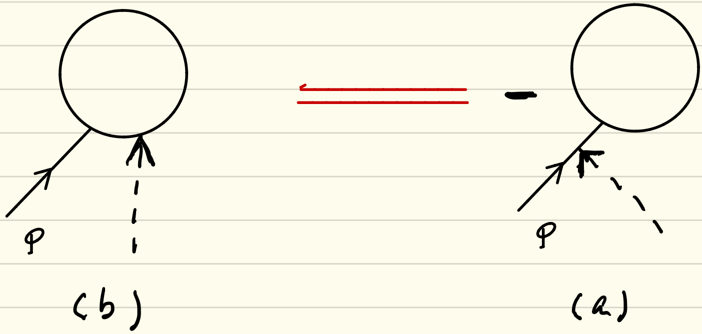


Fig. (a) $= (-)\dfrac{1}{n\cdot k} \Gamma(p+k) \dfrac{i\,\gamma\cdot(p+k)}{(p+k)^2+i\varepsilon} (-ig_s\,\gamma\cdot k\, T^a)\, u(p)$

$$= -\frac{g_s}{n\cdot k} \Gamma(p+k)\, T^a \left[1 - \frac{\gamma\cdot(p+k)}{(p+k)^2+i\varepsilon}\, \gamma\cdot p \right] u(p)$$

$\uparrow$

$\gamma\cdot p\, u(p) = 0 \quad \sim 0$

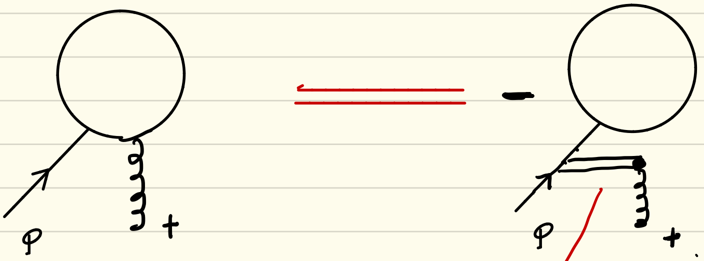
3.23.

$$\text{Fig. (a)} = -\frac{g_s}{n\cdot k}\,\Gamma(p+k)\,T^a\,u(p)$$ for on-shell quark.

In our case: $k^\mu \sim (1, \lambda^2, \lambda, \lambda)$,

$$G^\mu \sim (1, \lambda^2, \lambda, \lambda).$$

$$\Rightarrow \quad G^\mu(k) = \frac{k^\mu}{k^+} G^+ \left(1 + O(\lambda)\right),$$



(b) (a)

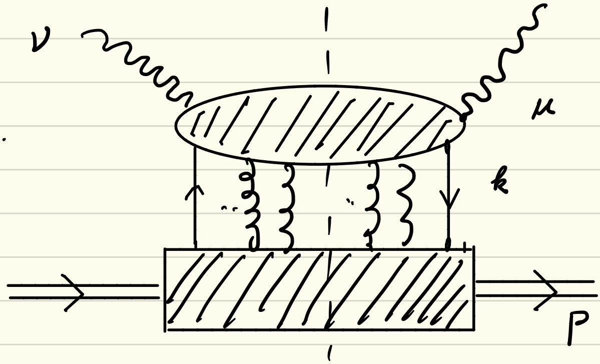
$$\frac{i}{k^+} \to \frac{i}{k^+ \pm i\varepsilon}$$

This can be generalised to insertion of any number of gluons. The sum gives gauge links.

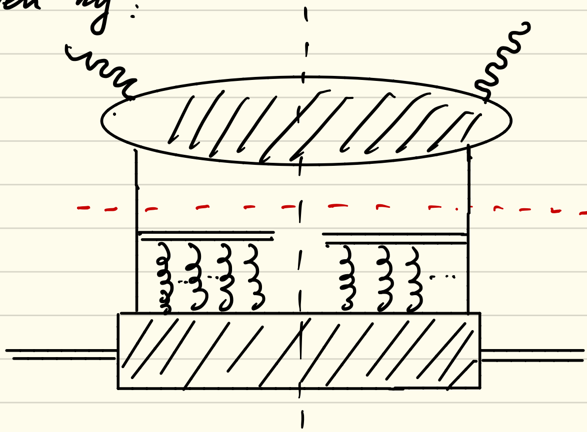
$\pm i\varepsilon$ irrelevant here

3.24

With Ward-Identity, the leading contribut from the sum of



is given by:

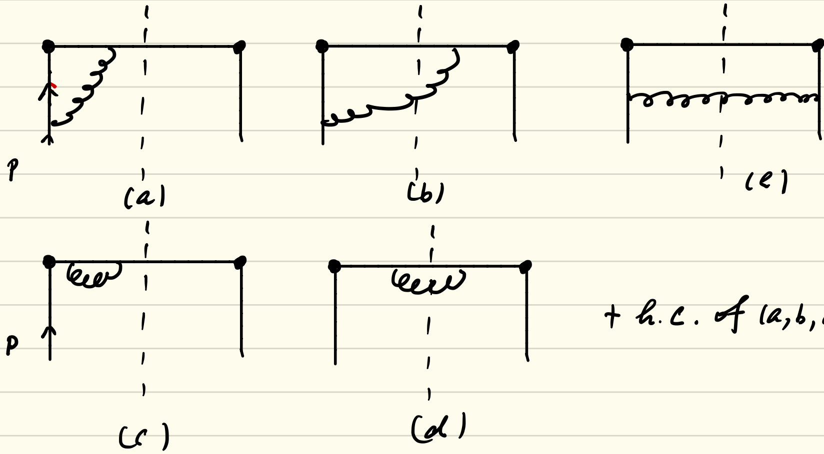


Top-part any order of α_s!

bottom part PDF, γ^--project.

Note: crucial that the quark lines stand for on-shell quarks.

At one-loop level, the top part:



+ h.c. of (a, b, c)

With the projection from the bottom:

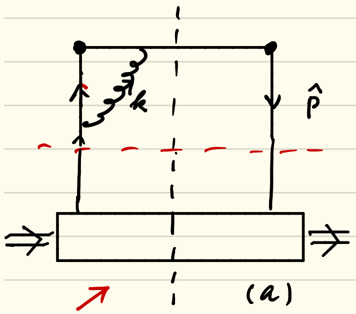
$$\frac{1}{2N_c}(\gamma^-)_{ji}\; f_{g/p}(z), \qquad \hat{p}^+ = z P^+ .$$

After the projection and taking $\hat{p}^\mu = (\hat{p}^+, 0, 0, 0)$,
the initial quark lines stand for on-shell quarks.

- We will use the subtractive approach (Collins's),
  it has a similarity to BHZP for U.V..

3.26

Consider Fig.(a):



The black box + gauge links

$$W_a^{\mu\nu} = \frac{1}{4N_c}\int d\hat{p}^+ \delta((\hat{p}+q)^2) \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\varepsilon} \mathrm{Tr}\Big[\gamma^\mu \gamma\cdot(\hat{p}+q) (-ig_s\gamma^\rho T^a) \frac{i\gamma\cdot(\hat{p}+q-k)}{(\hat{p}+q-k)^2+i\varepsilon} \gamma^\nu \frac{i\gamma\cdot(\hat{p}-k)}{(\hat{p}-k)^2+i\varepsilon} (-ig_s\gamma_\rho T^a)\gamma^-\Big] \cdot f_{q/p}(z),$$

$$\hat{p}^+ = zP^+.$$

Consider the momentum region $k^\mu \sim (1, \lambda^2, \lambda, \lambda)$, collinear to $\hat{p}$ or P. Expanding in λ, the leading order is:

$$W_{a,c}^{\mu\nu} = \underline{\frac{1}{4N_c}\int d\hat{p}^+ \delta((\hat{p}+q)^2)\, \mathrm{Tr}\left[\gamma^\mu \gamma\cdot(\hat{p}+q)\gamma^\nu\gamma^-\right]}$$

$$\cdot\left\{\int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\varepsilon}\mathrm{Tr}\left[\gamma^+ \frac{i\gamma\cdot(\hat{p}-k)}{(\hat{p}-k)^2+i\varepsilon}(-ig_s\gamma_\rho T^a)\gamma^-\right] \cdot \frac{i}{-k^++i\varepsilon}(-ig_s n^\rho T^a)\frac{1}{4N_c} f_{q/p}(z)\right\}.$$

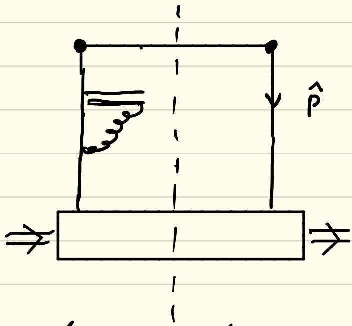
The integration over k is divergent,

Note $\underline{\;\cdots\cdots\;}$ is tree-level results. | collinear divergence!

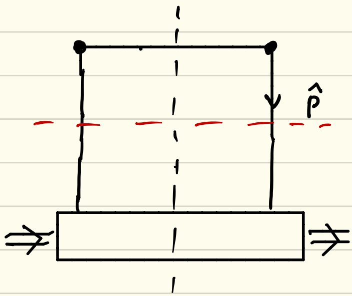
3.27

This collinear divergent contribution is represented by the diagram:

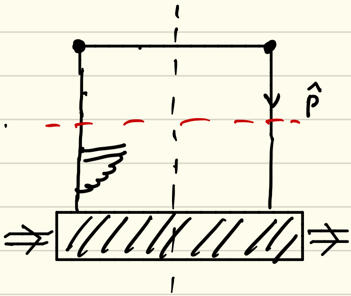
$W^{\mu\nu}_{a,c}$ =



In the tree-level, there is the same contribution:



contains.



Including $W^{\mu\nu}_{a,c}$ at one-loop results in a double counting. To avoid it, $W^{\mu\nu}_{a,c}$ must be subtracted.

(subtractive approach).

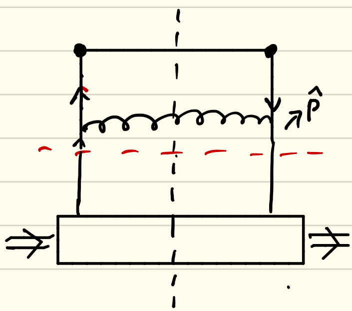
3.28

⇒ The contribution from Fig. (a) at one-loop is given by:

$$\boxed{W_a^{\mu\nu} - W_{a,c}^{\mu\nu}}$$

It is free from the collinear singularity from the region where the gluon is collinear to P !!

Fig. (e):



$$W_e^{\mu\nu} = \frac{1}{4N_c}\int d\hat{p}^+ \int \frac{d^4k}{(2\pi)^2}\, \delta\left((\hat{p}+q-k)^2\right)$$

$$\cdot \mathrm{Tr}\Big[(ig_s\gamma^\rho T^a)\frac{-i\gamma\cdot(\hat{p}-k)}{(\hat{p}-k)^2-i\varepsilon}$$

$$\gamma^\mu\gamma\cdot(\hat{p}+q-k)\gamma^\nu\frac{i\gamma\cdot(\hat{p}-k)}{(\hat{p}-k)^2-i\varepsilon}$$

$$(-ig_s\gamma_\rho T^a)\gamma^-\Big]$$

$$(-2\pi\delta(k^2))\; f_{q/p}(z)$$

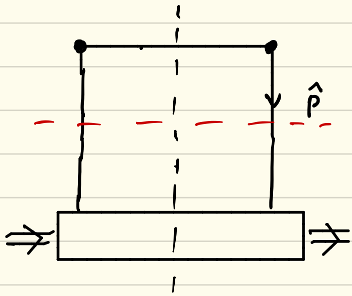
Consider the contribution from the collinear region with $k^\mu \sim (1, \lambda^2, \lambda, \lambda)$, 3.2f

The collinearly divergent contribute of Fig.(e):

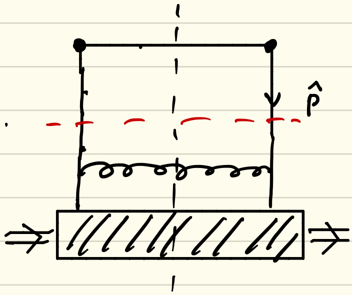
$$W^{\mu\nu}_{e,c} = \frac{1}{4N_c}\int d\hat{k}^+ \delta\left((\hat{k}+q)^2\right) \mathrm{Tr}\left[\gamma^\mu \gamma\cdot(\hat{k}+q)\gamma^\nu \gamma^-\right]$$
$$\left\{\frac{d^4k}{(2\pi)^4}\left(-2\pi\delta(k^2)\right) \mathrm{Tr}\left[\gamma^+ \frac{i\gamma\cdot(\hat{p}-k)}{(\hat{p}-k)^2+i\varepsilon}\left(-ig_s\gamma_\rho T^a\right)\right.\right.$$
$$\left.\cdot\gamma^- \frac{-i\gamma\cdot(\hat{p}-k)}{(\hat{p}-k)^2-i\varepsilon}\left(ig_s\gamma^\rho T^a\right)\right]\frac{1}{4N_c} f_{g/p}(z),$$

$\hat{k}^\mu = (\hat{p}^+ - k^+, 0, 0, 0)$.

Again, there is a double counting. For the collinear gluon, there is the same contribution in tree level result:



contains.



Subtraction is needed.

3.30

⇒ The contribution from Fig. (e) at one-loop

$$\boxed{W_e^{\mu\nu} - W_{e,c}^{\mu\nu}}$$

It is free from the collinear divergence !

Only Fig. (a), (b) an (c) contain the divergences when the exchanged gluon is collinear to P.

Doing the same for Fig. (b) ......

⇒ The contribution to $W^{\mu\nu}$ hasn't the collinear divergences.

There are I.R. divergences in each diagram.

They are cancelled in the sum, because we sum all final states, as disscued in Sect. 2.

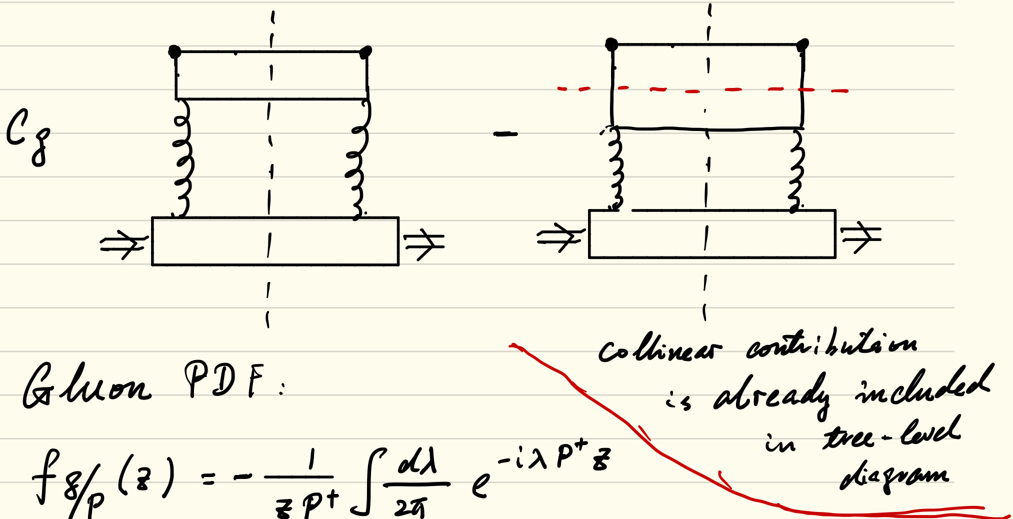
There are collinear divergences when the exchanged gluon is collinear to the final quark.

They are cancelled as I.R. one's

3.3.1.

Conclusion: The perturbative cofficient function C_g in $W^{\mu\nu}$ at one-loop is finite!

At one-loop, there are gluonic contributon



Gluon PDF:

$$f_{g/p}(z) = -\frac{1}{zP^+}\int\frac{d\lambda}{2\pi}e^{-i\lambda P^+ z}$$

$$\langle h(P)|\left(G^{+\mu}(\lambda n)V^+(\lambda n,\infty)\right)^a\left(V(0,\infty)G^{+\mu}(0)\right)^a|h(P)\rangle,$$

V: in adjoint reprensentation.

One can show C_g is finite at one-loop.

3.32

One can go iteratively beyond one-loop, and show the factorization:

$$F_2(x,Q^2) = x \sum_a \int_x^1 \frac{d\xi}{\xi} C_a\left(\frac{x}{\xi}, Q^2, \mu^2\right) f_{a/h}(\xi,\mu^2) + \cdots$$

$$= x \sum_a C_a \otimes f_{a/h} \left(1 + O\left(\frac{\Lambda^2}{Q^2}\right)\right),$$

The operators used to define PDF are twist-2 operators.

Theoretical predictions? $\mu \to \infty$, $\alpha_s \to 0$.

Bjorken scaling.

Evolution: DGLAP

$$\frac{\partial}{\partial \ln \mu^2} \begin{pmatrix} f_{q/p}(x) \\ f_{g/p}(x) \end{pmatrix} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} \begin{pmatrix} P_{qq}(z), & P_{qg}(z) \\ P_{gq}(z), & P_{gg}(z) \end{pmatrix} \begin{pmatrix} f_{q/p}(\xi) \\ f_{g/p}(\xi) \end{pmatrix},$$

$z = \frac{x}{\xi}$, $P_{ab}(z)$: splitting kernel.

P_{ab} can be calculate with perturbative theory, known at three-loop level.

3.33

$$F_i(x, Q^2) = \sum_a C_a(\mu^2, Q^2) \otimes f_{a/p}(\mu^2),$$

C_a depend on $\ln \frac{\mu^2}{Q^2}$, $C_a(\mu^2, Q^2) = C_a\left(\frac{\mu^2}{Q^2}\right)$,

Taking $\mu^2 = Q^2$, $F_i(x, Q^2) = \sum_a C_a(1) \otimes f_{a/p}(Q^2)$,

Q^2-dependence is determind by DGLAP.

Bjorken scaling: $F_i(x, Q^2) = F_i(x)$,

Scaling violation is predicted.

The prediction agrees with experiment. $\Longrightarrow$

Experiment of unpolarized DIS has told us:

1. Partons are quarks and gluons
2. Scaling violation from experiments is predicted correctly.
3. Extract PDF's for other usages.

"Classical test of QCD"

3.34

Theory VS. Experiment

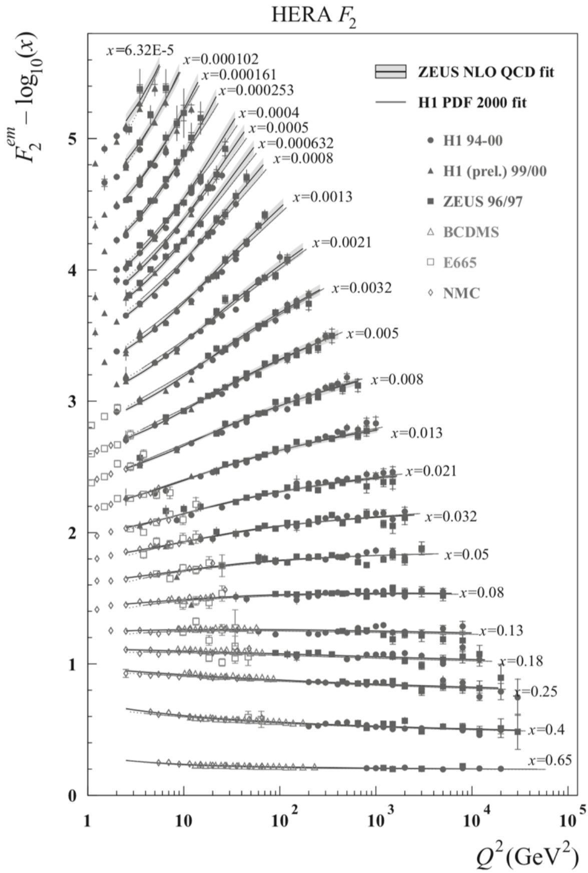


Fig. 2.7. Compilation of the world F_2 data for DIS on a proton. The proton F_2 structure function is plotted as a function of Q^2 for a range of values of x, as indicated next to the data. It can be seen that, except for very small x, F_2 is independent of Q^2, a manifestation of Bjorken scaling. (We thank Kunihiro Nagano for providing us with this figure.) A color version of this figure is available online at www.cambridge.org/9780521112574.

3.35

polarized DIS with proton:

The proton is polarized, a spin vector S

The decomposition:

$$W^{\mu\nu}(P,q) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x,Q^2) + \frac{\hat{p}^\mu \hat{p}^\nu}{P\cdot q} F_2(x,Q^2)$$

$$+ i\,\varepsilon^{\mu\nu\alpha\beta} \frac{q_\alpha}{P\cdot q} \Big[S_\beta\, g_1(x,Q^2)$$

$$+ (P\cdot q\, S_\beta - q\cdot S\, P_\beta)\, g_2(x,Q^2) \Big],$$

Two-additional structure functions.

g_1: can be measured with longitudinally polarized proton. Its factorization is similar to that of F_i with twist-2 operators.

experimental study of g_1:

"Spin crisics"

3.36

g_2: can be measured with transversely polarized proton. Its factorization is with twist-3 operators, complicated.

#

Momentum sum rule

Spin sum rule

⋮

3.37

Universarlity of PDF's: (very important)

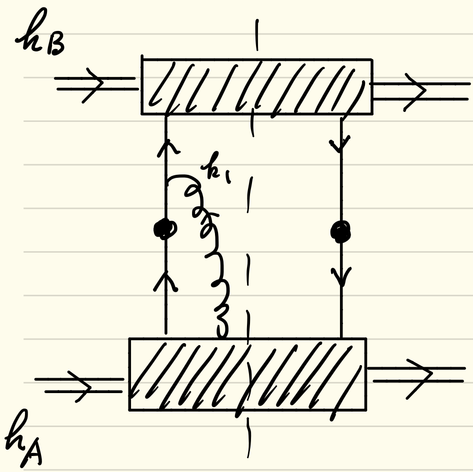
Drell-Yan process: (DY)

$$h_A(P_A) + h_B(P_B) \longrightarrow \gamma^* + X$$

$$\hookrightarrow \ell^+ \ell^-,$$

$$P_A^\mu \simeq (P_A^+, 0, 0, 0), \quad P_B^\mu \simeq (0, P_B^-, 0, 0)$$

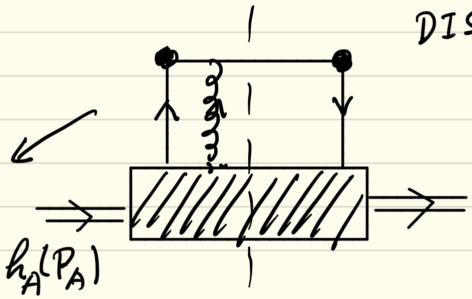
One can do the analysis as done before.



$\Rightarrow$ gives a factor

$$\frac{i}{-k_1^+ - i\varepsilon} \left(-i g_s n^\mu T^a\right)$$

DIS



$$\frac{i}{-k_1^+ + i\varepsilon} \left(-i g_s n^\mu T^a\right)$$

3.38

$\pm i\varepsilon$ has a physical meaning:

$+i\varepsilon$: final state interaction

This leads to that PDF defined in DIS is with the gauge link $V(x,\infty)$ pointing to the future.

$-i\varepsilon$: Initial state interaction

This leads to that PDF defined in DY is with the gauge link $V(x,-\infty)$ pointing to the past.

PDF in DY:

$$V(x,-\infty) = P\exp\left\{-ig_s\int_{-\infty}^{0} d\lambda\, G^+(\lambda n + x)\right\},$$

$$f_{q/p}(z)\Big|_{DY} = \frac{1}{2}\int\frac{d\lambda}{2\pi}\, e^{-i\lambda P_A^+ z}$$

$$\langle h_A|\,\bar{q}(\lambda n)\,V(\lambda n,-\infty)\,\gamma^+$$

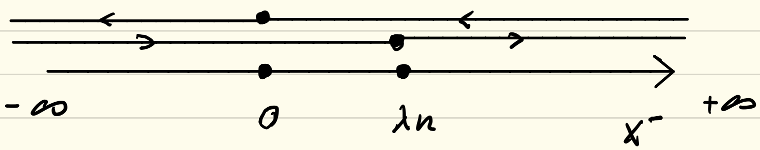
$$V^+(0,-\infty)\,q(0)\,|h_A\rangle$$ #

3.3f

Symmetry of Parity + Time-reversal

$$\Rightarrow \quad \boxed{f_{g/p}(z)\Big|_{DY} = f_{g/p}(z)\Big|_{DIS}}$$

or non-Abelian Stoke theorem



The area inside the close contour is zero!!

3.40

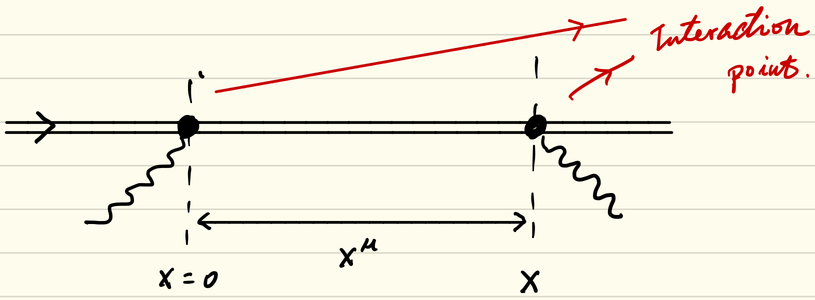
Physical picture of DIS:

Space-time structure of DIS:

rewrite:

$$W^{\mu\nu} = \frac{1}{2\pi} \operatorname{Im}\left[\int d^4x\, e^{iq\cdot x} \langle h(P)|\, T\left(J^\mu(x) J^\nu(0)\right) |h(P)\rangle\right]$$

Forward scattering:



Typical x-range, interact range, or observation range is given $q\cdot x \sim 1$.

We take the frame: The initial proton is in rest,

$$p^0 = m, \quad \vec{p} = 0.$$

the virtual photon moves in the $-z$-direct:

$$q^\mu = (q^0, 0, 0, q^3) = (q^0, 0, 0, -|q^3|),$$

$$q^+ = \frac{1}{\sqrt{2}}(q^0 - |q^3|), \quad q^- = \frac{1}{\sqrt{2}}(q^0 + |q^3|).$$

3.41

$Q^2 \to \infty$, or $Q^2 \gg m^2$,

$$2P\cdot q = 2m q^0 = \frac{Q^2}{x_B} \Rightarrow q^0 = \frac{Q}{2m x_B} Q \gg Q,$$

$$Q^2 = -q^2 = -\left((q^0)^2 - |q^3|^2\right) > 0 \Rightarrow |q^3| > q^0.$$

For q^μ: $q^0 \sim |q^3| \gg Q$.

$$q^- \sim \sqrt{2}\, q^0, \quad q^+ = \frac{-Q^2}{2q^-} \sim -\frac{Q^2}{2\sqrt{2}\, q^0} = -\frac{1}{\sqrt{2}} m x_B,$$

The typical range: $x^- \sim \dfrac{1}{|q^+|} \sim \dfrac{\sqrt{2}}{x_B m} \gtrsim \dfrac{1}{\Lambda}$.

$$x^+ \sim \frac{1}{q^-} \approx \frac{\sqrt{2}\, x_B m}{Q^2} \ll \frac{1}{\Lambda},$$

Λ: Λ_{QCD}, or m, soft scale,

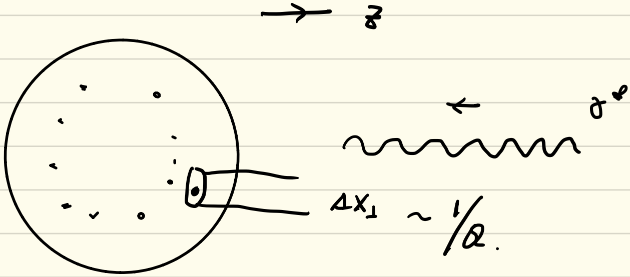
proton size $R \sim \dfrac{1}{\Lambda}$. $\quad x^- = \dfrac{1}{\sqrt{2}} \cancel{t}\, t_{\text{Ioffe}}$

Ioffe time.

Transverse range:

$x^2 > 0$ causality $\Rightarrow x_\perp^2 < 2x^+x^- \sim \dfrac{4}{Q^2} \ll \dfrac{1}{\Lambda^2}$.

3.42



$\Delta x_\perp$ small, dominant contribution is from the case of finding one-parton,

The probablity to find two partons is power suppressed by $1/Q$, part of high-twist effect.

Interaction time: partons inside a hadron in rest, the interaction time between partons: (soft).

$$t_p \sim \frac{1}{\Lambda} \sim R.$$

The time range between two interaction points:

$$t_{DIS} \sim \frac{1}{q^0} = \frac{2 x_B m}{Q^2} \ll \frac{1}{\Lambda}$$

$\Rightarrow$ The poton interacts with a free parton,
Physical reason why the factorizaton can be done.

3.43

An interesting observation:

$$x^- \sim \frac{1}{|q^+|} \sim \frac{\sqrt{2}}{x_B m} \gtrsim \frac{1}{\Lambda} \sim R$$

If x_B is small enough, $x^- > R$ or $x^- \gg R$.

At least, one interaction point is not located insider the hadron. How can the interaction happens outside the hadron?

Hot topic!! $\boxed{\text{Small-}x \text{ physics}}$

Section end.

4.1.

4. QCD Factorization in $e^+e^- \to h + X$

$$e^+e^- \to \gamma^*(q) \longrightarrow h(P) + X$$

$$q^2 > 0, \quad q^2 \gg \Lambda^2, \quad \Lambda \sim \Lambda_{QCD}$$

Similarly to DIS, we define the hadronic tensor:

$$W^{\mu\nu} = \int d^4x\, e^{iq\cdot x} \sum_X \langle 0| J^\mu(x) |h, X\rangle \langle h, X| J^\nu(0) |0\rangle,$$

Taking a frame:

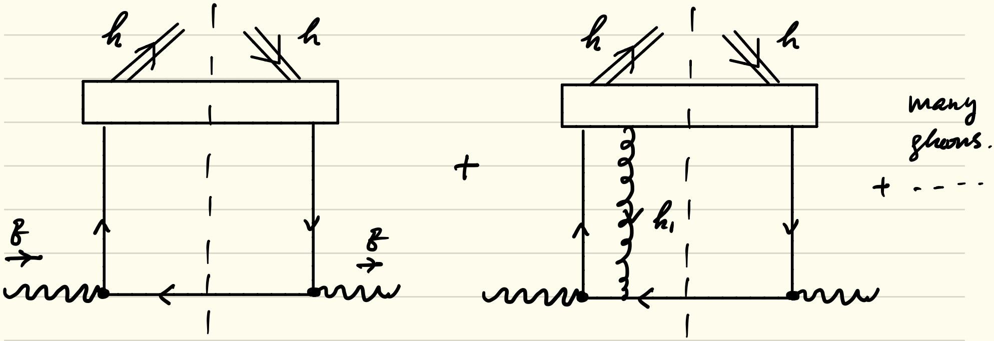
$$P^\mu \approx (P^+, 0, 0, 0), \quad q^\mu = (q^+, q^-, 0, 0)$$

The analysis is very similar to that of DIS, with the difference of jet functions or "black box".

The same power counting

At leading order:

4.2



A factor : $\dfrac{-i}{-k_1^+ + i\varepsilon}(-ig_s n^\rho)$

The sum of exchanges of gluons can be done with gauge links $V(x,\infty)$. This leads to the definition of parton Fragmentation Functions (FF's).

Quark FF:

$$D_{h/q}(z) = \frac{z}{4\pi}\int d\lambda\, e^{-i\lambda k^+} \frac{1}{2N_c} \sum_X \mathrm{Tr}\Big(\gamma^+ \langle 0|\, V^\dagger(0,\infty)\, q(0)\, |h(P), X\rangle \langle X, h(P)|\, \bar{q}(\lambda n)\, V(\lambda n,\infty)\, |0\rangle\Big),$$

$$k^+ = \frac{P^+}{z},$$

4.3.

it describs that a quark with the momentum k^+ fragments into a hadron with the momentum $zk^+ = P^+$.

$|z| \leq 1$, $z > 0$ for quark FF,

$z < 0$ for antiquark FF.

The gluon FF:

$$D_{h/g}(z) = -\frac{z}{4\pi (N_c^2 - 1) k^+} \int d\lambda \, e^{-i\lambda k^+}$$
$$\langle 0 | G^{+\mu}(0) \, \tilde{V}^+(0, \infty) | h(P), X \rangle$$
$$\langle X, h(P) | \tilde{V}(\lambda n, \infty) \, G^+{}_\mu(\lambda n) | 0 \rangle.$$

The FF' also depend on μ. Their evolutions are in the form of DGLAP. The leading order of evotutions are the same as those of PDF. The differences appear at two-loop.

4.4.

Statement of QCD factorization:

$$\frac{d\sigma(e^+e^- \to h+x)}{dz} = \sum_a \int \frac{d\xi}{\xi} H_a\left(\frac{z}{\xi}, Q^2, \mu^2\right) D_{h/a}(\xi, \mu^2) \cdot \left\{1 + O\left(\frac{\Lambda^2}{Q^2}\right)\right\}$$

$$\approx \sum_q \sigma(e^+e^- \to q\bar{q}) \left(D_{h/q}(z) + D_{h/\bar{q}}(z)\right) \cdot \left\{1 + O(\alpha_s) + O\left(\frac{\Lambda^2}{Q^2}\right)\right\}.$$

z: Energy fraction.

$$z = \frac{2E_h}{Q}, \quad Q^2 = s^2.$$

H_a: perturbative coefficent functions, finite!!

Universality of FF's ?

PT-symmetry does not apply, $e^{i\alpha}$

$$PT\,|h,x\rangle_{out} = |h,x\rangle_{in} \neq |h,x\rangle_{out}.$$

5.1.

5. TMD Factorization for SIDIS

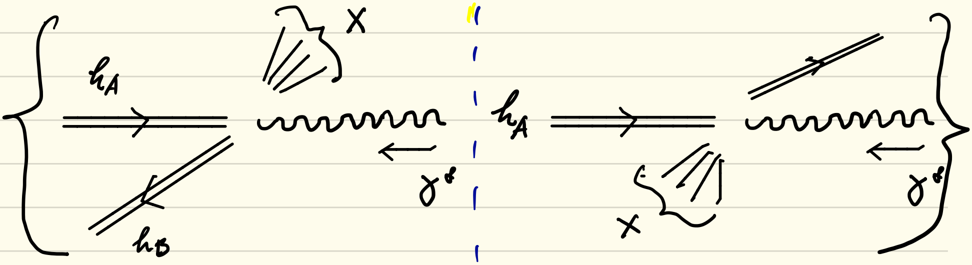
Semi-inclusive DIS (SIDIS):

$$e(k) + h_A(P) \longrightarrow e(k') + h_B(P_h) + X,$$

$q = k - k'$, $q^2 \to -\infty$

Kinematics: One-photon-exchange

$$\gamma^*(q) + h_A(P) \longrightarrow h_B(P_h) + X$$



Current fragmentation region

Target fragmentation region

"Fracture functions"

We don't consider this region.

$P^\mu \approx (P_A^+, 0, 0, 0)$,

$q^\mu = (q^+, q^-, 0, 0)$, $q^+ < 0$.

$$P_h^\mu \approx \left(\frac{P_{h\perp}^2}{2P_h^-}, P_h^-, P_{h\perp}^1, P_{h\perp}^2\right).$$

5.2.

$\Lambda \sim \Lambda_{QCD}, m_h \ldots$

soft scale.

Three kinematical regions.

a. $P_{h\perp}^2 \sim Q^2 \gg \Lambda^2$, collinear factorization

$$d\sigma \sim H \otimes f \otimes D \left\{ 1 + O\left(\frac{\Lambda^2}{Q^2}\right) \right\}$$

PDF FF

H: perturbative coefficient function, finite.

b. $Q^2 \gg P_{h\perp}^2 \gg \Lambda^2$, Collinear factorization, still.

But: large log's $\ln \frac{P_{h\perp}^2}{Q^2}$ in H,

resummation is needed.

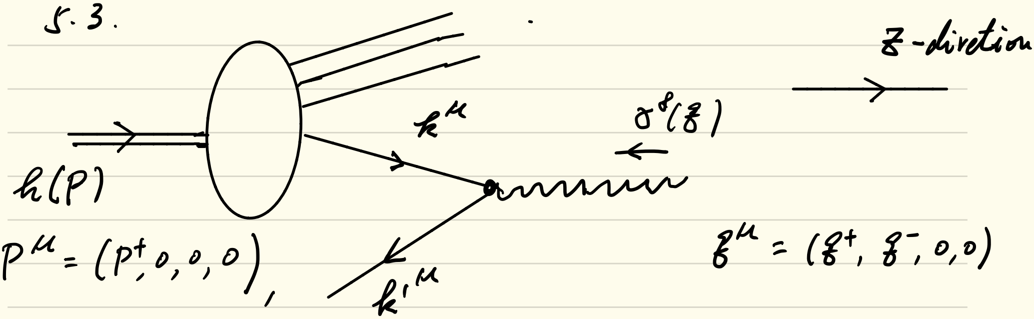
c. $Q^2 \gg P_{h\perp}^2 \sim \Lambda^2$, no collinear factorization.

small $P_{h\perp}$: sensitive to transverse momenta of partons, neglected in collinear factorizat.

Transe-Momentum-Dependent factorization

TMD for c. and b.

5.3.



$P^\mu = (P^+, 0, 0, 0)$, $q^\mu = (q^+, q^-, 0, 0)$

The parton inside h has the momentum

$$k^\mu = (k^+, k^-, \vec{k}_\perp) \sim (1, \Lambda^2, \Lambda, \Lambda),$$

The outgoing parton has also the transverse momentum $\vec{k}_\perp' = \vec{k}_\perp$

If we don't obersevе $\vec{k}_\perp'$, i.e, $\vec{k}_\perp'$ is integrated, we can make the apporximation by setting $\vec{k}_\perp = 0$. This brings an error at the order $k_\perp^2/Q^2 \sim \Lambda^2/Q^2 \ll 1$. like DIS.

If we do observe $\vec{k}_\perp'$, it implies that we detect $\vec{k}_\perp$ of the parton inside the hadron.

$\Rightarrow$ Here we can not set $\vec{k}_\perp = 0$. !!

5.3

Difficulty for defining TMD PDF:

$h(P)$, $\quad P^\mu \simeq (P^+, 0, 0, 0)$

We re-write the collinear PDF

$$f_{q/p}(x) = \int \frac{d\lambda}{4\pi} e^{-i\lambda k^+} \langle h | \bar{q}(\lambda n) V^+(\lambda n, \infty) \gamma^+ V(0,\infty) q(0) | h \rangle$$

$k^+ = xP^+$

$$= \int d^2 k_\perp \underline{\int \frac{d^3\tilde{\xi}}{2(2\pi)^3} e^{-i\tilde{\xi}\cdot k} \langle h | \bar{q}(\tilde{\xi}) V^+(\tilde{\xi}, \infty) \gamma^+ V(0,\infty) q(0) | h \rangle},$$

$\tilde{\xi}^\mu = (0, \xi^-, \vec{\xi}_\perp)$, $\xi^- = \lambda$, $\quad k^\mu = (k^+, 0, \vec{k}_\perp)$.

Define TMD PDF

$$f_{q/p}(x, \vec{k}_\perp) = \int \frac{d^3\tilde{\xi}}{2(2\pi)^3} e^{-i\tilde{\xi}\cdot k} \langle h | \bar{q}(\tilde{\xi}) V^+(\tilde{\xi}, \infty) \gamma^+ V(0,\infty) q(0) | h \rangle$$

?

But: The definition is inconsistent !!

5.4

The gauge link:

$$V(\xi,\infty) = P\exp\left\{-ig_s\int_0^\infty d\lambda\, G^+(\lambda n+\xi)\right\}$$

Under gaug transformation:

$$V^\dagger(\tilde{\xi},\infty)\,V(0,\infty) \Longrightarrow$$

$$\mathcal{U}(\tilde{\xi})\,V^\dagger(\tilde{\xi},\infty)\,\underbrace{\mathcal{U}^\dagger(\vec{\xi}_\perp,\,\xi^-=\infty)\,\mathcal{U}(0,\,\xi^-=\infty)}_{\neq 1\ !!}\,V(0,\infty)\,\mathcal{U}^\dagger(0)$$

$\Rightarrow$ No gauge invariance !!

In fact, in nonsingluar gauge, like Feynman gauge, $\mathcal{U}(\vec{\xi}_\perp,\,\xi^-=\infty) = 1$.

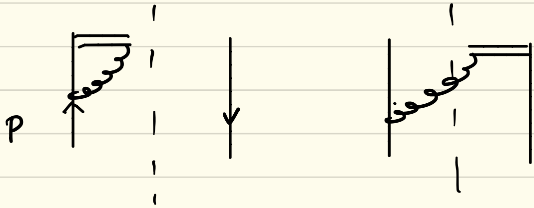
In other gauge, transverse gauge links at $\xi^-=\infty$ need to be added.

Singular gauge: $n\cdot G = 0$.

5.5.

Light-cone singularity:

If we calculate $f_{q/q}(x, \vec{k}_\perp)$ at one-loop,



(a) (b)

$$f_{q/q}(x, \vec{k}_\perp)\Big|_a \sim \delta(1-x)\, \delta^2(\vec{k}_\perp) \int_0^1 \frac{dy}{1-y},$$

↑ divergent

$$f_{q/q}(x, \vec{k}_\perp)\Big|_b \sim (x, \vec{k}_\perp)\, \frac{1}{1-x}, \qquad x \to 1$$

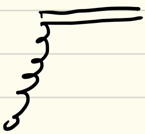
Collinear PDF

$$\int d^2k_\perp \left[f_{q/q}\Big|_a + f_{q/q}\Big|_b \right] = \text{"finite"}$$

The divergence: Light-cone singularity.

related to gauge links.

5.6.

 $\Rightarrow \dfrac{1}{n\cdot k + i\varepsilon} = \dfrac{1}{k^+ + i\varepsilon} \Rightarrow \dfrac{1}{1-x}$

$k^+ \to 0$, divergent.

One possible way to regularize the divergence:

Gauge link off light-cone:

$u^\mu = (u^+, u^-, 0, 0)$ instead of $n^\mu = (0,1,0,0)$

Def:

$$V_u(\xi, \infty) = P\exp\left\{-ig_s \int_0^\infty d\lambda\, u\cdot G(\lambda u + \xi)\right\},$$

$$f(x, k_\perp) = \int \frac{d^3\tilde{\xi}}{2(2\pi)^3}\, e^{-i\tilde{\xi}\cdot k} \langle h|\, \bar{q}(\tilde{\xi})\, V_u^\dagger(\tilde{\xi}, \infty)\, \gamma^+\, V_u(0,\infty)\, q(0)\, |h\rangle,$$

$u^+ \to 0$, but finite.

f depends on μ, $\quad \zeta_u^2 = \dfrac{4(u\cdot p)^2}{u^2} \approx \dfrac{2u^-}{u^+}(p^+)^2$

f depends on the energy of the hadron!!

5.7.

Evolution of TMD PDF:

$$\boxed{\mu \frac{\partial}{\partial \mu} f(x, k_\perp, \mu, \zeta_u) = 2\gamma_F \, f(x, k_\perp, \mu, \zeta_u)}$$

γ_F : Anomalous dimension of the quark field in the axial gauge $u \cdot G = 0$.

$$\gamma_F = \frac{3\alpha_s}{4\pi} C_F + O(\alpha_s^2) .$$

It is much more simple than DGLAP.

The reason:

$\int d^2k_\perp \, f(x, k_\perp)$ has more U.V. divergences, more U.V. subtraction.

$\Rightarrow$

$$\int d^2k_\perp \, f(x, k_\perp) \neq f_{q/p}(x) \quad !!$$

5.8:

The evolution of ζ_u: Collins-Soper Eq. !!

It takes simple form in b-space, $b^\mu = (b^1, b^2)$.

$$f(x, b, \mu, \zeta_u) = \int d^2k_\perp \, e^{i\vec{b}\cdot\vec{k}_\perp} f(x, k_\perp, \mu, \zeta_u)$$

$$\boxed{\zeta_u \frac{\partial}{\partial \zeta_u} f(x, b, \mu, \zeta_u) = \Big(K(\mu, b) + G(\mu, \zeta_u)\Big) f(x, b, \mu, \zeta_u)}$$

$$K(\mu, b) + G(\mu, \zeta_u) = -\frac{\alpha_s}{\pi} C_F \ln \frac{\zeta_u^2 b^2 e^{2\gamma_E - 1}}{4},$$

$$\mu \frac{\partial}{\partial \mu} K = -\gamma_K = -\mu \frac{\partial}{\partial \mu} G,$$

γ_K: cusp anomalous dimension

$$\gamma_K = \frac{\alpha_s}{\pi} 2C_F$$

CS equation $\Rightarrow$ CSS resummation

Very important! $\Rightarrow$ 6

5.9.

Similarly, we can define TMD FF's.

$h(P_h)$, $P_h^\mu \approx (0, P_h^-, 0, 0)$,

Def: $V_v(\xi, -\infty) = P\exp\left\{-ig_s\int_{-\infty}^{0} d\lambda\, v\cdot G(\lambda v+\xi)\right\}$,

$v^\mu = (v^+, v^-, 0, 0)$, $v^+ \gg v^-$.

(Kinematics).

TMD FF:

$$\hat{q}(z, k_\perp) = \frac{1}{2z}\int\frac{d^3\tilde{\xi}}{(2\pi)^3} e^{-i\tilde{\xi}\cdot k}\frac{1}{N_c}\sum_X \mathrm{Tr}$$

$$\langle 0|\gamma^+ V_v^\dagger(0)\, q(0)|hX\rangle\langle hX|\bar{q}(\tilde{\xi})V_v(\tilde{\xi})|0\rangle,$$

here: $\tilde{\xi}^\mu = (\xi^+, 0, \vec{\xi}_\perp)$, $k^\mu = (0, \frac{1}{z}P_h^-, -\frac{1}{z}\vec{P}_{h\perp})$,

expan...

$\hat{q}$ depends on μ, $\zeta_v^2 = \dfrac{4(v\cdot P_h)^2}{v^2}$

5.10

Factorization: Breit frame

$$h_A(P) + \gamma^*(q) \longrightarrow h_B(P_h) + X$$

$$q^2 = -Q^2 \to -\infty, \qquad P_{h\perp}^2/Q^2 \ll 1$$

Def:

$$x = \frac{-q^2}{2P\cdot q} = \frac{Q^2}{2P\cdot q} = -\frac{q^+}{P^+}, \qquad x_B.$$

$$z = \frac{P\cdot P_h}{P\cdot q} = \frac{P_h^-}{q^-}, \qquad P_{h\perp}^\mu = g_\perp^{\mu\nu} P_{h\nu}$$

The hadronic tensor:

$$W^{\mu\nu}(P, q, P_h) = \frac{1}{z}\int \frac{d^4k}{(2\pi)^4} e^{iq\cdot x} \sum_X$$

$$\langle h_A | J^\mu(x) | h_B X\rangle\langle X h_B | J^\nu(0) | h_A\rangle$$

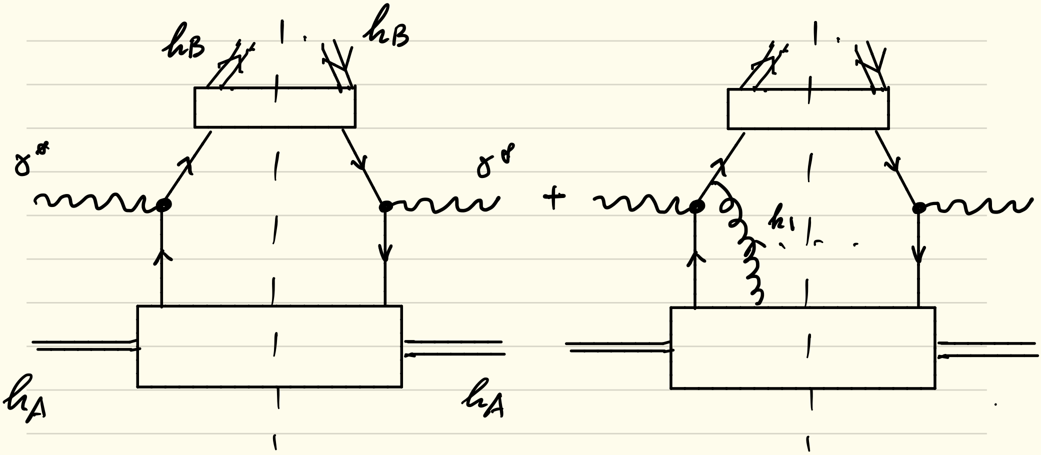
$$= -\frac{1}{2} g_\perp^{\mu\nu} F(x, z, P_{h\perp}, Q)$$

$$+ \text{“Power suppressed”}$$

$g_\perp^{\mu\nu} = g^{\mu\nu} - n^\mu \ell^\nu - n^\nu \ell^\mu$, can be difined covariantly.

5.11.

Factorization at tree-level:



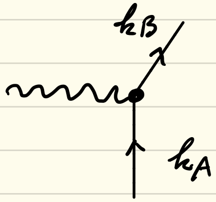
A factor:

$$\frac{i}{-k_1^+ - i\varepsilon}\left(-i g_s n^\mu T^a\right)$$

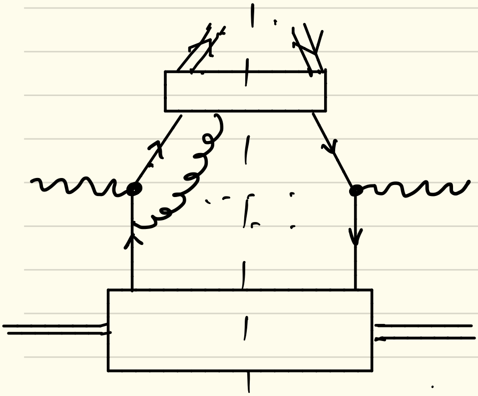
replace $k_1^+ \to n\cdot k$!!

Summing all G^+ gluon from the bottom

$\Rightarrow$ gauge link V_n,



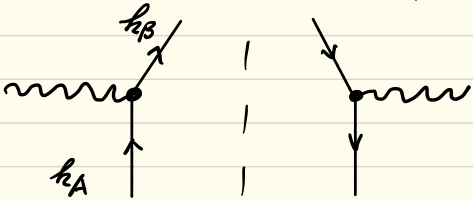
5.12.



summed with the gauge link V_v

Important: to calculate the middle part

In the middle part:



$k_A^\mu = (k_A^+, 0, 0, 0)$

$k_B^\mu = (0, k_B^-, 0, 0)$. on-shell amplitudes

$\Rightarrow$ gauge invariant

$k_{A\perp}$, $k_{B\perp}$ only included in the momentum conservation.

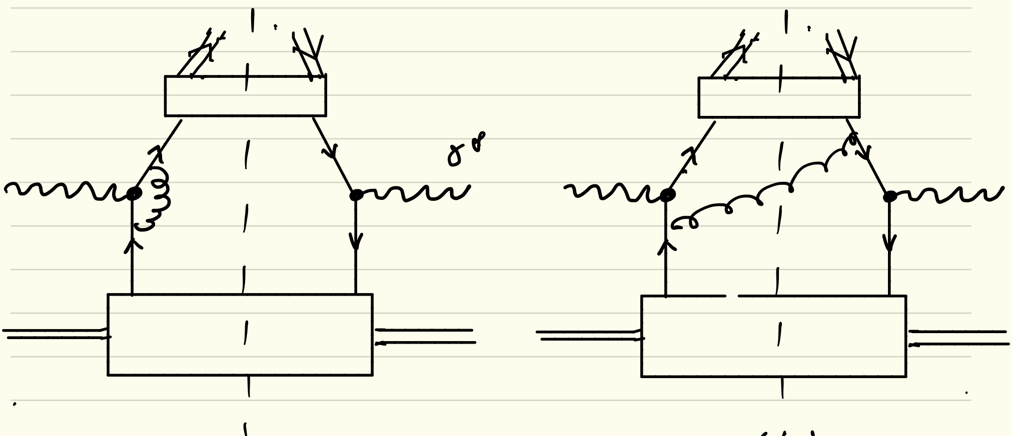
5.13.

⇛ Tree-level factorization:

$$F(x, z, P_{h\perp}, Q) = \int d^2k_{A\perp}\, d^2k_{B\perp}\, \delta^2(z\vec{k}_{B\perp} + \vec{k}_{A\perp} - \vec{P}_{h\perp})$$
$$f(x, k_{A\perp})\, \hat{g}(z, k_{B\perp})$$

But this is not correct beyond tree-level.

At one-loop:



(a) (b)

Because the sum of final states is incomplete, KLN theorem does not apply !!

5.14.

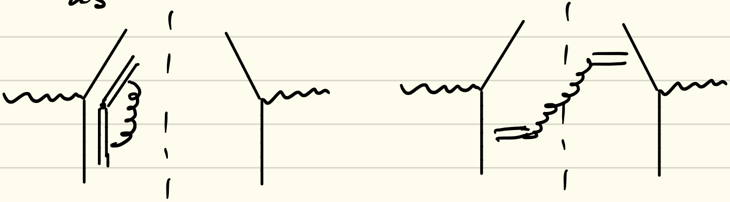
Following the analysis of DIS, one finds:

- The gluon collinear to h_A or k_A, with the momentum $k^\mu \sim (1, \lambda^2, \lambda, \lambda)$ is factorized into TMD PDF of h_A.

- The gluon collinear to h_B or k_B, with the momentum $k^\mu \sim (\lambda^2, 1, \lambda, \lambda)$ is factorized into TMD FF of h_B.

The contribution from the soft gluon with $k^\mu \sim (\lambda, \lambda, \lambda, \lambda)$ is still there and divergent, because no KLN.

The soft gluon Fig. (a) and (b) can be factorized as



(a) (b)

5.15

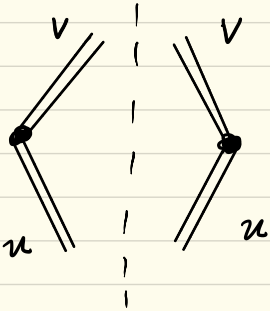
There are also extra soft-gluon-contributions in TMD PDF and TMD FF.

The soft factor: $\rho^2 = \frac{4(u\cdot v)^2}{u^2 v^2}$

$$S(\vec{\xi}_\perp, \mu, \rho) = \frac{1}{N_c} \mathrm{Tr} \langle 0 | V_v^\dagger(\vec{\xi}_\perp, -\infty) V_u^\dagger(\infty, \vec{\xi}_\perp) V_u(0,\infty) V_v(0,-\infty) | 0 \rangle ,$$

Diagram representation:

S :



gluons exchanged between double lines.

Def: $$\tilde{S}(\vec{k}_\perp, \mu, \rho) = \int \frac{d^2\xi_\perp}{(2\pi)^2} e^{i\vec{\xi}_\perp\cdot\vec{k}_\perp} \left[S(\vec{\xi}_\perp, \mu, \rho) \right]^{-1}$$

(−1) !!

5.16.

The correct factorization beyond tree-level:

$$\boxed{\begin{aligned} F(x,z,P_{h\perp},Q) = {} & H(Q,\mu,\zeta_u,\zeta_v) \\ & \int d^2k_{A\perp}\, d^2k_{B\perp}\, d^2k_\perp\, \delta^2(z\vec{k}_{B\perp} + \vec{k}_{A\perp} + \vec{k}_\perp - \vec{P}_{h\perp}) \\ & f(x,k_{A\perp},\mu,\zeta_u)\, \hat{g}(z,k_{B\perp},\mu,\zeta_v)\, \tilde{S}(k_\perp,\mu,\rho) \end{aligned}}$$

H: Perturbative coefficients, finite.

$$H = 1 + O(\alpha_s).$$

In b-space:

$$\begin{aligned} F(x,z,\vec{b},Q) = {} & H(Q,\mu,\zeta_u,\zeta_v)\, S^{-1}(b,\mu,\rho) \\ & f(x,b,\mu,\zeta_u)\, \hat{g}(z,b,\mu,\zeta_v) \end{aligned}$$

One may define:

$$f(x,b,\mu,\zeta_u) \to \frac{f(x,b,\mu,\zeta_u)}{\sqrt{S(b,\mu,\rho)}} \qquad \cdots$$

5.17.

Collinear PDF's vs TMD PDF's

Collinear PDF's: longitudinal motion of partons, one-dimension.

TMD PDF's: Three-dimensional motion of partons, more about inner structure.

At leading twist, only 3 PDF's of spin-1/2

$$\int \frac{d\lambda}{2\pi} e^{-i\lambda P^+ x} \langle h(P) | \bar{q}(\lambda n) \; q(0) | h(P) \rangle$$

$$= \frac{1}{2N_c} \left[\gamma^- q(x) + \gamma^5 \gamma^- \lambda q_L(x) - i \gamma^- \gamma_\perp^\mu \tilde{S}_{\perp\mu} q_T(x) \right] + \text{"} - - - \text{"}$$

high twist

λ: helicity of h

$\vec{S}_\perp$: transverse spin of h, $\tilde{S}_\perp^\mu = \varepsilon_\perp^{\mu\nu} S_{\perp\nu}$.

5.18

At leading power or twist-2, there are

8 TMD PDF's. !! TMD FF's

"TMD Physics"

Strong experimental programs at J-Lab, Compass and Belle, even BES II

$\longrightarrow$ Collins's effect

An important difference between collinear- and TMD parton distributions:

$f(x,\mu)$: does not depend on the momentum of the hadron

$f(x, k_\perp, \zeta_n^2, \mu)$ does through $\zeta_n^2 = \dfrac{4(n\cdot P)^2}{n^2}$.

This has a consequence ...

6. TMD Factorization of Drell-Yan Processes and $q_\perp$-resummation (CSS)

Drell-Yan process:

$$h_A(P_A) + h_B(P_B) \longrightarrow \gamma^*(q) + X$$

$$\hookrightarrow e^+e^-$$

momenta: $P_A^\mu \approx (P_A^+, 0, 0, 0)$, $\quad P_B^\mu \approx (0, P_B^-, 0, 0)$

$$q^\mu = (q^+, q^-, \vec{q}_\perp), \qquad y = \frac{1}{2}\ln\frac{q^+}{q^-} \qquad q^2 = Q^2$$

If we use collinear factorization, at $O(\alpha_s^0)$: $\quad \sigma_0 = \dfrac{4\pi^2\alpha^2}{9 s Q^2}$

$$\frac{d\sigma}{dQ^2\, dy\, dq_\perp^2} = \sigma_0 \sum_q q_A(x_1)\, \bar{q}_B(x_2)\, \delta^2(\vec{q}_\perp) + O(\alpha_s)$$

$$+ (A \rightleftarrows B)$$

$$x_1 = e^{y}\sqrt{\frac{Q^2}{s}}, \qquad x_2 = e^{-y}\sqrt{\frac{Q^2}{s}}$$

The distribution is nonzero only at $\vec{q}_\perp = 0$ and singular.

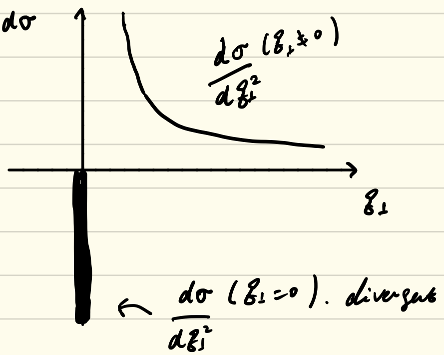
At order of α_s: $\quad q_\perp$ small, not zero

$$\frac{d\sigma}{dQ^2\, dy\, dq_\perp^2} \sim \alpha_s \frac{1}{q_\perp^2} \ln q_\perp^2 \qquad (q_\perp \neq 0)$$

$$\frac{d\sigma}{dQ^2\, dy\, dq_\perp^2} \sim \alpha_s\, \delta^2(q_\perp) \qquad (q_\perp = 0)$$

divergent coefficients.

6.2



Adding two parts together, performing $q_\perp$-integration, $d\sigma$ is finite.

Question: Can we make precise prediction about the $q_\perp$-distribution even at small $q_\perp$?

$\alpha_s \ln \frac{q_\perp}{Q}$: large log's !! If $Q \gg q_\perp$.

For $Q \gg q_\perp$, or $q_\perp \sim \Lambda_{QCD}$, Collinear factorization is not a good approach, because transverse momenta of partons can not be neglected !!

Collinear factorization:

$$q_A(x_1 P_A) + \bar{q}_B(x_2 P_B) \to \gamma^* \to e^+e^-$$

· transverse momenta are neglected.

6.3

TMD factorization is needed for the case $q_\perp \ll Q$.

$q_\perp \to 0$

$$\frac{d\sigma}{dQ^2\, dy\, d q_\perp^2} = \sigma_0 W(q_\perp, Q^2, x_1, x_2) \left\{ 1 + O\left(\frac{q_\perp}{Q}\right) \right\}$$

$$= \sigma_0 W(q_\perp, Q^2, x_1, x_2) + Y(q_\perp, Q^2, x_1, x_2)$$

↑ The most singular parts

b-space:

$$W(b, Q^2, x_1, x_2) = \int \frac{d^2 q_\perp}{(2\pi)^2} e^{i\vec{b}\cdot\vec{q}_\perp} W(q_\perp, Q^2, x_1, x_2)$$

TMD factorization: (similar to SIDIS, but ··).

$$W(b, Q^2, x_1, x_2) = H\, q_A(x_1, b, \zeta_u, \mu)\, \bar{q}_B(x_2, b, \zeta_v, \mu)\, \tilde{S}(b, \mu, \rho),$$

$$\zeta_u^2 = \frac{4(u\cdot P_A)^2}{u^2} \approx \frac{2u^-}{u^+}(P_A^+)^2, \quad \zeta_v^2 = \frac{4(v\cdot P_B)^2}{v^2} \approx \frac{2v^+}{v^-}(P_B^-)^2$$

Two vectors for two TMD's:

$$u^\mu = (u^+, u^-, 0, 0),\ u^+ \to 0, \qquad v^\mu = (v^+, v^-, 0, 0),\ v^- \to 0.$$

$$W(b, Q^2, x_1, x_2) = 1 + \alpha_s \left(\ln^2 bQ, \cdots \right)$$

↑

large double log's

6.4.

$$H = H(Q^2, \mu^2, \zeta_u^2, \zeta_v^2), \qquad \rho = \sqrt{\frac{u^- v^+}{u^+ v^-}}.$$

$\uparrow$

$$W(b, Q^2, x_1, x_2) = H \; \tilde{f}_A(x_1, b, \zeta_u, \mu) \, \tilde{f}_B(x_2, b, \zeta_v, \mu) \, \tilde{S}(b, \mu, \rho),$$

W: does not depend on ζ_u^2, ζ_v^2, we can take $\zeta_u^2 = \zeta_v^2 = \rho Q^2$,

$$\Rightarrow W(b, Q^2, x_1, x_2) = H(Q^2, \mu^2, \rho) \; \tilde{f}_A(x_1, b, \rho Q^2, \mu) \; \tilde{f}_B(x_2, b, \rho Q^2, \mu) \cdot \tilde{S}(b, \mu, \rho).$$

The Q^2-dependence of W is determined by the μ-dependence and Collins-Soper Eq..

6.5

From Collins-Soper Eq. (discussed before)

$$\boxed{\zeta_u \frac{\partial}{\partial \zeta_u} q_A(x, b, \mu, \zeta_u) = \left(K(\mu, b) + G(\mu, \zeta_u)\right) q_A(x, b, \mu, \zeta_u)}$$

$$K(\mu, b) + G(\mu, \zeta_u) = -\frac{\alpha_s}{\pi} C_F \ln \frac{\zeta_u^2 b^2 e^{2\gamma_E - 1}}{4},$$

$$\mu \frac{\partial}{\partial \mu} K = -\gamma_K = -\mu \frac{\partial}{\partial \mu} G,$$

$$\Rightarrow Q^2 \frac{\partial}{\partial Q^2} W(b, Q^2, x_1, x_2) = \left[K(b\mu, g_s(\mu)) + G'(Q/\mu, g_s(\mu))\right] \cdot W(b, Q^2, x_1, x_2),$$

$$K + G' = -\frac{\alpha_s}{\pi} C_F \ln \frac{Q^2 b^2 e^{2\gamma_E - 3/2}}{4}$$

($\leftarrow$ additional Q^2-dependence for μ)

large Q^2 → lower scale μ_L

The solution:

$(c_2 = 1)$.

6.6.

$$W(b, Q^2, x_1, x_2) = W(b, \mu_L^2, x_1, x_2)\, \exp\left\{ - S(Q^2, \mu_L^2, b) \right\}.$$

$$S(Q^2, \mu_L^2, b) = \int_{\mu_L}^{Q} \frac{d\hat{\mu}^2}{\hat{\mu}} \left[\ln\left(\frac{Q^2}{\hat{\mu}^2}\right) A(b\mu_L, \hat{\mu}) + B(1, b\mu_L, \hat{\mu}) \right],$$

$$A(b\mu_L, \hat{\mu}) = \gamma_K(\hat{\mu}) + \beta \frac{\partial}{\partial g} K(b\mu_L, g(\hat{\mu}))$$

$$B(1, b\mu_L, \hat{\mu}) = -2K(b\mu_L, g(\hat{\mu})) - 2G'(1, g(\hat{\mu}))$$

If we take $\mu_L = c_1/b$, $c_1 \sim O(1)$,

there are no large log's and double log's like $\ln b\mu_L$ or $\ln^2 b\mu_L$ in $W(b, \mu_L, x_1, x_2)$. $\Rightarrow$ CSS resummation!!

6.7

If b is small enough, TMD parton distribut can be factorized:

$$q_A(x, b, \zeta_u^2, \mu^2) = \int_x^1 \frac{dy}{y}\Big[C_q\left(\frac{x}{y}, b, \zeta_u^2, \mu\right) q_A(y, \mu)$$

$$+ C_g\left(\frac{x}{y}, b, \zeta_u^2, \mu\right) g_A(y, \mu)\Big]$$

C_q, C_g: perturbative coefficient functions

$q_A(y, \mu)$, $g_A(y, \mu)$: quark- and gluon distribution of h_A (collinear)

$\bar{q}_B(x, b, \zeta_v^2, \mu^2)$: similar.

In real applications, it is still complicated,

e.g., choices of parameters ... C_1, μ_L etc.

Nonperturbative effects:

$$S(Q, \mu_L, b) \to S(Q, \mu_L, b) + S_{NP}$$

...

6.1

6. . . . . .