1. Introduction

References: Simmons-Duffin 1602.07982
Rychkov 1601.05000
Poland, Rychkov, Vichi 1805.04405 (numerical conf. bootstrap)
Penedones 1608.04948 (AdS/CFT)
Bissi, Sinha, Zhou 2202.08475 (analytic bootstrap).

QFT: a universal language ($\underbrace{\text{particle physics, string}}$, CMT, etc)

assume Poincaré symmetry

translation. Rotation boost.

$SO(d-1,1) \xrightarrow{E.} SO(d)$

Consider enlarged spacetime symmetry

+ special conformal transformation & dilatation

→ Conformal symmetry

QFT ⟶ CFT

Many situations:

1. End points of RG flows
   - A theory w/ a mass gap (4d YM)
   - A theory w/ massless particles (QED)
   - A scale invariant theory w/ a continuous spectrum ✓

     ↓ enhance into conf.

2. Microscopic lattice systems at 2^{nd} order phase transition.

3d Ising . a model of classical spins $s_i = \{\pm 1\}$

$$Z = \sum_{\{s_i\}} \exp\left(-J \sum_{\langle ij \rangle} s_i s_j\right)$$

Tune J to get CFT, same as φ^4 ~~in~~ 3d

$$S = \int d^3x \; \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{4!} g \varphi^4$$

critical universality

3. AdS/CFT

4. Special motivations for this audience.

- 4d $\mathcal{N}=4$ SYM
- EEC : rep of spacetime symm on celestial sphere $\rightarrow$ (part. broken) conf. symm.

These lectures: CFTs in $d > 2$.

applies to $d = 2$ but only for global

2. Conformal symmetry.

Every local QFT:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

Noether theorem: "improvement term" ambiguities

Define instead as

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}$$

Consider g variation on correlators

$$\langle O_1(x_1)\ldots O_n(x_n)\rangle_g = \frac{1}{Z[g]}\int[d\phi]\, e^{-S[\phi,g]}\, O_1(x_1)\ldots O_n(x_n)$$

$$\langle O_1\ldots O_n\rangle_{g+\delta g} - \langle O_1\ldots O_n\rangle_g = \frac{1}{2}\int dx\sqrt{g}\,\delta g_{\mu\nu}(x)$$

$$\times\left(\langle T^{\mu\nu}(x)\, O_1(x_1)\ldots O_n(x_n)\rangle_g - \langle T^{\mu\nu}\rangle_g\langle O_1\ldots O_n\rangle_g\right)+\ldots$$

Around flat space

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x),$$

$$g_{\mu\nu}\to\tilde{g}_{\mu\nu} = g_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu.$$

Note inv under coor. transf.

$$\langle O_1(\tilde{x}_1)\ldots O_n(\tilde{x}_n)\rangle_{\tilde{g}} = \langle O_1(x_1)\ldots O_n(x_n)\rangle_g$$

we get

$$\sum_{i=1}^{n} \epsilon^{\mu}(x_i)\frac{\partial}{\partial x_i^{\mu}}\langle O_1(x_1)\cdots O_n(x_n)\rangle = -\int dx\, \epsilon_\nu(x)\langle \partial_\mu T^{\mu\nu}(x)\, O_1(x_1)\cdots O_n(x_n)\rangle$$

or,

$$\partial_\mu \langle T^{\mu\nu}(x) O_1(x_1)\cdots O_n(x_n)\rangle = -\sum_{i=1}^{n}\delta(x-x_i)\frac{\partial}{\partial x_i^{\nu}}\langle O_1(x_1)\cdots O_n(x_n)\rangle$$

(Ward identity)

$\rightarrow$ $\partial_\mu T^{\mu\nu} = 0$ up to contact terms.

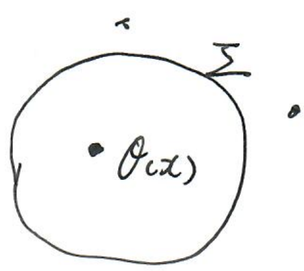
Topological operators and symmetries

Σ: a co-dimensional 1 surface

$$\underline{P^\nu(\Sigma)} = -\int_\Sigma dS^\mu\, T^{\mu\nu}(x)$$

$\hookrightarrow$ topological surface operator

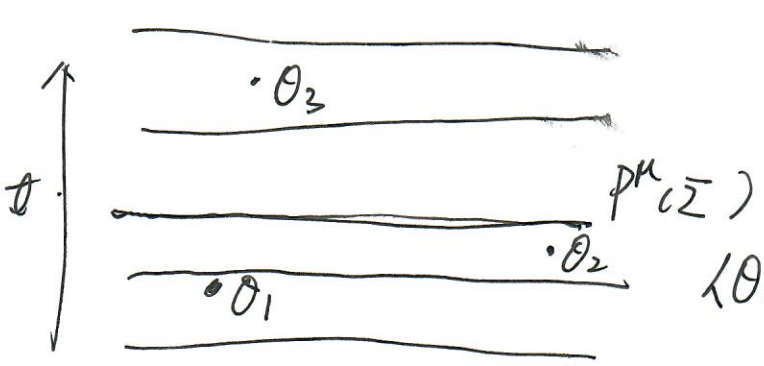
If



~~then~~ $\langle P^\mu(\Sigma)\, O(x)\cdots\rangle = \partial^\mu \langle O(x)\cdots\rangle$

path int. v.s. Hamiltonian formalism:

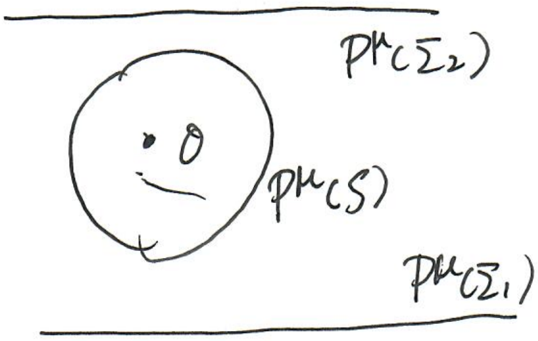
time = spacetime foliation. $\rightarrow$ quantization



correlator $\rightarrow$ time-ordered vev.

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle = \langle 0| T\{\hat{O}_1(t_1,\hat{\vec{x}}_1)\cdots \hat{O}_n(t_n,\hat{\vec{x}}_n)\}|0\rangle$$

T: larger t_i on the left



$$\langle P^\mu(S)\, O(x) \ldots \rangle$$
$$= \langle (P^\mu(\Sigma_2) - P^\mu(\Sigma_1))\, O(x) \ldots \rangle$$
$$= \langle 0 | T\{ [\underbrace{\hat{P}}_{\text{the momentum operator}}, \hat{O}(x)] \ldots \} | 0 \rangle$$

Use Ward id, we get

$$\langle 0 | T\{ [\hat{P}^\mu, \hat{O}(x)] \ldots | 0 \rangle = \partial^\mu \langle O(x) \ldots \rangle$$
$$= \partial^\mu \langle 0 | T\{ \hat{O}(x) \ldots \} | 0 \rangle$$

or $$[\hat{P}, \hat{O}(x)] = \partial^\mu \hat{O}(x)$$

From now on, we will drop the hat.

Same applies for any charge Q from a current J^μ.

We can also consider the finite version

$$O(x) = e^{x\cdot P} O(0)\, e^{-x\cdot P}$$

Consider a general current from $T^{\mu\nu}$

$$J_\epsilon^\mu = \epsilon_\nu(x)\, T^{\mu\nu}(x)$$

Conservation:

$$0 = \partial_\mu(\epsilon_\nu T^{\mu\nu}) = \frac{1}{2}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) T^{\mu\nu}$$

$\rightarrow$ Killing equation: $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 0$

Solutions in flat space:

$$p_\mu = \partial_\mu \rightarrow P_\mu, \qquad m_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu \rightarrow M_{\mu\nu}$$

Conformal symmetry

CFT is invariant under position dependent rescaling

$$\delta g_{\mu\nu}(x) = 2\omega(x)\, g_{\mu\nu}(x)$$

$\rightarrow$ $T^{\mu\nu}$ is traceless

$$T^{\mu}{}_{\mu}(x) = 0$$

$$\rightarrow \quad \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = c(x)\,\delta_{\mu\nu}$$

Conformal Killing equation.

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial\cdot\epsilon)\,\delta_{\mu\nu}$$

Ex: derive the condition

$$\partial_\alpha \partial_\beta \partial_\gamma \epsilon_\delta = 0 \qquad (d>2)$$

This condition implies ϵ_μ is at most quadratic.

New solutions:

$d = x^\mu \partial_\mu$ (dilatation) $\longrightarrow$ D

$K_\mu = 2x_\mu(x\cdot\partial) - x^2\partial_\mu$

(sp. conf. tr.) $\longrightarrow$ K_μ.

conf. algebra: raising/lowering operators

$$[D, P_\mu] = P_\mu \quad [D, K_\mu] = K_\mu, \quad [K_\mu, P_\nu] = 2\delta_{\mu\nu}D - 2M_{\mu\nu}$$

$$\left.\begin{matrix}[M_{\mu\nu}, P_\alpha] = \delta_{\nu\alpha}P_\alpha - \delta_{\mu\alpha}P_\nu \\ \quad K_\alpha \qquad\quad K_\alpha \qquad\quad K_\nu\end{matrix}\right\} \; P_\mu, K_\mu \text{ are vectors of } SO(d)$$

$$[M_{\alpha\beta}, M_{\mu\nu}] = \delta_{\beta\mu}M_{\alpha\nu} + \delta_{\alpha\nu}M_{\beta\mu} - \delta_{\alpha\mu}M_{\beta\nu} - \delta_{\beta\nu}M_{\alpha\mu}$$

Consider the linear combinations

$$L_{\mu\nu} = M_{\mu\nu}\,, \quad L_{-1,0} = D\,, \quad L_{0\mu} = \frac{1}{2}(P_\mu + K_\mu)\,, \quad L_{-1\mu} = \frac{1}{2}(P_\mu - K_\mu)$$

$$L_{ab} = -L_{ba}\,, \qquad a,b = -1, 0, \dots d$$

L_{ab} satisfy the commutation relations of $SO(1, d+1)$.

Isomorphic to rotations in $d+2$ dimensions.

The linear action will be useful in embedding space famalism.

3. Local operators

Classify using the conserved charges

Rotation: $$[M_{\mu\nu}, \mathcal{O}^a(0)] = \underbrace{(S_{\mu\nu})_b{}^a}_{\hookrightarrow\, SO(d) \text{ irrep}}\, \mathcal{O}^b(0)$$

$$M_{\mu\nu}\mathcal{O}(x) = M_{\mu\nu}\, \underline{e^{x\cdot P}\mathcal{O}(0)} = e^{x\cdot P}\left(e^{-x\cdot P} M_{\mu\nu} e^{x\cdot P}\right)\mathcal{O}(0)$$

$Q\mathcal{O} = [Q, \mathcal{O}]$

$$\overset{BCH}{=} e^{x\cdot P}(-x_\mu P_\nu + x_\nu P_\mu + M_{\mu\nu})\,\mathcal{O}(0)$$

$$= (x_\nu \partial_\mu - x_\mu \partial_\nu + S_{\mu\nu})\,\mathcal{O}(x)$$

Dilatation: $$[D, \mathcal{O}(0)] = \Delta\, \mathcal{O}(0)$$

$$D K_\mu \mathcal{O}(0) = ([D, K_\mu] + K_\mu D)\,\mathcal{O}(0) = (\Delta - 1) K_\mu \mathcal{O}(0)$$

Bounded from below $\to$ exist operators

$$[K_\mu, \mathcal{O}(0)] = 0 \qquad \underline{\text{primary operators}}$$

Descendant operators from P_μ

$$P_{\mu_1} \cdots\cdots P_{\mu_n} \mathcal{O}(0)$$

Ex: show

$$[K_\mu, \mathcal{O}(x)] = (K_\mu + 2\Delta x_\mu - 2x^\nu S_{\mu\nu})\mathcal{O}(x)$$

$$[Q_\epsilon, \mathcal{O}(x)] = \left(\epsilon\cdot\partial + \frac{\Delta}{d}(\partial\cdot\epsilon) - \frac{1}{2}(\partial^\mu x^\nu)S_{\mu\nu}\right)\mathcal{O}(x)$$

So far infinitesimal, now consider finite

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu \epsilon^\mu = \underbrace{\left(1+\frac{1}{d}(\partial\cdot\epsilon)\right)}_{\text{rescaling}}\underbrace{\left(\delta^\mu_\nu + \frac{1}{2}(\partial_\nu\epsilon^\mu - \partial^\mu\epsilon_\nu)\right)}_{\text{rotation}}$$

Exponentiating,

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x') R^\mu{}_\nu(x'), \quad R^T R = 1_{d\times d}$$

$$\delta_{\mu\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} = \Omega^2(x')\delta_{\alpha\beta}$$

Consider $U = e^{Q_\epsilon}$ and its infinitesimal action, we find

$$U\mathcal{O}^a(x)U^{-1} = \Omega^\Delta(x')\, \rho^a{}_b(R^{-1}(x'))\, \mathcal{O}^b(x')$$

Conformal correlators

Now we can consider the constraints of conformal symmetry on correlators of primary operators

Ward identity (scalars)

$$\langle \mathcal{O}_1(x_1)\ldots \mathcal{O}_n(x_n)\rangle = \langle U\mathcal{O}_1(x_1)U^{-1}\ldots\ldots U\mathcal{O}_n(x_n)U^{-1}\rangle = \Omega(x_1')^{\Delta_1}\cdots\Omega(x_n')^{\Delta_n}\langle \mathcal{O}_1(x_1')\ldots \mathcal{O}_n(x_n')\rangle$$

2pt functions:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\,\mathcal{O}_{\Delta_2}(x_2)\rangle = f(|x_1-x_2|)$$

$$\Big\downarrow\; U\mathcal{O}_\Delta(x)U^{-1} = \lambda^\Delta \mathcal{O}_\Delta(\lambda x)$$

$$= \frac{a}{|x_1-x_2|^{\Delta_1+\Delta_2}}$$

To use special conformal transformation, we use the trick of conformal inversion

$$I:\quad x^\mu \to \frac{x^\mu}{x^2}$$

$$\mathcal{O}_\Delta(x) = (x^2)^{-\Delta}\mathcal{O}_\Delta(x^\mu/x^2)$$

$$k_\mu = -I\circ P_\mu\circ I$$

$$(x_1'-x_2')^2 = \frac{(x_1-x_2)^2}{x_1^2x_2^2}$$

Correct transformation under I requires

either $a=0$ or $\Delta_1=\Delta_2$

3pt function

Fixed by same conditions up to an overal constant.

$$\langle \mathcal{O}_{\Delta_1}(x_1)\,\mathcal{O}_{\Delta_2}(x_2)\,\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\,x_{23}^{\Delta_2+\Delta_3-\Delta_1}\,x_{13}^{\Delta_1+\Delta_3-\Delta_2}}$$

A faster derivation will be given later.

4pt function:

No longer fixed by symmetry.

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

invariant conformal cross ratios

Conf. symm: $x_4 \to \infty$

$x_1 \to$ origin $\quad \to \quad z, \bar{z}$ independent d.o.f.

$x_3 = (1, 0, \ldots 0)$

$x_4 = (x, y, \ldots 0)$ $\quad z = x + iy$

$u = z\bar{z}, \quad V = (1-z)(1-\bar{z})$

4. Embedding space

Action of conformal transformation is linear in $\mathbb{R}^{1,d+1}$

$$P = (P^{-1}, P^0, P^1, \ldots P^d)$$

$$P^2 = -(P^{-1})^2 + (P^0)^2 + \sum_{\mu=1}^{d} P_\mu P^\mu$$

$$P \to gP, \quad g \in SO(1, d+1)$$

To get d physical coordinates, need to eliminate 2 dimensions

$$P^2 = 0 \qquad d+2 \to d+1$$

$$P \sim \lambda P, \ \lambda \neq 0 \qquad d+1 \to d$$

$x \in \mathbb{R}^d \iff$ a null ray in $\mathbb{R}^{1,d+1}$.

We will gauge fix rescaling as

$$P^+ = P^{-1} + P^0 = 1$$

Then x^μ parameterizes P^A as

$$P^A = \left(\frac{1+x^2}{2}, \frac{1-x^2}{2}, x^\mu \right)$$

To reproduce conformal transformations on x^μ, we need to make sure we have the same fixed gauge.

$$P \xrightarrow{g} gP \xrightarrow{\text{rescale}} gP/(gP)^+$$

Consider the following matrix g

$$g^A{}_B = \begin{bmatrix} \frac{1+\lambda^2}{2\lambda} & \frac{1-\lambda^2}{2\lambda} & 0 \\ \frac{1-\lambda^2}{2\lambda} & \frac{1+\lambda^2}{2\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$gP = \left(\frac{\lambda^{-1}+\lambda x^2}{2}, \frac{\lambda^{-1}-\lambda x^2}{2}, x^\mu \right)$$

To restore the gauge, we rescale by λ

$$\lambda gP = \left(\frac{1+\lambda^2 x^2}{2}, \frac{1-\lambda^2 x^2}{2}, \lambda x^\mu \right)$$

Clearly, this corresponds to a dilatation

$$x^\mu \to \lambda x^\mu$$

Ex: Find the matrices g to reproduce other conf. transf.

We can define operators in embedding space. Consider scalars for simplicity

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta} \mathcal{O}(P).$$

$$\mathcal{O}(P)\Big|_{P^+=1} = \mathcal{O}(x)$$

Under $P \to gP$,

$$\mathcal{O}(P) \to \mathcal{O}(gP) = ((gP)^+)^{-\Delta} \mathcal{O}(gP/(gP)^+)$$

In $\mathbb{R}^d$, we should have.

$$U_g \mathcal{O}(x) U_g^{-1} = \Omega(x')^{\Delta} \mathcal{O}(x')$$

This gives

$$\Omega(x') = ((gP)^+)^{-1}, \quad x'^{\mu} = \frac{(gP)^{\mu}}{(gP)^+}$$

To check the Weyl factor, we note

$$ds^2 = dP \cdot dP = (d(gP))^2 = (d(\lambda(P')P'))^2$$

$$= (\lambda dP' + P'(\nabla\lambda \cdot dP'))^2$$

$$= \lambda^2 (dP')^2$$

Using embedding space, it's easy to determine the 2pt and 3pt functions.

Two requirements:

1. Lorentz invariance: $-2P_i \cdot P_j = x_{ij}^2$
2. Individual rescaling: $\mathcal{O}_{\Delta_i}(\lambda_i P_i) = \lambda_i^{-\Delta_i} \mathcal{O}_{\Delta_i}(P_i)$

2pt:

$$\langle O_{\Delta_1}(P_1)\, O_{\Delta_2}(P_2)\rangle = \frac{1}{(-2P_1\cdot P_2)^{\alpha}}$$

$$\alpha = \Delta_1 = \Delta_2$$

3pt:

$$\langle O_{\Delta_1}(P_1)\, O_{\Delta_2}(P_2)\, O_{\Delta_3}(P_3)\rangle = \frac{C_{123}}{(-2P_1\cdot P_2)^{\alpha_3}(-2P_1\cdot P_3)^{\alpha_2}(-2P_2\cdot P_3)^{\alpha_1}}$$

$$\alpha_1 = \frac{\Delta_2+\Delta_3-\Delta_1}{2},\quad \alpha_2 = \frac{\Delta_1+\Delta_3-\Delta_2}{2},\quad \alpha_3 = \frac{\Delta_1+\Delta_2-\Delta_3}{2}.$$

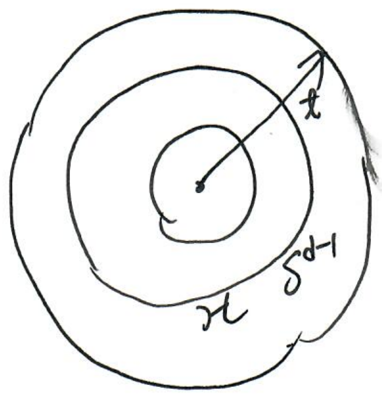
4pt:

$$U = \frac{(-2P_1\cdot P_2)(-2P_3\cdot P_4)}{(-2P_1\cdot P_3)(-2P_2\cdot P_4)} \qquad V = \frac{(-2P_1\cdot P_4)(-2P_2\cdot P_3)}{(-2P_1\cdot P_3)(-2P_2\cdot P_4)}$$

5. Radial quantization and state-operator correspondence.

So far our discussion is true for any quantization. Now let's pick a specific one.

scale symmetry. $\longrightarrow$



radial quantization

time-ordering $\to$ radial ordering

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle = \langle 0|\mathcal{R}\{O_1(x_1)\cdots O_n(x_n)\}|0\rangle$$

$$= \theta(|x_n - x_{n-1}|)\cdots\theta(|x_2-x_1|)\langle 0|O_n(x_n)\cdots O_1(x_1)|0\rangle$$

$$+ \text{ perms}$$

Operators → states

no operator → $|0\rangle$

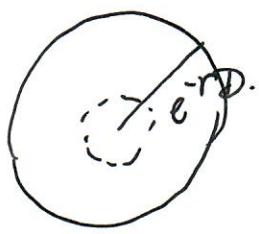
states → operators

For any $|\psi\rangle$, we can decompose it into eigenstates of the dilatation operator

$$D|\mathcal{O}_i\rangle = \Delta_i |\mathcal{O}_i\rangle$$

time translation : e^{-rD} → a factor on the state

shrink the ball



until it becomes a point.

↓

defines an operator at the origin

Using this correspondance and $|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle$

$K_\mu|0\rangle = D|0\rangle = M_{\mu\nu}|0\rangle = 0$

we get.

$$[K_\mu, \mathcal{O}(0)] = 0 \qquad\qquad K_\mu|\mathcal{O}\rangle = 0$$

$$[D, \mathcal{O}(0)] = \Delta\mathcal{O}(0) \quad \leftrightarrow \quad D|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle$$

$$[M_{\mu\nu}, \mathcal{O}(0)] = S_{\mu\nu}\mathcal{O}(0) \qquad M_{\mu\nu}|\mathcal{O}\rangle = S_{\mu\nu}|\mathcal{O}\rangle$$

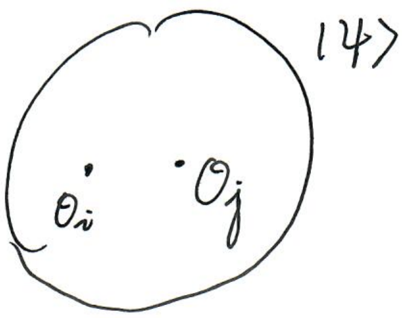
Starting from a conformal primary, the conformal multiplet is constructed by P_μ action

$$|O\rangle, \quad P_\mu|O\rangle, \quad P_\mu P_\nu|O\rangle, \ldots .$$

This is the same as acting w/ derivatives

$$\partial_\mu O(x)\Big|_{x=0}|0\rangle = [P_\mu, O(0)]|0\rangle = P_\mu|O\rangle.$$

6. Operator product expansion



$$O_i(x)O_j(0)|0\rangle = \sum_k C_{ijk}(x, P)O_k(0)|0\rangle$$

Holds as long as there are no other operators inside the sphere with radius $|x|$.

By state-operator correspondence, we can write.

$$O_i(x_1)O_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_2)O_k(x_2)$$

↳ Operator product expansion (OPE)

We can also write

$$O_i(x_1)\,O_j(x_2) = \sum_k C'_{ijk}(x_{13}, x_{23}, \partial_3)\,O_k(x_3)$$

Different C and C' but equivalent OPE.

C_{ijk} is determined by conformal symmetry (consider scalars for simplicity).

We first show

$$C_{ijk}(x,\partial) = |x|^{\Delta_k - \Delta_i - \Delta_j} (\alpha_0 + \alpha_1 x^\mu \partial_\mu + \alpha_2 x^\mu x^\nu \partial_\mu \partial_\nu + \alpha_3 x^2 \partial^2 + \cdots)$$

We act on the OPE w/ D.

LHS:

$$D\,\mathcal{O}_i(x)\,\mathcal{O}_j(0)|0\rangle = (x^\mu \partial_\mu + \Delta_i + \Delta_j)\,\mathcal{O}_i(x)\,\mathcal{O}_j(0)|0\rangle$$

$$= \sum_k (x^\mu \partial_\mu + \Delta_i + \Delta_j)\, C_{ijk}(x,P)\,\mathcal{O}_k(0)|0\rangle$$

RHS:

$$D \sum_k C_{ijk}(x,P)\,\mathcal{O}_k(0)|0\rangle = \sum_k [D, C_{ijk}(x,P)]\,\mathcal{O}_k(0)|0\rangle + \Delta_k\, C_{ijk}(x,P)\,\mathcal{O}_k(0)|0\rangle$$

This gives

$$(\underbrace{x^\mu \partial_\mu}_{\text{counts } x} + \Delta_i + \Delta_j - \Delta_k)\, C_{ijk}(x,P) = [\underbrace{D}_{\text{counts } P}, C_{ijk}(x,P)]$$

After extracting $|x|^{\Delta_k - \Delta_i - \Delta_j}$, the number of x matches the number of P in a small x expansion.

To fix the coefficients we use it to compute the 3pt function.

$$\langle \mathcal{O}_i(x_1)\, \mathcal{O}_j(x_2)\, \mathcal{O}_k(x_3)\rangle = C_{ijk}(x_{12}, \partial_2)\langle \mathcal{O}_k(x_2)\, \mathcal{O}_k(x_3)\rangle$$

We expand the identity in small $|x_{12}|/|x_{23}|$, and this gives the coefficients.

Consider $\Delta_i = \Delta_j = \Delta_\phi$, $\Delta_k = \Delta$

$$\frac{C_{ijk}}{x_{12}^{2\Delta_\phi - \Delta}\, x_{13}^{\Delta}\, x_{23}^{\Delta}} = C_{ijk}(x_{12}, \partial_2)\, x_{23}^{-2\Delta}$$

Ex: Show that we get the following leading coefficients

$$\alpha_0 = C_{ijk}\,, \quad \frac{\alpha_1}{\alpha_0} = \frac{1}{2}\,, \quad \frac{\alpha_2}{\alpha_0} = \frac{\Delta+2}{8(\Delta+1)}$$

$$\frac{\alpha_3}{\alpha_0} = -\frac{\Delta}{16(\Delta+1)\underline{\left(\Delta - \frac{d-2}{2}\right)}}$$

Also explain why this singularity is not a problem.

Ex: In 1d, we have

$$C_{ijk}(x,\partial) = C_{ijk}|x|^{\Delta - 2\Delta_\phi}(\beta_0 + \beta_1 x\partial + \beta_2 x^2\partial^2 + \beta_3 x^3\partial^3 + \dots)$$

Show

$$\beta_m = \frac{(\Delta)_m}{(1)_m(2\Delta)_m}$$

This gives $C_{ijk}(x,\partial) = C_{ijk}\, x^{\Delta - 2\Delta_\phi}\, {}_1F_1(\Delta, 2\Delta, x\partial)$

Knowing OPE, we can compute in principle any correlator

$$\langle O_1(x_1) \ldots\ldots O_n(x_n)\rangle = \sum_{k} C_{12k}(x_{12}, \partial_2)\langle O_k(x_2) \ldots\ldots O_n(x_n)\rangle$$

It terminates at 3pt function which is fixed by symmetry.

7. Conformal blocks

We now use OPE to compute 4pt functions.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{G(u,v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}$$

Consider 12 OPE

$$\phi(x_1)\phi(x_2) = \sum_{O} C_{\phi\phi O}\, C(x_{12}, \partial_2)\, O(x_2)$$

$\uparrow$ spin-l symmetric traceless

More symmetrically let's also do a 34 OPE.

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{O} C^2_{\phi\phi O}\, C(x_{12},\partial_2)\, C(x_{34},\partial_4) \langle O(x_2)\, O(x_4)\rangle$$

$$= \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{O} C^2_{\phi\phi O}\, g_{\Delta_O, l_O}(u,v).$$

$$g_{\Delta_O, l_O}(u,v) = x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}\, C(x_{12},\partial_2)\, C(x_{34},\partial_4) \langle O(x_2) O(x_4)\rangle$$

conformal block.

$$G(u,v) = \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}^2 \, g_{\Delta_{\mathcal{O}}, l_{\mathcal{O}}}(u,v).$$

Computing conformal blocks:

Via OPE:

Consider 1d for simplicity.

Only 1 cross ratio

$$z = \frac{x_{12} x_{34}}{x_{13} x_{24}}$$

$$u = z^2 \qquad v = (1-z)^2$$

Use the integral representation of ${}_1F_1$

$$C_{ijk}(x_{12}, \partial_2) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} x_{12}^{\Delta - 2\Delta_\phi} \int_0^1 dt \, t^{\Delta-1} (1-t)^{\Delta-1} e^{x_{12}\partial_2 t}$$

Act on the 3pt function $\langle \mathcal{O}(x_2) \phi(x_3) \phi(x_4) \rangle$

$$g^{1d}(z) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} x_{12}^{\Delta} x_{34}^{\Delta} \int_0^1 dt \, t^{\Delta-1} (1-t)^{\Delta-1} e^{x_{12}\partial_2 t} \frac{1}{x_{23}^{\Delta} x_{24}^{\Delta}}$$

$$= \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2} \int_0^1 dt \, t^{\Delta-1} (1-t)^{\Delta-1} \frac{x_{12}^{\Delta} x_{34}^{\Delta}}{(x_{23} + t x_{12})^{\Delta} (x_{24} + t x_{12})^{\Delta}}$$

Use the fact that it's a function of z.

Set $\{x_1, x_3, x_4\}$ to be $\{0, 1, \infty\}$

then $x_2 = z$

We get

$$g^{1d}_{\Delta}(z) = \frac{\Gamma(2\Delta)}{\Gamma(\Delta)^2}\int_0^1 dt\, t^{\Delta-1}(1-t)^{\Delta-1}\left(1-\frac{z}{z-1}t\right)^{-\Delta}\left(\frac{z}{1-z}\right)^{\Delta}$$

$$= \left(\frac{z}{1-z}\right)^{\Delta} {}_2F_1\left(\Delta,\Delta;2\Delta;\frac{z}{z-1}\right)$$

$$= z^{\Delta}\, {}_2F_1(\Delta,\Delta;2\Delta;z)$$

Used

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

Via conformal Casimir:

Consider radial quantization and $|x_{3,4}| > |x_{1,2}|$

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle$$

For each $\mathcal{O}$, define a projector

$$|\mathcal{O}| = \sum_{\alpha,\beta=\mathcal{O},P\mathcal{O},PP\mathcal{O}\ldots} |\alpha\rangle N^{-1}_{\alpha\beta}\langle\beta| \qquad N_{\alpha\beta} = \langle\alpha|\beta\rangle$$

$$1 = \sum_{\mathcal{O}} |\mathcal{O}|$$

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O}} \langle 0|\mathcal{R}\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\,\mathcal{R}\{\phi(x_1)\phi(x_2)\}|0\rangle$$

do.

Each term is a conformal block

$$\langle 0|R\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|R\{\phi(x_1)\phi(x_2)\}|0\rangle = \frac{C_{\phi\phi\mathcal{O}}^2}{x_{12}^{2\Delta\phi}x_{34}^{2\Delta\phi}} g_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}(u,v).$$

Consider the quadratic conformal Casimir

$$Cas = -\frac{1}{2}L^{AB}L_{AB}$$

$$= -\frac{1}{2}M^{\mu\nu}M_{\mu\nu} - \frac{1}{2}P\cdot K - \frac{1}{2}K\cdot P + D^2$$

$$Cas|\mathcal{O}\rangle = \lambda_{\Delta,l}|\mathcal{O}\rangle \qquad \lambda_{\Delta,l} = \Delta(\Delta-d) + l(l+d-2)$$

$$Cas|\mathcal{O}| = |\mathcal{O}|Cas = \lambda_{\Delta,l}|\mathcal{O}|$$

Denote the action of L_{AB} on $\phi(x_i)$ by $\mathcal{L}_{AB}^{(i)}$.

$$L_{AB}\phi(x_1)\phi(x_2)|0\rangle = \left([L_{AB},\phi(x_1)]\phi(x_2) + \phi(x_2)[L_{AB},\phi(x_1)]\right)|0\rangle$$
$$= \left(\mathcal{L}_{AB}^{(1)} + \mathcal{L}_{AB}^{(2)}\right)\phi(x_1)\phi(x_2)|0\rangle$$

$$Cas\,\phi(x_1)\phi(x_2)|0\rangle = -\frac{1}{2}\left(\mathcal{L}_{AB}^{(12)}\right)^2\phi(x_1)\phi(x_2)|0\rangle$$

$$\mathcal{L}_{AB}^{(12)} = \mathcal{L}_{AB}^{(1)} + \mathcal{L}_{AB}^{(2)}$$

$$-\frac{1}{2}(\mathcal{L}_{AB}^{(12)})^2\langle 0|R\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|R\{\phi(x_1)\phi(x_2)\}|0\rangle$$
$$= \langle 0|R\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|\,Cas\,R\{\phi(x_1)\phi(x_2)\}|0\rangle$$
$$= \lambda_{\Delta,l}\langle 0|R\{\phi(x_3)\phi(x_4)\}|\mathcal{O}|R\{\phi(x_1)\phi(x_2)\}|0\rangle$$

This implies

$$\mathcal{D}\, g_{\Delta,\ell}(u,v) = \lambda_{\Delta,\ell}\, g_{\Delta,\ell}(u,v)$$

where $\mathcal{D}$ is a differential operator in u and v.

The best way to find $\mathcal{D}$ is to use embedding space.

$$L_{AB} = P_A \frac{\partial}{\partial P^B} - P_B \frac{\partial}{\partial P^A}$$

$$P_{ij} = -2 P_i \cdot P_j = x_{ij}^2$$

$$u = \frac{P_{12} P_{34}}{P_{13} P_{24}} \qquad v = \frac{P_{14} P_{23}}{P_{13} P_{24}}$$

$$\mathcal{D} = \mathcal{D}_z + \mathcal{D}_{\bar{z}} + 2(d-2)\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right)$$

$$\mathcal{D}_z = 2\left(z^2(1-z)\partial_z^2 - z^2 \partial_z\right)$$

In 2d,

$$\mathcal{D} = \mathcal{D}_z + \mathcal{D}_{\bar{z}}$$

consequence of $SO(1,3) = SO(1,2) \times SO(1,2)$

$\mathcal{D}_z$ is the Casimir operator in 1d.

$$\mathcal{D}_z f(z) = 2h(h-1) f(z)$$

With the boundary condition $f(z) \sim z^h$ as $z \to 0$.
we find

$$f_h(z) = z^h\, {}_2F_1(h, h; 2h; z).$$