Generalized Unitarity and method of maximal cuts

Monday, February 3, 2025 4:29 PM

I will talk about constructing loop integrands using tree-level on-shell amplitudes

References:

- Bern, Carrasco, Chiodaroli, Johansson, Roiban, 1909.01358 (Appendix C)
- Bern, Huang, 1103.1869
- Bern, Dixon, Dunbar, Kosower, hep-ph/9403226
- Bern, Dixon, Dunbar, Kosower, hep-ph/9409265

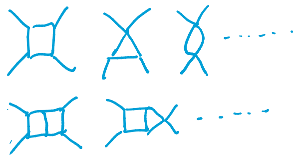
Basic idea:

$$A^{(n)} = \int \left[\prod_{i=1}^{n} \frac{d^D l_i}{(2\pi)^D}\right] \boxed{\sum_g \frac{N_g(l_i)}{D_g}} \rightarrow \text{loop integrand } I^{(n)}$$

NOT Feynman diagrams

Just represent propagator structures

Locality $\rightarrow$ $I^{(n)}$ can be represented by a sum of diagrams.



$I^{(n)}$ has simple poles when internal propagators are onshell

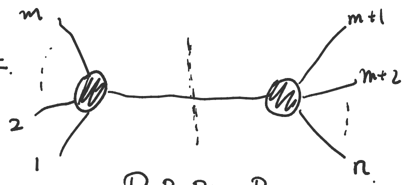
Up to a pure contact term, $I^{(n)}$ uniquely fixed by the residues.

integrate to zero $\int d^D l \, \mathrm{Poly}(l) = 0$

Unitarity $\rightarrow$ compute these residues by gluing tree level amplitudes. (Ghost free!)

Factorization:

$$\mathrm{Res}_{P^2=0}\, A(1,2,\ldots,n) =$$



$$= \sum_{\text{states}} A_L(1,2,\ldots m, P) A_R(-P, m+1, \ldots, n)$$

$P = P_1 + P_2 + \cdots + P_m$

$= -(P_{m+1} + P_{m+2} \cdots + P_n)$

reference vector

state sum: gluon $\displaystyle\sum_{\text{states}} \epsilon^\mu(P)\,\epsilon^\nu(-P) = -\left(\eta_{\mu\nu} - \frac{P_\mu q_\nu + P_\nu q_\mu}{P\cdot q}\right) \equiv -P_{\mu\nu}$.

$P_{\mu\nu}$ is a projector:

$P_\mu{}^\rho P_\rho{}^\nu = P_\mu{}^\nu$

$p^\mu P_{\mu\nu} = q^\mu P_{\mu\nu} = 0$

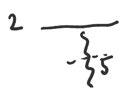
graviton $\displaystyle\sum_{\text{states}} \epsilon^{\mu\nu}(P)\,\epsilon^{\rho\sigma}(-P) = \frac{1}{2}\left(P_{\mu\rho}P_{\nu\sigma} + P_{\mu\sigma}P_{\nu\rho} - \frac{2}{D-2} P_{\mu\nu}P_{\rho\sigma}\right)$

The reference vector will cancel in the final result.

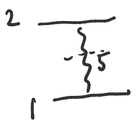
* Trivial at tree level: we can use $\eta_{\mu\nu}$ instead of $P_{\mu\nu}$ due to gauge invariance of A_L and A_R

Example:

$$= (\epsilon_5 \cdot P_1)^2.$$



$$= \sum_{\text{states}} (\epsilon_5 \cdot P_1)^2 (\epsilon_5 \cdot P_2)^2 = P_1^\mu P_1^\nu \frac{1}{2}\left(P_{\mu\rho}P_{\nu\sigma} + P_{\mu\sigma}P_{\nu\rho} - \frac{2}{D-2} P_{\mu\nu}P_{\rho\sigma}\right) P_2^\rho P_2^\sigma$$

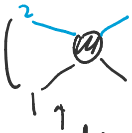


$$= \sum_{\text{states}} (\epsilon_5\cdot P_1)^2(\epsilon_5\cdot P_2)^2 = P_1^\mu P_1^\nu \frac{1}{2}\left(P_{\mu\rho}P_{\nu\sigma}+P_{\mu\sigma}P_{\nu\rho}-\frac{2}{D-2}P_{\mu\nu}P_{\rho\sigma}\right)P_2^\rho P_2^\sigma$$

$$= \left(2(P_1\cdot P_2)^2 - P_1^2P_2^2\right) \qquad (D=4)$$

$P_5^2=0 \rightarrow P_1P_5 = P_2P_5 = 0$

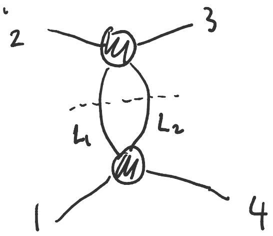
$$= m_1^2m_2^2\,(2y^2-1) \qquad y = u_1\cdot u_2.$$

Therefore $A\Big($  $\Big) = \dfrac{m_1^2m_2^2(2y^2-1)}{q^2} + \text{contact.}$

different flavors

minimally coupled to gravity

Cut:



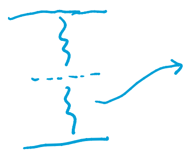
$$= \sum_{\text{states}} A(1, L_1, L_2, 4)\, A(-L_1, 2, 3, -L_2)$$

* In general, need to use the full projector $P_{\mu\nu}$

* While Reference vector will cancel on the cut (very nontrivially) they have nonzero contributions

Question: why gauge invariance does not trivially reduce $P_{\mu\nu}\rightarrow\eta_{\mu\nu}$ for cuts?

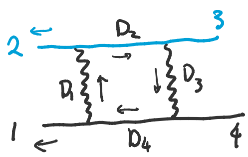
In practice, $P_{\mu\nu}\rightarrow\eta_{\mu\nu}$ if the constituent amplitudes satisfy "Generalized gauge invariance"

 $\rightarrow A\big|_{\epsilon_\ell\rightarrow\ell} = 0$ without using $\epsilon_i P_i = 0$

Kosmopoulos 2009.00141

Method of maximal cuts:

Two distinct scalars minimally coupled to gravity.



$D_1 = \ell^2$

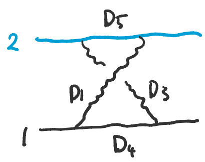
$D_2 = (P_2-\ell)^2 - m_2^2 = -2P_2\cdot\ell + \ell^2$

$D_3 = (\ell-q)^2$

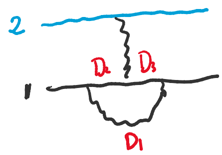
$D_4 = (P_1+\ell)^2 - m_1^2 = 2P_1\cdot\ell + \ell^2$

$P_1+P_4+q=0$

$P_2+P_3-q=0$

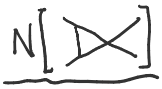


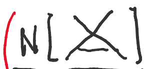
$D_5 = (\ell-P_3)^2 - m_2^2 = 2(P_2-q)\cdot\ell + \ell^2$

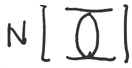
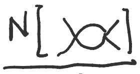


$$D_1 = \ell^2$$
$$D_2 = (P_1+\ell)^2 - m_1^2 = 2P_1\cdot\ell + \ell^2$$
$$D_3 = (P_4-\ell)^2 - m_1^2 = 2(P_1+\ell)\cdot\ell + \ell^2$$

$I^{(0)} = \dfrac{N[\,\cdot\,]}{D_1D_2D_3D_4}$  $+ \dfrac{N[\,\cdot\,]}{D_1D_5D_3D_4}$ 

$+ \dfrac{N[\,\cdot\,]}{D_1D_3D_4}$  $+ \dfrac{N[\,\cdot\,]}{D_1D_2D_3}$  $+ \Big[\dfrac{N[\,\cdot\,]}{D_2D_3D_4}$  $+ \dfrac{N[\,\cdot\,]}{D_1D_2D_4}$  $+ \dfrac{N[\,\cdot\,]}{D_5D_3D_4}$  $+ \dfrac{N[\,\cdot\,]}{D_4D_1D_3}$ 

$+ \Big(\dfrac{N[\,\cdot\,]}{D_1D_2D_3}$  $+ \text{flip} \Big) \Big]$

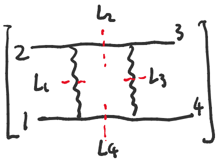
$+ \Big[\dfrac{N[\,\cdot\,]}{D_1D_3}$  $+ \dfrac{N[\,\cdot\,]}{D_2D_4}$  $+ \Big(\dfrac{N[\,\cdot\,]}{D_1D_3}$  $+ \dfrac{N[\,\cdot\,]}{D_1D_2}$  $+ \text{flip} \Big) \Big]$

+ (terms integrated to zero, like I_0 )

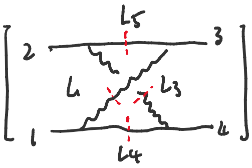
[...] terms do not contribute to classical physics

Maximal cuts

* Direct construction : glue 3-p tree amplitudes together (here)
* construct lower cuts and then take residues (see MMA notebook)

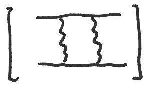
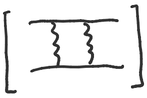
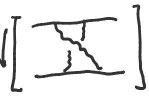
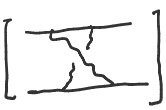


$$\text{cut}[\,\cdot\,] = \sum_{\text{states}} A(1, L_1, L_4)\, A(2, -L_1, L_2)\, A(-L_4, L_3, 4)\, A(-L_2, -L_3, 3)\Big|_{L_1^2=L_2^2=L_3^2=L_4^2=0}\,.$$



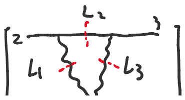
$$\text{cut}[\,\cdot\,] = \sum_{\text{states}} A(1, L_1, -L_4)\, A(2, -L_3, L_5)\, A(-L_4, L_3, 4)\, A(-L_5, -L_1, 3)\Big|_{L_1^2=L_3^2=L_4^2=L_5^2}\,.$$

Maximal cut = Numerator

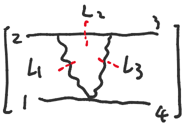
$N[\,\cdot\,]$  $\equiv \text{cut}[\,\cdot\,]$  $\qquad N[\,\cdot\,]$  $\equiv \text{cut}[\,\cdot\,]$ 

Lift off cuts

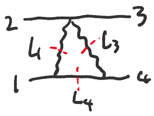
Next-to-Maximal cuts



$$\text{cut}[\,\cdot\,] = \sum A(1, L_1, L_3, 4)\, A(2, -L_1, L_2)\, A(-L_2, 3, -L_3)\Big|_{L_1^2=L_2^2=L_3^2=0}$$

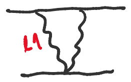
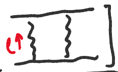
$$\mathrm{cut}\left[\right.$$ 

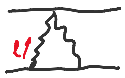
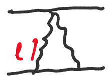
$$\left.\right] = \sum_{\text{States}} A(1, L_1, L_3, 4)\, A(2, -L_1, L_2)\, A(-L_2, 3, -L_3)\Big|_{L_1^2 = L_2^2 = L_3^2 = 0}$$

$$\mathrm{cut}\left[\right.$$ 

$$\left.\right] = \sum_{\text{states}} A(1, L_1, L_4)\, A(-L_4, L_3, 4)\, A(2, -L_1, -L_3, 3)\Big|_{L_1^2 = L_3^2 = L_4^2 = 0}$$

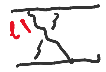
NMC − MC = numerator.

$N\Big[$  $\Big] \equiv \mathrm{cut}\Big[$  $\Big] - \dfrac{N\Big[\text{ }\Big]}{D_4} - \dfrac{N\Big[\text{ }\Big]}{D_4'}$  

$N\Big[$  $\Big] \equiv \mathrm{cut}\Big[$  $\Big] - \dfrac{N\Big[\text{ }\Big]}{D_2} - \dfrac{N\Big[\text{ }\Big]}{D_5}$  

Note: definition of ℓ should be consistent

$$\frac{N\Big[\text{ }\Big]}{D_4'} = \left.\frac{N\Big[\text{ }\Big]}{D_4}\right|_{\ell \to -(\ell - q)}$$

They must be local: No ℓ-depended denominators

Exercise: Do this by yourself!