Soft – Collinear Effective Theory (SCET) & Power Corrections.

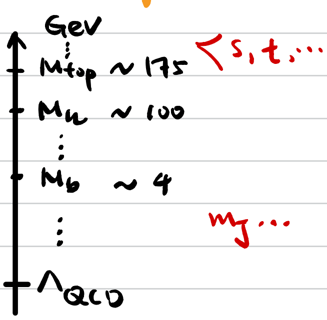
—— 2025/02 @ Fudan.

0. Motivation

1) EFT — modern tool to study scale – seperation in QFT

⇝ reduce multi-scale problems to a sequence of single-scale problems
⇝ systematically resum large logs of scale ratios via renormalization-group equations (RGE). Even for some very complicated cases: NGLs, SLLs, NLP logs, etc.

remark (rmk): Scale seperation is not only important for reducing multi-scale complexity, but also a necessity in reality with strongly interacting theories, e.g., QCD. (leading to factorization)



$$\lambda = E/\Lambda \ll 1$$

$$\sigma = \lambda^0 \cdot H^{LP} \cdot J_1^{LP} \otimes \cdots J_n^{LP} \otimes S^{LP}$$
$$+ \lambda^1 \cdot [H_1^{NLP} J_1^{NLP} \otimes \cdots \otimes J_n^{NLP} \otimes S^{NLP} + \cdots]$$
$$+ \cdots$$

H_i, J_i, S_i : α_s expansion

(Why Soft-Collinear Effective Theory (SCET)?)

1) Useful:

SCET = EFT for high-energy processes involving light particles.

@ LHC like colliders, full of energetic (almost) massless particles (jets)

→ as a systemmatic tool to study π, γ, jets ...

factorization formulae
↓
renormalization (anomalous dimensions)
↓
resummation (solve RGEs)

B – physics : a lot

collider physics :

i) cutting – edge precision preditions

e.g., Drell – Yan (like) — Becher, Neubert, Yang ...
ttbar — Li, Shao, Yang, Zhu ...
event-shapes — Pecaj, Scott, Wang, Yang ...
EEC — Zhu etc, ...
⋮ — Gao, Li, Zhu ...

ii) very complicated logarithms :
e.g. non-global log (NGL)
super-leading log (SLL) ...
Becher, Neubert, Shao, etc ...

iii) power corrections ...
Moult, Stewart, Zhu, etc.
Beneke ... Wang, etc. ...
Liu Neubert, XW, etc ...

iv) a lot of other fields : nuclear physics, DM physics ...

2) SCET is the collinear and soft version of QCD.
$\rightsquigarrow$ helpful to investigate IR structure of QCD.

e.g., Becher, Neubert + (Yang and etc.) '09 – '12

3) SCET formalism is not restricted to QCD, but also for gravity for example.

Beneke, Hager, Szafron '20 – '22

4) SCET is (kind of) Minkowski generalization of operator product expansion (OPE), which provides rigorous framework for an expansion in powers and logarithms for Euclidean processes.

It is closely related to method of regions (Beneke, Smirnov).
(for Feynman integrals)

conventional EFTs : based on (Euclidean) OPE
$\hookrightarrow$ integrate out heavy particles $(\Delta x \sim \frac{1}{M})$
SCET : based on method of regions
$\hookrightarrow$ integrate out "virtuality"

rmk: Method of region for Feynman integrals has not been proven rigorously so far as I understand, but it always work!

A similar technique, expansion by subgraph, which works for Euclidean FIs, can be proven rigorously.

See e.g., "Applied asymptotic expansions in momenta and masses" by V. A. Smirnov.

Method of region has received a lot attention recently, and has a lot of progress based on Newton polytope. Smirnov; Gardi, Ma and etc. ...

Due to the Minkowski nature, SCET, as an EFT, is complicated and "subtle". In particular, SCET involves particles with $p^2 \approx 0$, but p^μ has large components.

$\rightsquigarrow$ leads to non-locality on light-cones, even at leading power!

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a lot of good references on SCET:

position space {
1) Introduction to Soft-Collinear Effective Theory, a book by Becher, Broggio and Ferroglia, 1410.1892
2) 1803.04310, Becher
}

mom. space 3) Lectures on Soft-Collinear Effective Theory, by Stewart.

This lecture will only briefly review some basics of SCET and then use some concrete examples to have a look on some aspects of frontier of power corrections in SCET.

Some important comments:

Integrating out high-frequcy (virtuality ...) modes results in Wilson coefficients $C_i(\Lambda)$.

1) If the theory above Λ is perturbative, $C_i(\Lambda)$ can be derived from a perturbative "matching procedure" by requiring a finite set of matrix elements in the full theory & in the EFT to agree up to certain higher-order power corrections, which are neglected in the EFT.

   e.g. SCET $\longleftrightarrow$ QCD.

2) If the full theory is unknown, like SMEFT, then one treats $C_i(\Lambda)$ as unknown parameters, which must be extracted from experiments!

Outline

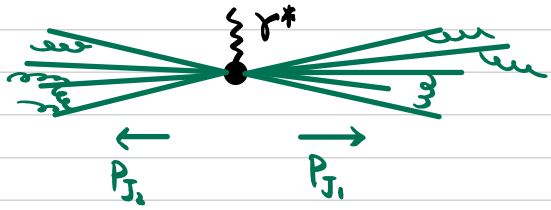
I. A brief Intro. of SCET
- Method of Region
- Basic Ingredients in SCET
- Construct LP $\mathcal{L}_{SCET}$
- Matching of 2-jet operator
- Soft Decoupling

II. NLP Factorization of $gg \to h$ $(h \to \gamma\gamma)$ by b-quark
- Region Analysis
- Heuristic pictures of Involved Operators
- Endpoint Divergence & the Way Out

III. Renormalization at NLP and Anomalous Dimensions
- Soft-quark function in $h \to \gamma\gamma$
- in $gg \to h$
- in Drell-Yan

I. A Brief Introduction of SCET Language

Consider $e^+e^- \longrightarrow 2$ jets :



large energy along jet axis, with small invariant :

$$P_{J_i}^2 := m_{J_i}^2 \ll s .$$

$$p_{J_1}^\mu = \left(E_1, 0, 0, \sqrt{E_1^2 - m_{J_1}^2}\right), \; p_{J_2}^\mu = \left(E_2, 0, 0, \sqrt{E_2^2 - m_{J_2}^2}\right); \; \boxed{E_i \approx \frac{\sqrt{s}}{2} \;\&\; m_{J_i}^2 \ll s}$$

Q: what is there to integrate out for such a Minkowski process?

Introduce small parameter : $\quad (m_{J_1} \sim m_{J_2} := m_J)$

$$\lambda := \frac{m_J}{\sqrt{s}} \ll 1$$

Define two reference light-like vectors along the jet directions :

$$\begin{cases} n_1^\mu = (1, 0, 0, 1) = \bar{n}_2^\mu \\ n_2^\mu = (1, 0, 0, -1) = \bar{n}_1^\mu \end{cases} \qquad n_i^2 = 0 \qquad n_1 \cdot n_2 = 2$$

Decompose a vector in the light-cone basis :

$$\begin{aligned} p^\mu &= n_1 \cdot p \, \frac{n_2^\mu}{2} + n_2 \cdot p \, \frac{n_1^\mu}{2} + p_\perp^\mu \\ &= n_1 \cdot p \, \frac{\bar{n}_1^\mu}{2} + \bar{n}_1 \cdot p \, \frac{n_1^\mu}{2} + p_\perp^\mu \\ &= \bar{n}_2 \cdot p \, \frac{n_2^\mu}{2} + n_2 \cdot p \, \frac{\bar{n}_2^\mu}{2} + p_\perp^\mu \end{aligned} \qquad \boxed{p_\perp \cdot n_i = 0}$$

For brevity, we use $n = n_1$, $\bar{n} = n_2$ for this 2-jet example.

$$n \cdot p \, \frac{\bar{n}}{2} + \bar{n} \cdot p \, \frac{n}{2} + p_\perp$$

$$\rightsquigarrow \begin{cases} n \cdot p_{J_1} = E_1 - \sqrt{E_1^2 - m_{J_1}^2} \simeq \dfrac{m_{J_1}^2}{2E_1} \simeq \dfrac{m_{J_1}^2}{\sqrt{s}} \sim \lambda^2 \sqrt{s} \\ \bar{n} \cdot p_{J_1} = \cdots \simeq 2E_1 \sim \lambda^0 \sqrt{s} \\ p_{J_1}^\perp = 0 . \end{cases}$$

Similarly, $\quad n \cdot p_{J_2} \sim \lambda^0 \sqrt{s}$, $\quad \bar{n} \cdot p_{J_2} \sim \lambda^2 \sqrt{s}$, $\quad p_{J_2}^\perp = 0$.

Individual partons inside the jets can carry momenta with same scaling rules, but they can have transverse components as long as

$$p_\perp^2 < m_{J_i}^2 .$$

$$\boxed{p_i^2 = n \cdot p_i \, \bar{n} \cdot p_i + p_{i,\perp}^2 \sim \lambda^2 s}$$

Thus, partons inside jet 1 :

$$(n \cdot p_i, \bar{n} \cdot p_i, p_i^\perp) \sim (\lambda^2, \lambda^0, \lambda)\sqrt{s} \longrightarrow \text{collinear}$$

partons inside jet 2 :

$$(n \cdot p_i, \bar{n} \cdot p_i, p_i^\perp) \sim (\lambda^0, \lambda^2, \lambda)\sqrt{s} \longrightarrow \text{anti-collinear}$$

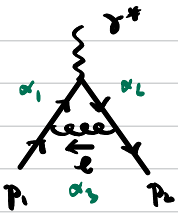
These active particles have much less virtuality than hard modes:

$$p_n \sim (\lambda^1, \lambda^0, \lambda^1)\sqrt{s}\,. \qquad \text{is } (\lambda^0, \lambda^0, 0)\sqrt{s} \text{ hard?}$$

Answer to Q: integrate out those hard fluctuations in virtual exchange! and the EFT d.o.f's would be collinear & anti-collinear modes.

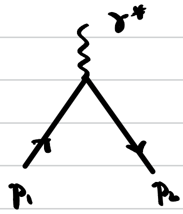
But this is NOT the whole story!

To see this point, we consider the classical example relevant to the 2-jet process at one loop: (off-shell) massless Sudakov form factor.



$$P_1^2 = -p_1^2 \ll (p_1 - p_2)^2 = Q^2$$
$$P_2^2 = -p_2^2 \ll Q^2$$

$+ \; Z_q \cdot$



$$Z_q = 1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon} + O(\alpha_s^2)$$

It suffices to consider the scalar case for illustration:

$$\to F = e^{\varepsilon\gamma_E} (\mu^2)^\varepsilon \int \frac{d^D \ell}{i\pi^{D/2}} \frac{-2p_1\cdot p_2}{[\ell^2 + i0]\cdot[(\ell+p_1)^2 + i0]\cdot[(\ell+p_2)^2 + i0]}$$

$$Q^2 = -q^2 - i0 < 0$$
$$P_i^2 = -p_i^2 - i0 > 0$$

due to off-shellness, the integral is IR finite. It is obviously UV finite!

$$\leadsto \boxed{F = \ln\frac{Q^2}{P_1^2}\ln\frac{Q^2}{P_2^2} + \frac{\pi^2}{3}} + O\left(\frac{P_i^2}{Q^2}, \varepsilon\right)$$

$$\nu - \frac{\ell+1}{2}D = -1+2\varepsilon$$
$$\nu - \frac{\ell}{2}D = 1+\varepsilon$$

$$F = (-1)\, e^{\varepsilon\gamma_E}\,\Gamma(1+\varepsilon) \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int_0^\infty d\alpha_3\, \delta(1-h(\alpha)) \frac{1}{(\mathcal{U}(\alpha))^{1-2\varepsilon}(\mathcal{F}(\alpha))^{1+\varepsilon}} \times \frac{2p_1\cdot p_2}{\mu^2}$$

with $\mathcal{U}(\alpha) = \alpha_1 + \alpha_2 + \alpha_3$, $\mathcal{F}(\alpha) = \alpha_1\alpha_2 \frac{-q^2}{\mu^2} + \alpha_2\alpha_3\frac{-p_2^2}{\mu^2} + \alpha_3\alpha_1\frac{-p_1^2}{\mu^2}$

exercise o

choose $\boxed{h(\alpha) = \alpha_3}$

$$\leadsto F\big|_{\varepsilon=0} = e^{\varepsilon\gamma_E}\,\Gamma(1+\varepsilon)\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2 \frac{1}{(1+\alpha_1+\alpha_2)(\alpha_1\alpha_2 + \alpha_1\chi_1 + \alpha_2\chi_2)}$$

$$\begin{cases} \chi_1 = \dfrac{P_1^2}{Q^2} \sim \lambda^2 \\ \chi_2 = \dfrac{P_2^2}{Q^2} \sim \lambda^2 \end{cases}$$

This is called Sudakov double logarithms, which are typical

n-loop $\qquad L^{2n} + L^{2n-1} + \cdots$

In CMS frame:

	$(n\cdot q,\ \bar{n}\cdot q,\ q_\perp)$	$= (1, 1, 0)\,Q$	hard
	$(n\cdot p_1,\ \bar{n}\cdot p_1,\ p_{1\perp})$	$\sim (\lambda^2, 1, 0)\,Q$	collinear
$P_1^2 \sim P_2^2 \sim \lambda^2 Q^2$	$(n\cdot p_2,\ \bar{n}\cdot p_2,\ p_{2\perp})$	$\sim (1, \lambda^2, 0)\,Q$	anti-collinear

1. Region Analysis

reproduce F by method of region : decompose the integral into a sum of simpler integrals depending on one single scale !

Method of Region : DR is necessary !

$$\int d^4\ell \longrightarrow \int d^D\ell \qquad D = D_{int} - 2\epsilon$$

① linearity

$$\int \frac{d^D\ell}{i\pi^{D/2}} \left(a f(\ell) + b g(\ell)\right) = a\int \frac{d^D\ell}{i\pi^{D/2}} f(\ell) + b\int \frac{d^D\ell}{i\pi^{D/2}} g(\ell)$$

② translation invariance

$$\int \frac{d^D\ell}{i\pi^{D/2}} f(\ell+p) = \int \frac{d^D\ell}{i\pi^{D/2}} f(\ell)$$

③ scaling behavior.

$$\int \frac{d^D\ell}{i\pi^{D/2}} f(\lambda\ell) = \lambda^{-D}\int \frac{d^D\ell}{i\pi^{D/2}} f(\ell).$$

④ normalization

$$\int \frac{d^D\ell}{i\pi^{D/2}} \exp(-\ell^2) = 1 .$$

Very important consequence : (often neglect)

$$\int \frac{d^D\ell}{i\pi^{D/2}} (-\ell^2)^a = \begin{cases} (-1)^{\frac{D}{2}}\,\Gamma(1-\frac{D}{2}) , & \text{if } \frac{D}{2}+a = 0 , \\ 0 , & \text{otherwise} . \end{cases}$$

scaleless integral

exercise 1 : prove the above from axioms of DR.

MoR : ① determine large and small scales in a FI ; $\hookrightarrow \lambda = \frac{\text{small}}{\text{large}} \ll 1$

② identify regions of loop momenta.

i) normally by experience :

ultrasoft region : $\ell \sim (\lambda^2, \lambda^2, \lambda^2)$, $\ell^2 \sim \lambda^4$

soft : $\ell \sim (\lambda, \lambda, \lambda)$, $\ell^2 \sim \lambda^2$

collinear : $\ell \sim (\lambda^2, 1, \lambda)$, $\ell^2 \sim \lambda^2$

anti-collinear : $\ell \sim (1, \lambda^2, \lambda)$, $\ell^2 \sim \lambda^2$

hard-collinear : $\ell \sim (\lambda, 1, \lambda)$, $\ell^2 \sim \lambda$

anti-hard-collinear : $\ell \sim (1, \lambda, \lambda)$, $\ell^2 \sim \lambda$
hard : $\ell \sim (1, 1, 1)$, $\ell^2 \sim \lambda^0$

ii) recently, identify geometrically ...

③ in each region, Taylor expand propagators.

④ perform the expanded integral in each region in the FULL space of loop momenta!

This is ensured by DR : overlap parts are scaless!

On the Sudakov Example

$p_1 \parallel n$
$p_2 \parallel \bar{n}$

① hard region : $\ell \sim (\lambda^0, \lambda^0, \lambda^0)\, Q$

$$(\ell + p_1)^2 = \ell^2 + 2p_1\cdot\ell + p_1^2 = \ell^2 + 2\ell\cdot\Big(\overset{\lambda^0}{n\cdot p_1 \frac{\bar{n}}{2}} + \overset{\lambda^2}{\bar{n}\cdot p_1 \frac{n}{2}} + \overset{\lambda}{p_\perp}\Big) + \overset{\lambda^2}{p_1^2}$$

$$= \ell^2 + \bar{n}\cdot p_1\, n\cdot\ell + O(\lambda)$$

$$(\ell + p_2)^2 = \ell^2 + n\cdot p_2\, \bar{n}\cdot\ell + O(\lambda) \qquad -2p_1\cdot p_2 = Q^2 + O(\lambda^2)$$

leading power (LP) ≥ next-to-leading power (NLP)

$$\rightsquigarrow F_h = e^{\epsilon\gamma_E} \mu^{2\epsilon} Q^2 \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{(\ell^2 + i0)(\ell^2 + \bar{n}\cdot p_1\, n\cdot\ell + i0)(\ell^2 + n\cdot p_2\, \bar{n}\cdot\ell + i0)} + O(\lambda)$$

$$= e^{\epsilon\gamma_E}\Gamma(1+\epsilon)\Big[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{\pi^2}{6} + O(\epsilon)\Big] + O(\lambda)$$

$\epsilon = \epsilon_{IR}$

② collinear region : $\ell \sim (\lambda^2, \lambda^0, \lambda)\, Q \sim p_1$

$$(\ell + p_1)^2 = (\ell + p_1)^2 = \ell^2 + 2p_1\cdot\ell + p_1^2 = \overset{\lambda^2}{\ell^2} + \overset{\lambda^2}{n\cdot p_1}\, \overset{\lambda^0}{\bar{n}\cdot\ell} + \overset{\lambda^0}{\bar{n}\cdot p_1}\, \overset{\lambda^2}{n\cdot\ell} + \overset{\lambda}{\ell_\perp}\cdot\overset{\lambda}{p_\perp} + \overset{\lambda^2}{p_1^2}$$

⤳ nothing to expand!

$$(\ell + p_2)^2 = \overset{\lambda^2}{\ell^2} + \overset{\lambda^0}{n\cdot p_2}\, \overset{\lambda^0}{\bar{n}\cdot\ell} + \overset{\lambda^2}{\bar{n}\cdot p_2}\, \overset{\lambda^2}{n\cdot\ell} + \overset{\lambda}{\ell_\perp}\cdot\overset{\lambda}{p_{2\perp}} + \overset{\lambda^2}{p_2^2}$$

$$= n\cdot p_2\, \bar{n}\cdot\ell + O(\lambda^2)$$

$$\rightsquigarrow F_c = e^{\epsilon\gamma_E} \mu^{2\epsilon} Q^2 \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{(\ell^2 + i0)((\ell + p_1)^2 + i0)(n\cdot p_2\, \bar{n}\cdot\ell + i0)} + O(\lambda^2)$$

$$= e^{\epsilon\gamma_E}\Gamma(1+\epsilon)\Big[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{p_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{p_1^2} - \frac{\pi^2}{6} + O(\epsilon)\Big] + O(\lambda^2)$$

$\epsilon = \epsilon_{IR}$

③ anti-collinear region : $\ell \sim (\lambda^0, \lambda^2, \lambda)\, Q \sim p_2$

similar $F_{\bar c} = e^{\epsilon\gamma_E}\Gamma(1+\epsilon)\left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{\mu^2}{p_2^2} - \frac{1}{2}\ln^2\frac{\mu^2}{p_2^2} - \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right] + \mathcal{O}(\lambda^2)$

Clearly $F_h + F_c + F_{\bar c} \neq F$, but rather

$$F_h + F_c + F_{\bar c} = e^{\epsilon\gamma_E}\Gamma(1+\epsilon)\Big[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{p_1^2 p_2^2}{Q^2\mu^2} + \frac{1}{2}\ln^2\frac{\mu^2}{Q^2} - \frac{1}{2}\ln^2\frac{\mu^2}{p_1^2} - \frac{1}{2}\ln^2\frac{\mu^2}{p_2^2}$$
$$- \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\Big] + \mathcal{O}(\lambda^2)$$

$$p_1^2 p_2^2/Q^2 \sim \lambda^4 Q^2$$

It turns out there is an extra region : $\ell \sim (\lambda^2, \lambda^2, \lambda^2)\, Q$ (ultrasoft) us

Note that

$$c + us \sim c$$
$$\bar c + us \sim \bar c$$

Hence exchanging ultrasoft particles between collinear and anti-collinear sectors does NOT violate momentum conservation!

④ ultrasoft region : $\ell \sim (\lambda^2, \lambda^2, \lambda^2)\, Q$

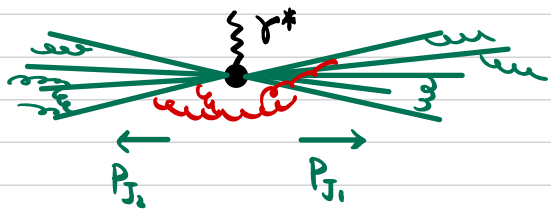
$$(\ell + p_1)^2 = \overset{\lambda^4}{\ell^2} + \overset{\lambda^2\ \lambda^2}{n\cdot p_1\, \bar n\cdot\ell} + \overset{\lambda^0\ \lambda^2}{\bar n\cdot p_1\, n\cdot\ell} + \overset{\lambda^2\ \lambda}{\ell_\perp\cdot p_1^\perp} + \overset{\lambda^2}{p_1^2}$$
$$= \bar n\cdot p_1\, n\cdot\ell + p_1^2 + \mathcal{O}(\lambda^3)$$

$$(\ell + p_2)^2 = n\cdot p_2\, \bar n\cdot\ell + p_2^2 + \mathcal{O}(\lambda^3)$$

$$\rightsquigarrow F_{us} = e^{\epsilon\gamma_E}\mu^{2\epsilon}Q^2\int\frac{d^d\ell}{i\pi^{d/2}}\frac{1}{(\ell^2+i0)(\bar n\cdot p_1\, \bar n\cdot\ell + p_1^2 + i0)(n\cdot p_2\, \bar n\cdot\ell + p_2^2 + i0)}$$
$$= e^{\epsilon\gamma_E}\Gamma(1+\epsilon)\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{\mu^2 Q^2}{p_1^2 p_2^2} + \frac{1}{2}\ln^2\frac{\mu^2 Q^2}{p_1^2 p_2^2} + \frac{\pi^2}{6} + \mathcal{O}(\epsilon)\right] + \mathcal{O}(\lambda^2)$$

$\epsilon = \epsilon_{UV}$

$\longrightarrow F_h + F_c + F_{\bar c} + F_{us} = F$!



Ultrasoft exchange is needed for color neutralization.

We will reproduce the above analysis in terms of SCET!

2. Construction of $\mathcal{L}_{SCET}$

QCD field $\to$ $\phi = \phi_{us} + \phi_s + \phi_c + \phi_{\bar{c}} + \cdots$ (if exist)

based on the scaling behavior of Fourier (momentum) modes!

For brevity, consider d.o.f's = us, c, $\bar{c}$, i.e., $\phi = \phi_{us} + \phi_c + \phi_{\bar{c}}$.

Q: where is ϕ_n ?

momentum conservation allows these interactions:

$$\underbrace{\phi_c \cdots \phi_c}_{n_c \geq 2} \underbrace{\phi_{us} \cdots \phi_{us}}_{n_{us} \gtrsim 1} \qquad \underbrace{\phi_{\bar{c}} \cdots \phi_{\bar{c}}}_{n_{\bar{c}} \geq 2} \underbrace{\phi_{us} \cdots \phi_{us}}_{n_{us} \gtrsim 1}$$

but not: $\phi_c \underbrace{\phi_{us} \cdots \phi_{us}}_{n_{us} \gtrsim 1}$ $\quad$ $\phi_{\bar{c}} \underbrace{\phi_{us} \cdots \phi_{us}}_{n_{us} \gtrsim 1}$

or: $\underbrace{\phi_c \cdots \phi_c}_{n_c \geq 2} \underbrace{\phi_{\bar{c}} \cdots \phi_{\bar{c}}}_{n_{\bar{c}} \geq 2} \underbrace{\phi_{us} \cdots \phi_{us}}_{n_{us} \gtrsim 1}$

$$\hookrightarrow \quad \mathcal{L}_{SCET}^{(us,c,\bar{c})} = \mathcal{L}_c + \mathcal{L}_{\bar{c}} + \mathcal{L}_{us} + \mathcal{L}_{c+us} + \mathcal{L}_{\bar{c}+us}.$$

To construct $\mathcal{L}_{SCET}$, we first need to understand collinear fields ψ_c & A_c. Since $p_c \sim (\lambda^2, \lambda^0, \lambda)$ is not homogeneous, the corresponding fields behave in a complicated way.

Collinear Quark Field

The spinor of a highly-energetic quark:

$$\not{p}\, u(p) = m\, u(p) \quad ; \quad m \ll E = p^0 \sim p^3$$

$$\hookrightarrow \not{n}\, u(p) \sim 0 + O(m/E)$$

A Dirac spinor contains more information than what we need!

Introduce 2 projectors: $\quad \hat{P}_n = \dfrac{\not{n}\not{\bar{n}}}{4}, \quad \hat{P}_{\bar{n}} = \dfrac{\not{\bar{n}}\not{n}}{4}.$

$$(\hat{P}_n^2 = \hat{P}_n, \quad \hat{P}_{\bar{n}}^2 = \hat{P}_{\bar{n}}, \quad \hat{P}_n + \hat{P}_{\bar{n}} = 1)$$

$$\boxed{\xi_n := \hat{P}_n \psi_c \quad , \quad \eta_n := \hat{P}_{\bar{n}} \psi_c}$$

Clearly $\psi_c = \xi_n + \eta_n$, and $\not{n}\xi_n = 0$, $\not{n}\eta_n \neq 0$. Hence ξ_n is the right d.o.f field we need for a collinear quark:

We can assign power counting for fields as well.

$$\langle 0|\hat{T}\{\psi_c(x)\bar{\psi}_c(0)\}|0\rangle = \int \frac{d^4p_c}{(2\pi)^4}\, e^{-ip_c\cdot x}\, \frac{i\not{p}_c}{p_c^2+i0} \quad \to n\cdot p_c \frac{\not{\bar{n}}}{2} + \bar{n}\cdot p_c \frac{\not{n}}{2} + \not{p}_c^{\perp}$$

$$\lambda^4 \qquad \lambda^0 \qquad \lambda^{-2}\times(\lambda^2, \lambda^0, \lambda)$$

ψ_c is not homogeneous in λ, as p_c !

Hence $\langle 0|\hat{T}\{\xi_n(x)\bar{\xi}_n(0)\}|0\rangle = \langle 0|\hat{T}\{\frac{\not{n}\not{\bar{n}}}{4}\psi_c(x)\bar{\psi}_c(0)\frac{\not{\bar{n}}\not{n}}{4}\}|0\rangle$

$$= \int \underbrace{\frac{d^4p_c}{(2\pi)^4}\, e^{-ip_c\cdot x}\, \frac{i\bar{n}\cdot p_c}{p_c^2+i0}}_{\lambda^2} \frac{\not{n}}{2}$$

$$\lambda^4 \qquad \lambda^0 \qquad \lambda^{-2}\cdot\lambda^0$$

$\hookrightarrow$ $\boxed{\xi_n \sim \lambda}$

Likewise $\quad \eta_n \sim \lambda^2 \quad$ exercise 2

$\uparrow$ power suppressed small component in ψ_c !

One can use ξ_n to express η_n via EOM (or by doing the functional integration in action)

Collinear Gauge Field

The easiest (maybe most physical) way to see the counting rules of A_c^μ is to note that the Lorentz index matches with $-i\partial^\mu = \hat{p}^\mu$. Hence gauge invariance dictates that $A_c^\mu \sim p_c^\mu$, such that

$$iD_c^\mu = i\partial^\mu + g_s A_c^{\mu,a}T^a \sim (\lambda^2, \lambda^0, \lambda)$$

is well-defined!

Ultrasoft Quark & Gauge Fields

$$\langle 0|\hat{T}\{q_{us}(x)\bar{q}_{us}(0)\}|0\rangle = \int \frac{d^4p_{us}}{(2\pi)^4}\, e^{-ip_{us}\cdot x}\, \frac{i\not{p}_{us}}{p_{us}^2+i0}$$

$$\lambda^8 \qquad \lambda^0 \qquad \lambda^2\cdot\lambda^{-4}$$

$\hookrightarrow$ $\boxed{q_{us} \sim \lambda^3}$

$\boxed{A_{us} \sim \lambda^2}$

Note that
1. q_{us} & A_{us} are homogeneous in λ ;
2. q_{us} is power suppressed than ξ_n ;
3. A_{us} looks suppressed than A_c, but $n\cdot A_{us} \sim n\cdot A_c$

$\mathcal{L}_{SCET}$ from $\mathcal{L}_{QCD}$ (focus on $\mathcal{L}^{(c)}_{SCET} + \mathcal{L}^{(c+us)}_{SCET}$)

$$\psi \longrightarrow \psi_c + q_{us} = \xi_n + \eta_n + q_{us}$$
$$A^\mu \longrightarrow A^\mu_c + A^\mu_{us}$$

$$\hookrightarrow \quad \bar\psi i\not{D}\psi \longrightarrow (\overset{\lambda}{\bar\xi_n} + \overset{\lambda^2}{\bar\eta_n} + \overset{\lambda^3}{\bar q_{us}})\, i\not{D}_{c+us}\, (\overset{\lambda}{\xi_n} + \overset{\lambda^2}{\eta_n} + \overset{\lambda^3}{q_{us}})$$

with $$i\not{D}_{c+us} = (\overset{\lambda^2}{in\cdot D_c} + \overset{\lambda^2}{g_s n\cdot A_{us}})\frac{\bar{\not{n}}}{2} + (\overset{1}{i\bar n\cdot D_c} + \overset{\lambda^2}{g_s \bar n\cdot A_{us}})\frac{\not{n}}{2} + \overset{\lambda}{i\not{D}_c^\perp} + \overset{\lambda^2}{g_s \not{A}^\perp_{us}}$$

$$\rightsquigarrow \quad \lambda^4 : \bar\xi_n \frac{\bar{\not{n}}}{2}(in\cdot D_c + g_s n\cdot A_{us})\xi_n + \underbrace{\bar\eta_n \frac{\not{n}}{2} i\bar n\cdot D_c \eta_n}_{\text{Gaussian}} + \underbrace{\bar\xi_n i\not{D}_c^\perp \eta_n + \bar\eta_n i\not{D}_c^\perp \xi_n}_{\text{mixing}}$$

$$\lambda^5 : \boxed{\bar q_{us} g_s \not{A}_c^\perp \xi_n} + \cdots$$

$$\vdots$$

$$\lambda^8 : \bar q_{us} i\not{D}_{us} q_{us}$$

There are also pure gauge terms, which are neglected here.

Rmk: One can not conclude that $\bar q_{us} i\not{D}_{us} q_{us}$ is highly power suppressed and just neglect it. Because we need to consider the $\int d^4x$ in the action!

$$\underset{\lambda^{-8}}{\int d^4x}\ \underset{\lambda^{8}}{\bar q_{us} i\not{D}_{us} q_{us}} \sim O(\lambda^0)$$

$$\tilde S^{(0)}_{SCET} = \overset{\lambda^{-4}}{\int d^4x} \Big[\overbrace{\bar\xi_n \frac{\bar{\not{n}}}{2} in\cdot D_c \xi_n + \bar\eta_n \frac{\not{n}}{2} \bar n\cdot iD_c \eta_n + \bar\xi_n i\not{D}_c^\perp \eta_n + \bar\eta_n i\not{D}_c^\perp \xi_n + \text{pure gluon}}^{\tilde{\mathcal{L}}^{(0)}_c \,:\, \lambda^4} \sim \lambda^0$$
$$+ \underbrace{\bar\xi_n \frac{\bar{\not{n}}}{2} g_s n\cdot A_{us} \xi_n + \text{pure gluon}}_{\mathcal{L}^{(0)}_{c+us} \,:\, \lambda^4} \sim \lambda^0 \Big]$$

LP interaction

$$+ \overset{\lambda^{-8}}{\int d^4x}\ \underbrace{\bar q_{us} i\not{D}_{us} q_{us} + \text{pure gluon}}_{\mathcal{L}^{(0)}_{us} \,:\, \lambda^8} \sim \lambda^0$$

$$\tilde S^{(1)}_{SCET} = \Big[\overset{\lambda^{-4}}{\int d^4x}\ \underbrace{\bar q_{us} g_s \not{A}_c^\perp \xi_n}_{\mathcal{L}_{\bar q \xi_n} \,:\, \lambda^5} \sim O(\lambda) + \cdots \Big]$$

NLP interaction

As mentioned before, ξ_n is the large component in ψ_c, instead of η_n. Besides, η_n is Gaussian. Hence one can perform the functional integral over η_n exactly:

$$\int D\xi_n D\eta_n \, e^{-i\tilde{S}_{SCET}[\xi_n, \eta_n \cdots]} = \int D\xi_n \, e^{-iS_{SCET}[\xi_n, \cdots]} .$$

Since it is Gaussian, it is equivalent to use the EoM of η_n given by

$$0 = \partial_\mu \cancel{\frac{\delta \mathcal{L}}{\delta(\partial_\mu \bar{\eta}_n)}} - \frac{\delta \mathcal{L}}{\delta \bar{\eta}_n} = \frac{\delta \mathcal{L}}{\delta \bar{\eta}_n} = \frac{\not{n}}{2} i\bar{n}\cdot D_c \eta_n + i\not{D}_c^\perp \xi_n$$

$\boxed{\text{exercise 3}}$

$$\hookrightarrow \quad \eta_n = -\frac{1}{i\bar{n}\cdot D_c + i0} \frac{\not{\bar{n}}}{2} i\not{D}_c^\perp \xi_n$$

exact, no approximation!

$$\tilde{\mathcal{L}}_c^{(0)} \longrightarrow \mathcal{L}_c^{(0)} = \bar{\xi}_n \frac{\not{\bar{n}}}{2} in\cdot D_c \xi_n - \bar{\xi}_n i\not{D}_c^\perp \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c + i0} i\not{D}_c^\perp \xi_n + \text{pure gluon}$$

To define $(i\bar{n}\cdot D_c + i0)^{-1}$ properly, we introduce the collinear Wilson line:

$$\boxed{W_c(x) = \hat{P}\exp\left(ig_s\int_{-\infty}^0 dt\, \underbrace{\bar{n}\cdot A_c}_{\bar{n}\cdot A_c \,\sim\, \lambda^0}(x+t\bar{n})\right)}$$

Which satisfies

$$\left(i\bar{n}\cdot D_c W_c(x)\right) = 0$$

Then at the diff. operator level,

$$W_c^\dagger \, i\bar{n}\cdot D_c W_c(x) = i\bar{n}\cdot\partial$$

$$\hookrightarrow \quad \frac{1}{i\bar{n}\cdot D_c + i0} = W_c \frac{1}{i\bar{n}\cdot\partial + i0} W_c^\dagger$$

$$\hookrightarrow \quad \bar{\xi}_n i\not{D}_c^\perp \frac{\not{\bar{n}}}{2} \frac{1}{i\bar{n}\cdot D_c + i0} i\not{D}_c^\perp \xi_n(x)$$

$$= \bar{\xi}_n i\not{D}_c^\perp \frac{\not{\bar{n}}}{2} W_c \frac{1}{i\bar{n}\cdot\partial + i0} W_c^\dagger i\not{D}_c^\perp \xi_n$$

$$= \left(\bar{\xi}_n i\not{D}_c^\perp W_c\right)(x) \frac{\not{\bar{n}}}{2} (-i)\int_{-\infty}^0 dt \left(W_c^\dagger i\not{D}_c^\perp \xi_n\right)(x+t\bar{n})$$

$$i\bar{n}\cdot\partial \, (-i)\int_{-\infty}^0 dt\, \phi(x+t\bar{n}) = \bar{n}\cdot\partial \int_{-\infty}^{\frac{n\cdot x}{2}} dt'\, \phi\left(\frac{\bar{n}x}{2}n + t'\bar{n} + x_\perp\right)$$

$$= \phi(x).$$

3. Multipole Expansion

If ultrasoft and collinear fields appear together, e.g., $\mathcal{L}_{c+us}$, one needs to multipole expand to be consistent with power counting.

$$\int d^4x\ \phi_{us}(x)\phi_c^2(x) = \int d^4x \int \frac{d^4p_{c,1}}{(2\pi)^4}\int\frac{d^4p_{c,2}}{(2\pi)^4}\int\frac{d^4p_{us}}{(2\pi)^4}\ \phi_c(p_{c,1})\phi_c(p_{c,2})\phi_{us}(p_{us})e^{-ix\cdot(\underbrace{p_{c,1}+p_{c,2}}_{p_c}+p_{us})}$$

$$p_{c,1}+p_{c,2} = p_c \sim (\lambda^2, 1, \lambda)$$

$$p_{us} \sim (\lambda^2, \lambda^2, \lambda^2)$$

$$\hookrightarrow \quad p_c+p_{us} \sim (\lambda^2, 1, \lambda) \rightarrow x \sim (1, \lambda^{-2}, \lambda^{-1})$$

i.e.
$$x^\mu = \underbrace{n\cdot x\frac{\bar{n}^\mu}{2}}_{x_+^\mu} + \underbrace{\bar{n}\cdot x\frac{n^\mu}{2}}_{x_-^\mu} + x_\perp^\mu$$

$$\qquad\qquad 1 \qquad\quad \lambda^{-2} \qquad \lambda^{-1}$$

$$p_c\cdot x = \underbrace{\underset{\lambda^2}{n\cdot p_c}\ \underset{\lambda^{-2}}{\bar{n}\cdot x}}_{\lambda^0} + \underbrace{\underset{\lambda^0}{\bar{n}\cdot p_c}\ \underset{\lambda^0}{n\cdot x}}_{\lambda^0} + \underbrace{\underset{\lambda}{p_c^\perp}\cdot \underset{\lambda^{-1}}{x^\perp}}_{\lambda^0}$$

But
$$p_s\cdot x = \underbrace{\underset{\lambda^2}{n\cdot p_s}\ \underset{\lambda^{-2}}{\bar{n}\cdot x}}_{\lambda^0} + \underbrace{\underset{\lambda^2}{\bar{n}\cdot p_s}\ \underset{\lambda^0}{n\cdot x}}_{\lambda^2} + \underbrace{\underset{\lambda^2}{p_s^\perp}\cdot \underset{\lambda^{-1}}{x^\perp}}_{\lambda^1}$$

$$= n\cdot p_s\,\bar{n}\cdot x + O(\lambda)$$

$$= \boxed{p_s\cdot x_-}$$

Hence $\boxed{\phi_{us}(x) \longrightarrow \phi_{us}(x_-)}$

4. Gauge Invariance

Gauge transformations involve gluon fields. For (anti-) collinear and ultrasoft fields, QCD gauge transformations should also split into (anti-) collinear and ultrasoft ones to not spoil power counting!

① collinear trans.

$$\xi_n \xrightarrow{U_c} U_c(x)\,\xi_n(x)$$

$$\bar{n}\cdot A_c \xrightarrow{U_c} U_c(x)\,\bar{n}\cdot A_c\,U_c^\dagger(x) + \frac{i}{g_s}U_c(x)\left(\bar{n}\cdot\partial\, U_c^\dagger(x)\right)$$

$$A_c^\perp \xrightarrow{U_c} U_c(x)\,A_c^\perp\,U_c^\dagger(x) + \frac{i}{g_s}U_c(x)\left(\partial_\perp U_c^\dagger(x)\right)$$

$$n\cdot A_c(x) + n\cdot A_{us}(x_-) \xrightarrow{U_c} U_c(x)(\cdots)U_c^\dagger(x) + \frac{i}{g_s}U_c(x)\left(n\cdot\partial\, U_c^\dagger(x)\right).$$

q_{us} & A_{us} do not transform!

$$W_c(x) \xrightarrow{U_c} U_c(x)W_c(x)\underbrace{U_c^\dagger(-\infty\bar{n})}_{1} = U_c(x)W_c(x).$$

4. Ultrasoft trans.

$$\xi_n \xrightarrow{U_{us}} U_{us}(x_-)\xi_n(x)$$

$$A_c \xrightarrow{U_{us}} U_{us}(x_-)A_c(x)U_{us}^\dagger(x_-)$$

$$q_{us} \xrightarrow{U_{us}} U_{us}(x)\, q_{us}(x)$$

$$A_{us} \xrightarrow{U_{us}} U_{us}(x)A_{us}(x)U_{us}^\dagger + \frac{i}{g_s}U_{us}(x)\left(\partial\, U_{us}^\dagger(x)\right)$$

$$W_c \xrightarrow{U_{us}} U_{us}(x_-)\, W_c(x)U_{us}^\dagger(x_-)$$

5. RPI

typy I	II	III
$n^\mu \to n^\mu + \Delta_\perp^\mu$	$n^\mu \to n^\mu$	$n^\mu \to \lambda n^\mu$
$\bar{n}^\mu \to \bar{n}^\mu$	$\bar{n}^\mu \to \bar{n}^\mu + \bar{\Delta}_\perp^\mu$	$\bar{n}^\mu \to \lambda^{-1}\bar{n}^\mu$
$\Delta_\perp \sim \lambda$	$\bar{\Delta}_\perp \sim \lambda$	

6. Soft Decoupling

$$\bar{\xi}_n(x)\frac{\bar{n}\!\!\!/}{2}\left(in\cdot\partial + g_s n\cdot A_c(x) + g_s n\cdot A_{us}(x_-)\right)\xi_n(x)$$

field redef.

$$\xi_n(x) = S_n(x_-)\xi_n^{(0)} \quad \& \quad A_c^\mu(x) = S_n(x_-)A_c^{\mu,(0)}(x)S_n^\dagger(x_-)$$

$$S_n(x) = \hat{P}\exp\left(ig_s\int_{-\infty}^{0}dt\, n\cdot A_{us}(x+nt)\right)$$

$$\left(in\cdot D_{us}S_n(x)\right) = 0$$

$$= \bar{\xi}_n^{(0)}(x)\frac{\bar{n}\!\!\!/}{2}S_n^\dagger(x_-)S_n(x_-)\left(in\cdot\partial + g_s n\cdot A_c^{(0)}(x)\right)\xi_n^{(0)}(x)$$

$$= \bar{\xi}_n^{(0)}(x)\frac{\bar{n}\!\!\!/}{2}\left(in\cdot\partial + g_s n\cdot A_c^{(0)}(x)\right)\xi_n^{(0)}(x)$$

$$= \bar{\xi}_n^{(0)}(x)\frac{\bar{n}\!\!\!/}{2}\, in\cdot D_c^{(0)}(x)\,\xi_n^{(0)}(x)$$

Decoupling does not mean soft gluon emission disappear. But rather, soft gluons are entirely described by Soft Wilson lines!

✗ Factorization and Renormalization of 2-Jet Current at LP

$$F_H \longrightarrow C_V(Q^2)$$
$$F_c \longrightarrow \langle p_1 | \chi_c^{(0)} | 0 \rangle = \langle p_1 | W_c^\dagger \xi_n^{(0)}(0) | 0 \rangle$$
$$F_{\bar c} \longrightarrow \langle p_2 | \bar\chi_{\bar c}^{(0)} | 0 \rangle = \langle p_2 | \bar\xi_{\bar n}^{(0)} W_{\bar c} | 0 \rangle$$
$$F_s \longrightarrow \langle 0 | \hat T \{ S_{\bar n}^\dagger(0)\, S_n(0) \} | 0 \rangle$$

$$J^\mu(0) = \bar\psi(0)\gamma^\mu\psi(0) = \bar\psi_{\bar c}(0)\gamma_\perp^\mu \psi_c(0)$$ match onto SCET operator

gauge $\downarrow$ invariance

$$\underbrace{\bar\xi_{\bar n}(0) W_{\bar c}(0)}_{\bar\chi_{\bar n}(0)} \gamma_\perp^\mu W_c^\dagger(0)\xi_n(0) \qquad \bar\chi_{\bar n}(0)\,\gamma_\perp^\mu\,\chi_n(0)$$

WLs here also have physical interpretation by tree level matching ...

Note $p_c \sim (\lambda^2, \lambda^0, \lambda)$, $p_{\bar c} \sim (\lambda^0, \lambda^2, \lambda)$

Hence $i\bar n\cdot D_c$ & $in\cdot D_{\bar c}$ are not λ suppressed at all! In principle these gauge-covariant parts should also appear in the matching procedure!

$$\rightsquigarrow \bar\xi_{\bar n}(0) W_{\bar c}(0)\gamma_\perp^\mu W_c^\dagger(0)\xi_n(0) \xrightarrow{\text{lift}} \left[\bar\xi_{\bar n}(0)(in\cdot D_{\bar c})^i\, W_{\bar c}(0)\right]\gamma_\perp^\mu\left[W_c^\dagger(0)(i\bar n\cdot D_c)^j\,\xi_n(0)\right]$$

$$= \left[(\bar\xi_{\bar n} W_{\bar c})(0)\,(-in\cdot\overleftarrow{\partial})^i\right]\gamma_\perp^\mu\left[(i\bar n\cdot\partial)^j\,(W_c^\dagger\xi_n)(0)\right] \qquad \boxed{\forall\ i, j}$$

translation on lightcones!

$$\rightsquigarrow \bar\psi_{\bar c}(0)\gamma_\perp^\mu\psi_c(0) \xrightarrow{\text{match}} \int ds\,dt\ \tilde C_V(s,t)\left[(\bar\xi_{\bar n} W_{\bar c})(tn)\right]\gamma_\perp^\mu\left[(W_c^\dagger\xi_n)(s\bar n)\right]$$

Acting on $\langle p_c | \cdots | p_{\bar c}\rangle$ and using translation invariance

$$= \underbrace{\int ds\,dt\ \tilde C_V(s,t)\, e^{itn\cdot\hat P_{\bar c}}\, e^{-is\bar n\cdot\hat P_c}}_{C_V(n\cdot\hat P_{\bar c}\ \bar n\cdot\hat P_c)} \left[(\bar\xi_{\bar n} W_{\bar c})(0)\right]\gamma_\perp^\mu\left[(W_c^\dagger\xi_n)(0)\right]$$

$$= C_V(Q^2)$$

soft-decoupling

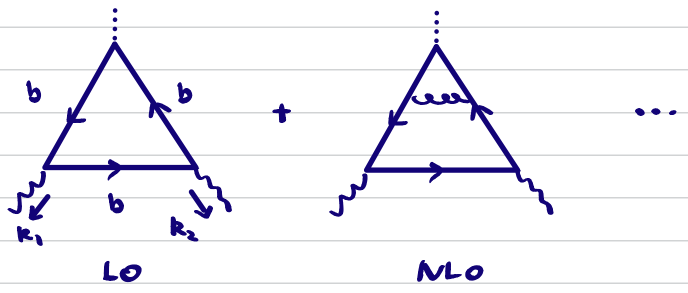
$$\bar\psi_{\bar c}(0)\gamma_\perp^\mu\psi_c(0) \xrightarrow{\text{match}} C_V(Q^2)\overbrace{\underbrace{\left[(\bar\xi_{\bar n}^{(0)} W_{\bar c})(0)\right]}_{\bar\chi_{\bar n}^{(0)}(0)}\gamma_\perp^\mu S_{\bar n}^\dagger(0) S_n(0)\underbrace{\left[(W_c^\dagger\xi_n^{(0)})(0)\right]}_{\chi_n^{(0)}(0)}}^{V^\mu_{SCET}(0)}$$

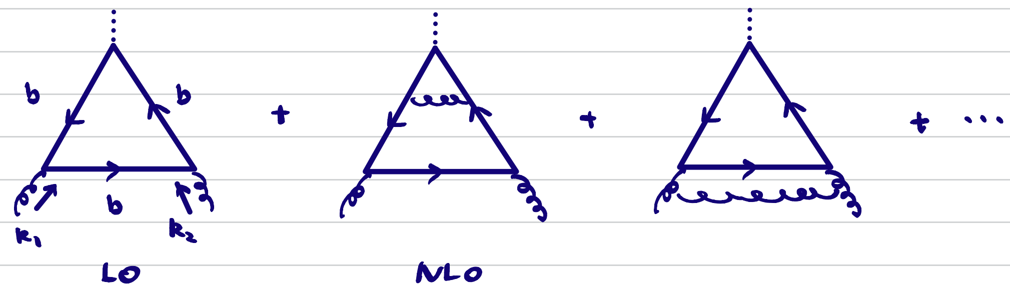
Reproduce from $V^\mu_{SCET}(0)$, depends on time!

II. NLP Factorization: A Brief Taste from $gg \xrightarrow{\text{b quark}} h$
$\left(h \longrightarrow gg \right)$

$m_b \ll m_H$, $\lambda = m_b/m_H \ll 1$

$k_1^2 = k_2^2 = 0$





$F_{h\to\gamma\gamma}$ & $F_{gg\to h}$ $\propto m_b$ due to Higgs coupling.

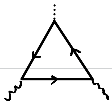
So they are zero in $m_b \to 0$ limit. That's why people normally consider $gg \xrightarrow{\text{top quark}} h$.

But loop corrections enhance the light-quark contribution to

$$\sim m_b \left(\log^2 \frac{m_H^2}{m_b^2} + \cdots \right)$$

↪ this log is large!

These corrections can be obtained systematically and more precisely.
Besides, these processes are benchmark to investigate NLP SCET formalism!

Region Analysis of 

This part devotes to region analysis of light-quark induced $h \to \gamma\gamma$ form factor to initiate the factorization pattern.

The LO starts at one loop already. For brevity, we neglect the numerators. Two-loop extension and some details here can be found in 1812.08818. See 2501.11824 for a recent calculation.

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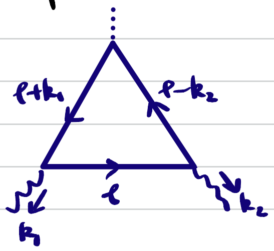
0. setup



$k_i^2=0,\ 2k_1\cdot k_2 = m_H^2$

$$I = e^{\varepsilon\gamma_E}(\mu^2)^{3-\frac{1}{2}D}\int\frac{d^D\ell}{i\pi^{D/2}}\frac{1}{[\ell^2-m^2]\cdot[(\ell+k_1)^2-m^2]\cdot[(\ell-k_2)^2-m^2]}$$

$$\begin{cases} \lambda := m/m_H \ll 1 & i0^+ \text{ implicit.} \\ k_1 = m_H\frac{n}{2}, & k_2 = m_H\frac{\bar{n}}{2} \end{cases}$$

direct calculation

$$I = (-1)\, e^{\varepsilon\gamma_E}\,\Gamma(1+\varepsilon)\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2\int_0^\infty d\alpha_3\,\delta(1-h(\alpha))\,\frac{1}{(\mathcal{U}(\alpha))^{1-2\varepsilon}(\mathcal{F}(\alpha))^{1+\varepsilon}}$$

$$\mathcal{U}(\alpha) = \alpha_1+\alpha_2+\alpha_3,\quad \mathcal{F}(\alpha) = -\frac{m_H^2}{\mu^2}\alpha_1\alpha_2 + (\alpha_1+\alpha_2+\alpha_3)^2\frac{m^2}{\mu^2}$$

$$\mathcal{U} = \sum_{T:\text{ spanning trees}}\ \prod_{e\notin T}\alpha_e,\quad \mathcal{F} = \sum_{\substack{F:\text{ spanning 2-forest} \\ = T_1 \cup T_2}}\frac{-P_F^2}{\mu^2}\prod_{e\notin F}x_e + \mathcal{U}\cdot\sum_{e=1}^{N}x_e\frac{m_e^2}{\mu^2}$$

choose $h(\alpha) = \alpha_1+\alpha_2+\alpha_3$

$$I = \left(\frac{\mu^2}{-m_H^2}\right)^{1+\varepsilon}\times(-1)\,e^{\varepsilon\gamma_E}\,\Gamma(1+\varepsilon)\int_0^1 d\alpha_1\int_0^{1-\alpha_1}d\alpha_2\,(\alpha_1\alpha_2-\lambda^2)^{-1-\varepsilon} \quad \cdots\cdots\cdots \quad (*)$$

(has chosen $h(\alpha)=\alpha_1+\alpha_2+\alpha_3$)

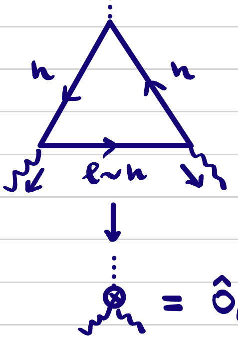
$$= \frac{\mu^2}{m_H^2}\times\frac{1}{2}\log^2\frac{-m_H^2}{m^2} + O(\lambda,\varepsilon)$$

$\downarrow$ set $\mu^2 = m_H^2$

rmk: full dependence can be expressed in terms of Li_2

$$= \frac{1}{2}\log^2\frac{-m_H^2}{m^2} + O(\lambda,\varepsilon).$$

1. hard region: $\ell \sim (1,1,1)$



$= \hat{O}_1$

$$\ell^2-m^2 = \ell^2 + O(\lambda^2);$$
$$(\ell+k_1)^2-m^2 = \ell^2+2k_1\cdot\ell + O(\lambda^2);$$
$$(\ell-k_2)^2-m^2 = \ell^2-2k_2\cdot\ell + O(\lambda^2);$$

$$I_h = (\mu^2)^{1+\varepsilon}e^{\varepsilon\gamma_E}\int\frac{d^D\ell}{i\pi^{D/2}}\frac{1}{[\ell^2]\cdot[\ell^2+2k_1\cdot\ell]\cdot[\ell^2-2k_2\cdot\ell]}$$

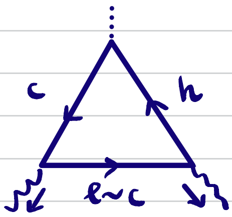
$$= -\left(\frac{\mu^2}{-m_H^2}\right)^{1+\varepsilon}\frac{e^{\varepsilon\gamma_E}\,\Gamma(1+\varepsilon)\,\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}. \qquad \varepsilon = \varepsilon_{IR}$$

Or we can start directly from (*)

$$\rightsquigarrow I_h = \left(\frac{\mu^2}{-m_H^2}\right)^{1+\varepsilon} \times (-1)\, e^{\varepsilon\gamma_E}\, \Gamma(1+\varepsilon) \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2\, (\alpha_1\alpha_2)^{-1-\varepsilon}$$

$$= -\left(\frac{\mu^2}{-m_H^2}\right)^{1+\varepsilon} \frac{e^{\varepsilon\gamma_E}\,\Gamma(1+\varepsilon)\,\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}.$$

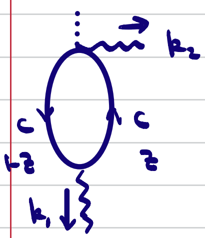
2. Collinear region : $\ell \sim (\lambda^2, 1, \lambda)$



$$\ell^2 - m^2 = \ell^2 - m^2$$
$$(\ell+k_1)^2 - m^2 = (\ell+k_1)^2 - m^2$$
$$(\ell-k_2)^2 - m^2 = -2k_2\cdot\ell + \mathcal{O}(\lambda^2) = -m_H\bar{n}\cdot\ell + \mathcal{O}(\lambda^2).$$

$$I_c = (\mu^2)^{1+\varepsilon}\cdot e^{\varepsilon\gamma_E}\int\frac{d^D\ell}{i\pi^{D/2}}\frac{1}{[\ell^2-m^2]\cdot[(\ell+k_1)^2-m^2]\cdot[-2k_2\cdot\ell]}$$

This integral is not regulated by $D=4-2\varepsilon$! $\rightsquigarrow$ rapidity div.
(need extra regulator)



$\hat{O}_2(z)$

$$I_c = (\mu^2)^{1+\varepsilon} e^{\varepsilon\gamma_E}\int_{-\infty}^{+\infty} dz \int\frac{d^D\ell}{i\pi^{D/2}}\frac{\delta(z+\bar{n}\cdot\ell/m_H)}{[\ell^2-m^2]\cdot[(\ell+k_1)^2-m^2]\cdot[z\, m_H^2]}$$

$$= \frac{\mu^2}{m_H^2}(\mu^2)^{\varepsilon}\cdot e^{\varepsilon\gamma_E}\int_{-\infty}^{+\infty}\frac{dz}{z}\int_0^1 d\alpha\int\frac{d^D\ell}{i\pi^{D/2}}\frac{\delta(z+\bar{n}\cdot\ell/m_H)}{[(\ell+\alpha k_1)^2-m^2]^2}$$

$$= \frac{\mu^2}{m_H^2}(\mu^2)^{\varepsilon}\cdot e^{\varepsilon\gamma_E}\int_{-\infty}^{+\infty}\frac{dz}{z}\int_0^1 d\alpha\int\frac{d^D\ell}{i\pi^{D/2}}\frac{\delta(z-\alpha+\bar{n}\cdot\ell/m_H)}{[\ell^2-m^2]^2}$$

$$= \frac{\mu^2}{m_H^2}\int_{-\infty}^{+\infty}\frac{dz}{z}\int_0^1 d\alpha\,\delta(z-\alpha)\, e^{\varepsilon\gamma_E}(\mu^2)^{\varepsilon}\int\frac{d^D\ell}{i\pi^{D/2}}\frac{1}{[\ell^2-m^2]^2}$$

$$= \frac{\mu^2}{m_H^2}\int_0^1\frac{dz}{z}\boxed{\left(\frac{\mu^2}{m^2}\right)^{\varepsilon} e^{\varepsilon\gamma_E}\Gamma(\varepsilon)}$$

z^{-1} is the hard func.

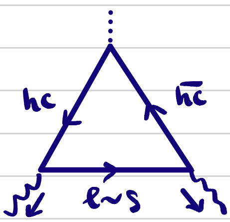
$\langle O_2(z)\rangle$

(rapidity div.)
endpoint div.

3. Anti-collinear region : $\ell \sim (1, \lambda^2, \lambda)$

likewise ...

4. Soft region: $\ell \sim (\lambda, \lambda, \lambda)$



$\ell + k_1 \sim (1, \lambda, \lambda)$ $\quad (\ell+k_1)^2 \sim \lambda \gg \ell^2$ hard-collinear (hc)

$(\ell-k_2)^2 \sim (\lambda, 1, \lambda)$ $\quad (\ell-k_2)^2 \sim \lambda \gg \ell^2$ anti-hard-collinear ($\overline{hc}$)

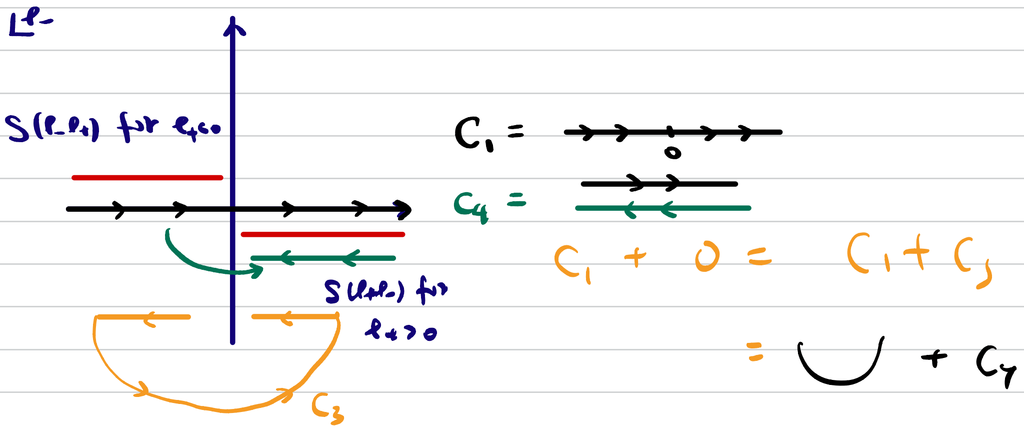
$$(\ell+k_1)^2 - m^2 = 2k_1\cdot\ell + \mathcal{O}(\lambda^2)$$
$$(\ell-k_2)^2 - m^2 = -2k_2\cdot\ell + \mathcal{O}(\lambda^2)$$

$$I_s = (\mu^2)^{1+\varepsilon}\cdot e^{\varepsilon\gamma_E}\int\frac{d^D\ell}{i\pi^{D/2}}\frac{1}{[\ell^2-m^2+i0]\cdot[m_H n\cdot\ell+i0]\cdot[-m_H\bar{n}\cdot\ell+i0]}$$

which has rapidity divergence as well.

$$I_s = \frac{1}{2}\frac{\mu^2}{m_H^2}(\mu^2)^\varepsilon e^{\varepsilon\gamma_E}\int_{-\infty}^{+\infty}\frac{d\ell_+}{\ell_+ + i0}\int_{-\infty}^{+\infty}\frac{d\ell_-}{-\ell_- + i0}\int\frac{d^{D-2}\ell_\perp}{i\pi^{D/2}}\frac{1}{\ell_+\ell_- - \ell_T^2 - m^2 + i0}$$

$$= \frac{\mu^2}{m_H^2}\int_{-\infty}^{+\infty}\frac{d\ell_+}{\ell_+ + i0}\int_{-\infty}^{+\infty}\frac{d\ell_-}{-\ell_- + i0}(\mu^2)^\varepsilon e^{\varepsilon\gamma_E}\Gamma(\varepsilon)\times\frac{-1}{2\pi i}\left(-\ell_-\ell_+ + m^2 - i0\right)^{-\varepsilon}$$



So, $I_s = \frac{\mu^2}{m_H^2}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_0^\infty\frac{d\ell_-}{\ell_-}(\mu^2)^\varepsilon e^{\varepsilon\gamma_E}\Gamma(\varepsilon)\frac{-1}{2\pi i}\left[(-\ell_-\ell_+ + m^2 + i0)^{-\varepsilon} - (-\ell_-\ell_+ + m^2 - i0)^{-\varepsilon}\right]$

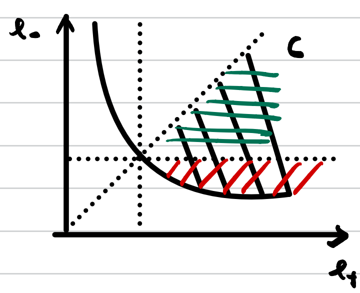
$$= \frac{\mu^2}{m_H^2}\int_0^\infty\frac{d\ell_+}{\ell_+}\int_\delta^\infty\frac{d\ell_-}{\ell_-}(\mu^2)^\varepsilon\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}(\ell_-\ell_+ - m^2)^{-\varepsilon}\,\theta(\ell_-\ell_+ - m^2)$$

cancellation of rapidity divergence

The full result, $I = \frac{1}{2}\log^2\frac{-m_H^2}{m^2} + O(\lambda,\varepsilon)$, does not have rapidity div.
When dividing into regions, rapidity divergence is inevitable, since DR can not distinguish $p_c^2 \sim p_s^2 \sim \lambda^2$ due to the same virtuality!
But the divergence must cancel in the end.
One can either introduce an extra regulator, e.g., analytic regulator, which will cancel after summing over all regions;
Or one can introduce a consistent cutoff ···, which we adopt below!

$$I_c^{(\delta)} = \int_\delta^1\frac{dz}{z}\left(\frac{\mu^2}{m^2}\right)^\varepsilon e^{\varepsilon\gamma_E}\Gamma(\varepsilon) = -\left(\frac{\mu^2}{m^2}\right)^\varepsilon e^{\varepsilon\gamma_E}\Gamma(\varepsilon)\ln\delta\,.$$

$(\delta = \ell_+^{max}/m_H)$



sector decomposition of $I_s = \underbrace{I_s(\ell_+>\ell_-)}_{I_{s,c}} + \underbrace{I_s(\ell_+<\ell_-)}_{I_{s,\bar{c}}}$

Soft & Collinear

We can skip to analyze the analytic structure by using reverse unitarity:

$$I_s = \frac{\mu^2}{m_H^2} \int_{-\infty}^{+\infty} \frac{dp}{p} \int_{-\infty}^{+\infty} \frac{d\alpha}{\alpha} (\mu^2)^\varepsilon e^{\varepsilon\gamma_E} \int \frac{d^D\ell}{i\pi^{D/2}} \frac{\delta(p - n\cdot\ell)\,\delta(\alpha + \bar{n}\cdot\ell)}{\ell^2 - m^2 + i0}$$

$$\Big\downarrow \quad \delta(\alpha + \bar{n}\cdot\ell) = \frac{1}{2\pi i}\left(\frac{1}{\alpha + \bar{n}\cdot\ell - i0} - \frac{1}{\alpha + \bar{n}\cdot\ell + i0}\right)$$

Then we calculate

$$\begin{aligned}
f_s &= (\mu^2)^\varepsilon e^{\varepsilon\gamma_E} \int \frac{d^D\ell}{i\pi^{D/2}} \frac{\delta(p - n\cdot\ell)}{(\ell^2 - m^2 + i0)(\alpha + \bar{n}\cdot\ell - i0)} \\
&= (\mu^2)^\varepsilon e^{\varepsilon\gamma_E} \int_0^\infty d\alpha \int \frac{d^D\ell}{i\pi^{D/2}} \frac{-m_H\,\delta(p - n\cdot\ell)}{[(\ell - \alpha k_0)^2 - (\alpha m_H a + m^2) + i0]^2} \\
&= (\mu^2)^\varepsilon e^{\varepsilon\gamma_E} \int_0^\infty d\alpha \int \frac{d^D\ell}{i\pi^{D/2}} \frac{-m_H\,\delta(p - \alpha m_H - n\cdot\ell)}{[\ell^2 - (\alpha m_H a + m^2) + i0]^2} \\
&= -(\mu^2)^\varepsilon e^{\varepsilon\gamma_E} \Gamma(\varepsilon)\,\Theta(p)\,[pa + m^2 - i0]^{-\varepsilon} \\
&= -(\mu^2)^\varepsilon e^{\varepsilon\gamma_E} \Gamma(\varepsilon)\,\Theta(p)\,[p(a - i0) + m^2]^{-\varepsilon}
\end{aligned}$$

$$\rightsquigarrow \quad I_s = \int_{-\infty}^{+\infty} \frac{dp}{p} \int_{-\infty}^{+\infty} \frac{d\alpha}{\alpha} \underbrace{\frac{1}{2\pi i}[f_s(p, \alpha - i0) - f_s(p, \alpha + i0)]}_{\text{disc. w.r.t. } \alpha \text{ (or } p\alpha \text{ b.c. } p>0)}$$

$\Theta(p)\,[p(\alpha - i0) + m^2]^{-\varepsilon}$ has disc. only if $p\alpha + m^2 < 0$

$$\begin{aligned}
&\frac{1}{2\pi i}[f_s(p, \alpha - i0) - f_s(p, \alpha + i0)] \\
&= e^{\varepsilon\gamma_E} \Gamma(\varepsilon)\,\Theta(p)\,\Theta(-p\alpha - m^2)\,(-p\alpha - m^2)^{-\varepsilon} \times \frac{\sin(\pi\varepsilon)}{\pi} \\
&= \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\,\Theta(p)\,\Theta(-p\alpha - m^2)\,(-p\alpha - m^2)^{-\varepsilon}.
\end{aligned}$$

$$I_s = \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+} \cdot \underbrace{1 \cdot 1}_{\text{jet func.}} \cdot \Big[\underbrace{(\mu^2)^\varepsilon \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} (\ell_+\ell_- - m^2)^{-\varepsilon}\,\Theta(\ell_+\ell_- - m^2)}_{\text{soft func.}}\Big]$$
