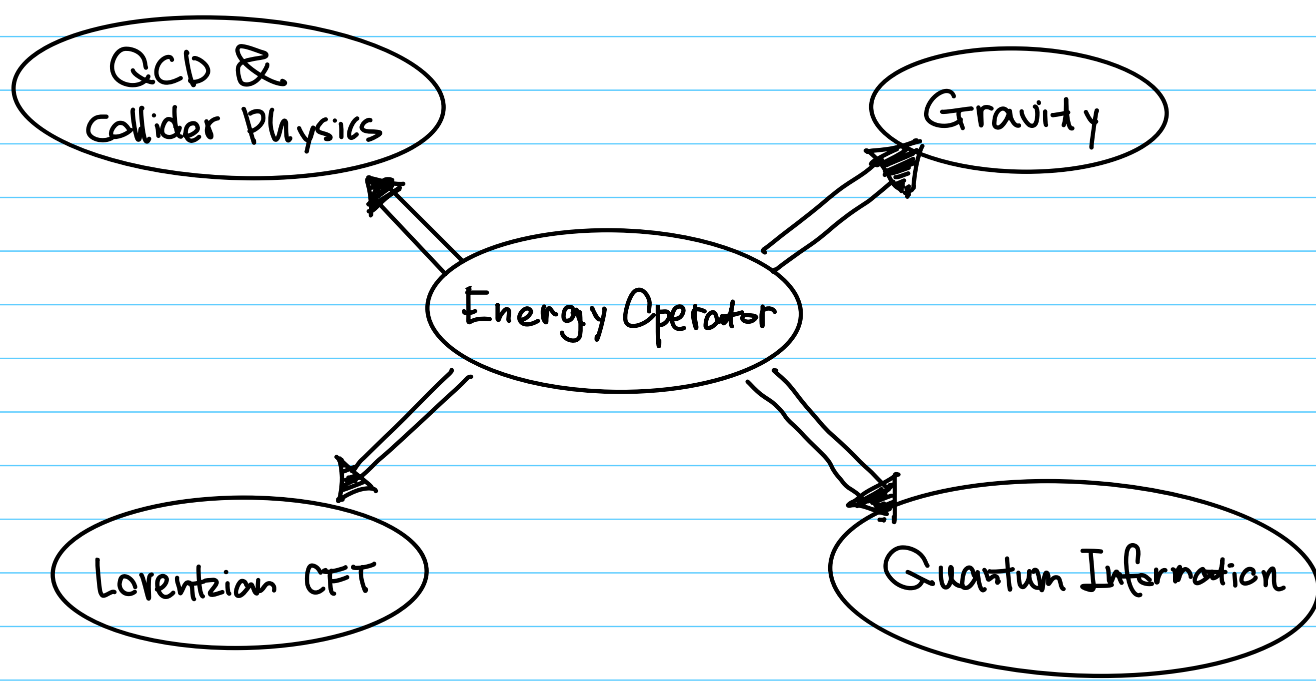
Basics of Energy Correlators

Out lines:

1. Introducing Energy Operators (ANEC), and light-ray operators
2. One point Energy flux; 3. Energy correlators.
4. Collinear limit: factorization and light-ray OPE.
5. Back-to-back limit: factorization / large spin perturbation theory.

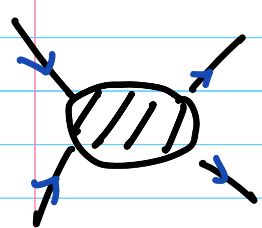
Lecture 1: Energy Operator and light-ray operators.



* What are good collider observables?

	Relation to field theory	Phenomelogy usefulness	Calculability	measurability
S-matrix	defined th. LSZ	hadron mass. coupling	Not IRC Safe require non-pert method.	low multiplicity
Jets	Difficult to write down operator def.	Boost object New phy.	mostly numeric	high multiplicity
(Energy) Correlators	correlator of ANECs.	Precision meas. scaling.	Analytic / numeric	high multiplicity

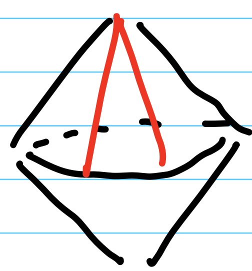
Scattering Amplitude



IRC unsafe

measurable for gapped theory

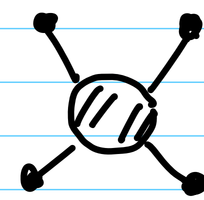
Energy Correlators



IRC safe

Measurable

local correlator



IRC safe

But not measurable at collider

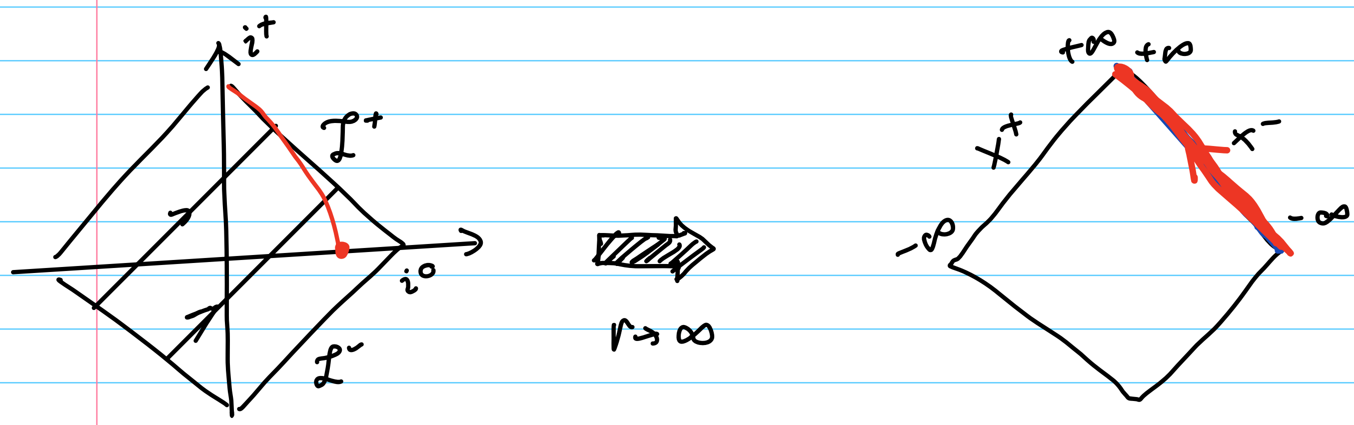
(1) New way to understand some subtle effects in QFT, CFT, Gravity

(2) New probe for phenomenology in the SM (and beyond)

* Definition of Energy Operator ; Penrose diagram

$$\mathcal{E}(n) = \lim_{r\to\infty} r^2 \int_0^\infty dt \; \bar{n}^i T_{0i}(t, r\vec{n})$$

Penrose diagram for Minkowski Space



(visualization of ANEC)

- Light-cone coordinate

$$x_+ = t+r \,, \qquad x_- = t-r \,, \qquad x = (t, \vec{n}r)$$

$$n^\mu = (1, \vec{n}) \,, \quad \bar{n} = (1, -\vec{n}) \qquad x_+ = x\cdot\bar{n} \,, \quad x_- = x\cdot n$$

$$x^\mu = \frac{x_-}{(n\cdot\bar{n})} n^\mu + \frac{x_-}{(n\cdot\bar{n})} \bar{n}^\mu + x_\perp{}^\mu$$

Large r limit $\Rightarrow$ $x^+ \to \infty$, x^- fixed.

$$t = \frac{1}{2}(x_+ + x_-) \qquad r = \frac{1}{2}(x_+ - x_-) = \frac{1}{(n\cdot\bar{n})}(x_+ - x_-)$$

$$\bar{n}^0 \bar{n}^i T_{0i} \Rightarrow \frac{1}{4}(n^\mu + \bar{n}^\mu)(\bar{n}^\nu - n^\nu)\, T_{\mu\nu}$$

$$= -\underset{\substack{\Uparrow \\ \text{vanish}}}{\frac{1}{4} T_{--}} + \frac{1}{4} T_{++}$$

$$\mathcal{E}(n) = \underbrace{\lim_{x^+\to\infty} (x_+)^2 \int_{-\infty}^{+\infty} dx_-}_{\text{Light transformation}} \; T_{++}(x^+ \bar{n}^\mu + x^- n^\mu) \frac{1}{(n\cdot\bar{n})^4}$$

- Collinear spin : $n^\mu \to \lambda n^\mu, \quad \bar{n}^\mu \to \lambda' \bar{n}^\mu$

$$\mathcal{E}(n) \to \lambda^{-3} \mathcal{E}(n)$$ [reparameterization invariance]

$$J_L = -3 = 1 - \Delta \qquad E^{-(2-3)} = E^{-2-J_L}$$

- mass dimension $[\mathcal{E}] = -3 + 4 = 1$

$$\Delta_L = 1 = J - 1$$

- $\mathbb{R}^{1,3}$ as embedding space of celestial sphere

* Energy Operator in free theory

- free scalar theory in $D=4$ (massless)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi$$

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}$$

$$T_{++} = \partial_+\phi\,\partial_+\phi = \bar{n}^\mu\bar{n}^\nu\partial_\mu\phi\,\partial_\nu\phi$$

Let's first consider the field at $\mathcal{I}^+$

$$\phi(x) = \int\frac{d^4k}{(2\pi)^4}(2\pi)\delta_+(k^2)\left[a_k e^{-ikx} + a_k^\dagger e^{ikx}\right]$$

$$= \frac{1}{2}\int\frac{dk_+\,dk_-\,d^2\vec{k}_\perp}{(2\pi)^3}\,\delta(k^+k^- - \vec{k}_\perp^{\,2})\left[a_k e^{-ikx} + \cdots\right]$$

$$= \frac{1}{2(2\pi)^3}\int_0^{+\infty}\frac{dk_+}{k_+}\,2\pi\int_0^\infty k_T\,dk_T$$

$\left(k_- = \frac{k_T^2}{k_+}\right)$
$$\times\left[a_k e^{-i\left(\frac{1}{2}k_+x_- + \frac{1}{2}k_-x_+\right)} + a_k^\dagger e^{\frac{i}{2}\left(k_+x_- + k_-x_+\right)}\right]$$

$$\propto \int_0^{+\infty}\frac{dk_+}{x_+}\left[a_k e^{-i\frac{1}{2}k_+x_-} + a_k^\dagger e^{i\frac{1}{2}k_+x_-}\right]$$

$$\Rightarrow T_{++} \propto \int_0^{+\infty}\frac{dp_+\,dq_+}{(x_+)^2}\,p_+q_+\Big[-a_pa_q e^{-i\frac{1}{2}(p_++q_+)x_-}$$

$$-a_p^\dagger a_q^\dagger e^{i\frac{1}{2}(p_++q_+)x_-} + a_p^\dagger a_q e^{i\frac{1}{2}(p_+-q_+)x_-} + a_p a_q^\dagger e^{\frac{i}{2}(q_+-p_+)x_-}\Big]$$

The integral on x_- gives two different delta:

$$\delta(p_+ + q_+) \qquad\qquad \delta(p_+ - q_+)$$
$$\Downarrow \qquad\qquad\qquad \Downarrow$$
$$p_+ = q_+ = 0 \qquad\qquad p_+ = q_+$$

$$\mathcal{E}(n) \propto \int_0^{+\infty} dp_+ \; (p_+)^2 \; a^\dagger(p_+ n)\, a(p_+ n)$$

- Physical interpretation:

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \, 2\omega_p \, \delta^{(3)}(\vec{p} - \vec{q})$$

$$\mathcal{E}(n) \, |p\rangle = p^0 \, \delta^{(2)}(\vec{n} - \hat{p}) \, |p\rangle$$

! The exponent of energy weighting is determined by J_L, not Δ_L

$$(p^0)^{-2-J_L}$$

$J_L = -3$ for ANEC

* Higher spin Light-ray operator; odd and even spin branch; Chew-fraustri plot

naive generalization

$$\phi\, \partial_+^2 \phi \xrightarrow[\text{transform}]{\text{Light}} \mathcal{E}$$

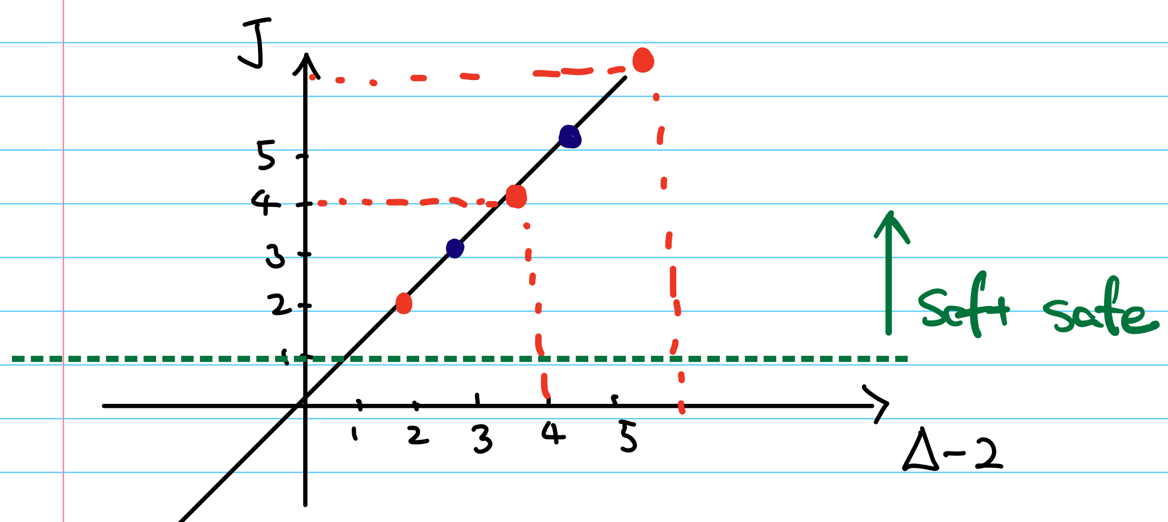
$$\phi\, \partial_+^3 \phi \longrightarrow Q^{[2]} \qquad \underbrace{Q^{[2]}|p\rangle = (p^0)^2 \delta^{(2)}(\ )|p\rangle}_{\text{in free theory}}$$

$$\phi\, \partial_+^4 \phi \longrightarrow Q^{[3]} \quad \ldots$$

$\vdots$

But $\phi\,\partial_+^3\phi$ doesn't exist as a local operator

because $\phi\,\partial_+^3\phi \underset{\text{IBP}}{\sim} -\partial_+\phi\,\partial_+^2\phi \sim \partial_+^2\phi\,\partial_+\phi$



odd spin through analytic continuation

(No local odd spin operator for free scalar)

$$\mathcal{E}^{[J-1]} \propto \int_0^\infty dp_+ \,(p_+)^J\, a^\dagger(p_+ n)\, a(p_+ n)$$

* Turn on interaction; Collinear Safety

- IRC Safety in pQCD

A physical obesrvable is a function of on-shell momentum

$$\mathcal{O}[\{k\}], \qquad \{k\} = (k_1, k_2, \cdots, k_n)$$

▲ Soft safety : $\mathcal{O}$ is insensitive to soft rad.

$$\lim_{k_i \to 0} \mathcal{O}[\{\cdots k_{i-1}, k_i, k_{i+1}, \cdots\}]$$

$$= \mathcal{O}[\{\cdots, k_{i-1}, k_{i+1}, \cdots\}]$$

▲ Collinear Safety : $\mathcal{O}$ is insensitive to coll. rad.

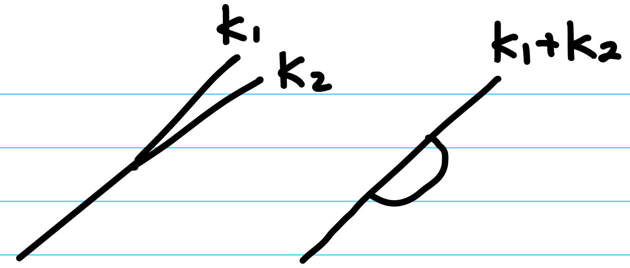
$$\lim_{k_i // k_{i+1}} \mathcal{O}[\{\cdots, k_i, k_{i+1}, \cdots\}]$$

$$= \mathcal{O}[\{\cdots, k_i + k_{i+1}, \cdots\}]$$

▲ IRC safe observables are finite observables in pQCD.

- ANEC is soft safe because of energy weighting.

- ANEC is also collinear safe by linearity

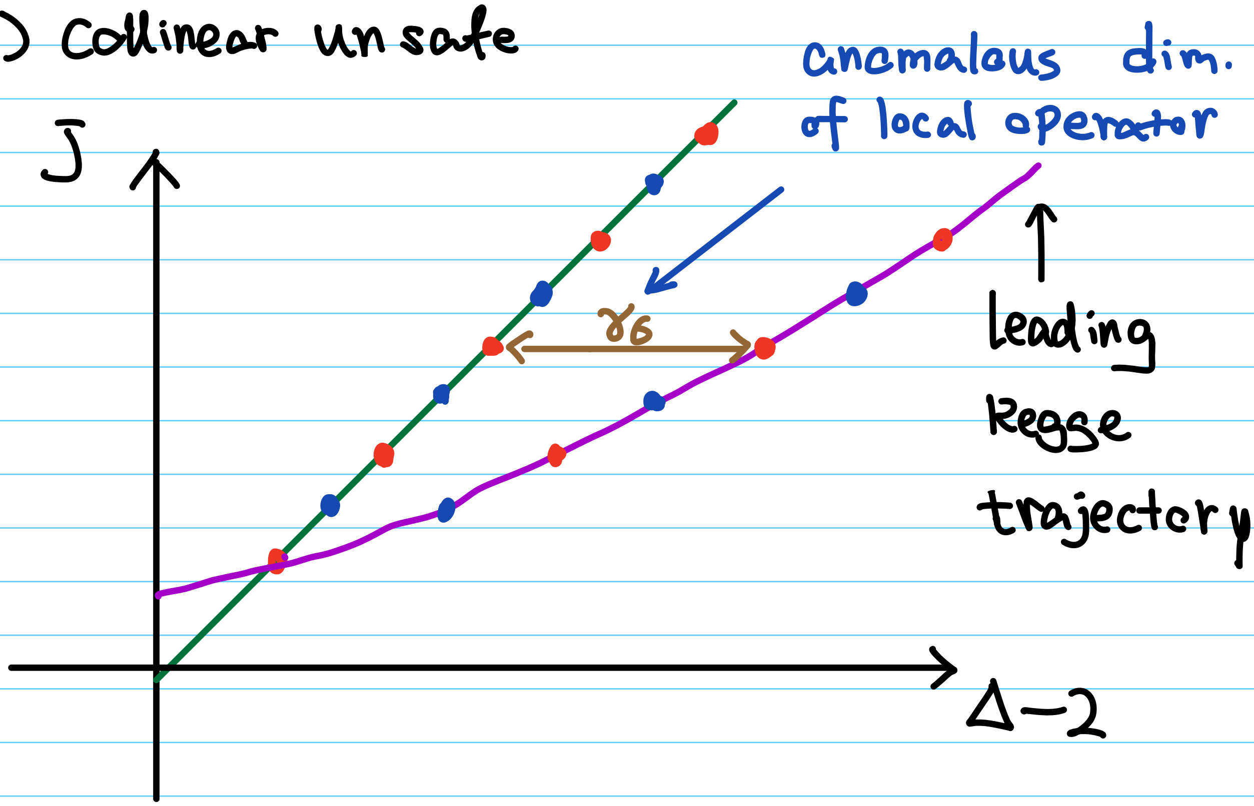


$$\frac{1}{\epsilon}(E_1+E_2) \qquad -\frac{1}{\epsilon}(E_1+E_2) \sim \text{finite}$$

- from local operator point of view, stress tensor are protected operator (conserved current)

- Higher spin twist 2 operators receive quantum corrections to its scaling dimension

$\Longleftrightarrow$ Collinear unsafe



- An important goal of Energy Correlator study is to carve out all the regge trajectory in the theory.

- Operator definition for higher spin light-rays.

try: $$\mathbb{Q}^{[J]} \propto \lim_{x_+\to\infty} (x_+)^2 \int_{-\infty}^{+\infty} dx_- \, \phi (D_+)^{J+1} \phi$$

dimension: $= \quad -2 \quad -1 \quad + J + 3 + \gamma_{J+1}$

$= J + \gamma_{J+1} \neq J$

revised: $$\mathbb{Q}^{[J]} \propto \lim_{x_+\to\infty} (x_+)^\tau \int_{-\infty}^{+\infty} dx_- \, \phi (D_+)^{J+1} \phi$$

(collinear) twist $\tau = \Delta_0 - J + \gamma_J$ ← Anomalous dim.

$\Uparrow$

engeering dimension

$\phi (\partial_+)^J \phi \quad \Longleftrightarrow \quad \mathcal{E}^{[J]}$

dimension : $\Delta = 2 + J + \gamma_{J+1}$ $\quad \Delta_L = J - 1$

collinear spin : J $\quad \Leftarrow J_L = -\tau - J$

$= 1 - \Delta$

! Light transform swap Δ and J

! The Q scaling behavior of $\langle \mathbb{Q}^{[J]} \rangle$ is determined by Δ_L

! The collinear spin determined by J_L (Energy weight)

* Interpretation in terms of PDFs and FFs.

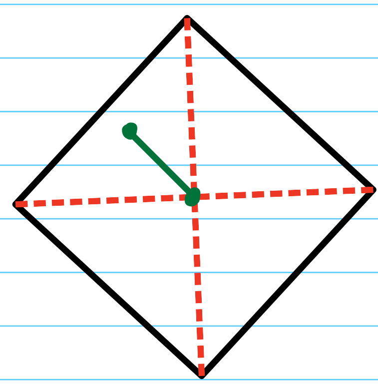
- Twist 2 family in QCD

local operator
$$\begin{cases} \text{quark:} & \bar{\psi}\,\gamma_+ (\partial_+)^{J-1}\,\psi \equiv O_q^J \\ \text{Gluon:} & F^{\mu}{}_{+}\,(\partial_+)^{J-2}\,F_{\mu+} \equiv O_g^J \quad \text{(unpolarized)} \end{cases}$$

- Parton distribution function

(quark)
$$f_q(x) = \int \frac{dy}{4\pi}\, e^{-iyxP_+} \langle P|\, \bar{\psi}(0)\,\gamma_+\,[0, y\bar{n}]\,\psi(y\bar{n})\, |P\rangle$$

illustrated with Penrose diagram



Mellin Moment

$$A_J = \int_0^1 dx\, x^{J-1} f(x)$$

$$\propto \underbrace{\langle P|\, \bar{\psi}(0)\,\gamma_+ (D_+)^{J-1}\,\psi(0)\,|P\rangle}_{\text{matrix element of local twist-2 operator}}$$

- DGLAP Equation for spacelike splitting

$$\frac{\partial f_q(x,Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dy}{y}\Big[P_{qq}\big(\tfrac{x}{y},\alpha_s(Q^2)\big) f_q(y,Q^2) + P_{qg}\big(\tfrac{x}{y},\alpha_s(Q^2)\big) f_g(y,Q^2)\Big]$$

In moment space

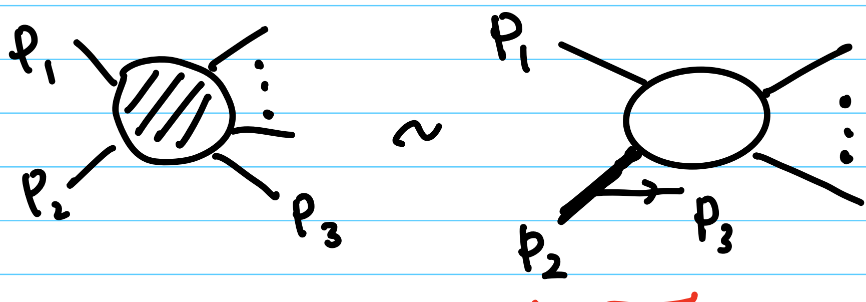
$$\frac{\partial A_J^q}{\partial \ln Q^2} = \gamma_{qq}(J)A^q(J) + \gamma_{qg}(J)A^g(J)$$

$$\gamma_{qq}(J) \equiv \int dx\, x^{J-1} P_{qq}(x)$$ moment of splitting func

One-loop splitting $P_{qq} \propto \frac{\alpha}{2\pi} P^{(0)}_{qq}$

$$P^{(0)}_{qq}(x) = C_F\left[\frac{1+x^2}{[1-x]_+} + \frac{3}{2}\delta(1-x)\right]$$

$\lim_{p_3 /\!/ p_2}$



Space-like: $(p_2-p_3)^2<0$

Split Amp

$1-x$ is the momentum fraction of p_3 with respect to p_2.

$P_{qq} \sim |\text{Split Amp}|^2$

Fragmentation function:

$$D_q(z, Q^2) = z \int \frac{dy}{4\pi} e^{-i \frac{y}{z} p^+}$$

$$\sum_X \langle 0 | \psi(0) [0, y\bar{n}] | hX \rangle \gamma_+ \langle Xh | \bar{\Psi}(y\bar{n}) | 0 \rangle$$

Evolved by TIME-LIKE DGLAP

$$\frac{\partial D_q(z, Q^2)}{\partial \ln Q^2} = \int_z^1 \frac{dy}{y} \Big[P_{qq}^{(T)}\left(\frac{z}{y}\right) D_q^h(y, Q^2)$$

$$+ P_{gq}^{(T)}\left(\frac{z}{y}\right) D_g^h(y, Q^2) \Big]$$

$P^{(T)}(z)$ are time-like Altarelli-Parisi kernel.

$$P_{qq}^{(T)}(z) = \frac{\alpha}{2\pi} P_{qq}^{(0,T)}(z) + \cdots$$

$$P_{qq}^{(0,T)}(z) = C_F \left[\frac{1+z^2}{[1-z]_+} + \frac{3}{2} \delta(1-z) \right]$$

Note that (crossing relation)

$x \Leftrightarrow \frac{1}{z}$

PDF

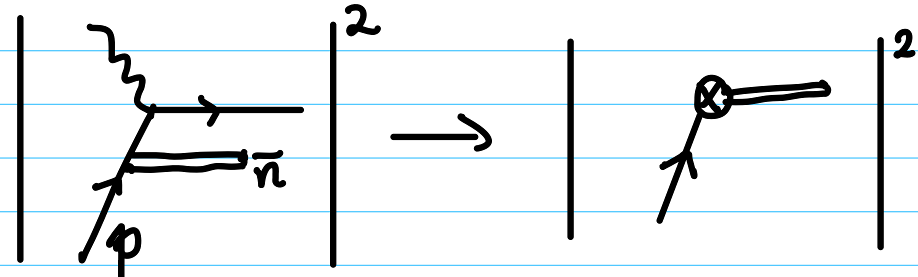
$x = \frac{k^+}{p^+}$ $\Leftarrow$ quark mom, $\Leftarrow$ Proton mom

FF

$z = \frac{q^+}{k^+}$ $\Leftarrow$ hadron, $\Leftarrow$ quark

- Example calculation of Splitting functions.

P_{qq} :



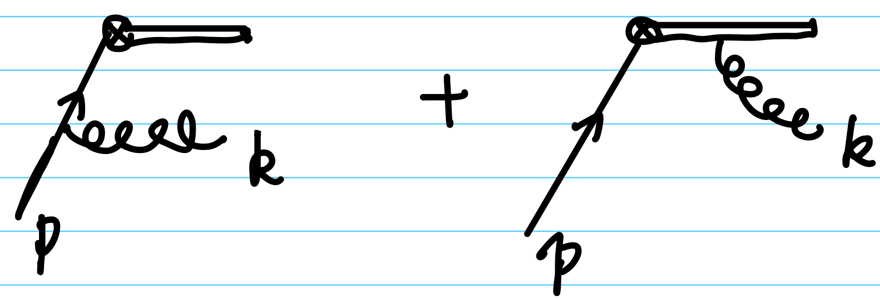
$= |M|^2_B$

$$W_n = P\exp\left[ig\int_{-\infty}^{0} ds\, \bar{n}\cdot A(x+\bar{n}s)\right]$$

$$|M_B|^2 = \mathrm{Tr}[\not{p}\cdot\bar{\not{n}}]\,\mathrm{Tr}[\mathbb{1}] = 4N_c\,\bar{n}\cdot p$$

Single splitting :

$M^{(1)} =$



$$= \bar{u}(\bar{n})\frac{i}{\not{p}-\not{k}}(ig)t^a\gamma_\mu u(p)\,\epsilon^{*\mu}(k)$$

$$-\bar{u}(\bar{n})u(p)\,g\,t^a\frac{\bar{n}_\mu}{\bar{n}\cdot k}\,\epsilon^{*\mu}(k)$$

$$|M^{(1)}|^2 = \frac{8g^2}{t}\left[\frac{1+z^2}{z-1} - \epsilon(z-1)\right]N_c C_F$$

$(p-k)^2$

$$\tilde{P}^{(1)}_{qq} = 2g^2 \left(\frac{z^2+1}{z-1} - \epsilon(z-1) \right) C_F$$

$$\tilde{P}^{(1)}_{qg} = 2g^2 \left(z^2 + (1-z)^2 - \epsilon \right) C_F$$

These are the unregularized splitting functions.

* Gribov-Lipatov Reciprocity

- One-loop

$$\gamma^T_{qq}(J) = \gamma^T_{qq}(J)$$

- Beyond one-loop, for the non-singlet

(non-singlet) $\gamma^T_{qq}(J) = \gamma^S_{qq}\left(J - \gamma^T_{qq}(J)\right)$

- For the singlet part, reciprocity holds for eigenvalues.

e.x.: check using results in 2006.10534

? How to understand this relation

to answer this question we need to study matrix element of Light-ray operators.

* (weighted) cross section as correlation function.

Wightman function

(0) $$\sigma_{tot} = \int d^4x\, e^{iq\cdot x} \langle \Omega | O^\dagger(x) O(0) | \Omega \rangle$$

e.g., O can be electromagnetic current operator.

In a CFT where O has definite scaling dim Δ,

$$\sigma_{tot} = \int d^4x\, e^{iqx} \frac{1}{\left(-(x^0 - i\varepsilon)^2 + \vec{x}^2\right)^\Delta}$$

$$= \Theta(q^0)\,\Theta(q^2)\,(q^2)^{\Delta-2} \frac{(\Delta-1)\,2\pi^3}{4^{\Delta-1}\,\Gamma(\Delta)^2}$$

$\geq 0 \implies \Delta \geq 1$ (unitary bound in CFT)

Calculation based on particle physics method.

$$\sigma_{tot} = \int d^4x\, e^{iqx} \sum\!\!\!\!\!\!\int_X \langle \Omega | O^\dagger(x) | X \rangle \langle X | O(0) | \Omega \rangle$$

by unitarity

$$= \mathrm{disc}_{q^2>0} \int d^4x\, e^{iqx} \langle \Omega | T\, O^\dagger(x)\, O(0) | \Omega \rangle$$

$$= \mathrm{disc}_{q^2>0} \int d^4x\, e^{iqx} \frac{1}{(-x^2)^\Delta}$$

using:

$$\frac{1}{(2\pi)^2 (-x^2)^\Delta} = \frac{\Gamma(2-\Delta)}{4^{\Delta-1}\Gamma(\Delta)} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} (-k^2)^{\Delta-2}$$

$$\sigma_{tot} \sim \text{disc}_{q^2>0} \frac{\Gamma(2-\Delta)}{4^{\Delta-1}\Gamma(\Delta)} \left(-q^2 - i\varepsilon\right)^{\Delta-2}$$

$$\sim \text{disc}_{q^2>0} \frac{(1-\Delta)\Gamma(1-\Delta)}{4^{\Delta-1}\,\Gamma(\Delta)} \left(-q^2 - i\varepsilon\right)^{\Delta-2}$$

$$\sim \text{disc}_{q^2>0} \;(1-\Delta)\frac{\Gamma(1-\Delta)\,\Gamma(\Delta)}{\Gamma(\Delta)^2} \left(-q^2 - i\varepsilon\right)^{\Delta-2}$$

$$\sim (1-\Delta)\frac{\pi}{\sin(\pi\Delta)\,\Gamma(\Delta)^2}\, \text{disc}_{q^2>0} \left(-q^2 - i\varepsilon\right)^{\Delta-2}$$

$$\text{disc}_{q^2>0} \left(-1 - i\varepsilon\right)^{\Delta-2}$$

$$= \text{disc}_{q^2>0} \exp\left[(\Delta-2)\ln(-1-i\varepsilon)\right]$$

$$= \text{disc}_{q^2>0} \exp\left[i(2-\Delta)\pi\right]$$

$$= 2\sin\left[(2-\Delta)\pi\right] = -2\sin(\Delta\pi)$$

$$\sigma_{tot} \sim \frac{(\Delta-1)}{4^{\Delta-1}\,\Gamma(\Delta)^2}\,\Theta(q^0)\,\Theta(q^2)\,(q^2)^{\Delta-2}$$

(1) $EF(\vec{n}) = \int d^4x \, e^{iqx} \langle \Omega | O^\dagger(x) \, \mathcal{E}(n) \, O(0) | \Omega \rangle$

(2) $EEC(z) = \int d^4x \, e^{iqx} \int d^2\Omega_1 \, d^2\Omega_2 \, \delta\left(z - \frac{n_1 \cdot n_2}{2}\right) \langle \Omega | O^\dagger(x) \mathcal{E}(n_1) \mathcal{E}(n_2) O(0) | \Omega \rangle$

(3) Obvious generalization.

! Energy operator annihilate vacuum

$$\mathcal{E}(n) | \Omega \rangle = 0$$

This is obvious from free theory.

In a CFT, let's consider

$$\langle \Omega | T_{++}(x_1) \, T_{++}(x_2) | \Omega \rangle$$

$= 4 C_T \dfrac{(x_+)^2}{x^{12}}$ where $x = x_1 - x_2$

Wightman ordering $x_\pm \to x_\pm - i\varepsilon$

$$\langle \Omega | T_{++}(x_1)\, \mathcal{E}(x_2) | \Omega \rangle$$

$$= \int_{-\infty}^{+\infty} dx_- \; 4 C_T \frac{(x_+)^2}{((x_+ - i\epsilon)(x_- - i\epsilon) + x_\perp^2)^6}$$

$= 0$ (Because the pole is on one side)

For a generic n-point function :

$$\langle O(x_1) O(x_2) \cdots O(x_n) \rangle$$

wightman prescription :

$$x_1^0 - i\epsilon_1 ,\; x_2^0 - i\epsilon_2 ,\; \cdots \; x_n^0 - i\epsilon_n$$

$$\epsilon_1 > \epsilon_2 \cdots > \epsilon_n$$

$\Rightarrow$ Matrix element of $\mathcal{E}$ is non-vanishing if it's sandwich by other local operator.

* One-point energy flux $\langle \mathcal{E}(n) \rangle_\psi$

- Scalar Source : $\mathcal{O}(x)$ e.g. ϕ^2

$$\langle \mathcal{E}(n) \rangle_{\mathcal{O}} \equiv \int d^4x \, e^{iqx} \langle \mathcal{O}^\dagger(x) \, \mathcal{E}(n) \, \mathcal{O}(0) \rangle$$

$$\mathcal{E}(n) = \lim_{x_+ \to \infty} x_+^2 \int dx_- \, T_{++}$$

$$\langle \mathcal{O}^\dagger(x_1) \, T_{++}(x_2) \, \mathcal{O}(x_3) \rangle$$

$$\sim \frac{1}{x_{13}^{2\Delta-2}} \left(\frac{x_{12,+}^2}{x_{12}^6 x_{23}^2} + 2 \frac{x_{12,+} x_{23,+}}{x_{12}^4 x_{23}^4} + \frac{x_{23,+}^2}{x_{12}^2 x_{23}^6} \right)$$

- Completely fixed by conformal symmetry, no free parameter!
- Wightman prescription:

$$x_{12,\pm} \to x_{12,\pm} - i\varepsilon, \quad x_{23,\pm} \to x_{23,\pm} - i\varepsilon$$

$$\frac{1}{x_{12}^2} = \frac{1}{\left((x_{12,+} - i\varepsilon)(\underset{\Uparrow}{x_{12,-}} - i\varepsilon) + x_{12,\perp}^2 \right)}$$

$x_{1,-} - x_{2,-}$

take residue of $x_{12,-}$ pole.

$$\int_{-\infty}^{+\infty} dx_{2-} \lim_{x_{2,+}\to\infty} x_{2,+}^2 \langle \mathcal{O}^+(x_1) T_{++}(x_2) \mathcal{O}(x_3) \rangle$$

$$\sim \frac{1}{x_{13}^{2\Delta-2}} \cdot \frac{1}{(x_{13,-})^3}$$

$$\int d^4x \, e^{iqx} \frac{1}{x_{13}^{2\Delta-2}} \cdot \frac{1}{(x_{13,-})^3}$$

$$\sim \frac{1}{(n\cdot q)^3} (q^2)^{\Delta-1} \sim \frac{q^0}{4\pi} \sigma_{tot}$$

CMS frame.

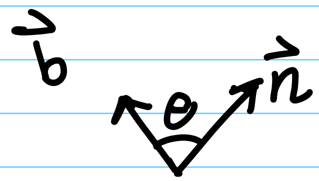
$$\int d\Omega_n \langle \mathcal{E}(n) \rangle_0 = \underbrace{q^0 \sigma_{tot}}_{\text{consequence of energy conservation}}$$

No free parameter in 1-point energy flux!

- VECTOR SOURCE.

$$\langle \mathcal{E}(n) \rangle_J = \text{F.T.} \langle \underset{\substack{\uparrow \\ \text{polarization}}}{\epsilon^*} \cdot J_1^\dagger \, \mathcal{E}(n) \, \epsilon \cdot \underset{\substack{\uparrow \\ \text{vector field}}}{J_3} \rangle$$

for a completely polarization vector : $\vec{\epsilon} = \vec{b}$



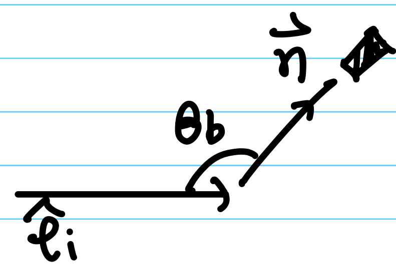
$$\langle \mathcal{E}(n) \rangle_J = \frac{q^0}{4\pi} \left[1 + a_2 \left(\cos^2\theta - \frac{1}{3} \right) \right]$$

In general, with degree of polarization P :

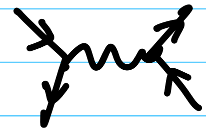
$$\epsilon_i \epsilon_j \sim \frac{(1-P)}{2} \left(\delta_{ij} - \hat{l}_i \hat{l}_j \right) + P b_i b_j$$

$\hat{l}_i$: direction of electron.

$$\langle \mathcal{E}(n) \rangle_{e^+e^- \to \gamma \to X} = \frac{q^0}{4\pi} \cdot \left[1 + a_2 \left(\frac{1}{2} \sin^2\theta_b - \frac{1}{3} \right) \right]$$



a_2 can be calculated in pQCD.

$M =$  $\quad \sum_{pol} |M|^2 \propto \frac{t^2+u^2}{s^2}$

$$t = -2p_1 \cdot p_3 \propto (1-\cos\theta_b)^2$$

$$u = -2p_1 \cdot p_4 \propto (1+\cos\theta_b)^2$$

$$t^2 + u^2 \propto 1 + \cos^2\theta_b = 2 - \sin^2\theta_b$$

$$\Rightarrow \quad 1 - \frac{1}{3}a_2 + \frac{1}{2}a_2 \sin^2\theta_b \propto 2 - \sin^2\theta_b$$

$$1 - \frac{1}{3}a_2 = -a_2 \Rightarrow a_2 = -\frac{3}{2}$$

QCD at NLO :

$$\int d\Omega_n \langle \mathcal{E}(n) \rangle_J = q^0 \left(1 + \frac{\alpha_s}{\pi} + \cdots \right)$$

$$\Rightarrow \quad a_2(\alpha_s) = -\frac{3}{2} + \frac{9\alpha_s}{2\pi} + \cdots$$

- Average Null Energy Condition:

$$\langle \mathcal{E}(\vec{n}) \rangle_\psi \geq 0 \quad \text{for any state } \psi$$

A "trivial" particle physics proof:

$$\langle \mathcal{E}(\vec{n}) \rangle \sim \underbrace{\int dPS \, |M|^2 E_p \delta^{(2)}(\hat{p}-\hat{n})}_{\text{manifestly positive}}$$

$\Rightarrow$ Conformal Collider Bound:

$$3 \geq a_2 \geq -\frac{3}{2}$$

Saturated by $j^\mu = (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$

Saturated by $j^\mu = \bar{\psi}\gamma^\mu \psi$

- In $N=1$ supersymmetry

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{q^0}{4\pi}\left(1 + 3\,\frac{c-a}{c}\left(\cos^2\theta - \frac{1}{3}\right)\right)$$

- Conformal anomaly:

$$T_\mu{}^\mu = \frac{c}{16\pi^2} W_{\mu\nu\rho\sigma} W^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E$$

In a supersymmetric theory with a current in the same supermultiplet of the stress tensor, a and c are encoded in $\langle JJT\rangle$ $\langle JTT\rangle$

In a free theory with $N=1$ chiral supermultiplet,

$$a = \frac{1}{48}, \quad c = \frac{1}{24} \qquad q^b = \frac{2}{3}, \quad q^{wf} = -\frac{1}{3}$$

$$a_2^{free} = 3\,\frac{(q^b)^2 - (q^{wf})^2}{(q^b)^2 + 2(q^{wf})^2} = 3\,\frac{c-a}{c}$$

- Exercise: for a stress tensor source,

$$\langle \mathcal{E}(\vec{n})\rangle = \frac{\langle 0|\,\epsilon^*_{ij} T_{ij}\, \mathcal{E}(\vec{n})\, \epsilon_{lk} T_{lk}\,|0\rangle}{\langle 0|\,\epsilon^*_{ij} T_{ij}\, \epsilon_{lk} T_{lk}\,|0\rangle}$$

$$= \frac{q^0}{4\pi}\left[1 + t_2\left(\frac{\epsilon^*_{il}\epsilon_{jl} n_i n_j}{\epsilon^*_{ij}\epsilon_{ij}} - \frac{1}{3}\right)\right.$$

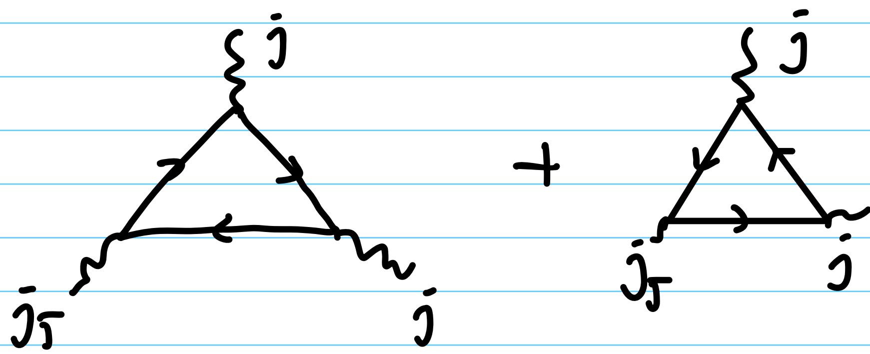
$$\left. + \, t_4\left(\frac{|\epsilon_{ij} n_i n_j|^2}{\epsilon_{ij}\epsilon^*_{ij}} - \frac{2}{15}\right)\right]$$

- Anomaly :

classical : $\partial_\mu j_5^\mu = 0$

Quantum : $\partial_\mu j_5^\mu \sim \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$

In standard QFT book, anomaly are usually calculated by :

① 

② Fujikawa method

- ③ Conformal Collider method :

$$\langle Q(\vec{n}) \rangle = \frac{\langle 0 | \vec{\epsilon}^* \cdot J_5^\dagger \, Q(\vec{n}) \, \vec{\epsilon} \cdot \vec{J} \, | 0 \rangle}{\langle 0 | \vec{\epsilon}^* \cdot J_5^\dagger \, \vec{\epsilon} \cdot \vec{J} \, | 0 \rangle}$$

$\Uparrow$

charge flux

$$= i\alpha \, \epsilon_{ijk} \, \epsilon_i^* \, \epsilon_j \, n_k \sim$$ Foward backward asymmetry

* Gribov-Lipatov reciprocity from one-point energy^n func.

$$\mathbb{O}_{J_L} \propto \lim_{x_+\to\infty} (x_+)^{I} \int_{-\infty}^{+\infty} dx_- \bar{\psi}\, \gamma_+ (D_+)^J \psi$$

for $J_L=-3$ reduces to Energy flux $\begin{cases} J_L = -3 \\ (p^+)^{-2-J_L} \text{ weight} \end{cases}$

- For generic J_L, $\mathbb{O}_{J_L}$ requires renormalization.

One point function

$\langle J\, \mathbb{O}_{J_L}\, J \rangle$:

$\mathbb{O}_{J_L}(\vec{n})$

$$= \sum_{i=q,g} \langle J\, \mathbb{O}_{J_L} |i\rangle\langle i| J \rangle$$

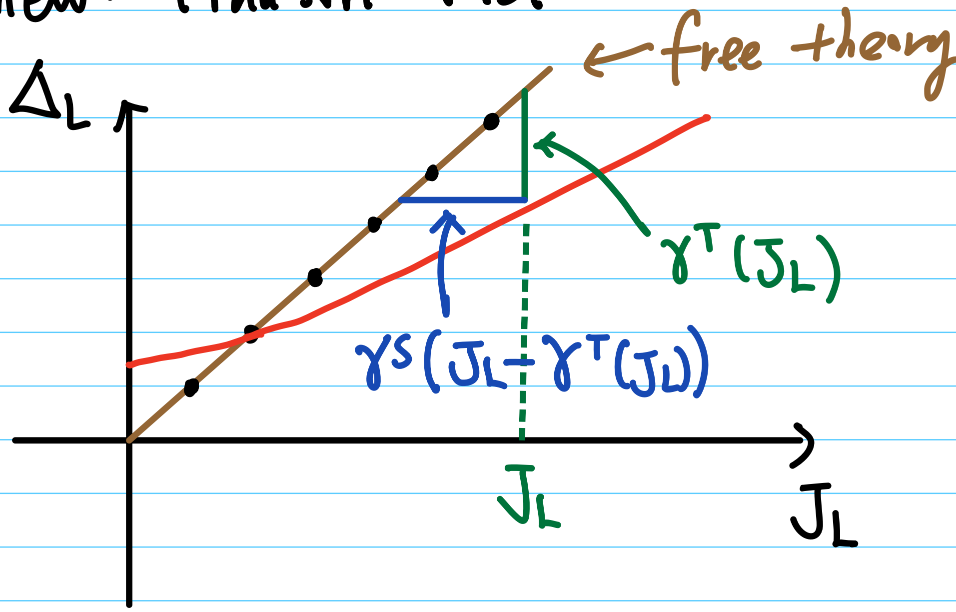
$$= \sum_X \int dPS\, |M|^2_{e^+e^-\to i+X}\, E_i\, \delta^{(2)}(\hat{p}_i - \hat{n})$$

This cross section is collinear unsafe, require renormalization of $\mathbb{O}_{J_L}$ by

$$Z_{qq}(J_L) \sim 1 + \gamma^T_{qq}(J_L)/\epsilon$$

$$Z_{gq}(J_L) \sim 1 + \gamma^T_{gq}(J_L)/\epsilon$$

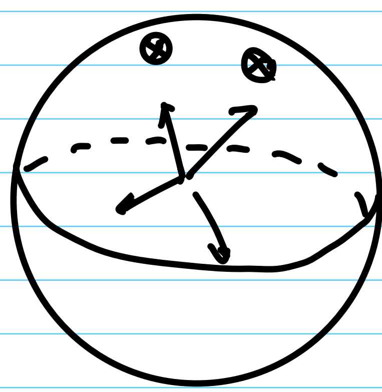
Chew-Frautschi Plot



But the free theory line is 45°.

$$\Rightarrow \quad \gamma^S(J_L-\gamma^T(J_L)) = \gamma^T(J_L) \quad !$$

EEC: $\langle \mathcal{E}(\vec{n}_2)\mathcal{E}(\vec{n}_3)\rangle_0$

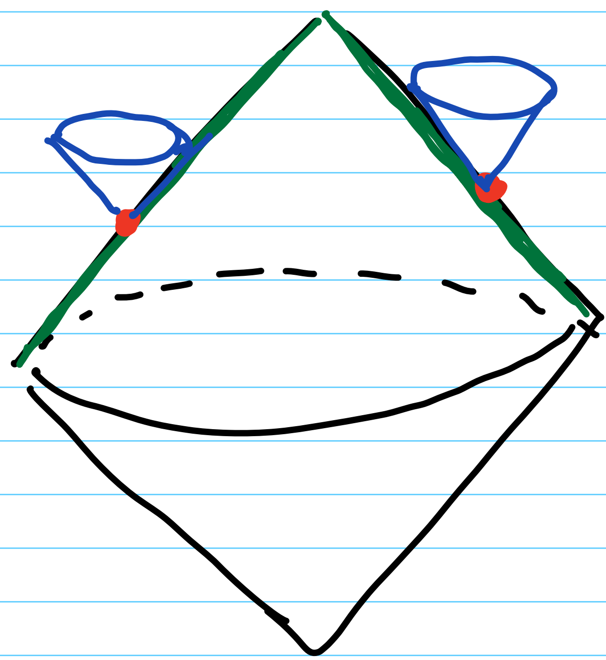


$$n_2 = (1, \vec{n}_2)$$
$$n_3 = (1, \vec{n}_3)$$

Average over orientation of the detector, the result only depends on $z = \frac{1-\cos\theta}{2} = \frac{1}{2} n_2 \cdot n_3$

- Commutativity :

$$\langle [\mathcal{E}_2, \mathcal{E}_3] \rangle_0 = 0$$



$$\langle \mathcal{E}\mathcal{E}\rangle_0 = \text{F.T.} \langle O_1 \mathcal{E}(n_2)\, \mathcal{E}(n_3)\, O_4 \rangle$$

$$= (\text{F.T.})(\text{L.T.}) \underbrace{\langle O_1 T_2 T_3 O_4 \rangle}_{\text{spinning 4pt function}}$$

Recall Xinan's lecture: In CFT with 4 scalar operator:

$$\langle O_1 O_2 O_3 O_4 \rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} G(u,v)$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

- complication from
  1) Spin Structure
  2) Causal Structure

$$x_{ij,\pm} \to x_{ij,\pm} - i\varepsilon \quad \text{for} \quad i<j$$

- In $N=4$ SYM ($\beta_{N=4}=0$)

$$\langle O T T O \rangle \xleftrightarrow[\text{conformal sym.}]{\text{super}} \langle O O O O \rangle$$

scalar operator in R current

- $\langle OOOO \rangle$ known to 3-loop, and to 11-loop integrand.

Two approaches:

(1) Mellin Amplitude:

2 cross ratio $\Leftrightarrow$ 2 Mellin Mandelstam.

$$\langle OOOO \rangle = \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{dj_1\, dj_2}{(2\pi i)^2} M(j_1, j_2, a)$$

$$(-x_{13}^2+i\epsilon x_{13}^0)^{-j_1-j_2}(-x_{24}^2+i\epsilon x_{24}^0)^{-j_1-j_2}(-x_{12}^2+i\epsilon x_{12}^0)^{j_1}$$

$$\times(-x_{34}^2+i\epsilon x_{34}^0)^{j_1}(-x_{23}^2+i\epsilon x_{23}^0)^{j_2}(-x_{14}^2+i\epsilon x_{14}^0)^{j_2}$$

Light transform become possible because the x_{ij} dependence are made manifest.

(2) Double discontinuity:

take 2-pt Wightman function as an example:

$\langle O(x_1)\, O(x_2) \rangle$ In the limit $x_2^0 \gg x_1^0$

$$= \frac{1}{(-x_{12}^2+i\epsilon x_{12}^0)^\Delta} = \frac{1}{(-x_{12}^2-i\epsilon)^\Delta}$$

$$\langle O(x_2)\, O(x_1) \rangle = \frac{1}{(-x_{21}^2+i\epsilon)^\Delta}$$

$\langle [\, O(x_1),\ O(x_2)\,] \rangle \sim$

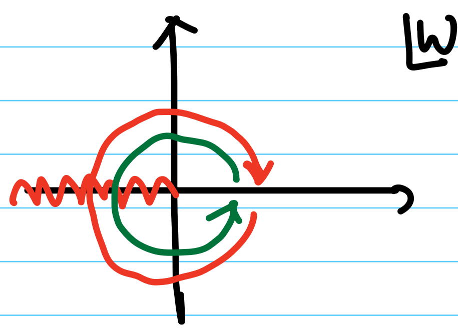
$= \mathrm{disc}_{x_{12}^2 = 0} \langle O(x_1)\, O(x_2) \rangle$

$\langle O_1\, \mathcal{E}_2\, \mathcal{E}_3\, O \rangle = \langle [\, O_1, \mathcal{E}_2\,][\,\mathcal{E}_3, O_4\,] \rangle$

$= \mathrm{L.T.}\ \langle [O_1, O_2][O_3, O_4] \rangle$

$= \mathrm{L.T.}\ \mathrm{dDisc}\, \langle O_1 O_2 O_3 O_4 \rangle$

$\mathrm{dDisc}\, f(w)$:



$= f^{\circlearrowright}(w) + f^{\circlearrowleft}(w) - 2 f(w)$

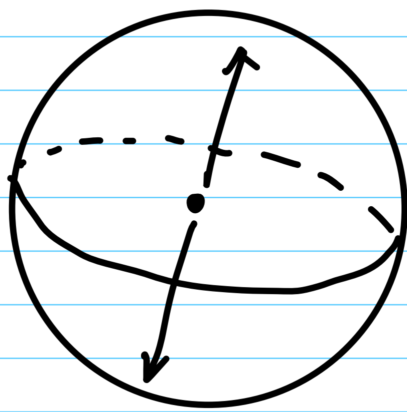
- EEC in QCD perturbation theory

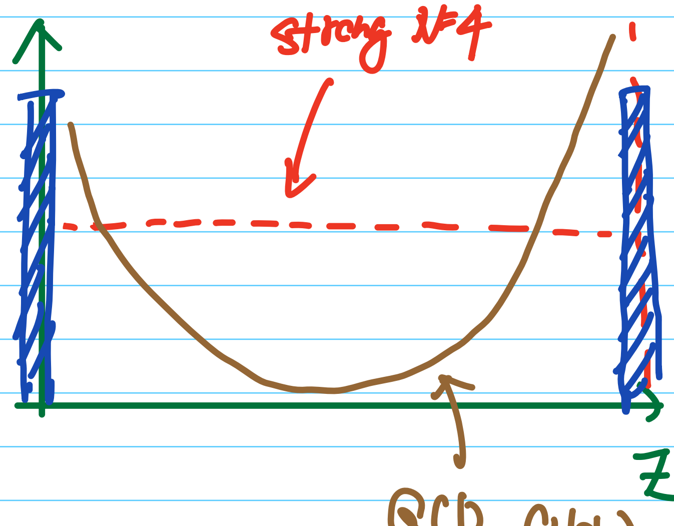
$$J^\mu = \bar{\psi}\gamma^\mu\psi .$$

$$\langle \mathcal{E}\mathcal{E}\rangle_J = (\text{F.T.}) \langle J\, \mathcal{E}\, \mathcal{E} \sum_X |X\rangle\langle X| J\rangle$$

$$= \sum_{a,b} \int dPS\, |\mathcal{M}|_{\gamma^* \to abX} \frac{E_a E_b}{Q^2} \delta\left(z - \frac{\hat{p}_a \cdot \hat{p}_b}{2}\right)$$

At LO: $\mathcal{M} \sim$  momentum conservation force back to back configuration

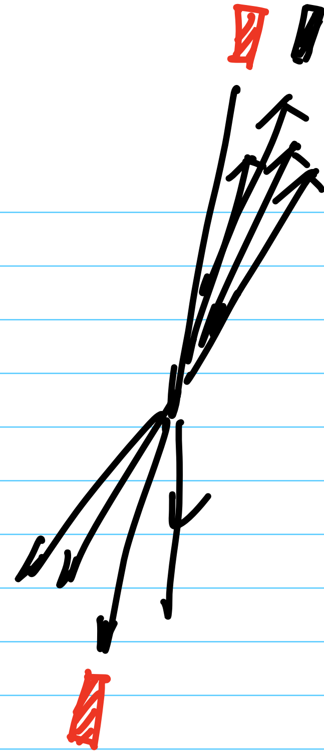




$$\langle EE\rangle_J = \frac{1}{2}\delta(z) + \frac{1}{2}\delta(1-z)$$

$$+ \frac{\alpha_s}{2\pi} C_F \frac{3-2z}{4(1-z)z^5}$$

$$\times \left[3z(2-3z) + 2(2z^2 - 6z + 3)\log(1-z)\right]$$



$z \to 0$: collinear limit

$$\langle EE \rangle_J \sim \alpha_s \frac{\#}{z}$$

$z \to 1$: back-to-back

$$\langle EE \rangle_J \sim \alpha_s \underbrace{\frac{\ln(1-z)}{1-z}}_{\text{Sudakov double log}}$$

- Relation to Event Shape.

Conjecture : $\langle \mathcal{E} \rangle$, $\langle \mathcal{E}\mathcal{E} \rangle$, $\langle \mathcal{E}\mathcal{E}\mathcal{E} \rangle$ ...
form a complete basis of IRC safe observable.

Example: C-parameter

$$C = \frac{3}{2Q^2} \int d^2\Omega_1\, d^2\Omega_2\, \mathcal{E}(n_1)\, \mathcal{E}(n_2) \sin^2\theta_{12}$$

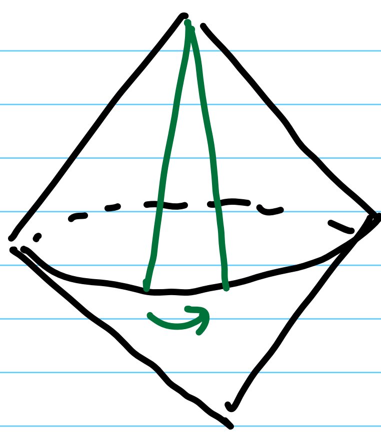
$$\langle C \rangle = \frac{3}{2Q^2} \int d^2\Omega_1\, d^2\Omega_2 \langle \mathcal{E}(n_1)\, \mathcal{E}(n_2) \rangle \sin^2\theta_{12}$$

* Collinear limit

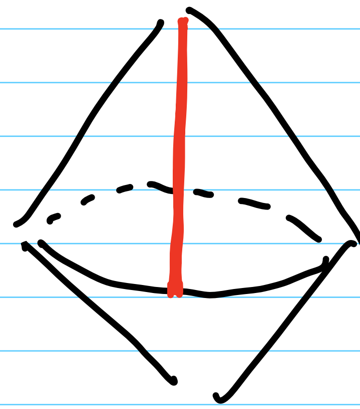
local OPE :

$$O(x_1)\;O(x_2) \sim \sum_k C_{12k} O_k$$

Light-ray OPE :



$$\Rightarrow \sum \frac{1}{\theta^*}$$



Our guiding principle is symmetry.

$$\mathbb{O}_{J_L} = \lim_{x_+ \to \infty} (x_+)^{\tau} \int dx_- \, \underbrace{\bar{\psi} \gamma_+ D_+^{J-1} \psi}_{\text{spin-}J \text{ local twist } 2 \text{ operator at even } J.}$$

$$\dim[\mathbb{O}_{J_L}] = \Delta_L = J + \tau - 1 - \tau = J - 1$$

$$\mathrm{Spin}[\mathbb{O}_{J_L}] = J_L = -J + 1 - \tau = 1 - \Delta$$

In particular,

$\dim[\mathcal{E}] = 2-1 = 1$

$\mathrm{Spin}[\mathcal{E}] = 1-4 = -3$

local operator doesn't exist

$$\mathcal{E} \times \mathcal{E} = \frac{1}{\theta^a}\, \mathbb{O}_{J_L}$$

$$\Downarrow$$

dim $\quad 2 = J-1 \quad \Rightarrow J=3$

spin $\quad -6 = -a+1-\Delta_{\mathbb{O}_{J_L}}$

$$\Delta_{\mathbb{O}_{J_L}} = J+\tau = 3+\tau$$

$$\Rightarrow \quad a = 7-3-\tau = 4-\tau = 2-\gamma$$

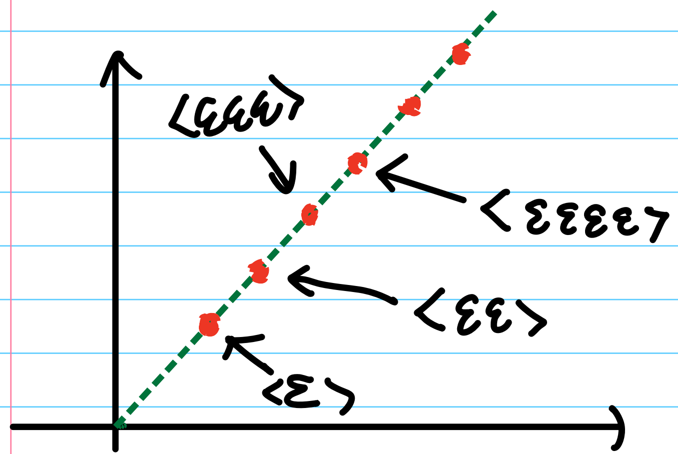
$$\mathcal{E} \times \mathcal{E} \sim \frac{1}{\theta^{2-\gamma}}\, \mathbb{O}_{J_L} \qquad J_L = 1-\Delta = -4-\gamma$$

where γ is anomalous dimension of LOCAL twist-2 operator at spin $J=3$. $\gamma > 0$

- Generalization:

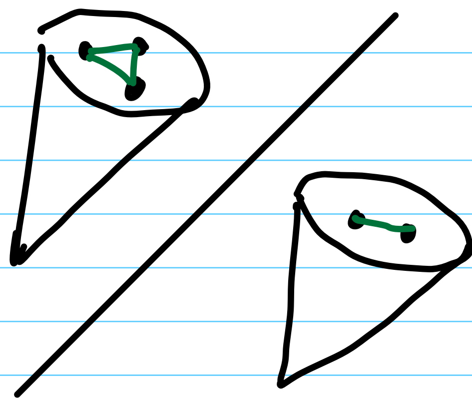
$$\mathcal{E}\times\mathcal{E}\times\mathcal{E} \sim \frac{1}{\theta^{2-\gamma(4)}}\, \mathbb{O}_{J_L}$$

Back to Chew-fraustri Plot



We can explore the C-F plot by multiple energy correlators!

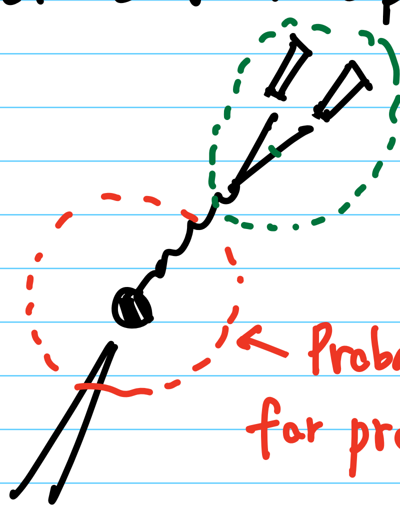
Ratio of energy correlators:



$$\sim \frac{1}{\theta^{\gamma(4)-\gamma(3)}}$$

most uncertainties cancel, leads to precision theory prediction.

- Factorization approach.



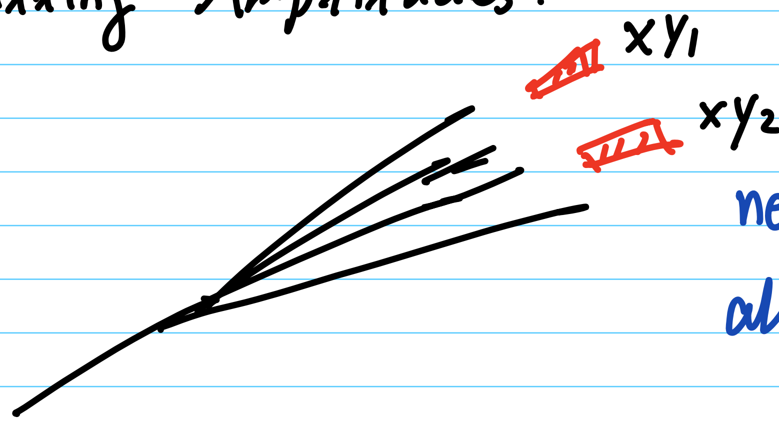
Probability for 2-particle energy correlation in a jet.

Probability for producing a parton

$$z = \frac{1-\cos\theta}{2} \sim \frac{\theta^2}{4}$$

$$\langle EE \rangle \sim \int_0^1 dx\, x^2 H(x)\, J(x, z)$$

In principle, J can be calculated using splitting Amplitudes.



need to sum over all pairs in the jet.

However, By RG invariance.

$$\frac{d\langle EE \rangle}{d\ln\mu} = 0$$

time-like splitting

$$\frac{dH(x)}{d\ln\mu} = \int_0^1 \frac{dy}{y} P_T(y)\, H\left(\frac{x}{y}\right)$$

$$\frac{dJ\left(\ln\frac{z\theta^2}{\mu^2}\right)}{d\ln\mu} = -\int_0^1 dy\, y^2 P_T(y)\, J\left(\ln\frac{zy^2\theta^2}{\mu^2}\right)$$

$$\langle EE \rangle = \int_c^1 dx\, x^2 H(x)\, J\left(\ln \frac{z x^2 Q^2}{\mu^2}\right)$$

Solving the RG equation for J can resum all the large logarithms in $\log z$.

- Puzzle : J is solved using Time-Like Splitting functions. But light-ray OPE tells us that $\log z$ can be resummed by space-like AD.

Assuming a power law solution for J :

$$J = z^a = \left(\frac{\theta^2 Q^2}{\mu^2}\right)^a \sim e^{-2a \ln \mu}$$

$$\frac{dJ}{d\ln\mu} = -2a\, J = -\int_0^1 dy\, y^2 P_T(y)\, y^{2a}\, J$$

$$-2a = \gamma_T(3-2a)$$

But $\quad \gamma_S(J) = \gamma_T(J + \gamma_S(J))$

$\Rightarrow \quad -2a = \gamma_S(J) \quad , \quad J = 3$

$\Rightarrow \quad J = \frac{1}{\theta^2}\, \theta^{\gamma_S(J)}$ in agreement with LR OPE !