LHC Theory Tools for Precision Gravitational Wave Physics

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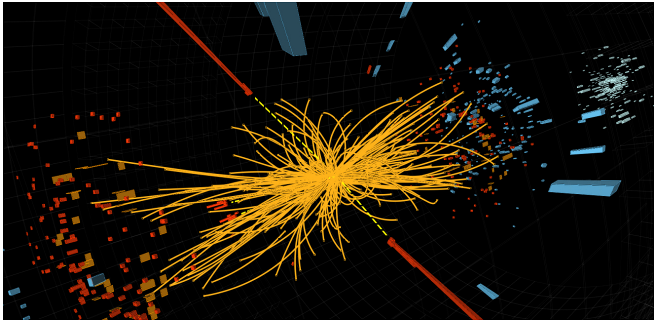
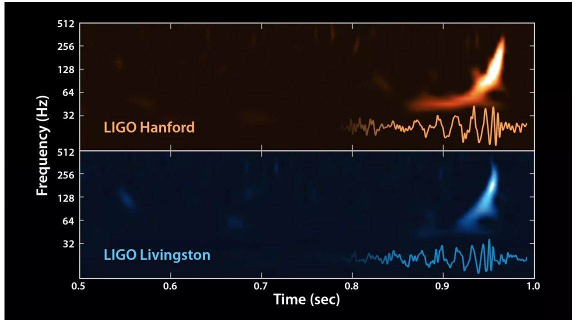
JHEP 08 (2023) 109 PRL 132 (2024) 221401 PRL 130 (2023) 101401 PRL 128 (2022) 161104
PLB 831 (2022) 137203 PRL 125 (2020) 261103 PRD 102 (2020) 124025 JHEP 06 (2021) 012

Precision era of fundamental physics





Two historic breakthroughs in science:

- Higgs bosons at the LHC (2012)
- Gravitational waves by the LIGO (2016)
- High-energy and gravitational physics entered a precision era!

Opened a new window on the Universe!

Precision era of fundamental physics

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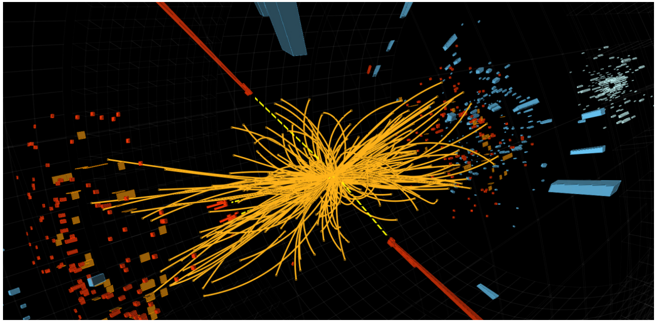

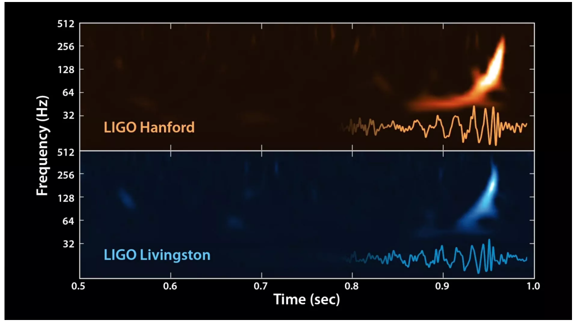

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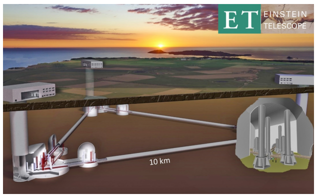
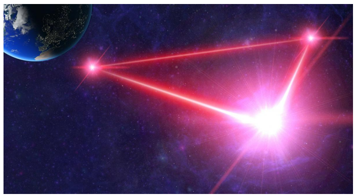
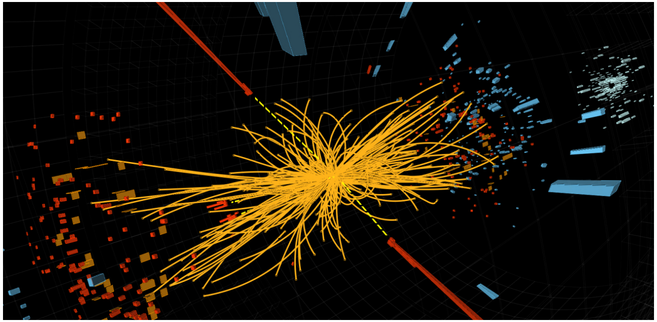
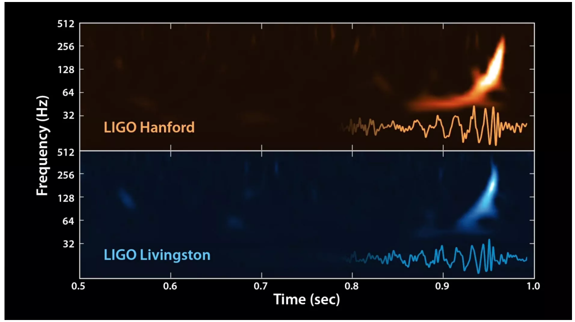




Precision era of fundamental physics





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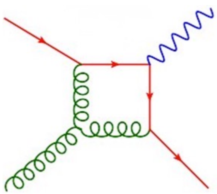
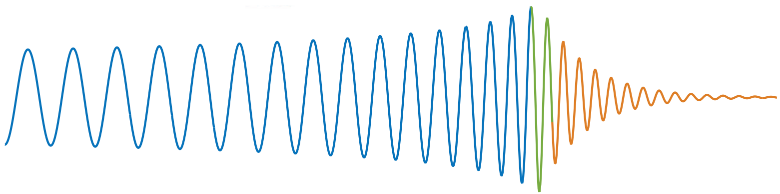
Modern techniques from LHC physics are playing a crucial role in precision GW physics!





Gravitational waves from binary coalescences

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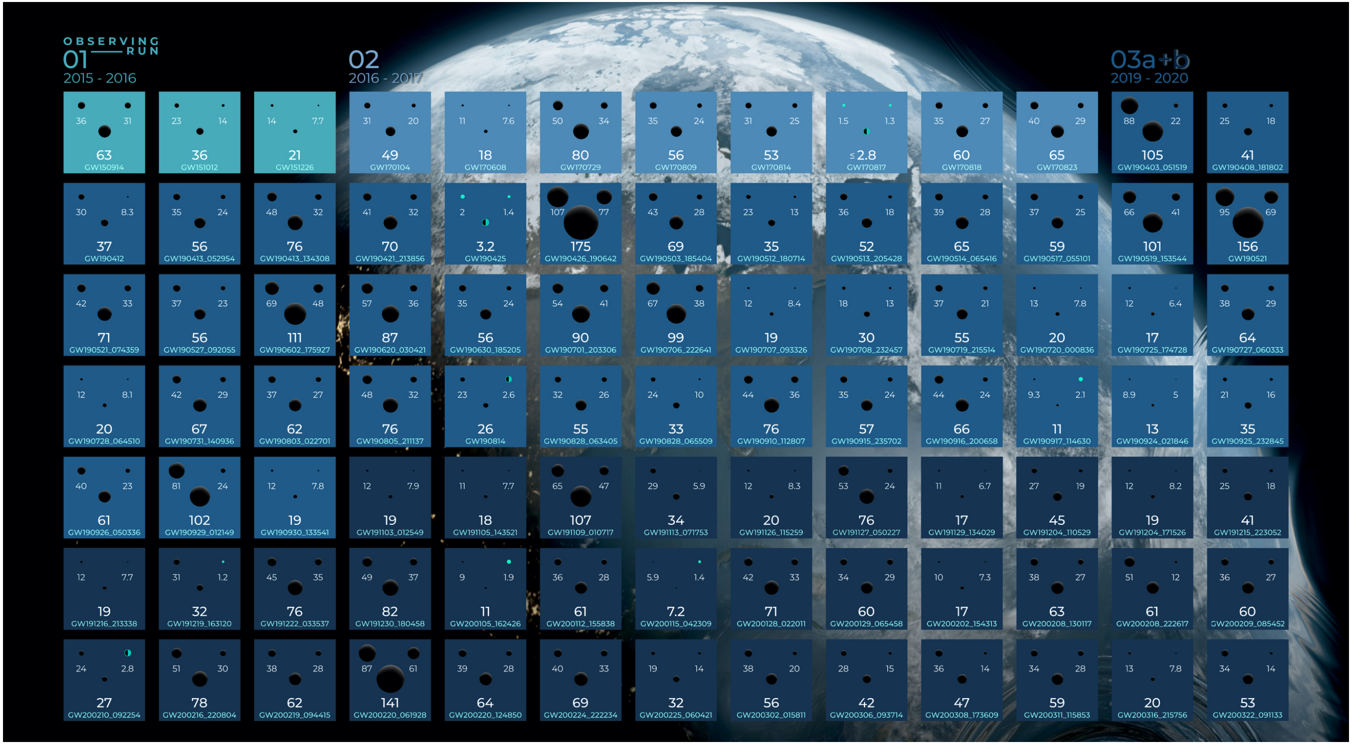

Credit: Carl Knox (OzGrav, Swinburne)

LIGO VIRGO KAGRA

GWTC-3: 90 GW events – the majority are binary black holes (BH), but also several binary neutron stars (NS) and mixed NS-BHs.

Gravitational waves from binary coalescences

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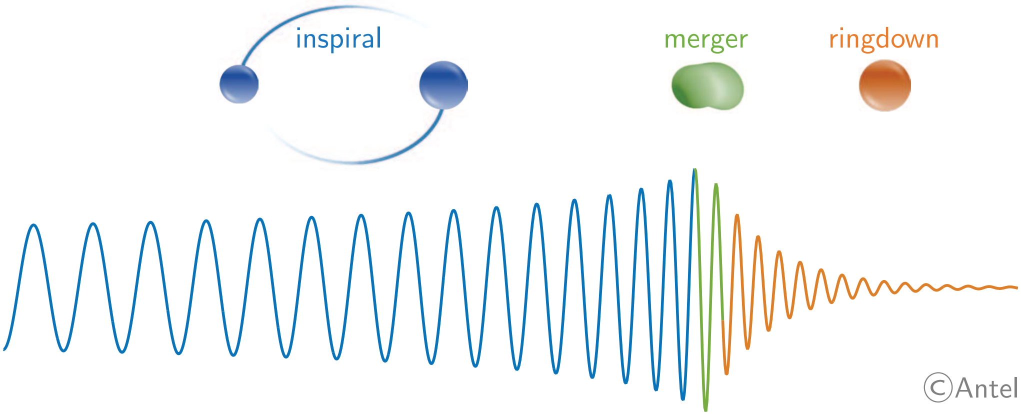

Merger: Numerical Relativity

Ringdown: black hole perturbation theory

Inspiral: the interaction between two bodies is weak

$$v^2 \sim \frac{GM}{r} \ll 1$$

- Numerical Relativity: accurately, but computationally expensive
- Analytic methods: corrections in v or G are perturbatively calculable

Post-Newtonian/post-Minkowskian expansion

  - LHC theory technology, QFT methodology, shown great power!

Effective Field Theory

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- Gravitational binary system

$$S_{\rm WL} = \sum_{i=1,2}\left[-\frac{m_i}{2}\int dt\, g_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu + \cdots\right] \qquad S_{\rm GR} = \frac{-1}{16\pi G}\int d^4x\sqrt{-g}\,R + \cdots$$

- Effective action for gravitational binary systems

Goldberger-Rothstein hep-th/0409156

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$e^{iS_{\rm eff}[x_a(\tau)]} = \int \mathcal{D}h_{\mu\nu}\, e^{iS_{\rm WL}+iS_{\rm GR}}$$

- Post-Minkowskian expand in powers of G

$$L_{\rm eff} = L_0 + GL_1 + G^2L_2 + \cdots \qquad L_0 = -\sum_i \frac{m_i}{2}\eta_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu$$

- The equations of motion for trajectories:

Kälin-Porto 2006.01184

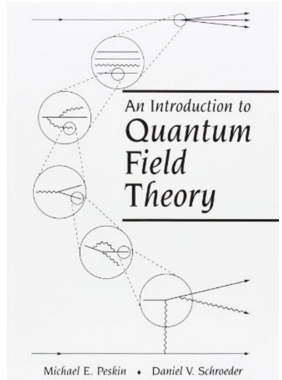
$$m_i\ddot{x}_i^\mu = -\eta^{\mu\nu}\sum_{n=1}^{\infty}G^n\left(\frac{\partial L_n}{\partial x_i^\nu} - \frac{d}{d\tau_i}\frac{\partial L_n}{\partial \dot{x}_i^\nu}\right) \qquad x_i^\mu = b_i^\mu + u_i^\mu\tau_i + \delta x_i^\mu(\tau_i) + \cdots$$

- Physical observables:

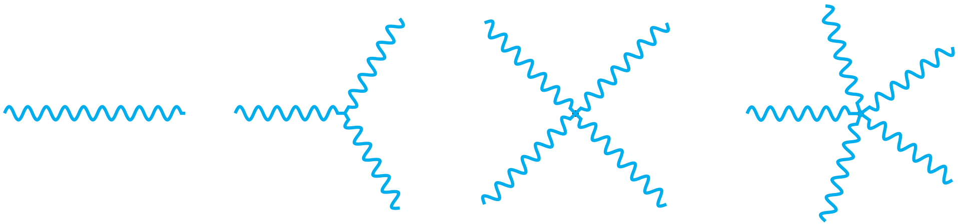
$$\Delta p_i^\mu = p_i^\mu(+\infty) - p_i^\mu(-\infty) = -\eta^{\mu\nu}\sum_{n=1}^{\infty}G^n\int_{-\infty}^{\infty}d\tau_i\left(\frac{\partial L_n}{\partial x_i^\nu}\right)$$

Effective Field Theory

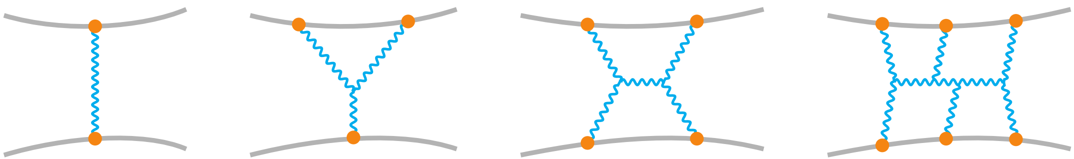
- Worldlines as classical sources in path integral:



- Hilbert-Einstein: $\mathcal{L}_{\text{HE}} = \mathcal{L}_{hh} + \mathcal{L}_{hhh} + \mathcal{L}_{hhhh} + \cdots$



- Classical physics: we use the saddle-point approximation in path integrals.



- Enjoy the advantages of quantum field theory methods and classical physics

  powerful and systematic & purely classical at all steps (simplicity)

Effective Field Theory

- Observables at $\mathcal{O}(G^N)$ Kälin-**ZL**-Porto PRL2020 Dlapa-Kälin-**ZL**-Porto PRL2022 JHEP2024

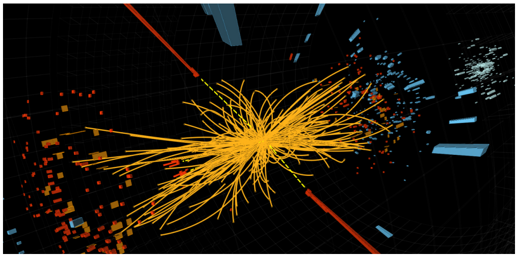
$$\Delta p_i^\mu \sim \int d^D q \, \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\delta(q\cdot u_2)}{|q^2|^\sharp} \int \left(\prod_{i=1}^{N-1} d^D\ell_i \, \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}} \right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3 \cdots}$$

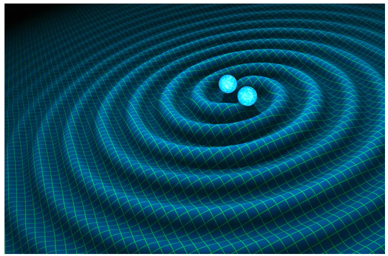
  - Graviton propagators:

$$\frac{1}{D_i} \longrightarrow \frac{1}{(\ell^0 \pm i0)^2 - \vec{\ell}^{\,2}} \quad \text{or} \quad \frac{1}{\ell^2 + i0}$$

  - Cut: always one delta function $\delta(\ell_i\cdot u_a)$ for each loop
  - Kinematics: $q\cdot u_a = 0,\ u_a^2 = 1,\ u_1\cdot u_2 = \gamma \implies$ single scale γ to all orders

- Multi-loop technology from particle physics can be used to solve gravitational problems!





LHC theory toolbox

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- Observables at $\mathcal{O}(G^N)$ Kälin-**ZL**-Porto PRL 2020 Dlapa-Kälin-**ZL**-Porto PRL 2022 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

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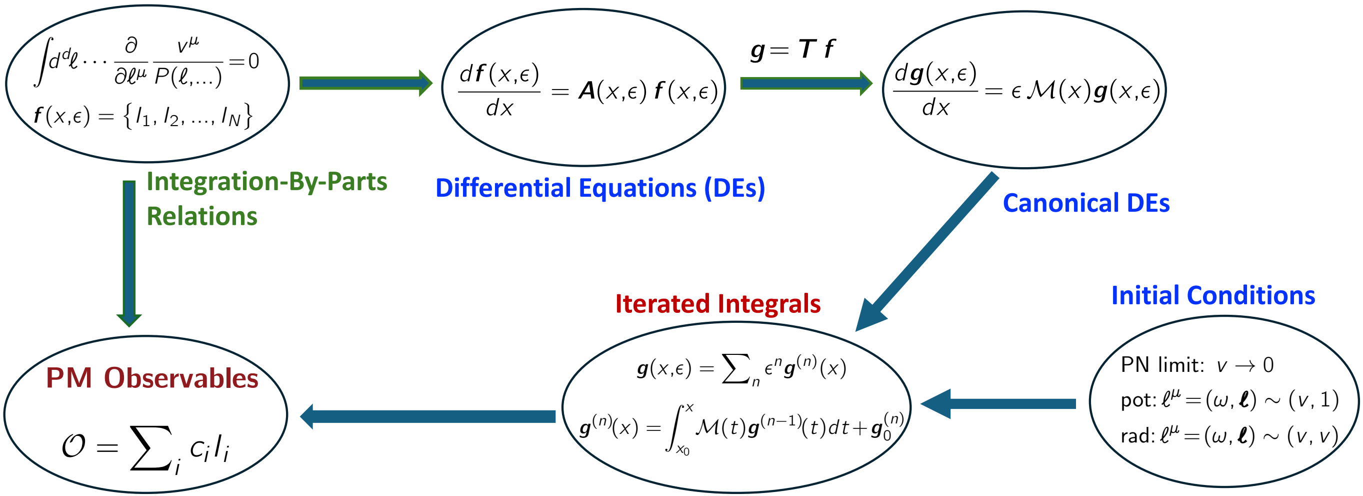
- Perturbative QFT toolbox: Dlapa-Kälin-**ZL**-Porto JHEP 2024



- Post-Minkowskian physics can be bootstrapped from Post-Newtonian data using DEs!

NLO: 2PM

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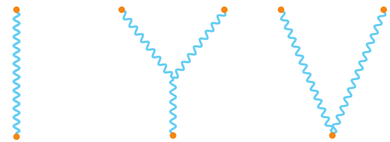
Spin interactions $g_{\mu\nu}e^\mu_a e^\nu_b = \eta_{ab}$

$$-\frac{1}{2}\left(\omega^{ab}_\mu S_{ab} v^\mu + \frac{1}{m} R_{\beta\rho\mu\nu} e^\alpha_a e^\beta_b e^\mu_c e^\nu_d S^{ab} S^{cd} v^\rho v_\alpha - \frac{C_{\rm ES}}{m} E_{\mu\nu} e^\mu_a e^\nu_b S^{ac} S_c{}^b + \cdots\right)$$

- EFT provides a systematic way to include spin effects.
- Needed one-loop integrals are simple

$$\int d^D\ell \, \frac{\delta(\ell\cdot u_1)}{[(\pm\ell\cdot u_2)]^{a_1}} \frac{1}{[\ell^2]^{a_2}\,[(\ell-q)^2]^{a_3}}$$



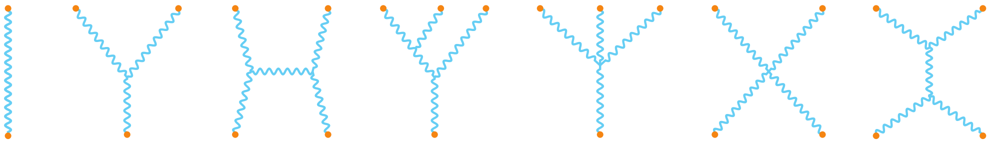
- Nontrivial to simplify complicated tensor expressions

Observables: **ZL**-Porto-Yang JHEP 2021

$$\Delta p^\mu_1 = \frac{\nu G^2 M^3}{|b|^3}\left[3D_1 \epsilon_{\alpha\rho\beta\sigma} \hat{b}^\mu \hat{b}^\alpha u^\beta_1 u^\sigma_2 a^\rho_1 + \cdots + \frac{D_{20}}{|b|} u^\mu_1 (a_1\cdot a_2) + \cdots + \frac{D_{14}}{|b|} u^\mu_2 a^2_1 + \cdots\right]$$

$$s_\mu = m a_\mu \equiv \frac{1}{2m}\epsilon_{\mu\nu\alpha\beta} p^\nu S^{\alpha\beta}$$

NNLO: 3PM



- $\mathcal{O}(G^3)$: two-loop integrals — Kälin-**ZL**-Porto PRL 2020 PRD 2020

$$\int \frac{d^D\ell_1 d^D\ell_2\, \delta(\ell_1\cdot u_1)\, \delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2]^{a_1}\, [\pm\ell_2\cdot u_1]^{a_2}} \frac{1}{[\ell_1^2]^{a_3}\, [\ell_2^2]^{a_4}\, [(\ell_1+\ell_2-q)^2]^{a_5}\, [(\ell_1-q)^2]^{a_6}\, [(\ell_2-q)^2]^{a_7}}$$

- The reduction and evaluation of integrals can be performed in standard techniques.

- Conservative dynamics at $\mathcal{O}(G^3)$: — Kälin-**ZL**-Porto PRL 2020

$$\Delta p_1^\mu = \frac{G^3 b^\mu}{|b^2|^2}\left(\frac{8m_1^2m_2^2(4\gamma^4-12\gamma^2-3)}{(\gamma^2-1)}\log\left(\gamma-\sqrt{\gamma^2-1}\right) - \frac{2m_1m_2(m_1^2+m_2^2)(16\gamma^6-32\gamma^4+16\gamma^2-1)}{(\gamma^2-1)^{5/2}} - \frac{4m_1^2m_2^2\gamma(20\gamma^6-90\gamma^4+120\gamma^2-53)}{3(\gamma^2-1)^{5/2}}\right) + \frac{3\pi}{2}\frac{(2\gamma^2-1)\,(5\gamma^2-1)}{(\gamma^2-1)^2}\frac{G^3m_1m_2(m_1+m_2)}{|b^2|^{3/2}}\left((m_1+\gamma m_2)u_2^\mu-(m_2+\gamma m_1)u_1^\mu\right)$$

- We provided the first confirmation for the result from a scattering amplitude clacualtion.
  Bern-Cheung-Roiban-Shen-Solon-Zeng 2019

- We also obtained the quadrupolar and octupolar tidal corrections at $\mathcal{O}(G^3)$.
  Kälin-**ZL**-Porto PRD 2020

NNNLO: 4PM

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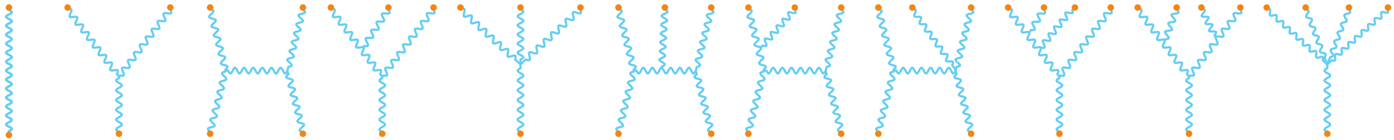

$\mathcal{O}(G^4)$: three-loop integrals   Dlapa-Kälin-**ZL**-Porto PRL 2022 PLB 2022 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\int d^D\ell_1 d^D\ell_2 d^D\ell_3 \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_2)\,\delta(\ell_3\cdot u_2)}{[\ell_1\cdot u_2]^{\alpha_1}\,[\ell_2\cdot u_1]^{\alpha_2}\,[\ell_3\cdot u_1]^{\alpha_3}}\,\frac{D_8^{-\nu_8}D_9^{-\nu_9}}{D_1^{\nu_1}D_2^{\nu_2}\cdots D_7^{\nu_7}}$$

$\{\ell_1^2, \ell_2^2, (\ell_1-q)^2, (\ell_2-q)^2, (\ell_3-q)^2, \ell_3^2, (\ell_1-\ell_2)^2, (\ell_2-\ell_3)^2, (\ell_3-\ell_1)^2\}$

IBP reduction:

conservative: $\mathcal{O}(10^2)$ master integrals   full: $\mathcal{O}(10^3)$ master integrals

Differential Equations   Dlapa-Kälin-**ZL**-Porto PRL 2022 PLB 2022 Dlapa-Henn-Wagner 2211.16357

$$\frac{d\vec{f}(x,\epsilon)}{dx} = \epsilon\,\mathcal{M}(x)\,\vec{f}(x,\epsilon)$$

- The majority can be solved in terms of multiple polylogarithms.
- Elliptic integrals appear in post-Minkwskian gravity for the first time.

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The full impulse at $\mathcal{O}(G^4)$: Dlapa-Kälin-**ZL**-Porto PRL 2022 JHEP 2023 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\Delta p_1^\mu\Big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b \frac{b^\mu}{|b|} + c_1 \frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2 \frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1} \right)$$

$$\begin{aligned}
\frac{c_b}{\pi} = & -\frac{3h_{34}m_2m_1(m_1^3+m_2^3)}{64v_\infty^5} + \frac{m_1^2m_{12}m_2^2}{4}\Bigg[\frac{3h_6\mathrm{K}^2(w_2)}{4v_\infty^3} - \frac{3h_8\mathrm{K}(w_2)\mathrm{E}(w_2)}{4v_\infty^3} + \frac{21h_5w_3\mathrm{E}^2(w_2)}{8v_\infty^3} - \frac{\pi^2h_{16}v_\infty}{4(\gamma+1)} + \frac{3\gamma h_{10}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{w_3v_\infty^2} \\
& + \log(v_\infty)\left(\frac{h_{32}}{2v_\infty^3} - \frac{3h_{14}\log(\frac{w_3}{2})}{v_\infty} - \frac{3\gamma h_{22}\log(w_1)}{2v_\infty^4}\right)\Bigg] + m_2^2m_1^3\Bigg[\frac{h_{52}}{48v_\infty^6} - \frac{h_{63}}{768\gamma^9w_3v_\infty^5} - \frac{3v_\infty(h_{40}\mathrm{Li}_2(w_2)+2w_3h_{33}\mathrm{Li}_2(-w_2))}{64w_3} \\
& + \frac{3h_{14}\log(\frac{w_3}{2})\log(w_3)}{4v_\infty} + \frac{\gamma h_{39}\log(w_1)}{8w_3^3v_\infty^2} + \frac{3\gamma h_{22}\log(w_3)\log(w_1) - h_{35}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{h_{56}\log(2) - h_{57}\log(w_3) + 2\gamma h_{55}\log(\gamma)}{32v_\infty^5} - \frac{\gamma h_{51}\log(w_1)}{16v_\infty^7}\Bigg] \\
& + m_1^2m_2^3\Bigg[\frac{h_{58}}{192\gamma^7v_\infty^5} + \frac{h_{53}}{48v_\infty^6} + \frac{\gamma h_{49}\log(w_1)}{16v_\infty^6} - \frac{2\gamma h_{50}\log(w_1) + 3\gamma^2h_{13}\log^2(w_1)}{32v_\infty^7} - \frac{h_{41}\log(\frac{w_3}{2})}{8v_\infty^4} + \frac{3\gamma\log(w_1)\big(5h_{26}\log(2)+8h_{12}\log(w_3)\big)}{8v_\infty^4} \\
& - \frac{h_{36}\log(w_3)}{4v_\infty^3} + \frac{\gamma h_{30}\log(\gamma)}{2v_\infty^3} + \frac{h_{37}\log(2)}{8v_\infty^3} + \frac{3(h_{17}w_3\mathrm{Li}_2(w_2) - 2h_{23}\mathrm{Li}_2(-w_2) + h_{15}\log^2(w_3) - h_9\log^2(2))}{8v_\infty} - \frac{3h_7\log(2)\log(w_3)}{v_\infty}\Bigg]
\end{aligned}$$

$$c_1 = m_1m_2^2\left(\frac{2h_{46}m_{12s}}{v_\infty^6} + \frac{9\pi^2h_1m_{12}^2}{32v_\infty^2}\right) + m_1^2m_2^3\left(\frac{4\gamma h_{47}}{3v_\infty^6} - \frac{8\gamma h_2\log(w_1)}{v_\infty^6} + \frac{16h_{25}\log(w_1)}{v_\infty^3} - \frac{8h_3}{3v_\infty^5}\right)$$

$$\begin{aligned}
c_2 = & -m_1^4m_2\left(\frac{9\pi^2h_1}{32v_\infty^2} + \frac{2h_{46}}{v_\infty^6}\right) + m_2^2m_1^3\Bigg[+\frac{h_{60}}{705600\gamma^8v_\infty^5} - \frac{4\gamma h_{48}}{3v_\infty^6} + \frac{3h_{38}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2})) - \gamma h_{21}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{16v_\infty^4} \\
& + \frac{3\gamma h_{31}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{8v_\infty^4} + \frac{h_{62}\log(w_1)}{6720\gamma^9v_\infty^6} + \frac{32\gamma^2h_{44}\log^2(w_1)}{v_\infty^7} + \frac{8\gamma\big(2h_4\log(2) - h_{27}\log(w_1)\big)\log(w_1)}{v_\infty^4} - \frac{32h_{29}\log(w_1)}{3v_\infty^3} + \frac{\pi^2h_{42}}{192v_\infty^4}\Bigg] \\
& + m_2^3m_1^2\Bigg[\frac{h_{59}}{1440\gamma^7v_\infty^5} - \frac{h_{19}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{8v_\infty^4} + \frac{h_{43}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{32v_\infty^4} - \frac{h_{20}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{4v_\infty^4} \\
& - \frac{h_{61}\log(w_1)}{480\gamma^8v_\infty^6} - \frac{16\gamma^2h_{11}\log^2(w_1)}{v_\infty^4} - \frac{32\gamma h_{45}\log^2(w_1)}{v_\infty^7} + \frac{16\gamma h_{28}\log(w_1)}{5v_\infty^3} - \frac{32h_{24}\log(2)\log(w_1)}{v_\infty^4} - \frac{\pi^2h_{18}}{48v_\infty^4} - \frac{2h_{54}}{45v_\infty^6}\Bigg]
\end{aligned}$$

with $\gamma \equiv u_1\cdot u_2$, $v_\infty = \sqrt{\gamma^2-1}$, $w_1 = \gamma - v_\infty$, $w_2 = \frac{\gamma-1}{\gamma+1}$, $w_3 = \gamma+1$, $h_i =$ polynomial in γ.

$$\mathrm{Li}_2(z) \equiv \int_0^z dx\,\frac{\log(1-x)}{-x}$$

$$K(z) \equiv \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

$$E(k) \equiv \int_0^1 dx\,\frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}$$

NNNLO: 4PM

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The full impulse at $\mathcal{O}(G^4)$: Dlapa-Kälin-**ZL**-Porto JHEP 2023 Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

$$\Delta p_1^\mu\big|_{\text{NNNLO}} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

- We obtained the full dynamics of binary inspirals at $\mathcal{O}(G^4)$ for the first time.
- Conservative part agrees perfectly with previous derivations.
  Bern-Parra-Martinez-Roiban-Ruf-Shen-Solon-Zeng 2022 Dlapa-Kälin-**ZL**-Porto PRL 2022
- Perfect agreement with the state-of-the-art PN computations
  Cho-Dandapat-Gopakumar 2021 Cho 2022 Bini-Geralico 2021 2022 Bini-Damour 2022
- Later, two new calculations confirmed our results.
  Damgaard-Hansen-Planté-Vanhove 2023 (exponentiation of amplitudes)
  Jakobsen-Mogull-Plefka-Sauer-Xu 2023 (worldline formalism)

Analytic vs Numerical Relativity

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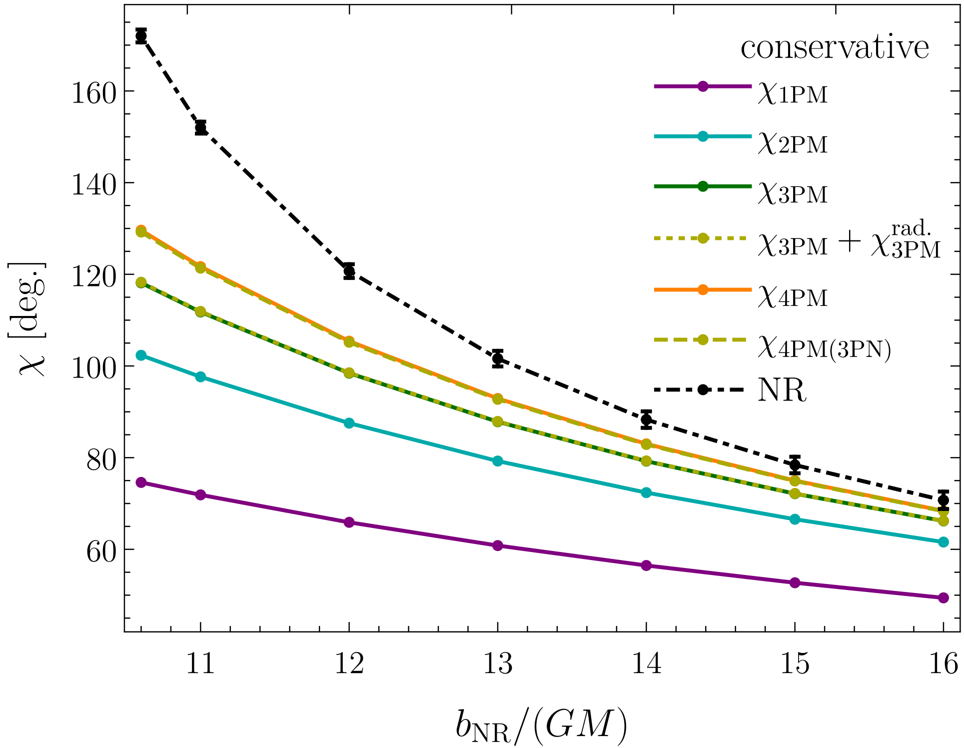

Khalil-Buonanno-Steinhoff-Vines 2204.05047

NNLO: local-in-time part

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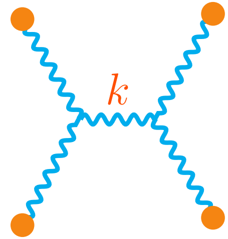
- The full result does not describe generic elliptic-like motion due to nonlocal-in-time effects.
  Damour-Jaranowski-Schäfer 2014 Galley-Leibovich-Porto-Ross 2015 Cho-Kälin-Porto 2021

- Nonlocal-in-time radial action:

$$\mathcal{S}_r^{(\text{nloc})} = -\frac{GE}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{dE}{d\omega}\log\left(\frac{4\omega^2}{\mu^2}e^{2\gamma_E}\right)$$



- The 4PM integrand can be built from 3PM diagrams. $\omega \equiv k\cdot u_{\text{com}}$

$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2][\ell_2\cdot u_1]}\frac{\log(\omega^2)}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$

- We managed to compute the integrals and obtained nonlocalin-time contribution:

$$\frac{\nu}{(\gamma^2-1)^2}\Bigg[h_1+\frac{\pi^2 h_2}{\sqrt{\gamma^2-1}}+h_3\log\frac{\gamma+1}{2}+\frac{h_4\text{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}+h_5\log\frac{\gamma-1}{8}+h_6\log^2\frac{\gamma+1}{2}+\frac{h_8\log(2)\text{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}$$
$$+h_7\,\text{arccosh}(\gamma)^2+h_9\log\frac{\gamma-1}{8}\log\frac{\gamma+1}{2}+\frac{h_{10}\log\frac{\gamma^2-1}{16}\text{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}+h_{11}\text{Li}_2\frac{\gamma-1}{\gamma+1}+h_{12}\frac{\text{arccosh}^2(\gamma)+4\text{Li}_2(\sqrt{\gamma^2-1}-\gamma)}{\sqrt{\gamma^2-1}}\Bigg]$$

Coefficients h_i: exact-ν (iterated elliptic integrals) and SF-expanded (30SF) forms
Dlapa-Kälin-**ZL**-Porto PRL 2024

- Using 6PN results in the literature, we constructed an improved bound Hamiltonian.

Conclusion & Outlook

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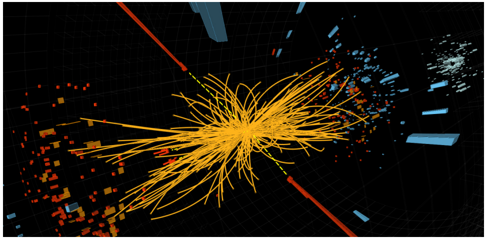
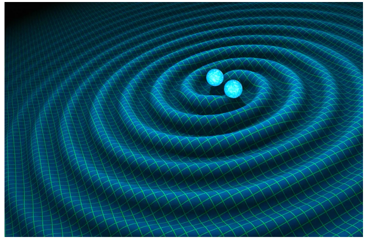
Modern techniques from Quantum Field Theory have already proven useful to improve theoretical predictions for gravitational-wave observables.



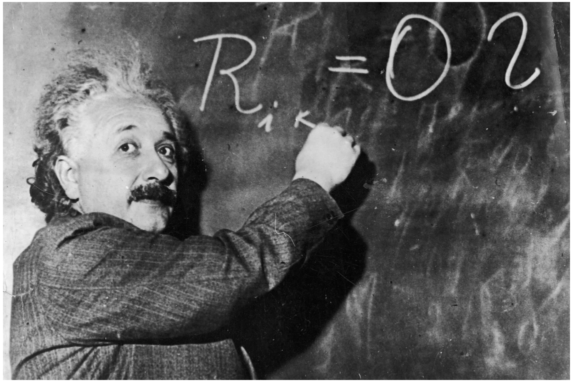


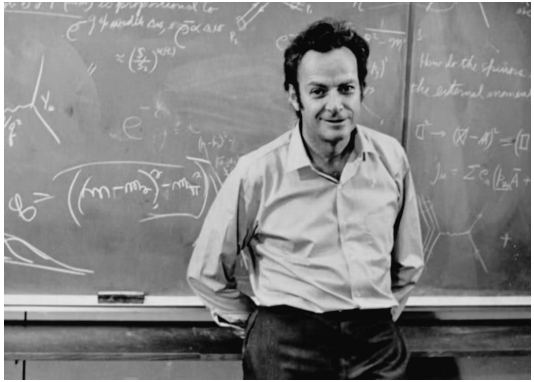
We have developed an efficient framework and made breakthroughs to NNNLO.

- Conservative spin & tidal effects to NLO JHEP 06 (2021) 012 PRD 102 (2020) 124025
- Conservative dynamics to NNLO PRL 125 (2020) 261103
- Conservative dynamics to NNNLO PLB 822 (2021) 136698 PRL 128 (2022) 161104 PRL 130 (2023) 101401
- Local-in-time & nonlocal-in-time separation PRL 132 (2024) 221401
- Novel techniques to evaluate loop integrals in gravity JHEP 07 (2023) 181 JHEP 08 (2023) 109

Gravitational-wave science is just starting! New discoveries rely highly on the precision of theoretical predictions.

Feynman integrals solve Einstein's equations!





谢谢！

Effective Field Theory

The in-in effective action is obtained by performing a closed-time-path integral

$$e^{i\mathcal{S}_{\text{eff}}[x_{a,1},x_{a,2}]} = \int \mathcal{D}h_1\mathcal{D}h_2\, e^{i(S_{\text{GR}}[h_1]-S_{\text{GR}}[h_2]+S_{\text{WL}}[h_1,x_{a,1}]-S_{\text{WL}}[h_2,x_{a,2}])}$$

It is convenient to use the Keldysh basis Galley PRL 110 (2013) 174301

$$h^-_{\mu\nu} = \frac{1}{2}(h_{1\mu\nu} + h_{2\mu\nu}) \qquad x^\alpha_{a,+} = \frac{1}{2}(x^\alpha_{a,1} + x^\alpha_{a,2})$$

$$h^+_{\mu\nu} = h_{1\mu\nu} - h_{2\mu\nu} \qquad x^\alpha_{a,-} = x^\alpha_{a,1} - x^\alpha_{a,2}$$

for which the matrix of (classical) propagators for gravitions becomes

$$i\begin{pmatrix} 0 & -\Delta_{\text{adv}}(x-y) \\ -\Delta_{\text{ret}}(x-y) & 0 \end{pmatrix}$$

The worldline equations of motion: Kälin-Neef-Porto JHEP 01 (2023) 140

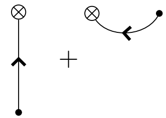
$$m_i\ddot{x}^\mu_i(\tau) = -\eta^{\mu\nu}\frac{\delta\mathcal{S}_{\text{eff, int}}[x_{a,\pm}]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\text{PL}} \qquad \Delta p^\mu_i = -\eta^{\mu\nu}\int_{-\infty}^{\infty} d\tau \frac{\delta\mathcal{S}_{\text{eff, int}}[x_{a,\pm}]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\text{PL}}$$

Physical Limit (PL): $x_{a,-} \to 0$, $x_{a,+} \to x_a$.

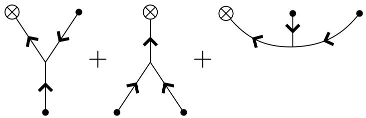
Effective Field Theory

- In practise, Feynman rules are still simple in the physical limit!
- Worldline source: $\downarrow k$  $= -\frac{im}{2M_{\rm Pl}} \int \mathrm{d}\tau\, e^{i\,k\cdot x} \dot{x}^{\mu} \dot{x}^{\nu}$
- Variation of worldline: $\downarrow k$  $= -\frac{im}{2M_{\rm Pl}} e^{i\,k\cdot x}\Big(i\,k^{\alpha}\dot{x}^{\mu}\dot{x}^{\nu} - i\,k\cdot\dot{x}\,\eta^{\mu\alpha}\dot{x}^{\nu} - \eta^{\mu\alpha}\ddot{x}^{\nu} - i\,k\cdot\dot{x}\,\eta^{\nu\alpha}\dot{x}^{\mu} - \eta^{\nu\alpha}\ddot{x}^{\mu}\Big)$
- Variation of effective action: 2207.00580 2304.01275

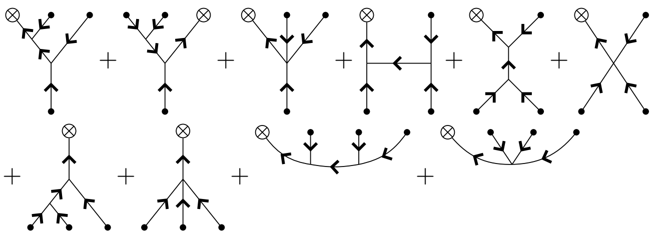
$$\left.\frac{\delta \mathcal{S}_{\rm eff}[x_{\pm}]}{\delta x^{\alpha}_{1-}}\right|^{G}_{\rm PL} =$$



$$\left.\frac{\delta \mathcal{S}_{\rm eff}[x_{\pm}]}{\delta x^{\alpha}_{1-}}\right|^{G^2}_{\rm PL} =$$



$$\left.\frac{\delta \mathcal{S}_{\rm eff}[x_{\pm}]}{\delta x^{\alpha}_{1-}}\right|^{G^3}_{\rm PL} =$$



Effective Field Theory

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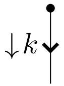
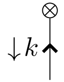
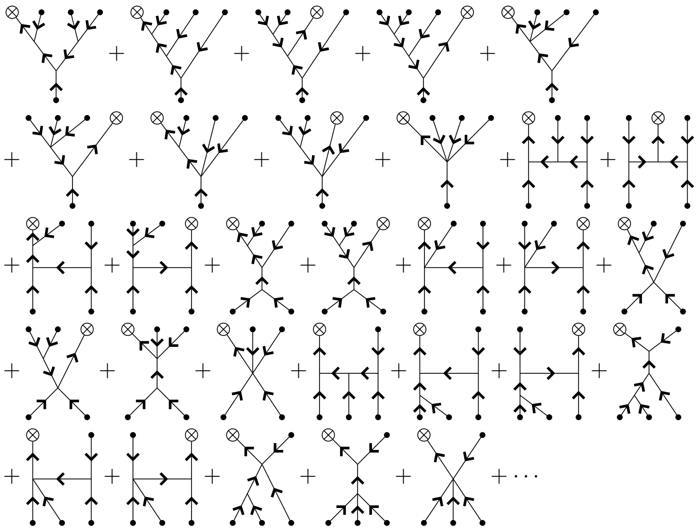
- In practise, Feynman rules are still simple in the physical limit!
- Worldline source:  $= -\dfrac{im}{2M_{\rm Pl}}\displaystyle\int \mathrm{d}\tau\, e^{i\,k\cdot x}\dot{x}^\mu \dot{x}^\nu$
- Variation of worldline:  $= -\dfrac{im}{2M_{\rm Pl}}\, e^{i\,k\cdot x}\Big(i\,k^\alpha \dot{x}^\mu \dot{x}^\nu - i\,k\cdot\dot{x}\,\eta^{\mu\alpha}\dot{x}^\nu - \eta^{\mu\alpha}\ddot{x}^\nu - i\,k\cdot\dot{x}\,\eta^{\nu\alpha}\dot{x}^\mu - \eta^{\nu\alpha}\ddot{x}^\mu\Big)$
- Variation of effective action: 2207.00580 2304.01275

$$\frac{\delta \mathcal{S}_{\rm eff}[x_\pm]}{\delta x^\alpha_{1-}}\bigg|^{G^4}_{\rm PL} =$$



$+\cdots$

Elliptic differential equations

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- From 4PM order, more complicated functions appear beyond polylogarithms.

- An elliptic example:

$$\frac{d}{dx}\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \frac{1-x^2}{2x(1+x^2)} & \frac{1+x^2}{4x(1-x^2)} & \frac{3x}{(1-x^2)(1+x^2)} \\ -\frac{1-x^2}{x(1+x^2)} & \frac{3(1+x^2)}{2x(1-x^2)} & -\frac{6x}{(1-x^2)(1+x^2)} \\ \frac{1-x^2}{x(1+x^2)} & -\frac{1+x^2}{2x(1-x^2)} & -\frac{1-4x^2+x^4}{x(1-x^2)(1+x^2)} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} + \mathcal{O}(\epsilon)$$

It can then be written as a third-order differential equation:

$$\left[\frac{d^3}{dx^3} - \frac{6x}{1-x^2}\frac{d^2}{dx^2} - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\frac{d}{dx} - \frac{1+x^2}{x^3(1-x^2)}\right] f_1(x) = 0$$

It is easy to find the three solutions:

$$x\,\mathrm{K}^2\,(1-x^2), \qquad x\,\mathrm{K}(1-x^2)\mathrm{K}(x^2), \qquad x\,\mathrm{K}^2(x^2)$$

Complete elliptic integrals: $\mathrm{K}(x) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-xt^2)}}$

Elliptic differential equations

With the knowledge of leading-ϵ solutions, one may transform the elliptic diagonal block into

$$\frac{d}{dx}\vec{g}(x,\epsilon) = \epsilon\,\tilde{D}_{\text{ell}}(x)\,\vec{g}(x,\epsilon) + \ldots$$

with

$$\tilde{D}_{\text{ell}} = \begin{pmatrix} -\frac{4\left(1+x^2\right)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)} & 0 \\ \frac{2\left(1+110x^2+x^4\right)\mathrm{K}^2\left(1-x^2\right)}{3\pi^2 x(1-x^2)} & -\frac{4\left(1+x^2\right)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)} \\ \frac{16\left(1+x^2\right)\left(1-18x+x^2\right)\left(1+18x+x^2\right)\mathrm{K}^4\left(1-x^2\right)}{27\pi^2 x(1-x^2)} & \frac{2\left(1+110x^2+x^4\right)\mathrm{K}^2\left(1-x^2\right)}{3\pi^2 x(1-x^2)} & -\frac{4\left(1+x^2\right)}{3x(1-x^2)} \end{pmatrix}$$

- Elliptic integrals appear in the transformation matrix: found by INITIAL
- Higer $\mathcal{O}(\epsilon)$: Iterated integrals involving elliptic kernels.