Scattering amplitude methods for classical gravity

Thursday, February 6, 2025 11:24 PM

References:

- Bern, Cheung, Roiban, Shen, Solon, Zeng, 1908.01493
- Kosower, Maybee, O'Connell, 1811.10950
- Snowmass White paper, 2203.13011, 2204.06547

Quantum field theory setup.

$$S = \int d^4x \sqrt{-g}\left[-\frac{R}{16\pi G} + \frac{1}{2}\sum_{a=1}^{2}\left(\partial_\mu\phi_i\,\partial^\mu\phi_i - m_i^2\phi_i^2\right)\right]$$

Hierachy of scales: $\lambda_{dB} \ll r_s \ll b$
(or λ_{Compton})

$\lambda_{dB} \ll r_s \rightarrow \frac{\hbar}{mv} \ll \frac{Gm}{c^2} \rightarrow \frac{Gm^2 v}{c^2} \gg \hbar$ For $v \sim c$, this is $m^2 \gg M_{PL}^2$.

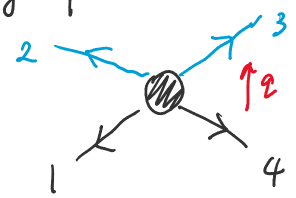
loop expansion assumes $m^2 \ll M_{PL}^2$.

quantum → classical in principle requires resummation.

$r_s \ll b \rightarrow \frac{GM}{bc^2} \ll 1$ $\quad \frac{GM}{bc^2}$ is the small parameter in the classical perturbative expansion

$\lambda_{dB} \ll b \rightarrow \hbar \ll mvb$ or $\frac{q}{mv} \ll 1$ Large angular momentum limit. (Bohr's correspondence principle)
keep only the leading order in small q expansion.

Two-body amplitude:



momentum transfer: $p_1 + p_4 + q = 0$
$p_2 + p_3 - q = 0$

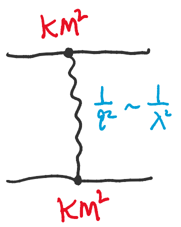
on-shell kinematics: $p_1^2 = p_4^2 = m_1^2$. $\quad p_1 \cdot q = -\frac{q^2}{2}$
$p_2^2 = p_3^2 = m_2^2$. $\quad p_2 \cdot q = +\frac{q^2}{2}$.

useful variables: $\bar{p}_1 = p_1 + \frac{q}{2}$. $\quad \bar{p}_2 = p_2 - \frac{q}{2}$
such that $\bar{p}_i \cdot q = 0 \quad \bar{m}_i^2 = m_i^2 + \frac{q^2}{4}$

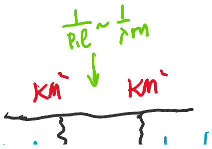
only integrate over soft region gravitons

$$q \sim \lambda q, \quad \ell \sim \lambda \ell. \quad p_i \sim p_i$$

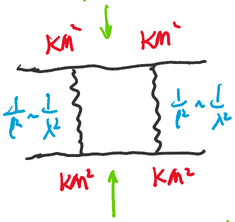
Method of regions expansion → keep only leading order in λ.



$\sim \frac{\kappa^2 M^4}{q^2} \sim \frac{GM^4}{q^2}$.



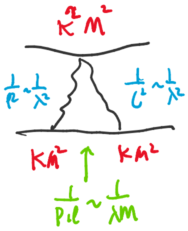
measure ↓ λ^4 $\quad$ graviton prop. $\frac{1}{\lambda^2}\frac{1}{\lambda^2}$ $\quad$ matter prop $\frac{1}{\lambda m}\frac{1}{\lambda m}$ $\quad \kappa^4 M^8 \sim \frac{G^2 M^6}{q^2} \sim \frac{GM^4}{q^2}(GMq)\frac{M}{q}$

$\int d^4\ell \sim \lambda^4 \int d^4\ell$. $\frac{1}{\ell^2} \sim \frac{1}{\lambda^2}$  $\frac{1}{\ell^2} \sim \frac{1}{\lambda^2}$ $= \lambda^4 \; \frac{1}{\lambda^2} \; \frac{1}{\lambda^2} \; \frac{1}{\lambda m} \frac{1}{\lambda m} \; \kappa^4 M^8 \sim \frac{G^2 M^6}{q^2} \sim \frac{GM^4}{q^2} (GMq) \frac{M}{q}$

super-classical

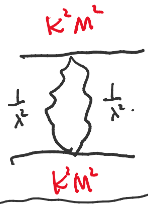
$$\frac{1}{(P_1+\ell)^2 - m_1^2 + i\epsilon} = \frac{1}{2P_1\cdot\ell + \ell^2 + i\epsilon} = \frac{1}{2P_1\cdot\ell + i\epsilon}\left(1 - \frac{\ell^2}{2P_1\cdot\ell + i\epsilon} + \cdots\right)$$

$\frac{1}{2P_1\cdot \ell+i\epsilon} \to \frac{1}{\lambda m}$

$\int d^4\ell \sim \lambda^4 \int d^4\ell$ $\frac{1}{\ell^2} \sim \frac{1}{\lambda^2}$  $\frac{1}{\ell^2} \sim \frac{1}{\lambda^2}$ $= \lambda^4 \cdot \frac{1}{\lambda^2} \frac{1}{\lambda^2} \cdot \frac{1}{\lambda m} \; \kappa^4 M^6 \sim \frac{G^2 M^5}{q} \sim \underbrace{\frac{GM^4}{q^2}(GMq)}_{\text{classical}}$

$\frac{1}{P_1\cdot\ell} \sim \frac{1}{\lambda m}$

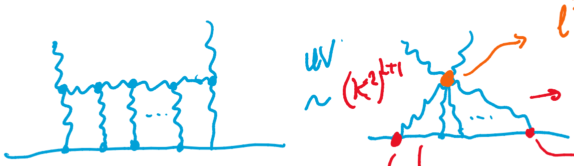
$\hookrightarrow$ in general $\frac{GM^4}{q^2}(GMq)^L$

$\int d^4\ell \sim \lambda^4$  $\frac{1}{\lambda^2}$ $\frac{1}{\lambda^2}$ $= \kappa^4 M^4 \sim G^2 M^4 \sim \frac{GM^4}{q^2}(GMq)\frac{q}{m}$

Quantum.

Exercise: IR divergence?

When does UV divergence appear?



measure ℓ^{4L}.

tensor reduction $\sim$ CT $= R R \partial\partial\phi \, \partial\partial\phi$ = 8 external momentum ℓ^{-8}

Altogether: $\ell^{4L} \cdot \ell^2 \frac{1}{(\ell^2)^{L+1}} \ell \frac{1}{\ell^8} = \ell^{2L-7}$

UV div appears at $L=4$ for Compton, $L=5$ for two-body.

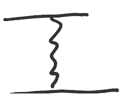
Counterterm $\quad C \, G^5 \, RR \, \partial\partial\phi \, \partial\partial\phi$

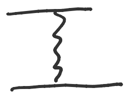
C: Love number.

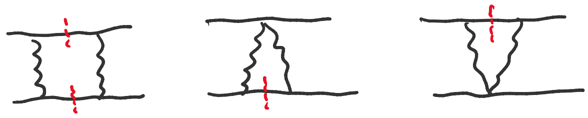
Claim: static Love numbers vanish for black holes.

To verify, need a 5-Loop two-body amplitude calculation.

Classical amplitude $\quad A(\bar{P}_1, \bar{P}_2, q) = A^{\text{tree}} + A^{1\text{-Loop}} + \cdots$

tree-level 

tree-level 

one-loop 

super-classical
IR divergent

$\delta(2p_1\cdot l)$

$$iA^{\text{tree}} = d_I \frac{1}{q^2} = -\frac{i\kappa^2 m_1^2 m_2^2 (2y^2-1)}{2q^2} \qquad y = \frac{\bar{p}_1\cdot\bar{p}_2}{\bar{m}_1\bar{m}_2} = \bar{u}_1\cdot\bar{u}_2$$

$$iA^{\text{one-loop}} = \underbrace{d_\square I_\square}_{-\frac{\kappa^4(2y^2-1)^2 m_1^4 m_2^4}{32} I_\square} + \underbrace{d_\triangle I_\triangle + d_\nabla I_\nabla}_{\frac{3i(5y^2-1)m_1^2m_2^2(m_1+m_2)}{512\sqrt{-q^2}}}$$

How to extract classical physics from classical amplitudes?

* EFT matching.
* Eikonal scattering
* Kosower-Maybee-O'Connell formalism.

EFT matching

Bottom-up construction of two-body effective interaction

$$V(\vec{r},\vec{p}) = \int_{\vec{q}} e^{i\vec{q}\cdot\vec{r}}\, V(\vec{p}+\vec{q},\vec{p}) = \sum_{n=1}^{\infty} \frac{G^n}{|\vec{r}|^n} C_n(\vec{p}^2)$$

Hard scale

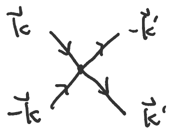
Isotropic gauge: no $\vec{p}\cdot\vec{r}$ term

no $\vec{p}\cdot\vec{q}$ term in amplitudes

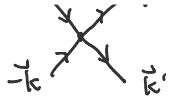
Consider the Lagrangian

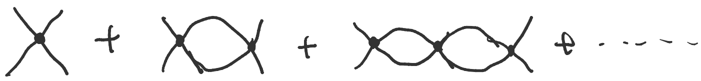
$$L = \int_{\vec{k}} \sum_{a=1}^{2} \xi_a^\dagger(-\vec{k})\left(i\partial_t - \sqrt{\vec{k}^2+m_a^2}\right)\xi_a(\vec{k}) - \int_{\vec{k},\vec{k}'} \xi_1^\dagger(\vec{k}')\xi_2^\dagger(-\vec{k}')V(\vec{k}',\vec{k})\xi_1(\vec{k})\xi_2(-\vec{k})$$

Feynman rules

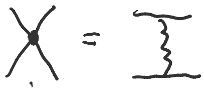
 $= \dfrac{i}{E-\sqrt{\vec{k}^2+m^2}+i\epsilon}$ 

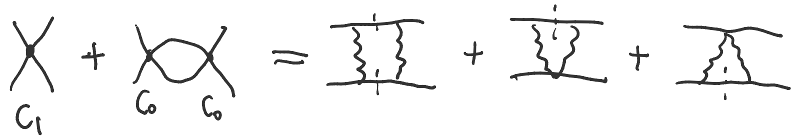
$= iV(\vec{k}',\vec{k})$

 $= \dfrac{i}{E-\sqrt{\vec{k}^2+m^2}+i\epsilon}$  $= iV(\vec{k}',\vec{k})$

$A^{EFT} =$ 

EFT matching : $A^{EFT} = \dfrac{A^{QFT}}{4E_1E_2}$.

tree-level  $\longrightarrow C_0 \sim d_I$

one-loop 

$d_\square \sim C_0^2$. iteration, carries not physical information

$C_1 \sim d_\Delta + d_\nabla$

* All order in velocity in PM expansion
* Local-in-time interactions directly applicable to bound orbits

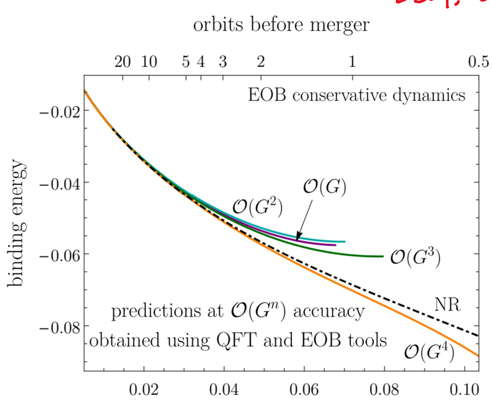
→ more accurate than PN expansions

* At 3-loop (4PM) and beyond, non-local-in-time interactions due to tail effect only apply to scattering problems
* Need a suitable prescription to remove the tail terms in PM expansion, and then put back the bound orbit tail terms (PN expanded)

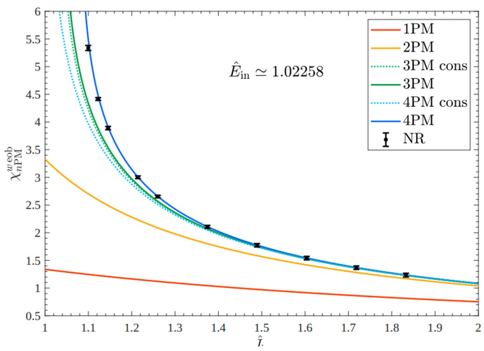
Current frontier : 4-Loop (5PM)

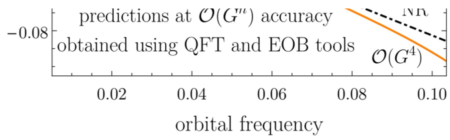
Excellent agreement with numerical GR.

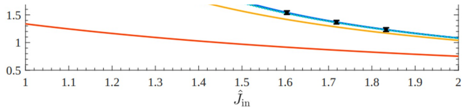
2204.05194



2211.01399





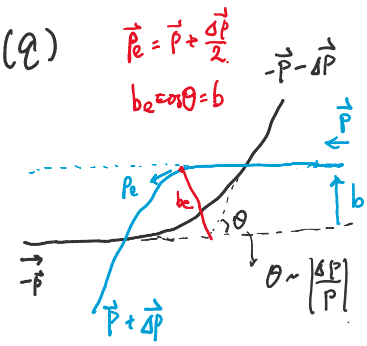


Eikonal scattering:

Fourier transform the classical amplitude to b-space

$$A(b) = \int \frac{d^4 q}{(2\pi)^4}\, 2\pi\delta(2\bar{p}_1 q)\, 2\pi\delta(2\bar{p}_2 q)\, e^{-iq\cdot b} A(q)$$

$$= \frac{1}{4m_1 m_2\sqrt{y^2-1}} \int \frac{d^2\vec{q}}{(2\pi)^2}\, e^{i\vec{q}\cdot\vec{b}} A(\vec{q})$$



conjecture: in the b-space

$$iA(b) = e^{i2\delta(b)} - 1 \quad \text{(in the classical limit)}$$

classical physics is encoded in the eikonal phase $\delta(b)$ ↳ (roughly) identified as the radial action in classical mechanics

$$\delta(b) = \underset{O(G)}{\delta^{(0)}} + \underset{O(G^2)}{\delta^{(1)}} + \delta^{(2)} + \cdots$$

$$iA^{\rm tree}(b) = i(2\delta^{(0)})$$

$$iA^{\text{1-loop}}(b) = \frac{1}{2}(2i\delta^{(0)})^2 + i(2\delta^{(1)})$$

$$iA^{\text{1-loop}}_{\text{super-classical}}(b) = \frac{1}{2}(2i\delta^{(0)})^2 \longrightarrow \text{does not carry classical information}$$

$$iA^{\text{1-loop}}_{\text{classical}}(b) = i(2\delta^{(1)})$$

Scattering angle: $\theta = \frac{\partial 2\delta(E,J)}{\partial J}$ (or impulse $\Delta p^\mu = \frac{\partial 2\delta}{\partial b^\mu}$)

Shapiro time delay: $T = \frac{\partial 2\delta(E,J)}{\partial E}$

Remark: the eikonal phase is a function of eikonal COM frame variables

$$P_{e,1}^\mu = P_1^\mu + \frac{\Delta P^\mu}{2} \qquad P_{e,2}^\mu = P_2^\mu - \frac{\Delta P^\mu}{2}$$

$$b_e \cos\frac{\theta}{2} = b$$

KMOC formalism (observable based)

KMOC formalism (observable based).

In quantum mechanics. observable $\longrightarrow$ Hermitian operator

measurement $\longrightarrow$ expectation value.

In a scattering process, we measure the change in the expectation value

$$\Delta O = \langle out|\hat{O}|out\rangle - \langle in|\hat{O}|in\rangle$$

$$\underset{\langle O\rangle_{in\text{-}in}}{\overset{\shortparallel}{}} = \langle in|S^\dagger \hat{O} S|in\rangle - \langle in|\hat{O}|in\rangle$$

$$= i\langle in|[O,T]|in\rangle + \langle in|T^\dagger[O,T]|in\rangle$$

In-state: two on-shell massive particles.

$$|in\rangle = \int d\Phi_1 d\Phi_2\, \phi(p_1)\phi(p_2) e^{ip_1b_1} e^{ip_2b_2} |p_1p_2\rangle$$

on-shell measure: $d\Phi = \frac{d^4p}{(2\pi)^4} 2\pi\delta(p^2-m^2)\Theta(p^0) = \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}$

on-shell state: $|p\rangle = a_p^\dagger|0\rangle$, relativistic normalization

$[a_p, a_{p'}^\dagger] = 2E_p(2\pi)^3\delta(\vec{p}-\vec{p}')$

wavefunction $\int d\Phi\, |\phi(p)|^2 = 1$

Notation: $\hat{d}^Dp = \frac{d^Dp}{(2\pi)^D}$ $\quad \hat{\delta}^D(x) = (2\pi)^D\delta^D(x)$

$$\langle in|[O,T]|in\rangle = \int d\Phi_1 d\Phi_2\, \underbrace{d\Phi_{1'} d\Phi_{2'}}_{}\, \phi(p_1)\phi(p_2)\phi^*(p_1')\phi^*(p_2') e^{\overset{-iq_1b_1 - iq_2b_2}{\overset{\shortparallel}{i(p_1-p_1')b_1 + i(p_2-p_2')b_2}}} \langle p_1'p_2'|[O,T]|p_1p_2\rangle$$

change variable. $p_1'-p_1 = q_1$, $p_2'-p_2 = q_2$.

$$\hat{d}^4q_1\, \hat{\delta}(2\bar{p}_1\cdot q_1)\hat{d}^4q_2\, \hat{\delta}(2\bar{p}_2\cdot q_2)\Theta(p_1^0)\Theta(p_2^0)$$

$\bar{p}_1 = p_1 + \frac{q_1}{2}$

$\bar{p}_2 = p_2 + \frac{q_2}{2}$

drop

Now consider classical limit. $q_1, q_2 \ll p_1, p_2$.

① $\Theta(p_1^0)$ identified as $\Theta(\bar{p}_1^0) \to$ trivialized by the linear on-shell condition $\delta(2\bar{p}_1\cdot q_1)$

drop

② $\phi(p_1)$ and $\phi^*(p_1')$ almost overlap; localized both in b and p space.

$$\int d\Phi_1\, \phi(p_1)\phi^*(p_1') \simeq \int d\Phi_1\, |\phi(p_1)|^2 = 1$$

classical scattering observables

$$\Delta O = \int \hat{d}^4q_1 \hat{d}^4q_2\, \hat{\delta}(2\bar{p}_1 q_1)\hat{\delta}(2\bar{p}_2 q_2) e^{-iq_1b_1 - iq_2b_2} \left[i\langle p_1'p_2'|[O,T]|p_1p_2\rangle + \langle p_1'p_2'|T^\dagger[O,T]|p_1p_2\rangle\right]$$

$$\Delta O = \int \hat{d}^4q_1\, \hat{d}^4q_2\, \hat{\delta}(2\bar{p}_1\cdot q_1)\,\hat{\delta}(2\bar{p}_2\cdot q_2)\, e^{-iq_1b_1 - iq_2b_2} \underbrace{\left[i\langle p_1' p_2' | [O,T] | p_1 p_2\rangle + \langle p_1' p_2' | T^\dagger [O,T] | p_1 p_2 \rangle \right]}$$

$p_1' = p_1 - q_1,\ p_2' = p_2 - q_2$

express in terms of $\bar{p}_1, \bar{p}_2$

expand to the LO of $q \ll p$.

Choice for O:

$P_1^\mu \ :\ P_1^\mu |p_1\rangle = p_1^\mu |p_1\rangle \ \longrightarrow$ impulse ΔP_1^μ.

$h_{\mu\nu} = \int d\Phi_k \,(a_k \epsilon_{\mu\nu} e^{-ikx} + a_k^\dagger \epsilon^*_{\mu\nu} e^{ikx}) \longrightarrow$ waveform $\langle h_{\mu\nu}\rangle$

$K^\mu \ :\ K^\mu |k_1 k_2 \ldots\rangle = (k_1 + k_2 + \cdots)^\mu |k_1 k_2 \ldots\rangle \longrightarrow$ radiated energy

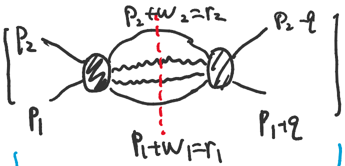
gravitons

Let's focus on impulse: $\Delta P_1^\mu = I^\mu_{amp} + I^\mu_{cut}$

$$I^\mu_{amp} = \int \hat{d}^4q_1\, \hat{d}^4q_2\, \hat{\delta}(2\bar{p}_1 q_1)\,\hat{\delta}(2\bar{p}_2 q_2)\, e^{-iq_1b_1 - q_2b_2} (p_1' - p_1)^\mu \underbrace{\langle p_1' p_2' | iT | p_1 p_2 \rangle}_{iA(p_1p_2 \to p_1' p_2')\,\hat{\delta}(q_1 - q_2)}$$

$$= \int \hat{d}^4 q\, \hat{\delta}(2\bar{p}_1 q)\, \hat{\delta}(2\bar{p}_2 q)\, e^{iq(b_2 - b_1)}\, q^\mu \underbrace{iA(p_1p_2 \to p_1 + q,\, p_2 - q)}_{\text{classical limit, use } \bar{p}_1, \bar{p}_2}$$

$$I^\mu_{cut} = \sum_X \!\!\!\!\!\int \int \hat{d}^4 q\, \hat{\delta}(2\bar{p}_1 q)\, \hat{\delta}(2\bar{p}_2 q)\, e^{iq(b_2 - b_1)}\, d\Phi_{r_1} d\Phi_{r_2}\, w_1^\mu \left[\right]$$



$A(p_1p_2 \to r_1 r_2, k_X)\, A(r_1 r_2 k_X \to p_1' p_2')$

$\times\, \hat{\delta}(w_1 + w_2 + k_X)$

graviton state

integrate over phase space

sum over polarizations

Schematically

$$\Delta P_1^\mu = \int \hat{d}^4 q\, \hat{\delta}(2\bar{p}_1 q)\, \hat{\delta}(2\bar{p}_2 q)\, e^{iq(b_2 - b_1)} \left[q^\mu\, iA \left[\right] + \int dLIPS\ \ell^\mu \text{ cut} \left[\right] \right]$$