Operators & Factorization (bare) of $gg \to h$

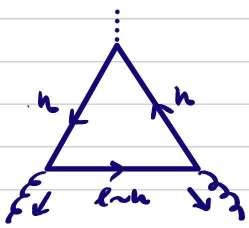
QCD
$\mu^2 \sim m_H^2$ integrate out h

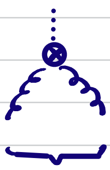
Step I SCET$_1$ ($hc, \overline{hc}, s$)
$\mu^2 \sim m_H m_b$ integrate out hc & $\overline{hc}$

Step II SCET$_2$ ($c, \bar{c}, s$)
$\mu^2 \sim m_b^2$

I will, however, skip this two-step matching procedure, which is more rigirous though, for brevity and heuristic purposes.
Instead, I will present operators and factorization inspired by region analysis.

$\ell \sim h \to$ all three propagators are hard and shrink to a point.

① 

$\stackrel{!}{=} H_1 \cdot$  $\hat{O}_1$

$$\hat{O}_1 = \frac{m_b}{g_s^2} h(0)\, \mathcal{G}^{\perp}_{n_1}(0) \cdot \mathcal{G}^{\perp}_{n_2}(0)$$

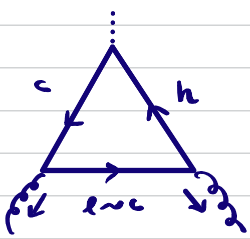
$\mathcal{G}^{\perp}_{n_i}$ is the n_i-hard-collinear gauge-inv. gluon building block.
(gluon field dressed with WL)

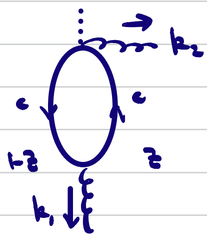
To calculate H_1, we sandwich $H_1 \cdot O_1$ between n_1 & n_2 on-shell states:

$$H_1 \langle k_1, k_2 | \hat{O}_1(0) | h(q) \rangle \qquad (k_i = \tfrac{1}{2} m_H n_i)$$

Since $k_1^2 = k_2^2 = 0$, $\langle \hat{O}_1(0) \rangle = m_b\, \varepsilon_\perp^*(k_1) \cdot \varepsilon_\perp^*(k_2)$ to all orders in α_s!
Then H_1 is nothing but the one-loop diagram on the left.

power counting: $m_b \sim \lambda$, $\mathcal{G}^{\perp}_{n_i} \sim \lambda \rightsquigarrow \hat{O}_1 \sim \lambda^3$

② 

$\stackrel{!}{=} H_2(z) \otimes$ 

$\downarrow$ dep. on m_H & z

$\hat{O}_2(z) \longrightarrow$ dep. on m_b & z

In this diagram, one propagator is hard and shrinks to a point.

$$\hat{O}_{2,n_1}(t) = h(0)\,\bar{\chi}_{n_1}(0)\,\gamma_\perp^\mu \frac{\bar{\not{n}}_1}{2}\,\chi_{n_1}(t\bar{n}_1)\,\mathcal{G}^\perp_{n_2,\mu}(0)$$

$$= h(0)\,\bar{\chi}_{n_1}(0)\,\not{\mathcal{G}}^\perp_{n_2}(0)\,\frac{\bar{\not{n}}_1}{2}\,\chi_{n_1}(t\bar{n}_1)$$

This is in the position space. We can convert it into the momentum space by Fourier transformation:

$$\bar{n}_1\cdot k_1 = m_H$$

$$\hat{O}_{2,n_1}(t) = \int d(\bar{n}_1\cdot k_1 z)\, e^{i\bar{n}_1\cdot k_1 z}\,\hat{O}_{2,n_1}(\bar{n}_1\cdot k_1, z)$$

Note that two collinear quark fields share the collinear momentum k_1.

$$z + (1-z) = 1.$$

$$\hat{O}_{2,n_1}(z) = h(0)\,\bar{\chi}_{n_1}(0)\,\not{\mathcal{G}}^\perp_{n_1}(0)\,\frac{\bar{\not{n}}_1}{2}\,\delta(\omega + i\bar{n}_1\cdot\partial)\,\chi_{n_1}(0)$$

$\hat{O}_{2,n_2}$ is defined similarly and its matrix element is identical to $\langle \hat{O}_{2,n_1}(z)\rangle$. So from now on, we use $\hat{O}_2(z)$ to denote both cases.

Calculate $H_2(z)$ & $\langle \hat{O}_2(z)\rangle$ to have a taste

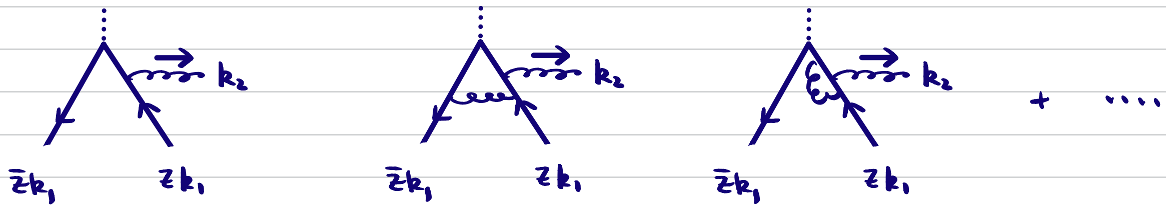
To extract $H_2(z)$, we compute the on-shell amplitude of

$$h \to g(k_2)\, b(\bar{z}k_1)\, \bar{b}(zk_1).$$

Then $\langle \hat{O}_2(z)\rangle$ takes the tree level result:

$$\mathcal{M}_{gb\bar{b}} = \int_0^1 dz'\, H_2(z')\,\langle b_i(\bar{z}k_1)\,\bar{b}_j(zk_1)\,g_q(k_2)|\,\hat{O}_2(z')\,|h(q)\rangle$$

$$= \frac{g_s}{m_H}\,H_2(z)\,\bar{u}(\bar{z}k_1)\,\not{\varepsilon}_\perp^*(k_2)\,T^a_{ij}\,\frac{\bar{\not{n}}_1}{2}\,v(zk_1)$$



LO NLO

$$H_2(z) = \frac{y_b}{\sqrt{2}}\left(\frac{1}{z} + \frac{1}{1-z}\right)\underbrace{\left\{1 + \frac{\alpha_s}{4\pi}\cdot\left[C_F\cdot f(z) + C_A\cdot g(z) + (z\longrightarrow 1-z)\right] + \mathcal{O}(\alpha_s^2)\right\}}_{\bar{H}_2(z)}$$

$$[\![\bar{H}_2(z)]\!] := \bar{H}_2(z)\Big|_{z\to 0} = \frac{y_b}{\sqrt{2}}\left\{1 + \frac{\alpha_s}{4\pi}\,e^{\epsilon\gamma_E}\,\frac{\Gamma(1+\epsilon)\,\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\,(-m_H^2)^{-\epsilon}\left[(C_F - C_A)\,\frac{2-4\epsilon-\epsilon^2}{z^\epsilon} - 2C_F(1-\epsilon)^2\right]\right\} + \mathcal{O}(\alpha_s^2)$$

endpoint behavior of $H_2(z)$

$\hat{O}_2(z)$ is gauge invariant. It is simplest to calculate in the light-cone gauge, $\bar{n}_1 \cdot G_{n_1} \equiv n_2 \cdot G_{n_1} = 0$, then all n_1-collinear WLs reduce to 1.

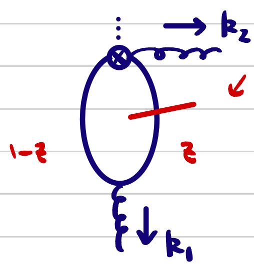
$$\hat{O}_2(z) \overset{\text{L.C.G.}}{=\!=} h(0)\, \boxed{\bar{\xi}_{n_1}(0)\, \gamma_\perp^\mu \frac{\bar{\slashed{n}}_1}{2}\, \delta(\omega + i\bar{n}_1\cdot\partial)\, \xi_{n_1}(0)}\, \boxed{G^\perp_{n_2,\mu}(0)}$$

n_1 sector $\qquad$ n_2 sector

no talk to each other after soft decoupling!

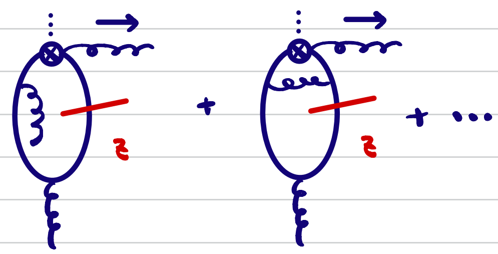
One can further set the light-cone gauge in n_2-collinear sector. Then $G^\perp_{n_2,\mu}(0) \longrightarrow g_s G^\perp_{n_2,\mu}(0)$.

LO of $\langle g_a(k_1)\, g_b(k_2) |\, \hat{O}_2(z)\, | h(q) \rangle = \langle \hat{O}_2(z) \rangle$



$\swarrow\ \delta(\omega + i\bar{n}_1\cdot\partial)$ cut constraint

LO



NLO

LO: $\langle g_a(k_1)\, g_b(k_2) |\, h(0)\, \bar{\xi}_{n_1}(0)\, \gamma_\perp^\mu \frac{\bar{\slashed{n}}_1}{2}\, \delta(\omega + i\bar{n}_1\cdot\partial)\, \xi_{n_1}(0)\, G^\perp_{n_2,\mu}(0)\, | h(q) \rangle$

$$= \langle g_a(k_1) |\, \bar{\xi}_{n_1}(0)\, \slashed{\epsilon}^*_\perp(k_2) \frac{\bar{\slashed{n}}_1}{2}\, \delta(\omega + i\bar{n}_1\cdot\partial)\, \xi_{n_1}(0)\, | 0 \rangle\, T^b$$

insert $\mathcal{L}_{int} = i g_s \int d^4x\, \bar{\psi}(x)\, \slashed{G}(x)\, \psi(x)$ and Wick contract.

$$\sim \int \frac{d^D\ell}{(2\pi)^D} \frac{\delta(2k_1\cdot\ell - z m_H^2) \times \mathrm{Tr}(\cdots)}{(\ell^2 - m_b^2)((\ell+k_1)^2 - m_b^2)} \longrightarrow m_b \left(\frac{\mu^2}{m_b^2}\right)^\epsilon e^{\epsilon\gamma_E}\, T(\epsilon)$$

only z & logs.

$$\langle \hat{O}_2(z) \rangle = m_b T_F \delta_{ab} \frac{\alpha_s}{4\pi} \Big\{ \underbrace{2 e^{\epsilon\gamma_E} \left(\frac{\mu^2}{m_b^2}\right)^\epsilon T(\epsilon)}_{\text{LO}} + \underbrace{\frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{m_b^2}\right)^{2\epsilon}}_{\text{NLO}} \times \big[C_F K_F(z) + C_A K_A(z) + (z \to 1-z) \big] + O(\alpha_s^2) \Big\}$$

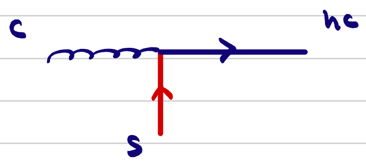
$$[\![\langle \hat{O}_2(z) \rangle]\!] := \langle \hat{O}_2(z) \rangle \Big|_{z\to 0} = \cdots$$

asymptotic of $\langle \hat{O}_2(z) \rangle$

③ Note when the quark is soft $(\lambda, \lambda, \lambda)$

$$c + s = (1, \lambda^2, \lambda) + (\lambda, \lambda, \lambda) = (1, \underset{\uparrow}{\lambda}, \lambda) = hc$$

virtuality $\sim \lambda \gg \lambda^2$!



a collinear gluon turns a soft quark into a hard-collinear quark.

At LP, this interaction is forbidden, since $q_s \sim \lambda^3$ is power suppressed.

At NLP, it is allowed, and is a common feature at NLP, through the following interaction insertion :

$$\boxed{\mathcal{L}_{q\xi_n}(x) = \underset{\lambda^3}{\bar{q}_s(x_-)} W_n^\dagger(x)\, i\underset{\lambda}{\not{D}_n^\perp} \underset{\lambda}{\xi_n}(x) = \bar{q}_s(x_-) \mathcal{A}^\perp_n(x) \chi_n(x)}$$

$$\rightsquigarrow \int \underset{\lambda^{-4}}{d^4x} \, (\mathcal{L}_{q\xi_n} + \text{h.c.}) \sim \lambda^{-4} \cdot \lambda^5 = \lambda \leftarrow \text{power suppressed}$$

compared with $\int d^4x \, \mathcal{L}_{LP} \sim \lambda^{-4} \cdot \lambda^4 = \lambda^0$!

To describe the soft quark contributions, we need to insert twice : one to convert n_1- collinear quark to the soft quark, then to convert the soft quark into n_2- collinear quark.

Then we end up with the following operator $\hat{O}_3(0)$

$$\boxed{\hat{O}_3(0) = \hat{T}\left\{ h(0) \bar{\chi}_{n_1}(0) \chi_{n_2}(0),\ i\int d^4x\, \mathcal{L}_{\bar{q}\xi_{n_1}}(x),\ i\int d^4y\, \mathcal{L}_{\xi_{n_2} q}(y) \right\} + \text{h.c.}}$$

Here χ_{n_1} & χ_{n_2} are hard-collinear quark fields, so scale as $\lambda^{\frac{1}{2}}$.

Then $O_3 \sim \lambda \cdot \lambda \cdot \lambda = \lambda^3$!

Then we perform soft decoupling of all fields, and organize different sectors to arrive at

$$\hat{O}_3 = h(0) \int d^Dx \int d^Dy \, \hat{T}\left\{ \mathcal{A}^{\perp, a}_{n_1}(x)\, \chi_{n_1}(x)\, \bar{\chi}_{n_1}(0) \right\}$$

$$\times \hat{T}\left\{ \chi_{n_2}(0)\, \bar{\chi}_{n_2}(y)\, \mathcal{A}^{\perp, b}_{n_2}(y) \right\}$$

$$\times \underbrace{\underset{\text{Tr}_c}{\hat{T}} \left\{ \bar{q}_s Y_{n_1}(x_-)\, T^a\, Y^\dagger_{n_1}(0)\, \frac{\not{n}_1 \not{n}_2}{4}\, Y_{n_2}(0)\, T^b\, Y_{n_2} q_s(y_+) \right\}}_{\hat{O}_g(x_-, y_+)}$$

$\leftarrow$ Lorentz & color index suppressed !

As mentioned previously, $p_{hc}^2 \sim \lambda \gg p_c^2 \sim p_s^2 \sim \lambda^2$,

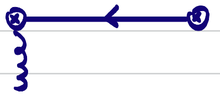
we need to integrate out hc & $\overline{hc}$ modes. The matching coefficients are the radiative jet functions:

$$\int d^Dx\, e^{ip_s\cdot x_-}\, \hat{T}\{\underbrace{\mathcal{A}^{\perp}_{n_1}\,\chi_{n_1}(x)\,\bar{\chi}_{n_1}(0)}_{hc\ \text{fields}}\} = \underset{\downarrow\, J_g(p_{hc}^2)}{J_g}\; \underbrace{\mathcal{A}^{\perp}_{n_1}}_{\text{collinear field}}(0)\,\frac{i\not{n}_1}{2n_1\cdot p_s}$$

To extract J_g, sandwich the above operator equation between $\langle g_a(k_1)| \cdots |0\rangle$, with $k_1^2=0$ & $2k_1\cdot p_s = \lambda m_H^2 \sim p_{hc}^2$.

For on-shell state $\langle g(k_1)|$, $\langle g_a(k_1)|\,\mathcal{A}^{\perp}_{n_1}(0)|0\rangle = g_s\,\epsilon^{*}_{\perp}(k_1)\,T^a$ to all orders of α_s!

J_g :



LO

NLO:

$+ \quad \cdots$

$$J_g(p^2) = 1 + \frac{\alpha_s}{4\pi}(C_F - C_A)\, e^{\epsilon\gamma_E}\,\frac{\Gamma(1+\epsilon)\Gamma^2(-\epsilon)}{\Gamma(2-2\epsilon)}\,(2-4\epsilon-\epsilon^2) + \mathcal{O}(\alpha_s^2).$$

The vacuum matrix element of the soft quark fields & soft WLs give the NLP soft function.

$$S_g(\ell_-\ell_+) = \frac{1}{2\pi i}\,\mathrm{Disc}_{\ell_-\ell_+}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} dt\, e^{is\ell_+ - it\ell_-}\,\langle 0|\,\hat{O}_g(s,t)\,|0\rangle$$

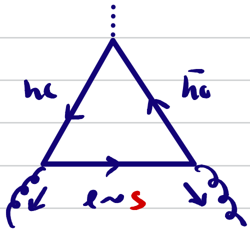
$$\text{LO}: \sim \frac{1}{2\pi i}\,\mathrm{Disc}_{\ell_-\ell_+}\int_{-\infty}^{+\infty} ds\int_{-\infty}^{+\infty} dt\, e^{is\ell_+ - it\ell_-}\int\frac{d^Dq}{(2\pi)^D}\, e^{-iq\cdot(sn_1 - tn_2)}\,\frac{-im_b}{q^2-m_b^2}\,T_F\,\delta_{ab}$$

$$\sim e^{\epsilon\gamma_E}\,\frac{1}{2\pi i}\,\mathrm{Disc}_{\ell_-\ell_+}\int\frac{d^{D-2}q_\perp}{(2\pi)^{D-2}}\,\frac{1}{\ell_+\ell_- - q_T^2 - m_b^2 + i0}$$

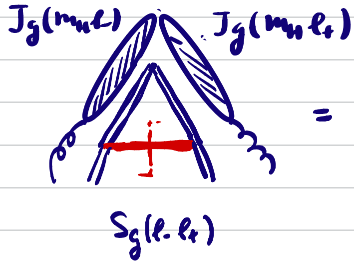
$$\sim e^{\epsilon\gamma_E}\int\frac{d^{D-2}q_\perp}{(2\pi)^{D-2}}\,\delta(\ell_+\ell_- - q_T^2 - m_b^2)$$

$$\sim \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\,(\ell_+\ell_- - m_b^2)^{-\epsilon}\,\theta(\ell_+\ell_- - m_b^2).$$

We will return to the soft function more explicitly later on.



$\stackrel{!}{=} H_3 \cdot$



$$= H_3 \int_0^{\infty} \frac{d\ell_-}{\ell_-} \int_0^{\infty} \frac{d\ell_+}{\ell_+} J_g(m_H \ell_-) J_g(-m_H \ell_+) S_g(\ell_- \ell_+)$$

rmk:

① T_2 & T_3 are both endpoint divergent!

endpoint $\begin{cases} \text{some are regulated in DR} \\ \text{the others not : rapidity.} \end{cases}$

② $$F_{gg\to h} = \underbrace{H_1 \langle O_1 \rangle}_{T_1} + \underbrace{2 H_2 \otimes \langle O_2 \rangle}_{T_2} + \underbrace{H_3 \overbrace{J_g \otimes J_g \otimes S_g}^{\langle O_3 \rangle}}_{T_3}$$

has extra IR poles in ε for onshell states $g_a(k_1)\, g_b(k_2)$.

(after dealing with endpoint div.)

Because external gluons carry color charges.

The extra IR poles can be described by

$$Z_{gg} = 1 - \frac{\alpha_s(\mu)}{4\pi}\left[2 C_A\left(\frac{1}{\varepsilon^2} - \frac{L_H}{\varepsilon}\right)\right] + O(\alpha_s^2)$$

as $Z_{gg}^{-1} F_{gg\to h}$ is ε-finite.

Factorization without Endpoint Div. & Re-factorization

A subtraction procedure based on re-factorization theorems has been proven to work to all orders in α_s in $h\to\gamma\gamma$ & $gg\to h$ (2009.04456, 2009.06779 & 2212.10447) and other processes, e.g., 2008.04943 & 2205.04479

> The philosophy is generic for NLP SCET and is the only way compatible with factorization & renormalization right now.

Sketch of the procedure:

$$
\left.\begin{aligned}
F_{gg\to h} = \; & H_1 \cdot \langle O_1 \rangle \\
& + 4\int_0^1 \frac{dz}{z}\, \bar{H}_2(z)\, \langle O_2(z)\rangle \\
& + H_3 \cdot \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+}\, J_g(m_H \ell_-)\, J_g(-m_H \ell_+)\, S_g(\ell_- \ell_+)
\end{aligned}\right\} \text{ill-defined due to endpoint div.}
$$

$$
\begin{aligned}
= \; & H_1 \cdot \langle O_1 \rangle \\
& + 4\int_0^1 \frac{dz}{z} \Big[\bar{H}_2(z)\, \langle O_2(z)\rangle - [\![\bar{H}_2(z)]\!]\, [\![\langle O_2(z)\rangle]\!] \Big] \longrightarrow \text{well-defined in endpoint region} \\
& + 4\int_0^1 \frac{dz}{z}\, [\![\bar{H}_2(z)]\!]\, [\![\langle O_2(z)\rangle]\!] \\
& + H_3 \cdot \int_0^\infty \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+}\, J_g(m_H \ell_-)\, J_g(-m_H \ell_+)\, S_g(\ell_- \ell_+)
\end{aligned}
$$

endpoint versions

$$
\begin{aligned}
= \; & (H_1 + \Delta H_1) \langle O_1 \rangle \\
& + 4\int_0^1 \frac{dz}{z} \Big[\bar{H}_2(z)\, \langle O_2(z)\rangle - [\![\bar{H}_2(z)]\!]\, [\![\langle O_2(z)\rangle]\!] \Big] \longrightarrow \text{well-defined in endpoint region} \\
& + H_3 \int_0^{m_H} \frac{d\ell_-}{\ell_-} \int_0^{m_H} \frac{d\ell_+}{\ell_+}\, J_g(m_H \ell_-)\, J_g(m_H \ell_+)\, S_g(\ell_- \ell_+) \longrightarrow \text{well-defined in endpoint region}
\end{aligned}
$$

endpoint versions

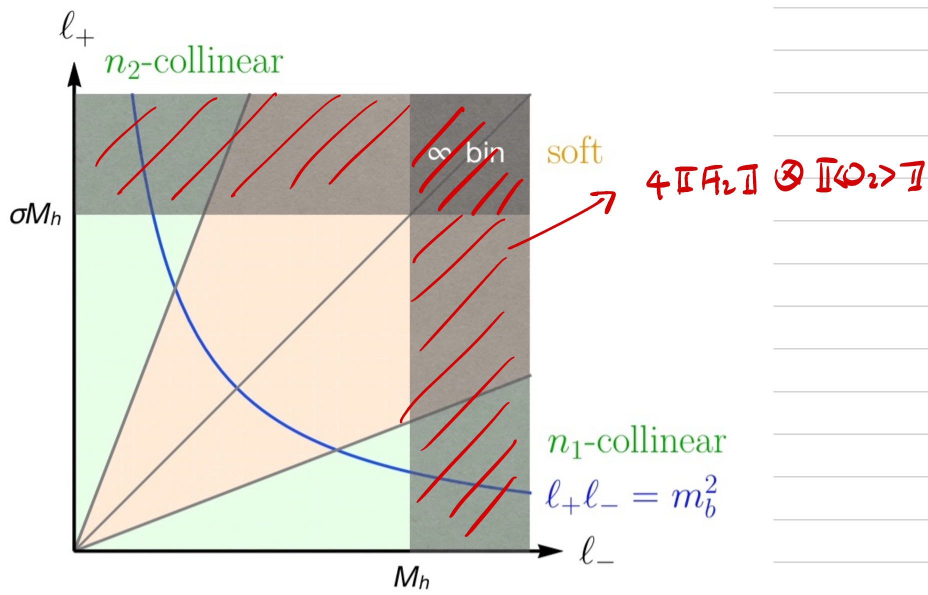
free of end. div respectively. No need of rapidity regulator!

The last formulae follows from the facts that

$$
\begin{cases}
[\![\bar{H}_2(z)]\!] = - J_g(m_H \ell_-)\, H_3 \\[2ex]
[\![\langle \hat{O}_2(z)\rangle]\!] = \frac{1}{2} \int_0^\infty \frac{d\ell_+}{\ell_+}\, J_g(-m_H \ell_+)\, S_g(\ell_- \ell_+)
\end{cases}
$$

they are called re.fact. conditions, and have been proven using SCET. see 2009.06779.

$$
\hookrightarrow 4 [\![\bar{H}_2]\!] \otimes [\![\langle O_2(z)\rangle]\!] = -2 H_3 \int_0^{m_H} \frac{d\ell_-}{\ell_-} \int_0^\infty \frac{d\ell_+}{\ell_+}\, J_g(-m_H \ell_+)\, J_g(m_H \ell_-)\, S_g(\ell_- \ell_+)
$$



$\Delta H_1 =$ over subtracted ∞-bin.

The above factorization is free of endpoint divergence. But all factorized pieces are at bare level. One needs renormalize them and check whether the subtractions procedure is compatible with $\overline{\text{MS}}$ renormalization.

2009.06779

However, it takes a lot to prove the compatibility, so I will skip this part.

not commute between renorm. & subtraction/cutoffs

leads to complicated structures of $H_1(\mu)$!

Factorization at Renormalized Level :

$$
\begin{aligned}
F_{gg\to h} = \; & H_1(\mu)\langle \hat{O}_1(\mu)\rangle \\
& + 4\int_0^1 \frac{dz}{z}\Big[\bar{H}_2(z,\mu)\langle \hat{O}_2(z,\mu)\rangle - [\![\bar{H}_2(z,\mu)]\!]\,[\![\langle \hat{O}_2(z,\mu)\rangle]\!]\Big] \\
& + H_3(\mu)\int_0^{M_h}\frac{d\ell_-}{\ell_-}\int_0^{\sigma M_h}\frac{d\ell_+}{\ell_+} J_g(M_h\ell_-,\mu)\, J_g(-M_h\ell_+,\mu)\, S_g(\ell_+\ell_-,\mu) .
\end{aligned}
$$

Here Z_{gg}^{-1} has been involved to absorb the extra ε_{IR} poles!

Similar for $F_{h\to\gamma\gamma}$.

All the Z factors (hence the anomalous dimensions) are either well-known or inferred from RG consistency, not calculated directly at operator level.

Renormalization & ADs of Soft-quark Functions

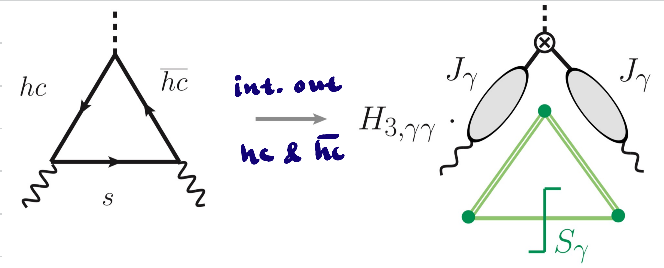
Soft-quark functions are essential ingredients in factorization formulae at NLP.

Goal & Outline:

This chapter devotes to renormalitation and ADs of soft-quark funcs.
We initiate with $h \to \gamma\gamma$ (b-quark loop) as a warm-up;
Then we move to $gg \to h$ (b-quark loop), which has a new feature;
Afterwards, we elaborate on the DY case, which is at Xs level.

1. $h \to \gamma\gamma$ (light-quark induced)

recall the factorization:

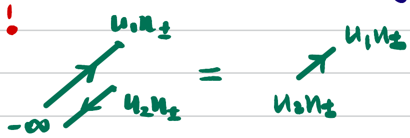


The soft function $S_\gamma(\ell_-\ell_+)$ is the Fourier trans. of the vacuum matrix element of the soft operator $O_\gamma(s,t)$:

$$\boxed{O_\gamma(s,t) = \hat{T}\left\{\bar{q}(tn_-)\, Y_{n_-}(t)\, Y^\dagger_{n_-}(0)\, \frac{\not{n}_-\not{n}_+}{4}\, Y_{n_+}(0)\, Y^\dagger_{n_+}(s)\, q(sn_+)\right\}}$$

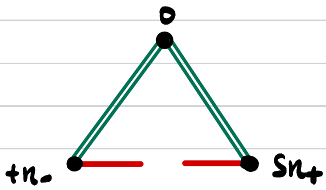
rmk: i) It is more natural and general to work at operator level, instead of matrix element.
ii) It is technically easier to work directly in the position space.

$Y_{n_-}(t)\,Y^\dagger_{n_-}(0)$ & $Y_{n_-}(t)\,Y^\dagger_{n_-}(0)$ can be combined as two finite-length Wilson lines, such that no IR div. exist!



$$[u_1 n_\pm, u_2 n_\pm] = Y_{n_\pm}(u_1)\, Y^\dagger_{n_\pm}(u_2) = \hat{P} \exp\left[i g_s T^a \int_{u_2}^{u_1} d\lambda\, n_\pm \cdot A^a(\lambda n_\pm)\right]$$

$$\rightsquigarrow \quad O_\gamma(s,t) = \hat{T}\left\{ \bar{q}(tn_-)\, [tn_-, 0]\, \frac{\not{n}_-\not{n}_+}{4}\, [0, sn_+]\, q(sn_+)\right\}$$

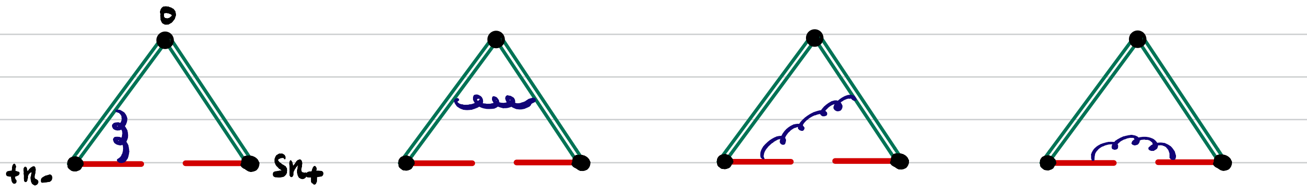


If we replace n_+-sector with a heavy quark field $h_v(0)$, with $v^2 = 1$, which is equiv. to a time-like infinite Wilson line $Y_v^\dagger(0)$.

then we obtain the operator relevant to the leading twist light-cone distribution amplitude (LCDA)!

$\rightsquigarrow$ $O_\gamma(s,t)$ looks like a "double copy" of that of LCDA.
We expect their ADs are related!

In the following, we first elaborate deriving the renormalization factor (hence the AD) at one loop order.
Then we obtain the AD at two loops without explicit calculations, but based on the "double copy" relation.



(a) (b) (c) (d)

(no IR poles, all UV poles in ε!) (no UV poles)

Calculating in the position space and at the operator level is much simpler!

Insert QCD interaction, open Wilson lines, and Wick contrations.

$$I_a = (ig_0)^2 \int d^Dz\, \bar{q}(z) \not{A}^a(z) T^a q(z)\, \bar{q}(tn_-) \int_0^1 du\, t\, n_-\cdot A^b(utn_-) T^b \frac{\not{n}_-\not{n}_+}{4} q(sn_+)$$

$$= -g_s^2 C_F \mu^{2\varepsilon} \frac{e^{\varepsilon\gamma_E}}{(4\pi)^\varepsilon} \int_0^1 du\, t \int \frac{d^Dp}{(2\pi)^D} \int \frac{d^D\ell}{(2\pi)^D} e^{it(\bar{u}n_-\cdot\ell - un_-\cdot p)} \frac{\bar{q}(p)\not{n}_-\not{\ell}}{\ell^2(\ell+p)^2} \frac{\not{n}_-\not{n}_+}{4} q(sn_+)$$

$\nwarrow$ introduce one Feynman para. α

$$= \frac{\alpha_s(\mu)}{4\pi} \frac{2C_F}{\varepsilon} \int_0^1 d\alpha \frac{\alpha}{1-\alpha} \left[\bar{q}(\alpha tn_-) - \bar{q}(tn_-)\right] \frac{\not{n}_-\not{n}_+}{4} q(sn_+)$$

exercise?

$\rightsquigarrow$ ε_{UV}, originates from $u \to 0$ in position space or $\ell \to \infty$ in momentum space.

by background field method

$$= \frac{\alpha_s(\mu)}{4\pi} \frac{2C_F}{\varepsilon} \int_0^1 d\alpha \frac{\alpha}{1-\alpha} \left[O_\gamma(s,\alpha t) - O_\gamma(s,t)\right] + O(\alpha_s^2)$$

$$= \frac{\alpha_s(\mu)}{4\pi} \frac{2C_F}{\varepsilon} \int_0^1 d\alpha \left(\frac{\alpha}{1-\alpha}\right)_+ O_\gamma(s,\alpha t) + O(\alpha_s^2).$$

$\hookrightarrow$ plus distribution, non-local.

cusp contribution

$$I_b = -2i g_s^2 C_F \mu^{2\epsilon} \frac{e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\, \bar{q}(tn_-) \frac{\not{n}_-\not{n}_+}{4} q(sn_+) \int_0^1 du \int_0^1 dv\; st \boxed{\int \frac{d^D\ell}{(2\pi)^D} \frac{e^{-i\ell\cdot(utn_- - vsn_+)}}{\ell^2}}$$

$$= -\frac{\alpha_s(\mu)}{4\pi} C_F \left[\frac{2}{\epsilon^2} + \frac{2}{\epsilon}\ln(st\mu^2 e^{2\gamma_E})\right] O_\gamma(s,t) + O(\epsilon^0)$$

ϵ_{UV}

from Wick contraction between $A^a(utn_-)$ & $A^b(vsn_+)$ from Wilson lines.

plot (d) does not have UV poles, due to off-light cone phase factor.
plot (c) does not contribute either in the Feynman gauge, but it does contribute in the general R_ξ gauge

$$I_c^{(\xi)} = -\frac{\alpha_s(\mu)}{4\pi\epsilon} C_F (1-\xi)\, O_\gamma(s,t) + O(\epsilon^0)$$

Counter terms $= -\left[I_a + I_b + \text{mirrors} + Z_q\big|_{\xi=1}\right]$

or $= -\left[I_a^{(\xi)} + I_b^{(\xi)} + I_c^{(\xi)} + \text{mirrors} + Z_q\right]$

$$q^{bare} = Z_q^{\frac{1}{2}} q^{ren} = \left[1 - \frac{\alpha_s(\mu)}{4\pi}\frac{C_F}{2\epsilon}\xi + O(\alpha_s^2)\right] q^{ren}$$

local

$$\rightsquigarrow O_\gamma(s,t;\mu) = O_\gamma^{bare}(s,t) + \frac{\alpha_s(\mu)}{4\pi} C_F \int_0^1 du \left[\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln(st\mu^2 e^{2\gamma_E}) + \frac{1}{2\epsilon}\right)\delta(1-u) - \frac{2}{\epsilon}\left(\frac{u}{1-u}\right)_+\right]\left(O_\gamma^{bare}(us,t) + O_\gamma^{bare}(s,ut)\right).$$

$$= \int_0^1 du\, Z_\gamma(u)\left[O_\gamma^{bare}(us,t) + O_\gamma^{bare}(s,ut)\right].$$

1) Nucl. Phys. B 129 (1977) 66
139 (1978) 545

AD in $\overline{MS}$ scheme in general (in the convolution sense)

2) hep-ph/0512208

$\hat{O}_{bare}(\{w\})$ is expressed in terms of renormalized fields (w./ Z_q, Z_A, Z_{α_s}). It depends on kinematic variable(s) $\{w\}$.

$$\boxed{\hat{O}(\{w\};\mu) = \int\{dw'\}\, Z(\{w\},\{w'\})\, \hat{O}_{bare}(\{w'\}) := Z \otimes \hat{O}_{bare}}$$

$$\hat{O}_{bare}(\{w\}) = Z^{-1}\otimes\hat{O} = \int\{dw'\}\, Z^{-1}(\{w\},\{w'\})\, \hat{O}(\{w'\};\mu)$$

$$(Z\otimes Z^{-1} = Z^{-1}\otimes Z = 1)$$

Operator mixing may happen.
$\hat{O}_i = Z_{ij}\otimes \hat{O}_j^{bare}$
Z then is matrix

In $\overline{MS}$ scheme, $Z = \delta(\{w\}-\{w'\}) + \sum_{n\geq 1}\frac{1}{\epsilon^n} Z^{(n)}(\{w\},\{w'\})$.

RGE of $\hat{O}(\{w\};\mu)$ reads:

$$\boxed{\frac{d}{d\ln\mu}\hat{O}(\{w\};\mu) = -\gamma\otimes\hat{O} = -\int\{dw'\}\,\gamma(\{w\},\{w'\})\,\hat{O}(\{w'\};\mu)}$$

$\rightsquigarrow$ Note that $0 = \frac{d}{d\ln\mu}\hat{O}_{bare}$.

Then $\frac{d}{d\ln\mu}\hat{O}(\{w\};\mu) = \int\{dw'\}\left[\frac{d}{d\ln\mu} Z(\{w\},\{w'\})\right]\hat{O}_{bare}(\{w'\})$

$$= \int\{dw'\}\left[\frac{d}{d\ln\mu} Z(\{w\},\{w'\})\right]\int\{dw''\} Z^{-1}(\{w'\},\{w''\})\,\hat{O}(\{w''\},\mu)$$

$$= -\int\{dw''\}\underbrace{\left[-\int\{dw'\}\left(\frac{d}{d\ln\mu} Z(\{w\},\{w'\})\right) Z^{-1}(\{w'\},\{w''\})\right]}_{\gamma(\{w\},\{w''\})}\hat{O}(\{w''\};\mu)$$

i.e., $\boxed{\gamma = -\left(\frac{d}{d\ln\mu} Z\right)\otimes Z^{-1}}$ ………………… $(*)$

It is ε finite, because $\frac{d}{d\ln\mu} = \frac{\partial}{\partial\ln\mu} + \frac{d\alpha_s(\mu)}{d\ln\mu}\frac{d}{d\alpha_s(\mu)}$ usual QCD beta func.

and $\frac{d}{d\ln\mu}\alpha_s(\mu) = -2\varepsilon\,\alpha_s(\mu) + \beta(\alpha_s)$

In reality, there is another formulation of $(*)$.

$$\gamma\otimes Z = \gamma + \gamma\otimes\sum_{n\geq1}\frac{Z^{(n)}}{\varepsilon^n} = -\frac{d}{d\ln\mu} Z = -\sum_{n\geq1}\frac{1}{\varepsilon^n}\left[\frac{\partial Z^{(n)}}{\partial\alpha_s}\frac{d\alpha_s}{d\ln\mu} + \frac{\partial Z^{(n)}}{\partial\ln\mu}\right]$$

$\rightsquigarrow \gamma = 2\alpha_s\frac{\partial}{\partial\alpha_s} Z^{(1)}$.

Back to $h\to\gamma\gamma$

$$Z^{(1)} = \frac{\alpha_s}{4\pi}C_F\left[\left(\ln(s+\mu^2 e^{i\pi/2}) + \tfrac{1}{2}\right)\delta(1-u) - 2\left(\frac{u}{1-u}\right)_+\right] + (s\leftrightarrow t)$$

$$\rightsquigarrow \gamma\otimes O_\gamma = -\frac{\alpha_s}{4\pi}4C_F\Big\{-\left(\ln(s+\mu^2 e^{i\pi/2}) + \tfrac{1}{2}\right)O_\gamma(s,t;\mu)$$
$$+\int_0^1 du\left(\frac{u}{1-u}\right)_+\left(O_\gamma(us,t;\mu) + O_\gamma(s,ut;\mu)\right)\Big\} + O(\alpha_s^2)$$

Exercise : (1) $\int_{-\infty}^{+\infty}\frac{dz}{2\pi}e^{iwz}\ln(i\mu(z-i0^+)e^{\gamma_E})\int_0^\infty dw' e^{-iw'(z-i0^+)}\tilde{O}(w')$

$$= \int_0^\infty dw'\left[-\delta(w-w')\ln\frac{w}{\mu} - w\left[\frac{\theta(w-w')}{w(w-w')}\right]_+\right]\tilde{O}(w')$$

(2) $\int_{-\infty}^{+\infty}\frac{dz}{2\pi}e^{iwz}\int_0^1 du\left(\frac{u}{1-u}\right)_+\int_0^\infty dw' e^{-iw'u(z-i0^+)}\tilde{O}(w')$

$$= \int_0^\infty dw'\left[\delta(w-w') + w\left[\frac{\theta(w'-w)}{w'(w'-w)}\right]_+\right]\tilde{O}(w')$$

$$S_\gamma(w) = \frac{1}{2\pi i}\mathrm{Disc}_{\ell_+\ell_-}\int_{-\infty}^{+\infty}ds\int_{-\infty}^{+\infty}dt\, e^{is\ell_+ - it\ell_-}\langle 0|O_\gamma(s,t)|0\rangle$$

$\hookrightarrow w = \ell_+\ell_-$ a consequence of reparameterization/Lorentz invariance

Then it follows :

$$\frac{d}{d\ln\mu} S_\gamma(w;\mu) = -\int_0^\infty dw' \, \gamma_\gamma(w,w') \, S_\gamma(w';\mu)$$

with $\gamma_\gamma(w,w') = -\frac{\alpha_s}{4\pi} 4C_F \left[\left(\ln\frac{w}{\mu^2} + \frac{3}{2} \right) \delta(w-w') + 2w\,T(w,w') \right] + \mathcal{O}(\alpha_s^2)$

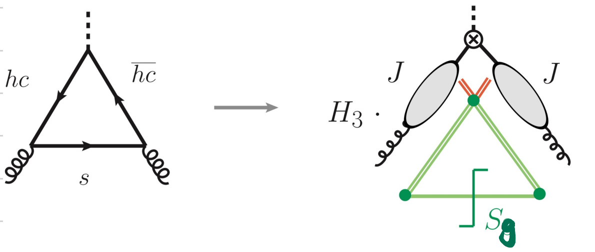
$$T(w,w') = \left[\frac{\theta(w-w')}{w(w-w')} + \frac{\theta(w'-w)}{w'(w'-w)} \right]_+$$

LN kernel in LCDA

$$\gamma_\gamma(w,w') = -\Big\{ \delta(w-w') \left[\Gamma_{\rm cusp}(\alpha_s) \ln\frac{w}{\mu^2} + \gamma_s(\alpha_s) \right] + 2w\,T(w,w')\, \Gamma_{\rm cusp}(\alpha_s) \Big\}$$

2. $gg \to h$ via light-quark loops.

recall the factorization



It looks rather similar as the case of $h \to \gamma\gamma$. But there is an essential distinction namely, the two external states carry color charges, such that the amplitude is not IR finite, but rather has extra IR poles in ε.

This distinction will require extra ingredients to derive the AD apart from the formalism developped in $h \to \gamma\gamma$.

That external states with color charges is very common in pQCD, e.g., DY at NLP later on. We will summarize the generality in the end.

The soft function $S_g(w = \ell_- \ell_+)$ is the Fourier trans. of the soft operator :

$$\boxed{\mathcal{O}_g^{\rm uns}(s,t) = \hat{T}\Big\{ \bar{q}(tn_-) Y_{n_-}(t)\, T^a\, Y^\dagger_{n_-}(0) \frac{\not{n}_- \not{n}_+}{4} Y_{n_+}(0)\, T^b\, T^\dagger_{n_+}(s)\, q(sn_+) \Big\}}$$

due to two insertions of color matrices, we can combine semi-infin. Wilson lines to finite lengths any longer!

A direct calculation reveals that fixed-order results based on the above not only have ε_{UV} poles but also ε_{IR} poles as expected, contrary to $h \to \gamma\gamma$ case, where only ε_{UV} poles exist!

$$Y_{n_\pm}(x)\, T^a\, Y^\dagger_{n_\pm}(x) = \mathcal{Y}^{ab}_{n_\pm}(x)\, T^b$$

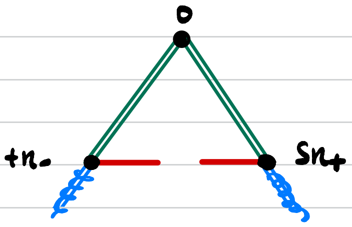
adjoint WL . $(T^a)_{bc} = -if^{abc}$

An energetic gluon comes from $-\infty$ to tn_-, turning the soft (anti-) quark into collinear one, represented by Wilson length to 0, where the hard interaction happens!

$$\rightsquigarrow \quad O_g^{\text{unsub}}(s,t) = \hat{T}\left\{ \bar{q}(tn_-)\, \mathcal{Y}_{n_-}^{ac}(t)\, T^c\, [tn_-,0]\, \frac{\not{n}_-\not{n}_+}{4}\, [0,sn_+]\, \mathcal{Y}_{n_+}^{bd}(s)\, T^d\, q(sn_+) \right\}$$

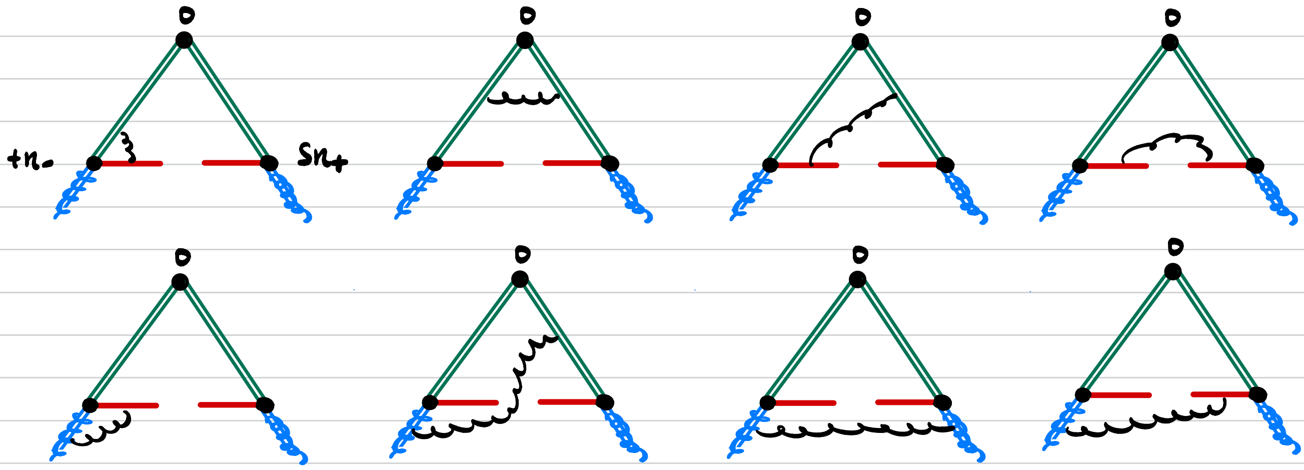
in the fact. fig $\longrightarrow$ $$= \hat{T}\left\{ \bar{q}(tn_-)\, [tn_-,0]\, \mathcal{Y}_{n_-}^{ac}(0)\, T^c\, \frac{\not{n}_-\not{n}_+}{4}\, \mathcal{Y}_{n_+}^{bd}(0)\, T^d\, [0,sn_+]\, q(sn_+) \right\}$$

We adopt the first line in the following.



semi-infinite adjoint Wilson lines.

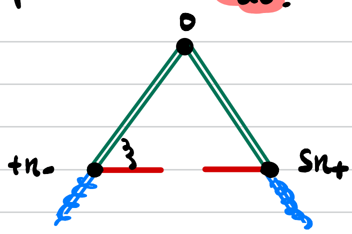
Correlations with those two Wilson lines have IR poles. But AD is related to ε_{UV} poles. How to proceed?



(a) (b) (c) (d)

The first row is the same as that of $h \to \gamma\gamma$, up to a replacement of color factor. All poles are ε_{UV}.

Exercise: Calculate



explicitly.

first diagram of the first row.

ε_{UV}

$$I^{(3)}_{\text{first-row}} = \frac{\alpha_s}{4\pi}\frac{2}{\varepsilon}\left(C_F - \frac{1}{2}C_A\right)\int_0^1 d\alpha \left(\frac{\alpha}{1-\alpha}\right)_+ \left(O_g^{\text{unsub}}(\alpha s,t) + O_g^{\text{unsub}}(s,\alpha t) \right)$$
$$- \frac{\alpha_s}{4\pi} \times \left\{ C_F\left[\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}\ln(st\mu^2 e^{2\gamma_E})\right] - \frac{C_A}{\varepsilon}(1-\xi) + \frac{C_F}{\varepsilon}\right\} O_g^{\text{unsub}}(s,t)$$

cusp diagram

vanish in Feynman gauge

Z_q

We focus on the second row: plots (a) − (d) now.

How IR divergence develops in appearance of semi-infinite Wilson lines?
Use plot (b) as an example. We stick with naive definition of $\mathcal{Y}$.

$$I_b^{(\text{naive})} = 2ig_s^2\mu^{2\varepsilon}\frac{C_A}{2}\frac{e^{\varepsilon\gamma_E}}{(4\pi)^\varepsilon}\int_{-\infty}^{0}d\lambda\int_0^1 dv\, s\int\frac{d^D\ell}{(2\pi)^D}\frac{e^{-i\ell\cdot[(t+\lambda)n_- - vsn_+]}}{\ell^2}$$

We can either integrate out λ first, which results in

$$\sim\int_0^1 dv\, s\int\frac{d^D\ell}{(2\pi)^D}\frac{e^{-i\ell\cdot(tn_- - vsn_+)}}{\ell^2\,(-in_-\cdot\ell)}$$

($i0^+$ are neglected here for brevity.)

there is divergence if $n_-\cdot\ell\to\infty$ (rapidity)
IR in momentum space

or integrate out ℓ first, which results in

$$\sim\int_{-\infty}^{0}d\lambda\int_0^1 dv\, s\,[(t+\lambda)vs]^{-1+\varepsilon}$$

there is IR divergence in position space if $\lambda\to-\infty$

We need to disentangle IR poles from ε_{UV} to derive well-defined UV renormalization & anomalous dimension!

One of the most natural way to regularize IR divergences is via off-shellness (virtuality).

Wilson lines in the soft operator originate from soft-decoupling trans. of energetic fields.

$p_c\sim$ collinear : $n_+\cdot p_c\sim Q \gg n_-\cdot p_c$; $\ell\sim$ soft (ultra-soft)

$$\frac{in_+\cdot p_c}{(p_c+\ell)^2+i0}\longrightarrow\frac{i}{n_-\cdot\ell+\frac{p_c^2}{n_+\cdot p_c}+i0} := \frac{i}{n_-\cdot\ell+\delta_-}$$

$$\underset{\lambda^0}{n_-\ell\, n_+\ell} + \ell_\perp^2 + p_c^2 + n_+\cdot p_c\, \underset{\lambda}{n_-\ell} + \overset{\lambda^2}{n_-\cdot p_c}\, \overset{\lambda}{n_+\ell}$$

$$\delta_- = \frac{p_c^2}{n_+\cdot p_c}+i0\,,\quad \text{likewise}\quad \delta_+ = \frac{p_{\bar c}^2}{n_-\cdot p_{\bar c}}+i0.$$

eikonal propagator related to Wilson line

$$\frac{i}{n_-\cdot \ell + i0^+} \leftrightarrow \hat{P} \exp\left[i g_s T^a \int_{-\infty}^{0} d\lambda\, e^{-i\lambda 0^+}\, n_\pm \cdot A^a(x+\lambda n_\pm)\right]$$

$= 1$ and normally suppressed

$$\boxed{\frac{i}{n_\pm\cdot \ell + \delta_\pm} \leftrightarrow \hat{P} \exp\left[i g_s T^a \int_{-\infty}^{0} d\lambda\, e^{-i\lambda \delta_\pm}\, n_\pm \cdot A^a(x+\lambda n_\pm)\right]}$$

Then semi-infinite WL has exponential suppression when $\lambda \to -\infty$.
IR poles are regulated as $\log(\delta_+/\mu)$.

In this context, we refine the define two semi-infinite adjoint WLs.

$$\mathcal{Y}^{ab}_{n_\pm} = \hat{P} \exp\left[-g_s f^{abc} \int_{-\infty}^{0} d\lambda\, e^{-i\lambda\delta_\pm}\, n_\pm \cdot A^c(x+\lambda n_\pm)\right].$$

Based on this definition, ε poles in plot (a)–(d) are all UV. But they has $\delta_\pm$ dependence!

For example, now

$$I_b^{(\delta_-)} = 2 i g_s^2 \mu^{2\varepsilon} \frac{C_A}{2} \frac{e^{\varepsilon\gamma_E}}{(4\pi)^\varepsilon} \int_{-\infty}^{0} d\lambda \int_0^1 dv\, s \int \frac{d^D\ell}{(2\pi)^D} \frac{e^{-i\ell\cdot[(t+\lambda)n_- - vsn_+] - i(t+\lambda)\delta_-}}{\ell^2}$$
$$\times\ \bar{q}(tn_-) T^a \frac{\not{n}_- \not{n}_+}{4} T^b q(sn_+)$$

$$= \frac{\alpha_s(\mu)}{4\pi} C_A e^{\varepsilon\gamma_E} \Gamma(1-\varepsilon) (s\mu^2)^\varepsilon \int_{-\infty}^{t} d\lambda \int_0^1 dv\, [\lambda v]^{-1+\varepsilon} e^{-i\lambda\delta_-} \times (\cdots)$$

$$\delta_- = \frac{p_c^2}{n_+\cdot p_c} + i0$$

$$= \frac{\alpha_s(\mu)}{4\pi} \frac{C_A}{\varepsilon}\left[\ln\frac{-\delta_-}{\mu} + \ln(-i\mu t e^{\gamma_E}) + O(\delta_-)\right] O_g(s,t) + O(\varepsilon^0).$$

ε_{UV}

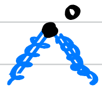
In total, the sum of the two rows read:

$$I^{(\delta_\pm)}_{O_g^{\rm uns}} = \frac{\alpha_s}{4\pi}\frac{2}{\varepsilon}\left(C_F - \frac{1}{2}C_A\right)\int_0^1 d\alpha \left(\frac{\alpha}{1-\alpha}\right)_+ \left(O_g^{\rm uns}(\alpha s, t) + O_g^{\rm uns}(s,\alpha t)\right)$$
$$+ \frac{\alpha_s}{4\pi}\left[\frac{C_A}{\varepsilon}\left(2 + \ln(st\mu^2 e^{2\gamma_E}) + 2\ln\frac{-\delta_-\delta_+}{\mu^2} - \ln\frac{\partial_s\partial_t}{\mu^2}\right)\right.$$
$$\left. - 2C_F\left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln(st\mu^2 e^{2\gamma_E})\right) - \frac{C_F}{\varepsilon}\right] O_g^{\rm uns}(s,t)$$

$\to Z_q$

We can not read off the Z factor, which should only be of UV nature, directly, since the above depends on $\delta_-\delta_+$ as IR regulators.

Those IR-related terms can be systematically subtracted and the subtraction procedure is consistent with factorization, which will be discussed later!

$$\hat{S}_g(0) = \mathcal{Y}^{ac}_{n_-}(0)\, \mathcal{Y}^{cb}_{n_+}(0) \quad :$$ 

define $\quad \delta^{ab} \langle 0 | \hat{T}\, \hat{S}_g(0) | 0 \rangle = \delta^{ab} R_+ R_-$

then $\quad \boxed{ O_g(s,t) := \dfrac{O_g^{\text{ums}}(s,t)}{\langle S_g(0) \rangle} = \dfrac{O_g^{\text{ums}}(s,t)}{R_+ R_-} . }$

Easy to find :

$$\langle \hat{S}_g(0) \rangle = R_+ R_- = 1 + \frac{\alpha_s}{4\pi} C_A \cdot \left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \ln(-\partial_-\partial_+/\mu^2) \right]$$

cancel with $\partial_-\partial_+$-dep. in O_g^{ums} !

$$\rightsquigarrow O_g(s,t;\mu) = O_g^{\text{bare}}(s,t) - \frac{\alpha_s(\mu)}{4\pi} \Big\{ \Big[\frac{2}{\epsilon^2}(C_A - C_F) - \frac{2}{\epsilon}\left(C_F - \frac{1}{2}C_A\right) \ln(s t \mu^2 e^{2\gamma_E})$$

in mom. space this becomes $\ln(\ell_-\ell_+/\mu^2) = \ln(w/\mu^2)$

$$- \frac{1}{\epsilon} C_A \ln\left(\frac{\partial_s \partial_t}{\mu^2}\right) - \frac{C_F - 2C_A}{\epsilon} \Big] O_g^{\text{bare}}(s,t)$$

$$+ \frac{2}{\epsilon}\left(C_F - \frac{1}{2}C_A\right) \int_0^1 d\alpha \left(\frac{\alpha}{1-\alpha}\right)_+ \left(O_g^{\text{bare}}(\alpha s, t) + O_g^{\text{bare}}(s, \alpha t) \right) \Big\} + \mathcal{O}(\alpha_s^2)$$

well defined !

Based on the same formulae, we can read off the AD γ_g.

In mom. space, it reads

$$\boxed{ \gamma_g(w,w') = -\frac{\alpha_s(\mu)}{4\pi} \left\{ \left[4(C_F - C_A) \ln\frac{w}{\mu^2} + 6C_F \right] \delta(w-w') + 8\left(C_F - \frac{1}{2}C_A\right) w\, \Gamma(w,w') \right\} }$$

Subtraction & Factorization.

Q1 : Why subtracted procedure produces the correct result ?

Q2 : We substract something, we must multiply it back. How can this be consistent with factorization ?

The two questions are closely related.

Matching eq. of the jet function :

$$\int d^D x\, e^{i p_s \cdot x_-}\, \hat{T} \left\{ W_n^\dagger\, i \not{D}_n^\perp\, \xi_n(x)\, \bar{\xi}_n W_n(0) \right\} = J_g(\bar{n}\cdot\hat{p}\, n\cdot p_s)\, \mathcal{A}_n^\perp(0)\, \frac{i\pi}{2 n\cdot p_s}$$

collinear gauge-invariant gluon field.

To extract J_g, we impose $\langle k | \cdots | 0 \rangle$, with $k^2 = 0$, $k \sim n$.

Then $\langle k | \mathcal{A}_n^{\perp}(0) | 0 \rangle \sim \not{\epsilon}_{\perp}$ to all orders of α_s, since $k^2 = 0$ all loop corrections are scaless.

In 2212.10447, two external gluons are onshell: $k_1^2 = k_2^2 = 0$.
But as mentioned, $\delta_\pm$ - regulators in WLs are dictated by off-shellness of external particles (gluons in the current case),
we should consider the factorization in the context of small off-shellness.

In this context, the factorization reads

$$(*) \cdots\cdots \mathcal{M}_{gg\to h} \cong \underbrace{H_3 \cdot \left[J_g(M_h \ell_-)\, J_g(-M_h \ell_+) \right] \otimes S_g^{uns}(\ell_-\ell_+) \langle A_g(k_1^2) \rangle \langle A_g(k_2^2) \rangle}_{T_3}$$

① If $k_1^2 = k_2^2 = 0$, then $\left\{ \begin{array}{l} \langle A_g(k_i^2) \rangle = 1 \text{ to all orders of } \alpha_s . \\ T_3 \text{ has IR poles of } \varepsilon \text{ in DR}. \end{array} \right.$
(no $\delta_\pm$ regulators)

Such IR poles in ε can be removed by Z_{gg}^{-1}, where Z_{gg} renormalizes UV poles of SCET gluonic two-jet operator:

$$O_{gg} = A_{\perp, n}^{\mu, a} A_{\perp \mu, n}^{a}(0)$$

$$Z_{gg} = 1 - \frac{\alpha_s}{4\pi} \left[2 C_A \left(\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \ln \frac{-M_h^2}{\mu^2} \right) \right] + O(\alpha_s^2)$$

② If $k_i^2 \neq 0$, then $\left\{ \begin{array}{l} \text{the amplitude } T_3 \text{ has NO IR poles in } \varepsilon \,! \\ \langle A_g(k_i^2) \rangle \neq 1 \text{ any more, but rather:} \end{array} \right.$
($k_i^2 \neq 0$ propagates as $\delta_\pm$ in WLs of O_g^{uns}) !

$$\langle A_g(k^2) \rangle = 1 + \frac{\alpha_s}{4\pi} C_A \left[\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon} \ln \frac{-k^2}{\mu^2} \right] + O(\alpha_s^2) .$$

Now we rewrite as

$$\begin{aligned} T_3 &= H_3 [J_g J_g] \otimes \frac{S_g^{uns}}{R_+ R_-} \left[\langle A_g(k_1^2) \rangle R_+ \right] \left[\langle A_g(k_2^2) \rangle \right] \\ &= H_3 [J_g J_g] \otimes S_g \underbrace{\left[\langle A_g(k_1^2) \rangle R_+ \right] \left[\langle A_g(k_2^2) \rangle \right]}_{= Z_{gg}^{-1}} . \end{aligned}$$

with $\frac{S_g^{uns}}{R_+ R_-} = S_g$.

One can read the formulae either in on-shell or off-shell cases.

3. Soft-Quark Function in Drell-Yan.

In the above two cases, we focus on amplitude-level soft functions @ NLP.
Now we turn the benchmark process, DY, whose soft function is at cross-section level.
Besides, we will find some universality.

In this section, we focus on the $g\bar{q}$-channel of DY, which is NLP.

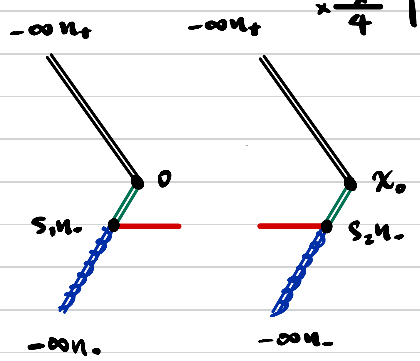
$$\Delta^{NLP}_{g\bar{q}}(z) = 2H(Q^2)\int \frac{d\omega_1}{\omega_1}\int\frac{d\omega_2}{\omega_2}\, J_g(\omega_1)\, J_g^*(\omega_2)\, S^{NLP}_{g\bar{q}}(\Omega;\omega_1,\omega_2)$$

The soft function $S^{NLP}_{g\bar{q}}$ is defined as three-fold Fourier trans. of the soft operator:

$$O^{NLP}_{g\bar{q}}(x_0;\{s\}) = \frac{g_s^2}{N_c C_F}\cdot \mathrm{Tr}_c\, \bar{\hat{T}}\Big[(\bar{q}_s Y_{n_-})(x_0+s_2 n_-)\, T^a\, Y^\dagger_{n_-}(x_0)\, Y_{n_+}(x_0)\Big]\frac{\not{n}_-}{4}$$
$$\hat{T}\Big[Y^\dagger_{n_+}(0)\, Y_{n_-}(0)\, T^a\, [Y^\dagger_{n_-} q_s](s_1 n_-)\Big]$$

uns

$$= \frac{g_s^2}{N_c C_F}\cdot \mathrm{Tr}_c\, \bar{\hat{T}}\Big[\bar{q}_s(x_0+s_2 n_-)\,(\mathcal{Y}^\dagger)^{ac}_{n_-}(x_0+s_2 n_-)\, T^c\, [x_0+s_2 n_-, x_0]\, Y_{n_+}(x_0)\Big]$$
$$\times\frac{\not{n}_-}{4}\,\hat{T}\Big[Y^\dagger_{n_+}(0)\,[0, s_1 n_-]\, T^d\, \mathcal{Y}^{da}_{n_-}(s_1 n_-)\, q_s(s_1 n_-)\Big]$$



$$S^{NLP}_{g\bar{q}}(\cdots) = \langle 0|\, O^{NLP}_{g\bar{q}}(\cdots)\,|0\rangle$$

LO:

$$S^{NLP}_{g\bar{q}} = \frac{\alpha_s}{4\pi} e^{\epsilon\gamma_E}\Gamma(2-\epsilon)\mu^{2\epsilon}\Big[i\Big(\frac{x^0}{2}+s_2-s_1-i0\Big)\Big]^{-2+\epsilon} \times \Big[i\frac{x^0}{2}\Big]^{-1+\epsilon}$$

Exercise: i) derive this;

ii) Do Fourier trans. to obtain the mom. space version:

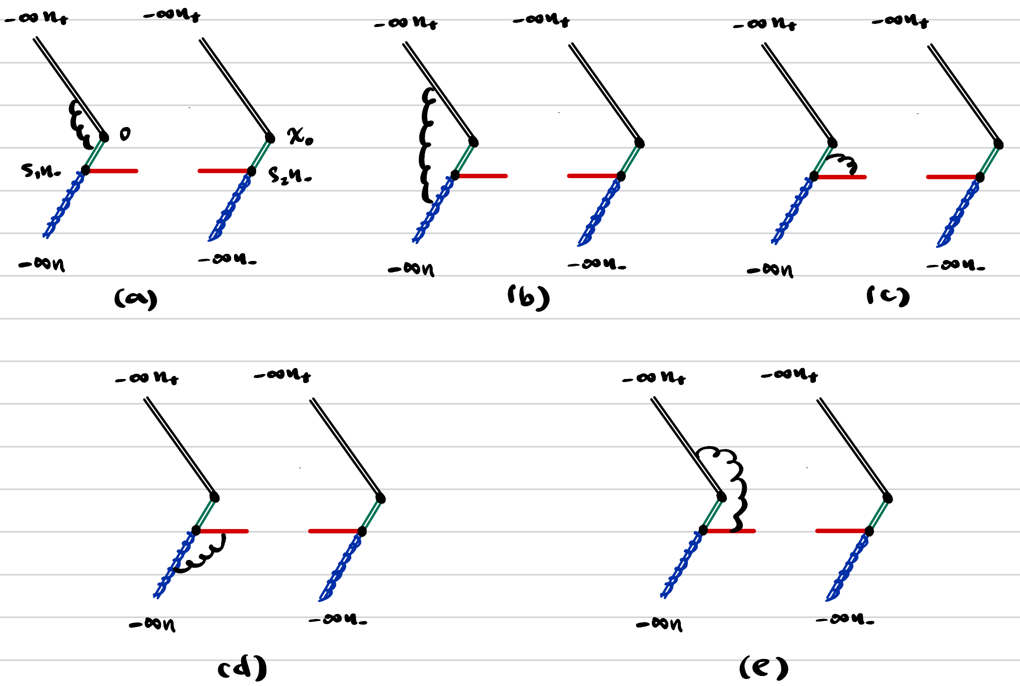
$$S^{NLP}_{g\bar{q}}(\Omega;\omega_1,\omega_2) = \frac{\alpha_s}{4\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\,\omega_1\left(\frac{\mu^2}{\omega_1(\Omega-\omega_1)}\right)^{\epsilon}\theta(\omega_1)\,\theta(\Omega-\omega_1)\,\delta(\omega_1-\omega_2)$$

Due to semi-infinite WLs, there are IR divergences in ϵ if without extra IR regulator!

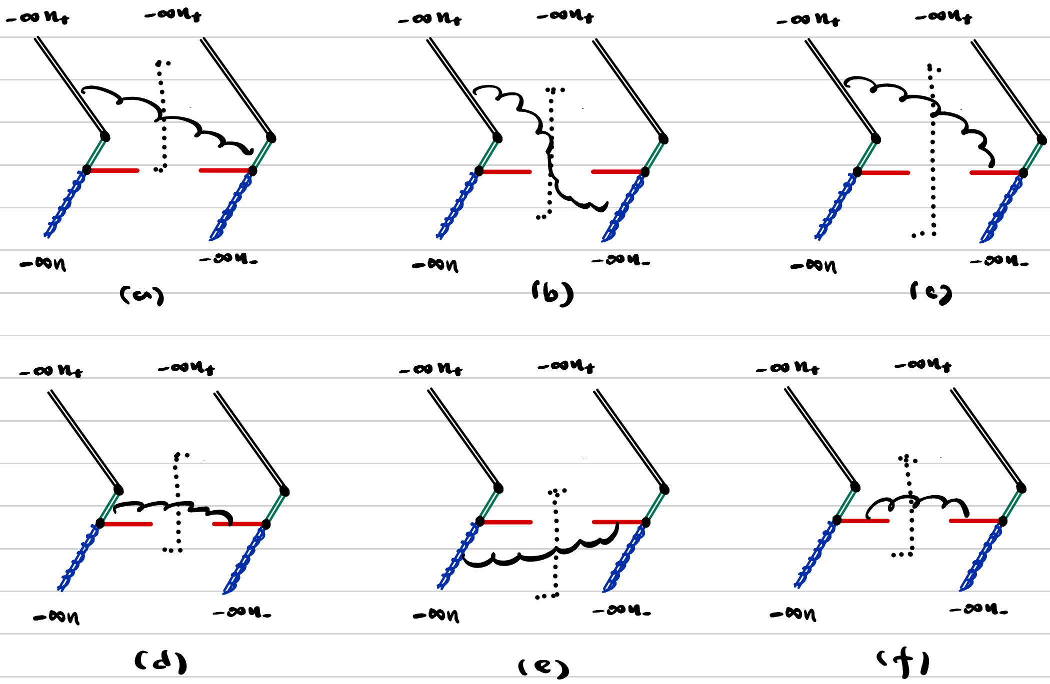
We adopt the same δ-regulator, which is related to off-shellness:

Unlike the $gg\to h$ case, there are both virtual & real corrections.

But in appearance of δ-regulator, there are no ε poles in real emission!



real corrections:



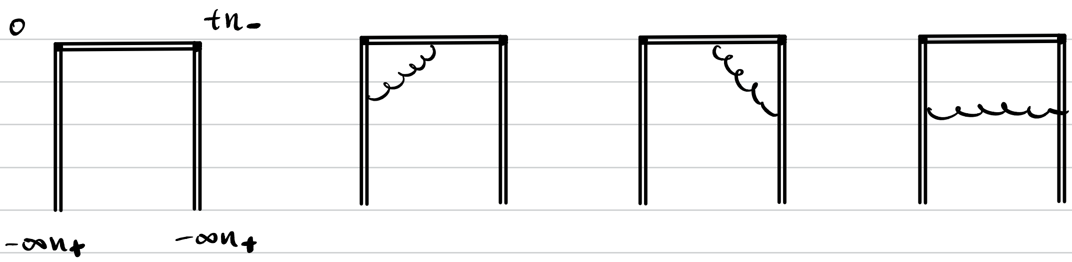
Exercise: show, for example, (a) does not have ε pole in presence of δ-regulator.

$$I^{(\delta_1)}_{\mathrm{us}} = I^{(\delta_1)}_{\mathrm{us,virt}} = \frac{\alpha_s}{4\pi \overset{\varepsilon_{UV}}{\varepsilon}} \left(C_F - \frac{1}{2}C_A\right) \int_0^1 d\alpha \left(\frac{\alpha}{1-\alpha}\right)_+ \left[O^{\mathrm{us}}_{g\bar{q}}(x_0, \alpha s_1, s_2) + O^{\mathrm{us}}_{g\bar{q}}(x_0, s_1, \alpha s_2)\right]$$
$$+ \frac{\alpha_s}{4\pi\varepsilon}\left\{C_A\left[\ln(\mu^2 e^{2\gamma_E} s_1 s_2) - 2\ln\frac{\mu^2}{|\delta_+|^2} - \ln\frac{\partial_{s_1}\partial_{s_2}}{\mu^2} + 2\right]\right.$$
$$\left. - C_F\left[\frac{4}{\varepsilon} + 2\ln\frac{\mu^2}{|\delta_+|^2} + 2\ln(\mu^2 e^{2\gamma_E} s_1 s_2)\right]\right\} O^{\mathrm{MP}}_{g\bar{q}}(x_0, s_1, s_2)$$
$$+ O(\alpha_s^2)$$

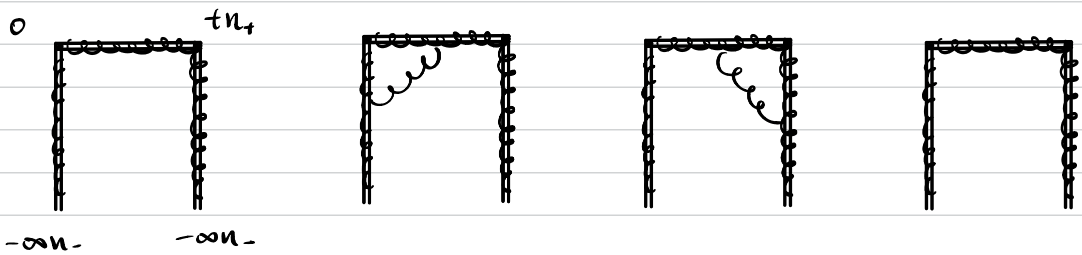
no x_0 dependence, hence no Ω-depe.! (at the moment)

$$\boxed{O^{NLP}_{g\bar q}(x_0,s_1,s_2) := \frac{O^{NLP}_{g\bar q,\text{uns}}(x_0,s_1,s_2)}{S_n^+(x_0)\,S_n^-(x_0)}}$$

S_n^+:



S_n^-:



$$S_n^+ = \langle 0|\hat W_n^+|0\rangle, \quad \hat W_n^+(t) = \frac{1}{N_c}\hat{\mathrm{Tr}}\,\hat T\left[Y^\dagger_{n_+}(tn_-)[tn_-,0]\,Y_{n_+}(0)\right]$$

$$S_n^- = \langle 0|\hat W_n^-|0\rangle, \quad \hat W_n^-(t) = \frac{1}{N_c^2-1}\hat T\left[(\mathcal Y^\dagger_{n_-})^{ab}(tn_+)[tn_+,0]^{bc}\,\mathcal Y^{ca}_{n_-}(0)\right]$$

We explain this later in terms of factorization & quark and gluon PDFs $x\to 1$ behavior!

$$S_n^+ = 1 + \frac{\alpha_s}{\pi}C_F\left[-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\left(\ln\frac{\mu}{|s_+|} + \ln(i\mu e^{\gamma_E}t)\right)\right] + O(\alpha_s^2)$$

$$S_n^- = 1 + \frac{\alpha_s}{\pi}C_A\left[-\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\left(\ln\frac{\mu}{|s_+|} + \ln(i\mu e^{\gamma_E}t)\right)\right] + O(\alpha_s^2)$$

ε_{UV}

$$\rightsquigarrow \left[Z_{g\bar q}O^{NLP}_{g\bar q,\text{bare}}\right](x_0,\{s\}) = O^{NLP}_{g\bar q,\text{bare}}(x_0,\{s\}) - \frac{\alpha_s}{4\pi\varepsilon}\times\Big\{-\beta_0 + C_F(4L_{x_0} - 2L_{s_1} - 2L_{s_2} - 1)$$
$$+ C_A\left[\frac{4}{\varepsilon} + 4L_{x_0} + L_{s_1} + L_{s_2} - \ln\frac{\partial_{s_1}\partial_{s_2}}{\mu^2} + 2\right]\Big\}\, O^{NLP}_{g\bar q,\text{bare}}(x_0,\{s\})$$
$$- \frac{\alpha_s}{4\pi\varepsilon}\left(C_F - \frac{1}{2}C_A\right)\int_0^1 d\alpha\left(\frac{\alpha}{1-\alpha}\right)_+\left(O^{NLP}_{g\bar q,\text{bare}}(x_0,\alpha s_1,s_2) + O^{NLP}_{g\bar q,\text{bare}}(x_0,s_1,\alpha s_2)\right)$$
$$+ O(\alpha_s^2).$$

$$L_{x_0} = \ln\left(i\mu e^{\gamma_E}(x^0/2 - i0)\right)$$
$$L_{s_1} = \ln\left(i\mu e^{\gamma_E}(s_1 + i0)\right)$$
$$L_{s_2} = \ln\left(-i\mu e^{\gamma_E}(s_2 - i0)\right)$$

Then we can read off the AD in position space directly from the above:

$$\frac{d}{d\ln\mu} O_{g\bar{q}}(x_0, \{s\}; \mu) = - \gamma_{g\bar{q}}^{NLP} \otimes O_{g\bar{q}}^{NLP}(x_0, \{s\}; \mu)$$

with
$$\gamma_{g\bar{q}}^{NLP} \otimes O_{g\bar{q}}^{NLP} = -\frac{\alpha_s}{4\pi}\Big\{ \Big[8(C_F + C_A) L_{x_0} - 4\left(C_F - \tfrac{1}{2}C_A\right)(L_{s_1} + L_{s_2}) - 2C_A \ln\frac{\partial_{s_1}\partial_{s_2}}{\mu^2}$$
$$+ 4C_A - 2C_F - 2\beta_0 \Big] O_{g\bar{q}}^{NLP}(x_0, \{s\}; \mu)$$
$$+ 4\left(C_F - \tfrac{1}{2}C_A\right) \int_0^1 d\alpha \left(\frac{\alpha}{1-\alpha}\right)_+ \left[O_{g\bar{q}}^{NLP}(x_0, \alpha s_1, s_2; \mu) + O_{g\bar{q}}^{NLP}(x_0, s_1, \alpha s_2; \mu) \right] \Big\}$$
$$+ \mathcal{O}(\alpha_s^2)$$

We can Fourier transform to the momentum space as:

$$\frac{d}{d\ln\mu} O_{g\bar{q}}^{NLP}(\Omega, \omega_1, \omega_2; \mu) = -\int_0^\infty d\Omega' \int_0^\infty d\omega_1' \int_0^\infty d\omega_2' \, \gamma_{g\bar{q}}^{NLP}(\Omega, \omega_1, \omega_2; \Omega', \omega_1', \omega_2')$$
$$O_{g\bar{q}}^{NLP}(\Omega', \omega_1', \omega_2'; \mu)$$

with

LP-like

$$\gamma_{g\bar{q}}^{NLP}(\Omega, \{\omega\}; \Omega', \{\omega'\}) = \frac{\alpha_s}{4\pi}\Big\{ 4(C_F + C_A)\left[\delta(\Omega - \Omega') \ln\frac{\Omega}{\mu} + \Omega \Gamma_<(\Omega, \Omega')\right] \delta(\{\omega\} - \{\omega'\})$$
$$- \delta(\Omega - \Omega')\,\delta(\omega_2 - \omega_2') \left[\left(4(C_F - C_A)\ln\frac{\omega_1}{\mu} + 3C_F - \beta_0\right)\delta(\omega_1 - \omega_1') + 4\left(C_F - \tfrac{1}{2}C_A\right)\omega_1 \Gamma(\omega_1, \omega_1') \right] \Big\}$$
$$+ (\omega_1 \leftrightarrow \omega_2, \ \omega_1' \leftrightarrow \omega_2') \quad + \quad \mathcal{O}(\alpha_s^2)$$

this is exactly the same as that of γ_g for $gg \to h$!

{ We have seen building blocks of ADs for soft-quark functions!
There are some extra relations between different ADs, see later on.

Now we explain the origin of the subtractions and how they are dictated by factorization.

The previous factorization is partonic:

$$\Delta_{g\bar{q}}^{NLP}(z) = 4H(Q^2) \int \frac{d\omega_1}{\omega_1} \int \frac{d\omega_2}{\omega_2} J_g(Q\omega_1) J_g^*(Q\omega_2) S_{g\bar{q}}^{NLP}(\Omega, \{\omega\})$$

The full factorization reads

$$\sigma_{g\bar{q}}^{NLP}(\tau) = \sigma_0 \int dz \int dx_1 \int dx_2 \, \delta(\tau - x_1 x_2 z) \, f_{\bar{q}/N_1}(x_1) \, f_{g/N_2}(x_2) \, \Delta_{g\bar{q}}^{NLP}(z)$$

which involves (anti)-quark & gluon PDFs.

H, J_g are matching coefficients, which are irrelevant for IR rearrangement.

The point is $\Delta^{NLP}_{g\bar{q}}$ in fact has extra IR poles, which cancel exactly with the PDFs! This is similar as $gg \to h$ case.

We denote those IR poles as $Z^{g\bar{q}}_{PDF}$

$$H[J_g J_g^*] \frac{S^{(S_t)}_{g\bar{q}}}{S_n^+ S_{\bar{n}}^-} S_n^+ Z_x^+ \; S_{\bar{n}}^- Z_A^-$$

$$= H[J_g J_g^*] \; S_{g\bar{q}} \cdot Z^{g\bar{q}}_{PDF}$$

AD of Soft-Quark Functions: A Glimpse of Hidden Structures

1. We have seen the operator "double copy" briefly between O_γ & the LCDA. Here we establish the relation between ADs more precisely.

Note that $\gamma_i(s,t) = \gamma_i(s) + \gamma_i(t)$, w.l $i = \gamma$ or g. at one loop explicitly.