# Lectures on perturbative computations in the Color Glass Condensate

Farid Salazar

## CONTENTS

## . RESOURCES

- Kovchegov and Levin's book of High Energy QCD [1]. Comprehensive review of small-x physics, follows the light-cone perturbation theory approach.
- Lappi's doctoral training notes for ECT*. Excellent discussion on the initial conditions for heavy-ion physics (not discussed in these lectures). `https://indico.ectstar.eu/event/14/sessions/58/attachments/259/330/Lappi_dtp_notes.pdf`
- Classic reviews on the Color Glass Condensate [2] and [3]. More recent review with extended discussion of phenomenology at HERA, RHIC, LHC and the EIC [4]
- Conception of the classical description of Color Glass Condensate and the separation of degrees of freedom: [5–7]. The renormalization of the classical theory: JIMWLK equations [8–13]. A projectile perspective on renormalization: the Balitsky-Kovchegov equation and Balitsky hierarchy [14, 15]
- You can also find online lectures by Iancu `https://www.youtube.com/watch?v=f-4tRHjwmns&t=372s` and Venugopalan `https://www.youtube.com/watch?v=LyjSYjEeG0U`
- Seminal paper on the relation between CGC and TMDs [16]

## I. LECTURE I: CGC BASICS

We begin this section by reviewing the light-cone coordinates and QED+QCD Feynman rules that will be used throughout these lectures. We then introduce the basic elements of the CGC EFT: separation of degrees of freedom into sources and fields. In the classical approximation, we relate sources and fields by solving the Yang-Mills equations for a fast-moving current along the light cone. We close this lecture by deriving the effective vertices for the interaction of quarks and gluon in the presence of the classical color field.

### A. Conventions

#### 1. *Lightcone coordinates*

We work in lightcone coordinates,

$$x^+ = \frac{1}{\sqrt{2}}\left(x^0 + x^3\right), \quad x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^3\right), \tag{1}$$

with the transverse momenta components the same as Minkowski space. Four-vectors are defined as $a^\mu = (a^+, a^-, \boldsymbol{a}_\perp)$, where $\boldsymbol{a}_\perp$ denote the two-dimensional transverse components. The magnitude of the two-dimensional vector $\boldsymbol{a}_\perp$ is denoted as $a_\perp$. Following these conventions, the scalar product of two vectors is $a_\mu b^\mu = a^+b^- + a^-b^+ - \boldsymbol{a}_\perp \cdot \boldsymbol{b}_\perp$.

The same convention is used for the gamma matrices $\gamma^+$ and $\gamma^-$, with the anti-commutation relations satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4\,, \tag{2}$$

where the only non-zero entries in the metric are $g^{+-} = g^{-+} = 1$ and $g^{ij} = -\delta^{ij}$.

We will define two light-cone vectors:

$$n^\mu = \delta^{\mu +}\,, \tag{3}$$

$$\bar{n}^\mu = \delta^{\mu -}\,. \tag{4}$$

Note $n.\bar{n} = 1$ and $n.n = \bar{n}.\bar{n} = 0$.

We will denote the transverse components of a vector as $a_{\mathrm{T}}$:

$$a^\mu_{\mathrm{T}} = (0, 0, \boldsymbol{a}_\perp)\,, \tag{5}$$

hence, $a_{\mathrm{T}\mu} a^\mu_{\mathrm{T}} = -\boldsymbol{a}^2_\perp$.

Throughout these notes we will assume the proton/nucleus with momentum $P$ moves always along the plus light-cone direction (we will neglect its mass):

$$P^\mu = P^+ n^\mu\,. \tag{6}$$

#### 2. *QCD+QED Feynman Rules*

We will label below spinor and vector indices respectively as $(\sigma, \bar{s})$ and $(\mu, \nu)$, and color indices in SU(3) in the fundamental and adjoint presentation as $(i, j)$ and $(a, b, c)$ respectively. It will be convenient to work in a light-cone gauge $n.A = 0$.

The free massless quark and gluon Feynman propagators are,

$$S^0_{\sigma\bar{s},ij}(l) = \frac{i \not{l}_{\sigma\bar{s}}}{l^2 + i\epsilon}\delta_{ij}\,, \tag{7}$$

$$G^0_{\mu\nu,ab}(l) = \frac{i\Pi_{\mu\nu}(l)}{l^2 + i\epsilon}\delta_{ab}\,, \tag{8}$$

We also define the gluon polarization tensor $\Pi_{\mu\nu}$ which appears in the free gluon propagator as

$$\Pi_{\mu\nu}(l) = -g_{\mu\nu} + \frac{l_\mu n_\nu + n_\mu l_\nu}{n.l} \,. \tag{9}$$

The polarization vector for an on-shell gluon with non-zero transverse momentum $\boldsymbol{l}_\perp$ is

$$\epsilon^\mu(l, \lambda = \pm 1) = \left( \frac{\boldsymbol{\epsilon}_\perp^{\pm 1} \cdot \boldsymbol{l}_\perp}{l^-}, 0, \boldsymbol{\epsilon}_\perp^{\pm 1} \right) \,, \tag{10}$$

where we only have the two physical transverse polarizations. For circularly polarized photons, the 2-dimensional vector $\boldsymbol{\epsilon}_\perp^\lambda$ is given by

$$\boldsymbol{\epsilon}_\perp^{\pm 1} = \frac{1}{\sqrt{2}}(1, i\lambda) \tag{11}$$

The photon quark-antiquark and the gluon-quark-antiquark vertices read

$$V^{\gamma q\bar{q}}_{\mu,\sigma\bar{s}} = -ieq_f(\gamma_\mu)_{\sigma\bar{s}} \ ; \ V^{gq\bar{q},a}_{\mu,\sigma\bar{s},ij} = ig(\gamma_\mu)_{\sigma\bar{s}} t^a_{ij} \,, \tag{12}$$

where $e$ is the electromagnetic coupling constant, $q_f$ is the fractional charge of the quark, $g$ is the strong coupling, and $t^a_{ij}$ is a generator in the fundamental representation. At one loop order in our computation we do not need the cubic and quartic gluon vertices except in the cubic coupling of gluons to the background field, represented below by the gluon effective vertex.

The polarization vector for a photon with virtuality $Q^2 = -q^2$ with momentum

$$q = \left( -\frac{Q^2}{2q^-}, q^-, \mathbf{0}_\perp \right) \tag{13}$$

is given by

$$\epsilon^\mu(q, \lambda = 0) = \left( \frac{Q}{q^-}, 0, \mathbf{0}_\perp \right) \,, \tag{14}$$

$$\epsilon^\mu(q, \lambda = \pm 1) = \left( 0, 0, \boldsymbol{\epsilon}_\perp^{\pm 1} \right) \,, \tag{15}$$

where $\lambda = 0$ denotes the longitudinal polarization, $\lambda = \pm 1$ denote the two transverse polarizations, and the two-dimensional vector $\boldsymbol{\epsilon}_\perp^{\pm 1} = \frac{1}{\sqrt{2}}(1, \pm i)$. Observe that the longitudinal polarization vector vanishes for real photons $Q = 0$.

### B. Degrees of freedom: Sources and fields

The CGC is an effective field theory for high-energy QCD. For a hadron moving in the plus light-cone direction with large momentum $P^+$ probed at the scale $x_0 P^+$, with $x_0 \ll 1$, the CGC separates the partonic content of hadrons according to their longitudinal momentum $k^+ = xP^+$ where $x$ refers to the longitudinal momentum fraction of the parton probed in the nucleus/nucleon. Partons carrying large longitudinal momentum fraction $x \gtrsim x_0$ (large-$x$ partons) are treated as static and localized color sources $\rho$. Heisenberg's uncertainty principle justifies this view: the degree of localization of partons $\Delta z^-$ is much smaller than the longitudinal resolution $1/(x_0 P^+)$ of the probe:

$$\Delta z^- \sim \frac{1}{k^+} = \frac{1}{xP^+} \ll \frac{1}{x_0 P^+} \,. \tag{16}$$

Similarly, the time scale $\Delta z^+$ for the evolution of these large-$x$ partons is much larger than the time scale of the probe $\tau \sim \frac{2x_0 P^+}{k_\perp^2}$, where $\boldsymbol{k}_\perp$ is the transverse momentum of the produced quark:

$$\Delta z^+ \sim \frac{1}{k^-} = \frac{2k^+}{k_\perp^2} = \frac{2xP^+}{k_\perp^2} \gg \frac{2x_0 P^+}{k_\perp^2} \,. \tag{17}$$

From the point of the small-$x$ partons, large-$x$ partons are localized in the longitudinal direction $z^-$ and frozen in light-cone time $z^+$. Large-$x$ partons ($x \gtrsim x_0$) are integrated out and their net effect is effectively treated by introducing a stochastic color charge density $\rho$ which is described by a non-perturbative gauge invariant weight functional $W_{x_0}[\rho]$. The partons possessing a small momentum fraction $x \lesssim x_0$ are treated as a delocalized dynamical field $A^{\mu,a}(z)$ (small-$x$ partons) generated by the color charge current $\rho$. Mathematically this is achieved by the following path-integral formulation:

$$\langle\langle \mathcal{O} \rangle\rangle = \int [\mathcal{D}\rho]\, W_{x_0}[\rho] \left\{ \frac{\int^{x_0 P^+} [\mathcal{D}A] \mathcal{O}[A] e^{iS[A,\rho]}}{\int^{x_0 P^+} [\mathcal{D}A] e^{iS[A,\rho]}} \right\} , \tag{18}$$

where $\mathcal{O}$ is an observable of interest (e.g. correlator, scattering amplitude, cross-section, etc). The invariance of the expectation value $\langle\langle \mathcal{O} \rangle\rangle$ on the separation scale choice of $x_0$ leads to the RG evolution of the weight-functional.

The action $S[A,\rho]$ is the Yang-Mills action endowed with the coupling of the gauge field with the sources $\rho$:

$$S[A,\rho] = S[A] + i \int \mathrm{d}^4x A^\mu(x) j_\mu(x) \tag{19}$$

Thus the computation of an observable follows a two-step process:

1. Compute the path integral (with cut-off $x_0 P^+$ in the plus longitudinal momentum) for a given configuration of color sources $\rho$ representing the large-$x$ partons (with longitudinal momentum $k^+ > x_0 P^+$).

$$\frac{\int^{x_0 P^+} [\mathcal{D}A] \mathcal{O}[A] e^{iS[A,\rho]}}{\int^{x_0 P^+} [\mathcal{D}A] e^{iS[A,\rho]}} \tag{20}$$

   The sources are sampled from a (non-perturbative) gauge invariant weight functional $W_{x_0}[\rho]$.

2. Average over the gauge invariant weight functional (so called CGC average).

### C. Semi-classical approximation

For a hadron/nucleus moving close to the light-cone in the plus direction, these sources generate a current independent of the light-cone time $z^+$:

$$J^{\mu,a}(z) = \delta^{\mu +} \rho^a(z^-, \boldsymbol{z}_\perp) , \tag{21}$$

where the support of $\rho$ along the minus light-cone direction is localized near the origin.

The gauge field $A^\mu$ representing small-$x$ partons can be obtained by solving classical Yang-Mills equations:

$$[D_\mu, F^{\mu\nu}] = J^\nu \tag{22}$$

The classical treatment of the gauge field is justified when the occupation number of color charge density is large. This approximation is appropriate when $x_0$ is sufficiently small and/or when the nucleus is large so that the large-$x$ partons that have been integrated out form a large current.

The independence of $z^+$ of the current in Eq. (21) is consistent with the conservation equation $[D_\mu, J^\mu] = 0$ when working in an appropriate gauge ($A^- = 0$). For this choice of gauge condition, the classical gauge field adopts a simple solution:

$$A^{\mu,a}_{\mathrm{cl}}(z) = \delta^{\mu +} \alpha^a(z^-, \boldsymbol{z}_\perp) , \tag{23}$$

where $\alpha^a(z^-, \boldsymbol{z}_\perp)$ satisfies the two-dimensional Poisson equation $\nabla^2_\perp \alpha^a = -\rho^a$ (for more details see Sec. I E.).

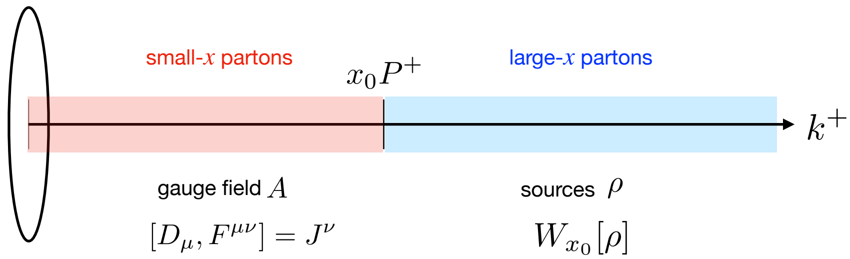


FIG. 1. In the CGC EFT partons are organized as color sources or fields according to their longitudinal momentum fraction $x$ relative to the characteristic momentum fraction of the probe $x_0$. Sources are stochastic and their distribution is characterized by a gauge invariant weight functional $W_{x_0}[\rho]$ (represented in blue). The gauge field is a solution to Yang-Mills equations in the presence of the sources (represented in red).

### D. The gauge invariant weight functional

The most widely used choice for the weight function is the McLerran-Venugopalan (MV) model [5, 6]. For a sufficiently large nucleus, the MV model invokes the central limit theorem, thus constructing a distribution following Gaussian statistics (for a detailed exposition see [17]):

$$W_{x_0}[\rho] = \mathcal{N} \exp\left\{ -\frac{1}{2} \int \mathrm{d}x^- \mathrm{d}^2\boldsymbol{x}_\perp \frac{\rho^a(x^-, \boldsymbol{x}_\perp)\rho^a(x^-, \boldsymbol{x}_\perp)}{\lambda_{x_0}(x^-)} \right\} . \tag{24}$$

The function $\lambda_{x_0}(x^-)$ is related to the transverse color charge density distribution inside the nucleus. An energetic probe will interact coherently with the partons encountered along its longitudinal trajectory. Considering the contribution from the valence quarks only one finds the quantity

$$\mu^2 = \int \mathrm{d}x^- \lambda_{x_0}(x^-) = \frac{2\pi g^2 A}{R_A^2} \sim A^{1/3} , \tag{25}$$

where $A$ is the nuclear mass number, $R_A \sim A^{1/3}$ is the nuclear radius and $g$ is the strong coupling constant. This new quantity $\mu^2$ is closely related to the saturation scale $Q_s^2$ as we will see in section II where we introduce the high energy correlators.

### E. Classical solution to Yang-Mills equations

The YM equations are

$$[D_\mu, F^{\mu\nu}] = J^\nu \tag{26}$$

where the covariant derivative is $D_\mu = \partial_\mu - igA_\mu$, and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]$

Let's solve them for a current of the form:

$$J^\mu(z) = \delta^{\mu+}\rho(z^-, \boldsymbol{z}_\perp) \tag{27}$$

Ansatz:

$$A^\mu(z) = \delta^{\mu+}\alpha(z^-, \boldsymbol{z}_\perp) \tag{28}$$

Then

$$\begin{aligned} F^{ij} &= F^{i-} = 0 \\ F^{+-} &= -\partial^- A^+(z^-, \boldsymbol{z}_\perp) = 0 \\ F^{i+} &= \partial^i \alpha(z^-, \boldsymbol{z}_\perp) \end{aligned} \tag{29}$$

The equations for $\nu = -$ or $\nu = j$ are trivially satisfied. For $\nu = +$

$$[D_\mu, F^{\mu+}] = \partial_i F^{i+} = \partial_i \partial^i \alpha(z^-, \boldsymbol{z}_\perp) = \rho(z^-, \boldsymbol{z}_\perp) \tag{30}$$

Hence, the equation for the field $\alpha$ is simply the two-dimensional Poisson's equation

$$\nabla_\perp^2 \alpha(z^-, \boldsymbol{z}_\perp) = -\rho(z^-, \boldsymbol{z}_\perp) \tag{31}$$

Notice that this solution satisfies the Lorenz gauge condition

$$\partial_\mu A^\mu = 0 \tag{32}$$

and also the so called light-cone gauge condition $A^- = 0$. Note that in this gauge, the current (covariant) conservation is trivially satisfied

$$D_\mu J^\mu = \partial_+ \rho(z^-, \boldsymbol{z}_\perp) = 0\,. \tag{33}$$

### F. Derivation of quark CGC effective vertex

We begin with the generic effective vertex for the interaction of a quark with a classical background field $A_{\rm cl}^{\mu,a}(z)$:

$$V^{q,{\rm A}}(l,l') = ig\gamma_\mu \widetilde{A}_{\rm cl}^{\mu,a}(l-l')t^a \tag{34}$$

where $\widetilde{A}_{\rm cl}^{\mu,a}(q)$ the Fourier transform (momentum space) of the classical field:

$$\widetilde{A}_{\rm cl}^{\mu,a}(q) = \int {\rm d}^2\boldsymbol{z}_\perp e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{z}_\perp} \int {\rm d}z^+ e^{iq^-z^+} \int {\rm d}z^- e^{iq^+z^-} A_{\rm cl}^{\mu,a}(z)\,. \tag{35}$$

#### 1. *One scattering with the background field*

For a back-ground field of the form $A_{\rm cl}^\mu(z) = \delta^{\mu+} A_{\rm cl}^+(z^-, \boldsymbol{z}_\perp)$, independent of $z^+$ we find

$$\widetilde{A}_{\rm cl}^{\mu,a}(q) = 2\pi\delta(q^-) \int {\rm d}^2\boldsymbol{z}_{\boldsymbol{1}\perp} e^{-i\boldsymbol{q}_\perp\cdot\boldsymbol{z}_{\boldsymbol{1}\perp}} \int {\rm d}z^- e^{iq^+z^-} A_{\rm cl}^{+,a}(z_1^-, \boldsymbol{z}_{\boldsymbol{1}\perp})\,. \tag{36}$$

where the delta function $\delta(q^-)$ arises since the background field is independent of $z^+$. Then the interaction vertex for one scattering takes the form

$$V_1^{q,A}(l,l') = 2\pi\delta(l^- - l'^-)\gamma^- \int {\rm d}^2\boldsymbol{z}_{\boldsymbol{1}\perp} e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot\boldsymbol{z}_{\boldsymbol{1}\perp}} (ig) \int {\rm d}z^- e^{i(l^+-l'^+)z_1^-} A_{\rm cl}^{+,a}(z_1^-, \boldsymbol{z}_{\boldsymbol{1}\perp}) t^a\,. \tag{37}$$

In the eikonal approximation $l_\perp, l'_\perp \ll l^-$, we make the approximation $e^{i(l^+-l'^+)z_1^-} \approx 1$; hence

$$V_{1{\rm eik}}^{q,A}(l,l') = 2\pi\delta(l^- - l'^-)\gamma^- \int {\rm d}^2\boldsymbol{z}_{\boldsymbol{1}\perp} e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot\boldsymbol{z}_{\boldsymbol{1}\perp}} (ig) \int {\rm d}z^- A_{\rm cl}^{+,a}(z_1^-, \boldsymbol{z}_{\boldsymbol{1}\perp}) t^a\,. \tag{38}$$

#### 2. *Two scatterings with the background field*

The effective vertex for two scatterings is defined as

$$V_2^{q,A}(l,l') = \int \frac{{\rm d}^4 l_1}{(2\pi)^4} V^{q,A}(l,l_1) S^0(l_1) V^{q,A}(l_1,l') \tag{39}$$

Inserting the expressions:

$$V_2^{q,A}(l,l') = \int \frac{\mathrm{d}^4 l_1}{(2\pi)^4}\delta(l^- - l_1^-)\gamma^- \frac{i\not{l}_1}{l_1^2+i\epsilon}\delta(l_1^- - l'^-)\gamma^- \\ \times 2\pi(ig)\int \mathrm{d}^2\boldsymbol{z_{2\perp}} e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l_{1\perp}})\cdot \boldsymbol{z_{2\perp}}}\int \mathrm{d}z_2^- e^{i(l^+ - l_1^+)z_2^-} A_{\mathrm{cl}}^{+,a}(z_2^-,\boldsymbol{z_{2\perp}})t^a \\ \times 2\pi(ig)\int \mathrm{d}^2\boldsymbol{z_{1\perp}} e^{-i(\boldsymbol{l_{1\perp}} - \boldsymbol{l}'_\perp)\cdot \boldsymbol{z_{1\perp}}}\int \mathrm{d}z_1^- e^{i(l_1^+ - l'^+)z_1^-} A_{\mathrm{cl}}^{+,b}(z_1^-,\boldsymbol{z_{1\perp}})t^b . \tag{40}$$

Integrate over $l_1^-$ with the help of the delta function, and use $\gamma^-\not{l}_1\gamma^- = 2l_1^-\gamma^-$

$$V_2^{q,A}(l,l') = -g^2(2\pi)\delta(l^- - l'^-)\gamma^- \int \mathrm{d}^2\boldsymbol{z_{2\perp}} e^{-i\boldsymbol{l}_\perp\cdot \boldsymbol{z_{2\perp}}}\int \mathrm{d}z^- e^{il^+z_2^-}\int \mathrm{d}^2\boldsymbol{z_{1\perp}} e^{i\boldsymbol{l}'_\perp\cdot \boldsymbol{z_{1\perp}}}\int \mathrm{d}z_1^- e^{-il'^+z_1^-} \\ \times \frac{1}{(2\pi)^2}\int \mathrm{d}^2\boldsymbol{l_{1\perp}} e^{i\boldsymbol{l_{1\perp}}\cdot(\boldsymbol{z_{2\perp}}-\boldsymbol{z_{1\perp}})}\int \frac{\mathrm{d}l_1^+}{2\pi}\frac{2il_1^-}{l_1^2+i\epsilon} e^{-il_1^+(z_2^- - z_1^-)} A_{\mathrm{cl}}^{+,a}(z_2^-,\boldsymbol{z_{2\perp}}) A_{\mathrm{cl}}^{+,b}(z_1^-,\boldsymbol{z_{1\perp}})t^a t^b , \tag{41}$$

The integration over $l_1^+$ is performed via contours employing Cauchy's theorem, we find

$$\int \frac{\mathrm{d}l_1^+}{2\pi}\frac{2il_1^- e^{-il_1^+(z_2^- - z_1^-)}}{l_1^2+i\epsilon} = \int \frac{\mathrm{d}l_1^+}{2\pi}\frac{ie^{-il_1^+(z_2^- - z_1^-)}}{\left[l_1^+ - \frac{\boldsymbol{l_{1\perp}}^2 - i\epsilon}{2l^-}\right]} = \Theta(z_2^- - z_1^-)e^{-i\frac{\boldsymbol{l_{1\perp}}^2}{2l^-}(z_2^- - z_1^-)} . \tag{42}$$

Mathematically, the step function $\Theta(z_2^- - z_1^-)$ appears because the pole is the lower-half plane (since $l^- > 0$), and when $z_2^- < z_1^-$ the contour can be closed in the upper half plane, only when $z_2^- > z_1^-$ the integral can be closed in the lower half plane. Physically, the quark propagating in the minus light-cone direction imposes an ordering in the interactions with the background field (note that this is also a result of the eikonal approximation).

$$V_2^{q,A}(l,l') = (ig)^2 2\pi\delta(l^- - l'^-)\gamma^- \int \mathrm{d}^2\boldsymbol{z_{2\perp}} e^{-i\boldsymbol{l}_\perp\cdot \boldsymbol{z_{2\perp}}}\int \mathrm{d}^2\boldsymbol{z_{1\perp}} e^{i\boldsymbol{l}'_\perp\cdot \boldsymbol{z_{1\perp}}}\int \mathrm{d}z_2^- e^{il^+z_2^-}\int^{z_2^-} \mathrm{d}z_1^- e^{-il'^+z_1^-} \\ \times A_{\mathrm{cl}}^{+,a}(z_2^-,\boldsymbol{z_{2\perp}}) A_{\mathrm{cl}}^{+,b}(z_1^-,\boldsymbol{z_{1\perp}})t^a t^b \frac{1}{(2\pi)^2}\int \mathrm{d}^2\boldsymbol{l_{1\perp}} e^{i\boldsymbol{l_{1\perp}}\cdot(\boldsymbol{z_{2\perp}}-\boldsymbol{z_{1\perp}})} e^{-i\frac{\boldsymbol{l_{1\perp}}^2}{2l^-}(z_2^- - z_1^-)} \tag{43}$$

In the eikonal approximation we can neglect the phase $e^{-i\frac{\boldsymbol{l_{1\perp}}^2}{2l^-}(z_2^- - z_1^-)} \approx 1$, this allow to easily perform the integration over $\boldsymbol{l_{1\perp}}$, which results in a delta function. Physically, this implies that in the eikonal approximation, the quark does not change its transverse location between the interactions.

$$V_2^{q,A}(l,l') = (ig)^2 2\pi\delta(l^- - l'^-)\gamma^- \int \mathrm{d}^2\boldsymbol{z_{2\perp}} e^{-i\boldsymbol{l}_\perp\cdot \boldsymbol{z_{2\perp}}}\int \mathrm{d}^2\boldsymbol{z_{1\perp}} e^{i\boldsymbol{l}'_\perp\cdot \boldsymbol{z_{1\perp}}}\int \mathrm{d}z_2^- e^{il^+z_2^-}\int^{z_2^-} \mathrm{d}z_1^- e^{-il'^+z_1^-} \\ \times A_{\mathrm{cl}}^{+,a}(z_2^-,\boldsymbol{z_{2\perp}}) A_{\mathrm{cl}}^{+,b}(z_1^-,\boldsymbol{z_{1\perp}})t^a t^b \delta^{(2)}(\boldsymbol{z_{2\perp}}-\boldsymbol{z_{1\perp}}) . \tag{44}$$

Performing the integral over $\boldsymbol{z_{2\perp}}$ we find

$$V_2^{q,A}(l,l') = 2\pi\delta(l^- - l'^-)\gamma^- \int \mathrm{d}^2\boldsymbol{z_{1\perp}} e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot \boldsymbol{z_{1\perp}}}(ig)^2\int \mathrm{d}z_2^- e^{il^+z_2^-}\int^{z_2^-} \mathrm{d}z_1^- e^{-il'^+z_1^-} A_{\mathrm{cl}}^{+,a}(z_2^-,\boldsymbol{z_{1\perp}}) A_{\mathrm{cl}}^{+,b}(z_1^-,\boldsymbol{z_{1\perp}})t^a t^b \tag{45}$$

Within the eikonal approximation, we can also ignore the phases $e^{il^+z_2^-}, e^{-il'^+z_1^-} \approx 1$ With the help of the path ordering function we find

$$V_{2\mathrm{eik}}^{q,A}(l,l') = 2\pi\delta(l^- - l'^-)\gamma^- \int \mathrm{d}^2\boldsymbol{z_{1\perp}} e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot \boldsymbol{z_{1\perp}}}\mathcal{P}\left\{\frac{1}{2}(ig)^2\int \mathrm{d}z_2^-\int \mathrm{d}z_1^- A_{\mathrm{cl}}^{+,a}(z_2^-,\boldsymbol{z_{1\perp}}) A_{\mathrm{cl}}^{+,b}(z_1^-,\boldsymbol{z_{1\perp}})t^a t^b\right\} \tag{46}$$

### 3. Multiple scatterings with the background field and resummation

It is easy to observe that for an $n$ number of interactions

$$V^{q,A}_{\text{neik}}(l,l') = 2\pi\delta(l^- - l'^-)\gamma^- \int \mathrm{d}^2\boldsymbol{z}_{\mathbf{1}\perp} e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot \boldsymbol{z}_{\mathbf{1}\perp}} \mathcal{P}\left\{ \frac{1}{n!}(ig)^n \prod_{k=1}^{n} \int \mathrm{d}z_k^- A^{+,a_k}_{\text{cl}}(z_k^-, \boldsymbol{z}_{\mathbf{1}\perp}) t^{a_k} \right\} \tag{47}$$

Now, it is evident that the resummation of multiple scatterings of a quark with the background field (including no scattering) can be exponentiated. The derivation for the scattering of the anti-quark with the background field is almost identical. One finds:

$$\mathcal{T}^q(l,l') = 2\pi\delta(l^- - l'^-)\gamma^- \,\mathrm{sgn}(l^-) \int \mathrm{d}^2\boldsymbol{z}_{\mathbf{1}\perp} e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot \boldsymbol{z}_{\mathbf{1}\perp}} V^{\mathrm{sgn}(l^-)}(\boldsymbol{z}_{\mathbf{1}\perp})\,, \tag{48}$$

where

$$V(\boldsymbol{z}_{\mathbf{1}\perp}) = \mathcal{P}\exp\left( ig \int_{-\infty}^{\infty} dz^- A^{+,c}_{\text{cl}}(z^-, \boldsymbol{z}_{\mathbf{1}\perp}) t^c \right). \tag{49}$$

### 4. CGC Feynman rules

We define lightlike Wilson lines in the fundamental and adjoint representations appearing in the effective CGC vertices are given by the SU(3) matrices

$$V_{ij}(\boldsymbol{z}_\perp) = \mathcal{P}\exp\left( ig \int_{-\infty}^{\infty} dz^- A^{+,c}_{\text{cl}}(z^-, \boldsymbol{z}_\perp) t^c_{ij} \right), \tag{50}$$

$$U_{ab}(\boldsymbol{z}_\perp) = \mathcal{P}\exp\left( ig \int_{-\infty}^{\infty} dz^- A^{+,c}_{\text{cl}}(z^-, \boldsymbol{z}_\perp) T^c_{ab} \right), \tag{51}$$

where $t^c_{ij}$ and $T^c_{ab}$ are the generators of SU(3) in the fundamental and adjoint representations respectively. $A^+_{\text{cl}}$ is the back-ground gauge field of the classical small $x$ gluon field in Lorenz gauge. Here $\mathcal{P}$ stands for path ordering such that the operator at $z = -\infty$ is in the rightmost position, while that at $z = +\infty$ is in the leftmost position.

Note that the Wilson lines (at a fixed transverse position) are unitary color matrices:

$$V(\boldsymbol{z}_\perp)V^\dagger(\boldsymbol{z}_\perp) = V^\dagger(\boldsymbol{z}_\perp)V(\boldsymbol{z}_\perp) = \mathbb{1}_{N_c}\,, \tag{52}$$

$$U(\boldsymbol{z}_\perp)U^\dagger(\boldsymbol{z}_\perp) = U^\dagger(\boldsymbol{z}_\perp)U(\boldsymbol{z}_\perp) = \mathbb{1}_{N_c^2-1}\,. \tag{53}$$

The CGC effective vertices for the eikonal interaction of the quark (moving with large minus lightcone momentum component) with the background of the nucleus (moving with large plus light-cone momentum component) is given by

$$\mathcal{T}^q_{\sigma\sigma',ij}(l,l') = (2\pi)\delta(l^- - l'^-)\gamma^-_{\sigma\sigma'} \,\mathrm{sgn}(l^-) \int \mathrm{d}^2\boldsymbol{z}_\perp e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot \boldsymbol{z}_\perp} V^{\mathrm{sgn}(l^-)}_{ij}(\boldsymbol{z}_\perp)\,, \tag{54}$$

and similarly the eikonal interaction of the gluon (moving with large minus lightcone momentum component) with the background of the nucleus (moving with large plus light-cone momentum component) reads

$$\mathcal{T}^g_{\mu\nu,ab}(l,l') = -(2\pi)\delta(l^- - l'^-)(2l^-)g_{\mu\nu} \,\mathrm{sgn}(l^-) \int \mathrm{d}^2\boldsymbol{z}_\perp e^{-i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot \boldsymbol{z}_\perp} U^{\mathrm{sgn}(l^-)}_{ab}(\boldsymbol{z}_\perp)\,, \tag{55}$$

where $l$ and $l'$ are the incoming and outgoing momenta of the quark/gluon. The superscript $\mathrm{sgn}(l^-)$ denotes the color matrix or its inverse $V^{+1}(\boldsymbol{z}_\perp) = V(\boldsymbol{z}_\perp)$ and $V^{-1}(\boldsymbol{z}_\perp) = V^\dagger(\boldsymbol{z}_\perp)$, where the latter follows from the unitarity of $V(\boldsymbol{z}_\perp)$, and similarly for $U(\boldsymbol{z}_\perp)$.

### G. Exercises

- Compute the effective vertex gluon scattering. What happens with the four-gluon QCD vertex?
- Show that the propagators or vertices between two effective CGC propagators are now allowed.
- Start from MV model correlator for sources and derive the correlator for the dipole.

## II. LECTURE II: COMPUTATION OF OBSERVABLES

In this lecture, I will review the computation of two classical observables in the CGC EFT: i) single-inclusive hadron production in proton-nucleus collision, and deep inelastic scattering, and show that they both feature the two-point correlator of Wilson lines. Then, I will discuss the computation a semi-inclusive dijet production in DIS which will feature a new correlator: the quadrupole.

### A. From amplitude to cross-section

For a (semi)-inclusive process of the form:

$$P_0(q) + A(P_A) \to \sum_k P_k(p_k) + X \tag{56}$$

1. Compute the scattering amplitude $\mathcal{S}[\rho]$ in the presence of the background field representing the nucleus $A$ using the momentum space Feynman rules (endowed with the CGC effective vertices) outlined in the previous lecture.
2. Subtract the non-scattering contribution (setting all Wilson lines to unity) and factorize an overall minus momentum conserving delta function.

$$2\pi\delta\left(\sum_{j=1}^{N_i} q_j^- - \sum_{k=1}^{N_f} p_k^-\right)\mathcal{M}[\rho] = \mathcal{S}[\rho=0] - \mathcal{S}[\rho] \tag{57}$$

3. The unpolarized cross-section for the semi-inclusive process: $P_0(q) + A(P_A) \to \sum_k P_k(p_k) + X$ is given by

$$\mathrm{d}\sigma = \underbrace{2\pi\delta\left(q^- - \sum_{k=1}^{N_f} p_k^-\right)}_{\substack{\text{momentum conserving}\\ \text{delta function}}} \underbrace{\frac{1}{(2q^-)}}_{\text{flux factor}} \underbrace{\left[\sum_{\substack{\text{color}\\ \text{spin}}} \left\langle \mathcal{M}[\rho]\mathcal{M}^\dagger[\rho]\right\rangle_{x_0}\right]}_{\text{amplitude squared}} \underbrace{\prod_{k=1}^{N_f} \frac{\mathrm{d}p_k^- \mathrm{d}^2\boldsymbol{p_{k\perp}}}{2p_k^-(2\pi)^3}}_{\text{phase space factor}} \tag{58}$$

For the special cases:

- $P_0(q) + A(P_A) \to P_1(p) + X$

$$\mathrm{d}\sigma = 2\pi\delta(q^- - p^-)\frac{1}{2q^-} \underbrace{\left[\sum_{\substack{\text{color}\\ \text{spin}}} \left\langle \mathcal{M}[\rho]\mathcal{M}^\dagger[\rho]\right\rangle_{x_0}\right]}_{\text{amplitude squared}} \frac{\mathrm{d}p^- \mathrm{d}^2\boldsymbol{p}_\perp}{2p^-(2\pi)^3} \tag{59}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z\mathrm{d}^2\boldsymbol{p}_\perp} = \frac{1}{4q^-p^-(2\pi)^2} \underbrace{\left[\sum_{\substack{\text{color}\\ \text{spin}}} \left\langle \mathcal{M}[\rho]\mathcal{M}^\dagger[\rho]\right\rangle_{x_0}\right]}_{\text{amplitude squared}} \delta(1-z) \tag{60}$$

  where $z = p^-/q^-$ is the light-cone momentum fraction of the initial particle carried by the final state particle.

- $P_0(q) + A(P_A) \to P_1(k) + P_2(p) + X$

$$\ldots \tag{61}$$

### B. Single inclusive hadron production in proton-nucleus collisions

The scattering amplitude is given by

$$\mathcal{S}[\rho] = \bar{u}(p,s)\mathcal{T}(p,q)u(q,\bar{s}) \tag{62}$$

The physical amplitude (subtracting no scattering contribution and factoring an overall conserving minus momentum delta function)

$$\mathcal{M}[\rho] = \bar{u}(p,s)\gamma^- u(q,\bar{s}) \int \mathrm{d}^2\boldsymbol{z}_\perp e^{-i(\boldsymbol{p}_\perp - \boldsymbol{q}_\perp)\cdot\boldsymbol{z}_\perp} \left(V(\boldsymbol{z}_\perp) - \mathbb{1}\right) \tag{63}$$

Then

$$\frac{1}{2N_c}\sum_{\substack{s\bar{s}\\ \text{colors}}} \langle \mathcal{M}[\rho]\mathcal{M}[\rho]^\dagger\rangle_{x_0} = \frac{1}{2N_c}\mathrm{Tr}\left[\not{p}\gamma^-\not{q}\gamma^-\right]\int \mathrm{d}^2\boldsymbol{z}_\perp \mathrm{d}^2\boldsymbol{z}'_\perp e^{-i(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot(\boldsymbol{z}_\perp - \boldsymbol{z}'_\perp)}\mathrm{Tr}\left[V(\boldsymbol{z}_\perp)V^\dagger(\boldsymbol{z}'_\perp)\right] + C\delta^{(2)}(\boldsymbol{p}_\perp - \boldsymbol{q}_\perp) \tag{64}$$

$$= 4p^-q^- \int \mathrm{d}^2\boldsymbol{b}_\perp \int \mathrm{d}^2\boldsymbol{r}_\perp e^{-i(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot\boldsymbol{r}_\perp} S^{(2)}\left(\boldsymbol{b}_\perp + \frac{\boldsymbol{r}_\perp}{2}, \boldsymbol{b}_\perp - \frac{\boldsymbol{r}_\perp}{2}\right) \tag{65}$$

where the contribution that arises from zero or one Wilson has been separated, we will ignore this contribution. We have defined the dipole correlator:

$$S^{(2)}\left(\boldsymbol{b}_\perp + \frac{\boldsymbol{r}_\perp}{2}, \boldsymbol{b}_\perp - \frac{\boldsymbol{r}_\perp}{2}\right) = \frac{1}{N_c}\left\langle \mathrm{Tr}\left[V(\boldsymbol{x}_\perp)V^\dagger(\boldsymbol{y}_\perp)\right]\right\rangle_{x_0} \tag{66}$$

The differential cross-section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z\mathrm{d}^2\boldsymbol{p}_\perp} = \delta(1-z)\int \frac{\mathrm{d}^2\boldsymbol{b}_\perp \mathrm{d}^2\boldsymbol{r}_\perp}{(2\pi)^2} e^{-i(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot\boldsymbol{r}_\perp} S^{(2)}\left(\boldsymbol{b}_\perp + \frac{\boldsymbol{r}_\perp}{2}, \boldsymbol{b}_\perp - \frac{\boldsymbol{r}_\perp}{2}\right) \tag{67}$$

Thus

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z\mathrm{d}^2\boldsymbol{p}_\perp} = \delta(1-z)\tilde{S}^{(2)}(\boldsymbol{q}_\perp - \boldsymbol{p}_\perp) \tag{68}$$

where

$$\tilde{S}^{(2)}(\boldsymbol{q}_\perp - \boldsymbol{p}_\perp) = \int \frac{\mathrm{d}^2\boldsymbol{b}_\perp \mathrm{d}^2\boldsymbol{r}_\perp}{(2\pi)^2} e^{-i(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot\boldsymbol{r}_\perp} S^{(2)}\left(\boldsymbol{b}_\perp + \frac{\boldsymbol{r}_\perp}{2}, \boldsymbol{b}_\perp - \frac{\boldsymbol{r}_\perp}{2}\right) \tag{69}$$

### C. Total cross-section in deep inelastic scattering (DIS)

Optical theorem

$$2\Re[\langle \mathcal{M}^{\gamma^* A\to\gamma^* A}[\rho]\rangle_{x_0}] = 2q^- \sigma_{tot}^{\gamma^*} \tag{70}$$

We will work in a frame where

$$q^\mu = \left(-\frac{Q^2}{2q^-}, q^-, \mathbf{0}_\perp\right) \tag{71}$$

The forward scattering amplitude reads:

$$\mathcal{S}^\lambda[\rho] = (-1)\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\frac{\mathrm{d}^4 l'}{(2\pi)^4}\mathrm{Tr}[S^0(l)(-iee_f\not\epsilon)(q,\lambda)S^0(l-q)\mathcal{T}(l-q,l'-q)S^0(l'-q)(-iee_f\not\epsilon^*(q,\lambda))S^0(l')\mathcal{T}(l',l)] \tag{72}$$

Reduced amplitude:

$$(2\pi)\delta(q^- - q^-)\mathcal{M}^\lambda[\rho] = \mathcal{S}^\lambda[\rho=0] - \mathcal{S}^\lambda[\rho] \tag{73}$$

$$\langle \mathcal{M}^\lambda[\rho]\rangle_{x_0} = (2q^-)N_c\int \mathrm{d}^2\boldsymbol{b}_\perp \mathrm{d}^2\boldsymbol{r}_\perp D(\boldsymbol{r}_\perp,\boldsymbol{b}_\perp)\mathcal{A}^\lambda(\boldsymbol{r}_\perp) \tag{74}$$

where color correlator

$$D(\boldsymbol{r}_\perp,\boldsymbol{b}_\perp) = 1 - S^{(2)}\left(\boldsymbol{b}_\perp + \frac{\boldsymbol{r}_\perp}{2}, \boldsymbol{b}_\perp - \frac{\boldsymbol{r}_\perp}{2}\right) \tag{75}$$

$$\mathcal{A}^\lambda(\boldsymbol{r}_\perp) = -\frac{(ee_f)^2}{2\pi}\int \frac{\mathrm{d}^4 l}{(2\pi)^3}\frac{\mathrm{d}^4 l'}{(2\pi)^3}\frac{(2q^-)\delta(l^- - l'^-)e^{i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot\boldsymbol{r}_\perp}A^\lambda(l,l')}{[l^2+i\epsilon]\,[(l-q)^2+i\epsilon]\,[l'^2+i\epsilon]\,[(l'-q)^2+i\epsilon]} \tag{76}$$

and

$$A^\lambda(l,l') = \frac{1}{(2q^-)^2}\mathrm{Tr}[\not l\not\epsilon(q,\lambda)(\not l - \not q)\gamma^-(\not l' - \not q)\not\epsilon^*(q,\lambda)\not l'\gamma^-] \tag{77}$$

Note that despite the fact that this is a leading order computation, notice that it requires the evaluation of a loop. This is a consequence of multiple scattering. In what follows we perform the integration over the loop momentum $l$. First, we will do the integration over $l^-$, then $l^+$, and finally over $\boldsymbol{l}_\perp$.

#### 1. *Integrating over $l^-$ and $l^+$*

Integration over $l'^-$ can be carried out easily using the delta function. We define $z = l^-/q^-$ then we have

$$\begin{aligned}\mathcal{A}^\lambda(\boldsymbol{r}_\perp) = &-(ee_f)^2\int_0^1 \frac{\mathrm{d}z}{4\pi}\int \frac{\mathrm{d}^2\boldsymbol{l}_\perp \mathrm{d}^2\boldsymbol{l}'_\perp}{(2\pi)^4}e^{i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)}A^\lambda(\boldsymbol{l}_\perp,\boldsymbol{l}'_\perp,z)\\ &\times\int \frac{\mathrm{d}l^+}{2\pi}\frac{(2q^-)}{[l^2+i\epsilon]\,[(l-q)^2+i\epsilon]}\int \frac{\mathrm{d}l'^+}{2\pi}\frac{(2q^-)}{[l'^2+i\epsilon]\,[(l'-q)^2+i\epsilon]}\end{aligned} \tag{78}$$

where we used the fact that $A^\lambda(l,l')$ is independent of $l^+$ and $l'^+$. Using

$$\int \frac{\mathrm{d}l^+}{2\pi}\frac{(2q^-)}{[l^2+i\epsilon]\,[(l-q)^2+i\epsilon]} = \frac{i}{\bar{Q}^2 + \boldsymbol{l}_\perp^2} \tag{79}$$

Thus

$$\mathcal{A}^{\lambda}(\boldsymbol{r}_\perp) = (ee_f)^2 \int_0^1 \frac{\mathrm{d}z}{4\pi} \int \frac{\mathrm{d}^2\boldsymbol{l}_\perp \mathrm{d}^2\boldsymbol{l}'_\perp}{(2\pi)^4} \frac{e^{i(\boldsymbol{l}_\perp - \boldsymbol{l}'_\perp)\cdot \boldsymbol{r}_\perp} A^{\lambda}(\boldsymbol{l}_\perp, \boldsymbol{l}'_\perp, z)}{\left[\bar{Q}^2 + \boldsymbol{l}_\perp^2\right]\left[\bar{Q}^2 + \boldsymbol{l}'^2_\perp\right]} \tag{80}$$

where

$$\bar{Q}^2 = z(1-z)Q^2 \tag{81}$$

$$z = \frac{l^-}{q^-} \tag{82}$$

## 2. Perturbative factor

Useful identities

$$\not{l}\not{\epsilon}(q, \lambda = \pm 1)(\not{l} - \not{q})\gamma^- = -2\boldsymbol{l}^i_\perp \not{l}\gamma^- \left(\boldsymbol{\epsilon}^{\pm 1, i}_\perp + \frac{1}{2z}\gamma^i\gamma^j \boldsymbol{\epsilon}^{\pm 1, j}_\perp\right) + C^i_1 \gamma^i \gamma^- \tag{83}$$

$$(\not{l}' - \not{q})\not{\epsilon}^*(q, \lambda' = \pm 1)\not{l}'\gamma^- = -2\boldsymbol{l}'^m_\perp \left(\boldsymbol{\epsilon}^{\pm 1*, m}_\perp + \frac{1}{2z}\gamma^n\gamma^m \boldsymbol{\epsilon}^{\pm 1*, n}_\perp\right)\not{l}'\gamma^- + C^m_2 \gamma^m \gamma^- \tag{84}$$

Then

$$\begin{aligned} A^{\lambda=\pm 1}(\boldsymbol{l}_\perp, \boldsymbol{l}'_\perp, z) &= \frac{1}{(2q^-)^2} 2\boldsymbol{l}^i_\perp 2\boldsymbol{l}'^m_\perp \mathrm{Tr}\left[\not{l}\gamma^- \left(\boldsymbol{\epsilon}^{\pm 1, i}_\perp + \frac{1}{2z}\gamma^i\gamma^j \boldsymbol{\epsilon}^{\pm 1, j}_\perp\right)\left(\boldsymbol{\epsilon}^{\pm 1*, m}_\perp + \frac{1}{2z}\gamma^n\gamma^m \boldsymbol{\epsilon}^{\pm 1*, n}_\perp\right)\not{l}'\gamma^-\right] \\ &= 2z^2 \boldsymbol{l}^i_\perp \boldsymbol{l}'^m_\perp \mathrm{Tr}\left[\left(\boldsymbol{\epsilon}^{\pm 1, i}_\perp + \frac{1}{2z}\gamma^i\gamma^j \boldsymbol{\epsilon}^{\pm 1, j}_\perp\right)\left(\boldsymbol{\epsilon}^{\pm 1*, m}_\perp + \frac{1}{2z}\gamma^n\gamma^m \boldsymbol{\epsilon}^{\pm 1*, n}_\perp\right)\right] \end{aligned} \tag{85}$$

Using

$$\begin{aligned} &\mathrm{Tr}\left[\left(\boldsymbol{\epsilon}^{\lambda, i}_\perp + \frac{1}{2z}\gamma^i\gamma^j \boldsymbol{\epsilon}^{\lambda, j}_\perp\right)\left(\boldsymbol{\epsilon}^{\lambda*, m}_\perp + \frac{1}{2z}\gamma^n\gamma^m \boldsymbol{\epsilon}^{\lambda*, n}_\perp\right)\right] \\ &= \frac{2}{z^2}\left[z^2 + (1-z)^2\right]\boldsymbol{\epsilon}^{\lambda, i}_\perp \boldsymbol{\epsilon}^{\lambda*, m}_\perp \end{aligned} \tag{86}$$

Then

$$A^{\lambda=\pm 1}(\boldsymbol{l}_\perp, \boldsymbol{l}'_\perp, z) = 4\left[z^2 + (1-z)^2\right](\boldsymbol{l}_\perp \cdot \boldsymbol{\epsilon}^{\lambda}_\perp)(\boldsymbol{l}'_\perp \cdot \boldsymbol{\epsilon}^{\lambda*}_\perp) \tag{87}$$

Thus

$$\mathcal{A}^{\lambda}(\boldsymbol{r}_\perp) = (ee_f)^2 \int_0^1 \frac{\mathrm{d}z}{4\pi} 4\left[z^2 + (1-z)^2\right] \int \frac{\mathrm{d}^2\boldsymbol{l}_\perp}{(2\pi)^2} \frac{e^{i\boldsymbol{l}_\perp\cdot \boldsymbol{r}_\perp}(\boldsymbol{l}_\perp \cdot \boldsymbol{\epsilon}^{\lambda}_\perp)}{\left[\bar{Q}^2 + \boldsymbol{l}^2_\perp\right]} \int \frac{\mathrm{d}^2\boldsymbol{l}'_\perp}{(2\pi)^2} \frac{e^{-i\boldsymbol{l}'_\perp\cdot \boldsymbol{r}_\perp}(\boldsymbol{l}'_\perp \cdot \boldsymbol{\epsilon}^{\lambda*}_\perp)}{\left[\bar{Q}^2 + \boldsymbol{l}'^2_\perp\right]} \tag{88}$$

Using

$$\int \frac{\mathrm{d}^2\boldsymbol{l}_\perp}{(2\pi)} \frac{e^{i\boldsymbol{l}_\perp\cdot \boldsymbol{r}_\perp}\boldsymbol{l}_\perp}{\left[\boldsymbol{l}^2_\perp + \bar{Q}^2\right]} = \frac{i\boldsymbol{r}_\perp}{r_\perp}\bar{Q}K_1(\bar{Q}r_\perp) \tag{89}$$

$$\mathcal{A}^{\lambda}(\boldsymbol{r}_\perp) = \frac{4\alpha_{em}e_f^2}{\pi} \int_0^1 \frac{\mathrm{d}z}{4\pi}\left[z^2 + (1-z)^2\right] \frac{(\boldsymbol{r}_\perp \cdot \boldsymbol{\epsilon}^{\lambda}_\perp)(\boldsymbol{r}_\perp \cdot \boldsymbol{\epsilon}^{\lambda*}_\perp)}{r^2_\perp} \bar{Q}^2 K_1^2(\bar{Q}r_\perp) \tag{90}$$

where $\alpha_{em} = e^2/(4\pi)$

$$\mathcal{A}^{T}(\boldsymbol{r}_\perp) = \frac{1}{2}\sum_\lambda \mathcal{A}^{\lambda}(\boldsymbol{r}_\perp) = \frac{2\alpha_{em}e_f^2}{\pi} \int_0^1 \frac{\mathrm{d}z}{4\pi}\left[z^2 + (1-z)^2\right]\bar{Q}^2 K_1^2(\bar{Q}r_\perp) \tag{91}$$

Finally

$$\langle \mathcal{M}^{\lambda=T}[\rho]\rangle_{x_0} =(2q^-)\frac{2\alpha_{em}e_f^2N_c}{\pi}\int \mathrm{d}^2\boldsymbol{b}_\perp \mathrm{d}^2\boldsymbol{r}_\perp D(\boldsymbol{r}_\perp,\boldsymbol{b}_\perp)\int_0^1 \frac{\mathrm{d}z}{4\pi}\Big[z^2+(1-z)^2\Big]\bar{Q}^2K_1^2(\bar{Q}r_\perp) \tag{92}$$

Thus

$$\sigma_{tot} = \frac{\alpha_{em}e_f^2N_c}{\pi}\int \mathrm{d}^2\boldsymbol{b}_\perp \mathrm{d}^2\boldsymbol{r}_\perp 2D(\boldsymbol{r}_\perp,\boldsymbol{b}_\perp)\int_0^1 \frac{\mathrm{d}z}{4\pi}\Big[z^2+(1-z)^2\Big]\bar{Q}^2K_1^2(\bar{Q}r_\perp) \tag{93}$$

- Discuss interpretation in terms of LCPT
- Plot LCWF of photon to qqbar, role of virtuality as transverse resolution scale

### D. Universality

- Point out the universality of dipole amplitude
- Model for the dipole: GBW and MV model, basic properties of the dipole distribution: color transparency and saturation
- Pheno of DIS: geometric scaling

### E. Semi-inclusive dijet production in DIS

#### 1. *Amplitude*

Let $z = k^-/q^-$. We denote the longitudinal polarization as $\lambda = 0$, and the two transverse polarization as $\lambda = \pm 1$. Assume $q^- > 0$, $k^- > 0$, and $p^- > 0$.

The scattering amplitude for dijets at LO is given by

$$\mathcal{S}^\lambda_{s\bar{s}}[\rho] = \int \frac{d^4l}{(2\pi)^4}\bar{u}(k,s)\mathcal{T}^q(k,l)S^0(l)\left(-iee_f\not{\epsilon}(q,\lambda)\right)S^0(l-q)\mathcal{T}^q(l-q,-p)v(p,\bar{s}) \tag{94}$$

Subtracting the non-interaction piece and factoring and overall delta function in longitudinal momenta

$$(2\pi)\delta(k^-+p^--q^-)\mathcal{M}^\lambda_{s\bar{s}} = \mathcal{S}^\lambda_{s\bar{s}}[\rho=0]-\mathcal{S}^\lambda_{s\bar{s}}[\rho] \tag{95}$$

We have

$$\mathcal{M}^\lambda_{s\bar{s}}[\rho] = (2q^-)\int \mathrm{d}^2\boldsymbol{x}_\perp \mathrm{d}^2\boldsymbol{y}_\perp e^{-i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}e^{-i\boldsymbol{p}_\perp\cdot\boldsymbol{y}_\perp}\mathcal{C}(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp)\mathcal{N}^\lambda_{s\bar{s}}(\boldsymbol{r}_{xy}) \tag{96}$$

with color structure

$$\mathcal{C}(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp) \equiv \left[\mathbb{1}-V(\boldsymbol{x}_\perp)V^\dagger(\boldsymbol{y}_\perp)\right] \tag{97}$$

and perturbative factor

$$\mathcal{N}^\lambda_{s\bar{s}}(\boldsymbol{r}_\perp) = -i(ee_f)\int \frac{\mathrm{d}^4l}{(2\pi)^3}e^{i\boldsymbol{l}_\perp\cdot\boldsymbol{r}_\perp}\frac{(2q^-)N^\lambda_{s\bar{s}}(l)\delta(k^--l^-)}{(l^2+i\epsilon)((l-q)^2+i\epsilon)} \tag{98}$$

$$N^\lambda_{s\bar{s}}(l) = \frac{1}{(2q^-)^2}\left[\bar{u}(k,s)\gamma^-\not{l}\not{\epsilon}(q,\lambda)(\not{l}-\not{q})\gamma^-v(p,\bar{s})\right] \tag{99}$$

Again $N^\lambda_{s\bar{s}}(l)$ is independent of $l^+$ then

$$\mathcal{N}^\lambda_{s\bar{s}}(\boldsymbol{r}_\perp) = -i(ee_f)\int \frac{\mathrm{d}^2\boldsymbol{l}_\perp}{(2\pi)^2}\frac{e^{i\boldsymbol{l}_\perp\cdot\boldsymbol{r}_\perp}N^\lambda_{s\bar{s}}(\boldsymbol{l}_\perp,z)}{\boldsymbol{l}_\perp^2+\bar{Q}^2} \tag{100}$$

$$\mathrm{d}\sigma = 2\pi\delta(q^- - k^- - p^-)\frac{1}{2q^-}\sum_{s\bar{s},\text{colors}}\left\langle \mathcal{M}^\lambda_{s\bar{s}}[\rho]\left(\mathcal{M}^\lambda_{s\bar{s}}\right)^\dagger[\rho]\right\rangle_{x_0}\frac{\mathrm{d}k^-\mathrm{d}^2\boldsymbol{k}_\perp}{2k^-(2\pi)^3}\frac{\mathrm{d}p^-\mathrm{d}^2\boldsymbol{p}_\perp}{2p^-(2\pi)^3} \tag{101}$$

Then

$$\frac{\mathrm{d}\sigma^\lambda}{\mathrm{d}y_q\mathrm{d}y_{\bar{q}}\mathrm{d}^2\boldsymbol{k}_\perp\mathrm{d}^2\boldsymbol{p}_\perp} = \delta(1-z-\bar{z})\frac{1}{(2q^-)^2}\sum_{s\bar{s},\text{colors}}\left\langle \mathcal{M}^\lambda_{s\bar{s}}[\rho]\left(\mathcal{M}^\lambda_{s\bar{s}}\right)^\dagger[\rho]\right\rangle_{x_0}\frac{1}{2(2\pi)^5} \tag{102}$$

$$\begin{aligned}\sum_{s\bar{s},\text{colors}}\left\langle \mathcal{M}^\lambda_{s\bar{s}}[\rho]\left(\mathcal{M}^\lambda_{s\bar{s}}\right)^\dagger[\rho]\right\rangle_{x_0} &= (2q^-)^2(4\pi)\int \mathrm{d}^2\boldsymbol{x}_\perp\mathrm{d}^2\boldsymbol{y}_\perp\mathrm{d}^2\boldsymbol{x}'_\perp\mathrm{d}^2\boldsymbol{y}'_\perp e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)}e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{y}_\perp-\boldsymbol{y}'_\perp)}\\ &\times \Xi(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp;\boldsymbol{y}'_\perp,\boldsymbol{x}'_\perp)\mathcal{H}^\lambda(\boldsymbol{r}_\perp,\boldsymbol{r}'_\perp)\end{aligned} \tag{103}$$

where

$$\Xi(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp;\boldsymbol{y}'_\perp,\boldsymbol{x}'_\perp) = 1 - S^{(2)}(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp) - S^{(2)}(\boldsymbol{y}'_\perp,\boldsymbol{x}'_\perp) - S^{(4)}(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp,\boldsymbol{y}'_\perp,\boldsymbol{x}'_\perp) \tag{104}$$

$$\mathcal{H}^\lambda(\boldsymbol{r}_\perp,\boldsymbol{r}'_\perp) = \frac{N_c}{4\pi}\sum_{s\bar{s}}\mathcal{N}^\lambda_{s\bar{s}}(\boldsymbol{r}_\perp)\left(\mathcal{N}^\lambda_{s\bar{s}}(\boldsymbol{r}'_\perp)\right)^\dagger = \frac{(ee_f)^2N_c}{4\pi}\int\frac{\mathrm{d}^2\boldsymbol{l}_\perp\mathrm{d}^2\boldsymbol{l}'_\perp}{(2\pi)^4}\frac{e^{i\boldsymbol{l}_\perp\cdot\boldsymbol{r}_\perp}e^{-i\boldsymbol{l}'_\perp\cdot\boldsymbol{r}'_\perp}\Gamma^\lambda(\boldsymbol{l}_\perp,\boldsymbol{l}'_\perp,z)}{(\boldsymbol{l}_\perp^2+\bar{Q}^2)(\boldsymbol{l}'^2_\perp+\bar{Q}^2)} \tag{105}$$

and

$$\begin{aligned}\Gamma^\lambda(\boldsymbol{l}_\perp,\boldsymbol{l}'_\perp,z) &= \frac{1}{(2q^-)^4}\mathrm{Tr}\left[\not{k}\gamma^-\not{l}\not{\epsilon}(q,\lambda)(\not{l}-\not{q})\gamma^-\not{p}\gamma^-(\not{l}'-\not{q})\not{\epsilon}^*(q,\lambda)\not{l}'\gamma^-\right]\\ &= \frac{z(1-z)}{(2q^-)^2}\mathrm{Tr}\left[\not{l}\not{\epsilon}(q,\lambda)(\not{l}-\not{q})\gamma^-(\not{l}'-\not{q})\not{\epsilon}^*(q,\lambda)\not{l}'\gamma^-\right]\end{aligned} \tag{106}$$

Then

$$\Gamma^\lambda(\boldsymbol{l}_\perp,\boldsymbol{l}'_\perp,z) = 4z(1-z)\left[z^2+(1-z)^2\right](\boldsymbol{l}_\perp\cdot\boldsymbol{\epsilon}^\lambda_\perp)(\boldsymbol{l}'_\perp\cdot\boldsymbol{\epsilon}^{\lambda*}_\perp) \tag{107}$$

Thus

$$\mathcal{H}^\lambda(\boldsymbol{r}_\perp,\boldsymbol{r}'_\perp) = \frac{\alpha_{em}e_f^2N_c}{\pi^2}z(1-z)\left[z^2+(1-z)^2\right]\frac{(\boldsymbol{r}_\perp\cdot\boldsymbol{\epsilon}^\lambda_\perp)(\boldsymbol{r}'_\perp\cdot\boldsymbol{\epsilon}^{\lambda*}_\perp)}{r_\perp r'_\perp}\bar{Q}K_1(\bar{Q}r_\perp)\bar{Q}K_1(\bar{Q}r'_\perp) \tag{108}$$

$$\mathcal{H}^T(\boldsymbol{r}_\perp,\boldsymbol{r}'_\perp) = \frac{\alpha_{em}e_f^2N_c}{2\pi^2}z(1-z)\left[z^2+(1-z)^2\right]\frac{(\boldsymbol{r}_\perp\cdot\boldsymbol{r}'_\perp)}{r_\perp r'_\perp}\bar{Q}K_1(\bar{Q}r_\perp)\bar{Q}K_1(\bar{Q}r'_\perp) \tag{109}$$

Thus

$$\begin{aligned}\frac{\mathrm{d}\sigma^\lambda}{\mathrm{d}y_q\mathrm{d}y_{\bar{q}}\mathrm{d}^2\boldsymbol{k}_\perp\mathrm{d}^2\boldsymbol{p}_\perp} &= \delta(1-z-\bar{z})\int\frac{\mathrm{d}^2\boldsymbol{x}_\perp\mathrm{d}^2\boldsymbol{y}_\perp\mathrm{d}^2\boldsymbol{x}'_\perp\mathrm{d}^2\boldsymbol{y}'_\perp}{(2\pi)^4}e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{x}'_\perp)}e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{y}_\perp-\boldsymbol{y}'_\perp)}\\ &\times\Xi(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp;\boldsymbol{y}'_\perp,\boldsymbol{x}'_\perp)\mathcal{H}^\lambda(\boldsymbol{r}_\perp,\boldsymbol{r}'_\perp)\end{aligned} \tag{110}$$

### F. Exercises

- Compute DIS for longitudinally polarized case (MUST DO)
- Integrate one particle to compute SIDIS. Momentum and coordinate space (MUST DO)

- Compute total DIS cross-section from two-particle production in DIS by integrating out the phase space (proof of optical theorem)
- Compute quark + photon production in pA collision (MUST DO)
- Compute heavy-quark pair production
- Compute dijet production in proton-nucleus collisions. There are three channels: $q \to qg$, $g \to q\bar{q}$ and $g \to gg$

## III. LECTURE III: CGC TRANSVERSE MOMENTUM DEPENDENT (TMD) FACTORIZATION

Change of variables:

$$\boldsymbol{P}_\perp = (1-z)\boldsymbol{k}_\perp - z\boldsymbol{p}_\perp \tag{111}$$

$$\boldsymbol{q}_\perp = \boldsymbol{k}_\perp + \boldsymbol{p}_\perp \tag{112}$$

The corresponding coordinate space conjugates:

$$\boldsymbol{r}_\perp = \boldsymbol{x}_\perp - \boldsymbol{y}_\perp \tag{113}$$

$$\boldsymbol{b}_\perp = z\boldsymbol{x}_\perp + (1-z)\boldsymbol{y}_\perp \tag{114}$$

$$\boldsymbol{r}'_\perp = \boldsymbol{x}'_\perp - \boldsymbol{y}'_\perp \tag{115}$$

$$\boldsymbol{b}'_\perp = z\boldsymbol{x}'_\perp + (1-z)\boldsymbol{y}'_\perp \tag{116}$$

$$\tag{117}$$

Then we can write the cross-section as

$$\frac{\mathrm{d}\sigma^\lambda}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}} \mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{p}_\perp} = \delta(1-z-\bar{z}) \int \frac{\mathrm{d}^2\boldsymbol{b}_\perp \mathrm{d}^2\boldsymbol{r}_\perp \mathrm{d}^2\boldsymbol{b}'_\perp \mathrm{d}^2\boldsymbol{r}'_\perp}{(2\pi)^4} e^{-i\boldsymbol{P}_\perp\cdot(\boldsymbol{r}_\perp - \boldsymbol{r}'_\perp)} e^{-i\boldsymbol{q}_\perp\cdot(\boldsymbol{b}_\perp - \boldsymbol{b}'_\perp)}$$
$$\times\, \Xi\left(\boldsymbol{b}_\perp + (1-z)\boldsymbol{r}_\perp, \boldsymbol{b}_\perp - z\boldsymbol{r}_\perp; \boldsymbol{b}'_\perp - z\boldsymbol{r}'_\perp, \boldsymbol{b}'_\perp + (1-z)\boldsymbol{r}'_\perp\right) \mathcal{H}^\lambda(\boldsymbol{r}_\perp, \boldsymbol{r}'_\perp) \tag{118}$$

### A. Dijet production in momentum space expression

$$\frac{\mathrm{d}\sigma^\lambda}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}} \mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{p}_\perp} = \delta(1-z-\bar{z}) \int \frac{\mathrm{d}^2\boldsymbol{x}_\perp \mathrm{d}^2\boldsymbol{y}_\perp \mathrm{d}^2\boldsymbol{x}'_\perp \mathrm{d}^2\boldsymbol{y}'_\perp}{(2\pi)^4} e^{-i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}_\perp - \boldsymbol{x}'_\perp)} e^{-i\boldsymbol{p}_\perp\cdot(\boldsymbol{y}_\perp - \boldsymbol{y}'_\perp)}$$
$$\times\, \Xi(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; \boldsymbol{y}'_\perp, \boldsymbol{x}'_\perp) \frac{(ee_f)^2 N_c}{4\pi} \int \frac{\mathrm{d}^2\boldsymbol{l}_\perp \mathrm{d}^2\boldsymbol{l}'_\perp}{(2\pi)^4} \frac{e^{i\boldsymbol{l}_\perp\cdot\boldsymbol{r}_\perp} e^{-i\boldsymbol{l}'_\perp\cdot\boldsymbol{r}'_\perp} \Gamma^\lambda(\boldsymbol{l}_\perp, \boldsymbol{l}'_\perp, z)}{(\boldsymbol{l}_\perp^2 + \bar{Q}^2)(\boldsymbol{l}'^2_\perp + \bar{Q}^2)} \tag{119}$$

Can be written as

$$\frac{\mathrm{d}\sigma^\lambda}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}} \mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{p}_\perp} = \delta(1-z-\bar{z}) \int \frac{\mathrm{d}^2\boldsymbol{l}_\perp \mathrm{d}^2\boldsymbol{l}'_\perp}{(2\pi)^4} \int \frac{\mathrm{d}^2\boldsymbol{x}_\perp \mathrm{d}^2\boldsymbol{y}_\perp \mathrm{d}^2\boldsymbol{x}'_\perp \mathrm{d}^2\boldsymbol{y}'_\perp}{(2\pi)^4} \Xi(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp; \boldsymbol{y}'_\perp, \boldsymbol{x}'_\perp)$$
$$\times\, e^{-i(\boldsymbol{k}_\perp - \boldsymbol{l}_\perp)\cdot\boldsymbol{x}_\perp + i(\boldsymbol{k}_\perp - \boldsymbol{l}'_\perp)\cdot\boldsymbol{x}'_\perp} e^{-i(\boldsymbol{p}_\perp + \boldsymbol{l}_\perp)\cdot\boldsymbol{y}_\perp + i(\boldsymbol{p}_\perp + \boldsymbol{l}'_\perp)\cdot\boldsymbol{y}'_\perp} \frac{(ee_f)^2 N_c}{4\pi} \frac{\Gamma^\lambda(\boldsymbol{l}_\perp, \boldsymbol{l}'_\perp, z)}{(\boldsymbol{l}_\perp^2 + \bar{Q}^2)(\boldsymbol{l}'^2_\perp + \bar{Q}^2)} \tag{120}$$

Change of variables:

$$\boldsymbol{\ell}_\perp + \frac{\boldsymbol{q}_\perp}{2} = \boldsymbol{k}_\perp - \boldsymbol{l}_\perp \to \boldsymbol{l}_\perp = \boldsymbol{P}_\perp - \boldsymbol{\ell}_\perp \tag{121}$$

$$\boldsymbol{\ell}'_\perp + \frac{\boldsymbol{q}_\perp}{2} = \boldsymbol{k}_\perp - \boldsymbol{l}'_\perp \to \boldsymbol{l}'_\perp = \boldsymbol{P}_\perp - \boldsymbol{\ell}'_\perp \tag{122}$$

Then

$$\frac{\mathrm{d}\sigma^\lambda}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}} \mathrm{d}^2\boldsymbol{k}_\perp \mathrm{d}^2\boldsymbol{p}_\perp} = \delta(1-z-\bar{z}) \int \frac{\mathrm{d}^2\boldsymbol{\ell}_\perp \mathrm{d}^2\boldsymbol{\ell}'_\perp}{(2\pi)^4} \mathcal{G}(\boldsymbol{q}_\perp, \boldsymbol{\ell}_\perp, \boldsymbol{\ell}'_\perp) \mathcal{H}(Q, \boldsymbol{P}_\perp; \boldsymbol{\ell}_\perp, \boldsymbol{\ell}'_\perp) \tag{123}$$

where

$$\mathcal{H}^\lambda(Q, \boldsymbol{P}_\perp; \boldsymbol{\ell}_\perp, \boldsymbol{\ell}'_\perp) = \alpha_{em} e_f^2 N_c \frac{\Gamma^\lambda\left(\boldsymbol{P}_\perp - \boldsymbol{\ell}_\perp, \boldsymbol{P}_\perp - \boldsymbol{\ell}'_\perp, z\right)}{\left[(\boldsymbol{P}_\perp - \boldsymbol{\ell}_\perp)^2 + \bar{Q}^2\right]\left[(\boldsymbol{P}_\perp - \boldsymbol{\ell}'_\perp)^2 + \bar{Q}^2\right]} \tag{124}$$

$$\mathcal{G}(\boldsymbol{q}_\perp,\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)=\int\frac{\mathrm{d}^2\boldsymbol{x}_\perp\mathrm{d}^2\boldsymbol{y}_\perp\mathrm{d}^2\boldsymbol{x}'_\perp\mathrm{d}^2\boldsymbol{y}'_\perp}{(2\pi)^4}\Xi(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp;\boldsymbol{y}'_\perp,\boldsymbol{x}'_\perp)$$
$$\times\, e^{-i\left(\frac{\boldsymbol{q}_\perp}{2}+\boldsymbol{\ell}_\perp\right)\cdot\boldsymbol{x}_\perp+i\left(\frac{\boldsymbol{q}_\perp}{2}+\boldsymbol{\ell}'_\perp\right)\cdot\boldsymbol{x}'_\perp}e^{-i\left(\frac{\boldsymbol{q}_\perp}{2}-\boldsymbol{\ell}_\perp\right)\cdot\boldsymbol{y}_\perp+i\left(\frac{\boldsymbol{q}_\perp}{2}-\boldsymbol{\ell}'_\perp\right)\cdot\boldsymbol{y}'_\perp} \tag{125}$$

or

$$\mathcal{G}(\boldsymbol{q}_\perp,\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)=\int\frac{\mathrm{d}^2\boldsymbol{b}_\perp\mathrm{d}^2\boldsymbol{r}_\perp\mathrm{d}^2\boldsymbol{b}'_\perp\mathrm{d}^2\boldsymbol{r}'_\perp}{(2\pi)^4}\Xi(\boldsymbol{x}_\perp,\boldsymbol{y}_\perp;\boldsymbol{y}'_\perp,\boldsymbol{x}'_\perp)e^{-i\boldsymbol{q}_\perp\cdot(\boldsymbol{b}_\perp-\boldsymbol{b}'_\perp)}e^{-i\boldsymbol{\ell}_\perp\cdot\boldsymbol{r}_\perp}e^{-i\boldsymbol{\ell}'_\perp\cdot\boldsymbol{r}'_\perp} \tag{126}$$

The perturbative coefficient for transversely polarized:

$$\mathcal{H}^T(Q,\boldsymbol{P}_\perp;\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)=\alpha_{em}e_f^2N_c\frac{2z(1-z)\left[z^2+(1-z)^2\right](\boldsymbol{P}_\perp-\boldsymbol{\ell}_\perp)\cdot\left(\boldsymbol{P}_\perp-\boldsymbol{\ell}'_\perp\right)}{\left[(\boldsymbol{P}_\perp-\boldsymbol{\ell}_\perp)^2+\bar{Q}^2\right]\left[\left(\boldsymbol{P}_\perp-\boldsymbol{\ell}'_\perp\right)^2+\bar{Q}^2\right]} \tag{127}$$

Discuss convolution momenta very difficult to grasp flow. Motivate the momentum change of variables, violation of kT factorization.

## B. TMD expansion: the WW gluon TMD

Discuss the physical situation and kinematics, in relation to experiments.

Consider the limit $\boldsymbol{q}_\perp^2, Q_s^2 \ll Q^2, \boldsymbol{P}_\perp^2$. Explain physically why $|\boldsymbol{\ell}_\perp\pm\boldsymbol{q}_\perp/2|\sim|\boldsymbol{\ell}'_\perp\pm\boldsymbol{q}_\perp/2|\sim Q_s$. Taylor expand perturbative factor around $\boldsymbol{\ell}_\perp^2\sim 0, \boldsymbol{\ell}'^2_\perp\sim 0$, then

$$\frac{\mathrm{d}\sigma^\lambda}{\mathrm{d}y_q\mathrm{d}y_{\bar q}\mathrm{d}^2\boldsymbol{k}_\perp\mathrm{d}^2\boldsymbol{p}_\perp}=\delta(1-z-\bar z)\mathcal{H}^\lambda(Q,\boldsymbol{P}_\perp;0,0)\int\frac{\mathrm{d}^2\boldsymbol{\ell}_\perp\mathrm{d}^2\boldsymbol{\ell}'_\perp}{(2\pi)^4}\mathcal{G}(\boldsymbol{q}_\perp,\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)+\dots \tag{128}$$

Observe

$$\int\frac{\mathrm{d}^2\boldsymbol{\ell}_\perp}{(2\pi)^2}e^{-i\boldsymbol{\ell}_\perp\cdot\boldsymbol{r}_\perp}=\delta^{(2)}(\boldsymbol{r}_\perp) \tag{129}$$

$$\int\frac{\mathrm{d}^2\boldsymbol{\ell}_\perp\mathrm{d}^2\boldsymbol{\ell}'_\perp}{(2\pi)^4}\mathcal{G}(\boldsymbol{q}_\perp,\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)=\int\frac{\mathrm{d}^2\boldsymbol{b}_\perp\mathrm{d}^2\boldsymbol{b}'_\perp}{(2\pi)^4}\Xi(\boldsymbol{b}_\perp,\boldsymbol{b}_\perp;\boldsymbol{b}'_\perp,\boldsymbol{b}'_\perp)e^{-i\boldsymbol{q}_\perp\cdot(\boldsymbol{b}_\perp-\boldsymbol{b}'_\perp)}=0 \tag{130}$$

since $\Xi(\boldsymbol{b}_\perp,\boldsymbol{b}_\perp;\boldsymbol{b}'_\perp,\boldsymbol{b}'_\perp)=0$ due to unitarity of Wilson lines. Linear term also vanishes, need quadratic term in expansion $\boldsymbol{\ell}_\perp\boldsymbol{\ell}'_\perp$:

$$\frac{\mathrm{d}\sigma^\lambda}{\mathrm{d}y_q\mathrm{d}y_{\bar q}\mathrm{d}^2\boldsymbol{k}_\perp\mathrm{d}^2\boldsymbol{p}_\perp}=\delta(1-z-\bar z)\frac{\partial^2\mathcal{H}^\lambda(Q,\boldsymbol{P}_\perp;\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)}{\partial\boldsymbol{\ell}^i_\perp\partial\boldsymbol{\ell}'^j_\perp}\bigg|_{\boldsymbol{\ell}_\perp=\boldsymbol{\ell}'_\perp=0}\int\frac{\mathrm{d}^2\boldsymbol{\ell}_\perp\mathrm{d}^2\boldsymbol{\ell}'_\perp}{(2\pi)^4}\boldsymbol{\ell}^i_\perp\boldsymbol{\ell}'^j_\perp\mathcal{G}(\boldsymbol{q}_\perp,\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)+\dots \tag{131}$$

Using

$$\int\frac{\mathrm{d}^2\boldsymbol{\ell}_\perp}{(2\pi)^2}\boldsymbol{\ell}^i_\perp e^{-i\boldsymbol{\ell}_\perp\cdot\boldsymbol{r}_\perp}=i\partial^i_\perp\delta^{(2)}(\boldsymbol{r}_\perp) \tag{132}$$

Then

$$\int\frac{\mathrm{d}^2\boldsymbol{\ell}_\perp\mathrm{d}^2\boldsymbol{\ell}'_\perp}{(2\pi)^4}\boldsymbol{\ell}^i_\perp\boldsymbol{\ell}'^j_\perp\mathcal{G}(\boldsymbol{q}_\perp,\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)=-\int\frac{\mathrm{d}^2\boldsymbol{b}_\perp\mathrm{d}^2\boldsymbol{b}'_\perp}{(2\pi)^4}\frac{1}{N_c}\mathrm{Tr}\left[\left(V(\boldsymbol{b}_\perp)\partial^i_\perp V(\boldsymbol{b}_\perp)\right)\left(V(\boldsymbol{b}'_\perp)\partial^j_\perp V(\boldsymbol{b}'_\perp)\right)\right]e^{-i\boldsymbol{q}_\perp\cdot(\boldsymbol{b}_\perp-\boldsymbol{b}'_\perp)}$$
$$=\frac{1}{2N_c}\alpha_sG^{ij}_{WW}(\boldsymbol{q}_\perp) \tag{133}$$

Then we have

$$\frac{\mathrm{d}\sigma^{\lambda}}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}} \mathrm{d}^2\boldsymbol{P}_\perp \mathrm{d}^2\boldsymbol{q}_\perp} = H^{\lambda}_{ij}(Q,\boldsymbol{P}_\perp) G^{ij}_{WW}(\boldsymbol{q}_\perp) \tag{134}$$

where

$$H^{\lambda}_{ij}(Q,\boldsymbol{P}_\perp) = \frac{1}{2}\alpha_s \delta(1-z-\bar{z}) \frac{\partial^2 \mathcal{H}^{\lambda}(Q,\boldsymbol{P}_\perp;\boldsymbol{\ell}_\perp,\boldsymbol{\ell}'_\perp)}{\partial \boldsymbol{\ell}^i_\perp \partial \boldsymbol{\ell}'^j_\perp}\Bigg|_{\boldsymbol{\ell}_\perp=\boldsymbol{\ell}'_\perp=0} \tag{135}$$

Using

$$\frac{\partial}{\partial \boldsymbol{\ell}^i_\perp}\left[\frac{(\boldsymbol{P}_\perp - \boldsymbol{\ell}_\perp)^k}{(\boldsymbol{P}_\perp - \boldsymbol{\ell}_\perp)^2 + \bar{Q}^2}\right]\Bigg|_{\boldsymbol{\ell}_\perp=0} = -\frac{1}{\boldsymbol{P}^2_\perp + \bar{Q}^2}\left[\delta^{ik} - \frac{2\boldsymbol{P}^i_\perp \boldsymbol{P}^k_\perp}{(\boldsymbol{P}^2_\perp + \bar{Q}^2)}\right] \tag{136}$$

for transversely polarized we find

$$H^{T}_{ij}(Q,\boldsymbol{P}_\perp) = \alpha_s \alpha_{em} e_f^2 \delta(1-z-\bar{z}) \frac{z(1-z)\left[z^2+(1-z)^2\right]}{(\boldsymbol{P}^2_\perp + \bar{Q}^2)^2}\left[\delta^{ij} - \frac{4\bar{Q}^2 \boldsymbol{P}^i_\perp \boldsymbol{P}^j_\perp}{(\boldsymbol{P}^2_\perp + \bar{Q}^2)^2}\right] \tag{137}$$

#### 1. *Unpolarized vs linearly decomposition and correlations*

Decompose the WW into trace and traceless pieces, corresponding to unpolarized and linearly polarized contributions:

$$G^{ij}_{WW}(\boldsymbol{q}_\perp) = \frac{1}{2}\delta^{ij} G_{WW}(\boldsymbol{q}_\perp) + \frac{1}{2}\left[\frac{2\boldsymbol{q}^i_\perp \boldsymbol{q}^j_\perp}{\boldsymbol{q}^2_\perp} - \delta^{ij}\right] h_{WW}(\boldsymbol{q}_\perp) \tag{138}$$

Then we have

$$\frac{\mathrm{d}\sigma^{T}}{\mathrm{d}y_q \mathrm{d}y_{\bar{q}} \mathrm{d}^2\boldsymbol{P}_\perp \mathrm{d}^2\boldsymbol{q}_\perp} = \alpha_s \alpha_{em} e_f^2 N_c \delta(1-z-\bar{z}) \frac{z(1-z)\left[z^2+(1-z)^2\right](\boldsymbol{P}^4_\perp + \bar{Q}^4)}{(\boldsymbol{P}^2_\perp + \bar{Q}^2)^4}$$
$$\times \left[G_{WW}(\boldsymbol{q}_\perp) - \frac{2\bar{Q}^2 \boldsymbol{P}^2_\perp}{\boldsymbol{P}^4_\perp + \bar{Q}^4} h_{WW}(\boldsymbol{q}_\perp)\cos(2\Phi_{\boldsymbol{P}_\perp \boldsymbol{q}_\perp})\right] \tag{139}$$

Discuss physical meaning of unpol and lineary polarized contributions. Bound $h < G$.

- Numerical results for difference between TMD and full CGC

### C. TMD expansion: dipole TMD

The differential cross-section for quark+photon production in $pA$ reads:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y_q \mathrm{d}y_\gamma \mathrm{d}^2\boldsymbol{p}_\perp \mathrm{d}^2\boldsymbol{p}_{\boldsymbol{\gamma}\perp}} = \frac{\alpha_{em}}{2\pi^2}\sum_q e_q^2 x_q f_{q/p}(x_q) \widetilde{S}^{(2)}(\boldsymbol{p}_\perp + \boldsymbol{p}_{\boldsymbol{\gamma}\perp}) \frac{(1-z)z^2\left[1+(1-z)^2\right](\boldsymbol{p}_\perp + \boldsymbol{p}_{\boldsymbol{\gamma}\perp})^2}{\left(z(\boldsymbol{p}_\perp + \boldsymbol{p}_{\boldsymbol{\gamma}\perp}) - \boldsymbol{p}_{\boldsymbol{\gamma}\perp}\right)^2 \boldsymbol{p}_{\boldsymbol{\gamma}\perp}^2} \tag{140}$$

where $z = p^-_\gamma / k^-$

TMD expansion:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y_q \mathrm{d}y_\gamma \mathrm{d}^2\boldsymbol{P}_\perp \mathrm{d}^2\boldsymbol{q}_\perp} = \sum_q x_q f_{q/p}(x_q) \frac{\alpha_s \alpha_{em} e_q^2}{N_c} \frac{(1-z)z^2\left[1+(1-z)^2\right]}{\boldsymbol{P}^4_\perp} G_{dip}(\boldsymbol{q}_\perp) \tag{141}$$

where

$$\alpha_s G_{dip}(\boldsymbol{q}_\perp) = \frac{N_c}{2\pi^2}\boldsymbol{q}^2_\perp \widetilde{S}^{(2)}(\boldsymbol{q}_\perp) \tag{142}$$

### D. Gauge link structure: dipole vs WW gluon TMD

- Discuss the difference gauge link structure due to initial vs final state interactions
- Show that CGC definition of WW coincides with operator definition
- Show the difference between dipole and WW type (focus un unpolarized), and the dilute limit they are equal. Show results for evolution/geometric scaling.
- Mention more complicated gauge link structure for other processes

### E. Beyond Leading Order

- Discuss joint small-x and Sudakov resummation, need for kinematical constraint. Numerical results.
- Give counting for the coefficient of the double log (anomalous dimensions?)

### F. Exercises

- TMD factorization for SIDIS
- Repeat above considering massive quarks
- Determine the small-x TMD operators by examining the gauge link structure
- Take the correlation limit for dijet production in the proton-nucleus collision
- Consider the Improved TMD factorization

# IV. LECTURE IV: SMALL-X EVOLUTION

## A. Dipole evolution: the Balitsky-Kovchegov equation

- Start from dipole consider real and virtual emissions in the slow gluon limit (briefly mention Low's theorem)
- Obtain BK equations for dipole. Discuss properties. Fixed points, cancellation of divergences. Proportional to Nc. Physical meaning of the virtual and real terms.
- Obtain BFKL equation in the dilute limit

## B. Running coupling and kinematic constraint?

- Balitsky prescription for running of the coupling. Other prescriptions.

## C. Solutions to the BK equation

## D. Multi-pole evolution: the JIMWLK hierarchy

- Combinatorics for JIMWLK equation
- Shift kernel for WW emission
- Langevin equation for JIMWLK equation for Wilson line. Walk in SU(3) space. RGE of gauge field or sources.

## E. Numerical results

## F. Exercises

- Analytic solutions?
- Derive BK equation from pA single inclusive (universality)
- Derive evolution equation for the monopole and quadrupole

# V. LECTURE V: DIFFRACTION

## A. Basics

- Color singlet, rapidity gap, HERA observations. Coherent vs incoherent in the Good-Walker picture. Impact parameter dependence (ansatz for evolution, IR regulator)

## B. Classic observable: DVCS

- Compute DVCS following forward scattering amplitude result
- Discuss t-dependence and bT, tomography. Connection to GPDs?

## C. Vector meson production, dijet production

- Pheno result for vector meson production, massive quarks
- Dijet production and connection to Wigner in appropriate limit

## D. Diffractive TMD factorization

- SIDIS diffractive factorization

## Appendix A: Useful two-dimensional integrals

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