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Complete one-loop analytic and expansion formulae for the muon magnetic dipole moment

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Mainly based on the paper:
Shi-Ping He, Handbook of the analytic and expansion formulae for the muon magnetic dipole moment, arXiv:2308.07133.

Contents

- Introduction
  - muon $(g-2)$
  - general interactions
- Contributions to $(g-2)_\mu$ at one loop
  - general structure
  - analytic form in different representations
  - singularities
- Expansion of the loop functions
  - scalar loop functions
  - vector loop functions
- Applications and examples
- Summary and conclusions

Introduction

pioneer works

J. S. Schwinger, On Quantum electrodynamics and the magnetic moment of the electron, Phys. Rev. 73, 416-417 (1948).

W. A. Bardeen, R. Gastmans and B. E. Lautrup, Static quantities in Weinberg's model of weak and electromagnetic interactions, Nucl. Phys. B 46, 319-331 (1972).

R. Jackiw and S. Weinberg, Weak interaction corrections to the muon magnetic moment and to muonic atom energy levels, Phys. Rev. D 5, 2396-2398 (1972).

K. Fujikawa, B. W. Lee and A. I. Sanda, Generalized Renormalizable Gauge Formulation of Spontaneously Broken Gauge Theories, Phys. Rev. D 6, 2923-2943 (1972).

J. P. Leveille, The Second Order Weak Correction to (G-2) of the Muon in Arbitrary Gauge Models, Nucl. Phys. B 137, 63-76 (1978).

reviews papers

F. Jegerlehner and A. Nyffeler, The Muon g-2, Phys. Rept. 477, 1-110 (2009), arXiv:0902.3360.

A. Freitas, J. Lykken, S. Kell and S. Westhoff, Testing the Muon g-2 Anomaly at the LHC, JHEP 05, 145 (2014) [erratum: JHEP 09, 155 (2014)], arXiv:1402.7065 [hep-ph].

F. S. Queiroz and W. Shepherd, New Physics Contributions to the Muon Anomalous Magnetic Moment: A Numerical Code, Phys. Rev. D, 095024 (2014), arXiv:1403.2309 [hep-ph].

M. Lindner, M. Platscher and F. S. Queiroz, A Call for New Physics : The Muon Anomalous Magnetic Moment and Lepton Flavor Violation, Phys. Rept. 731, 1-82 (2018), arXiv:1610.06587 [hep-ph].

P. Athron, C. Balazs, D. H. J. Jacob, W. Kotlarski, D. Stockinger and H. Stockinger-Kim, New physics explanations of a_μ in light of the FNAL muon g-2 measurement, JHEP 09, 080 (2021), arXiv:2104.03691 [hep-ph].

our papers on new physics

S. P. He, Leptoquark and vectorlike quark extended models as the explanation of the muon g-2 anomaly, Phys. Rev. D 105, 035017 (2022) [erratum: Phys. Rev. D 106, 039901 (2022)], arXiv:2112.13490 [hep-ph].

S. P. He, Leptoquark and vectorlike quark extended models as the explanation of the muon g-2 anomaly, PoS ICHEP2022, 137 (2022), arXiv:2211.04941 [hep-ph].

S. P. He, Leptoquark and vector-like quark extended model for simultaneous explanation of W boson mass and muon g-2 anomalies, Chin. Phys. C 47, 043102 (2023), arXiv:2205.02088 [hep-ph].

S. P. He, Scalar leptoquark and vector-like quark extended models as the explanation of the muon g-2 anomaly: bottom partner chiral enhancement case, Chin. Phys. C 47, 073101 (2023), arXiv:2211.08062 [hep-ph].

L. Cai, C. Han, S. P. He and P. Wu, Hybrid Type-II and Type-III seesaw model for the muon g-2 anomaly, arXiv:2408.15910 [hep-ph].

Handbook of the analytic and expansion formulae for the muon magnetic dipole moment

Shi-Ping He (APCTP, Pohang) (Aug 14, 2023)
e-Print: 2308.07133 [hep-ph]

- Muon (g-2) is one of the most important probe of new physics
- Systematic analytical results and approximations are still absent
  - expansion at special mass scale
  - less discussion on singularities
- Although one-loop amplitudes can be derived fully automatically, a handbook is useful to check.

FeynRules→FeynArts → FeynCalc/FormCalc → Package-X

model by model — less discussions on general physical implications canonical interactions ✓

quasi-canonical interactions ✓

non-canonical contributions

field redefinition, total derivative, equation of motion

- Canonical interactions ✓

- symmetry: Lorentz and $SU_C(3) \otimes U_Q(1)$
- renormalizable

  - $\mu f S(V)$ interactions

$$\bar{\mu}(y_L \omega_- + y_R \omega_+) f S + \bar{\mu}\gamma^\mu (g_L \omega_- + g_R \omega_+) f V_\mu + \text{h.c.}$$

chiral basis

  - standard QED interactions

$$eQ_f \bar{f}\gamma^\mu f A_\mu + ieQ_S[S^\dagger(\partial^\mu S) - (\partial^\mu S)^\dagger S]A_\mu + ieQ_V\{[V_{\mu\nu}(V^\mu)^\dagger - (V_{\mu\nu})^\dagger V^\mu]A^\nu + V_\mu (V_\nu)^\dagger A^{\mu\nu}\}$$

unitary gauge: no $\gamma V^- S^+$

- Quasi-canonical interactions ✓

  - interactions involving Weyl and Majorana fermions

  - interactions involving other spin half fermions

$\bar{\chi} f S^*$ $\xrightarrow{m_\mu \to m_\chi}$ dark matter MDM

electric charge conservation condition:

$$Q_f + Q_S = -1 \to Q_f + Q_{S^*} = Q_f - Q_S = Q_\chi = 0$$

  - interactions involving charge conjugate fields

$\bar{\mu} f^C S$ $\xrightarrow[C \text{ transformation}]{\psi \equiv f^c}$ $Q_\psi = Q_{f^c} = -Q_f, \quad m_\psi = m_f$

  - interactions involving equation of motion

$$(\partial_\mu S)\bar{\mu}\gamma^\mu\gamma^5\mu = \partial_\mu(S\bar{\mu}\gamma^\mu\gamma^5\mu) - S\partial_\mu(\bar{\mu}\gamma^\mu\gamma^5\mu)$$

- Non-canonical contributions
  - high spin particles
  - high-dimensional operators
    - four-fermion interactions $(\overline{L_{L,a}}\sigma^{\mu\nu}e_R)\epsilon_{ab}(\overline{Q_{L,a}}\sigma_{\mu\nu}u_R)$
    - dipole interactions $\bar{\mu}\sigma^{\mu\nu}fV_{\mu\nu}$
    - non-standard scalar QED interactions $i[(\partial^\mu S)^\dagger(\partial^\nu S)-(\partial^\nu S)^\dagger(\partial^\mu S)]A_{\mu\nu}$
    - non-standard vector QED interactions $ieQ_{W^+}[g_\gamma(W^+_{\mu\nu}W^{-,\mu}-W^-_{\mu\nu}W^{+,\mu})A^\nu+\kappa_\gamma W^+_\mu W^-_\nu A^{\mu\nu}+\frac{\lambda_\gamma}{m_W^2}W^+_{\mu\nu}W^{-,\nu\rho}A_\rho{}^\mu]$
    - two field strength interactions $SA_{\mu\nu}Z^{\mu\nu}$
  - high-loop contributions
  - beyond local QFT: non-local QED, gravity

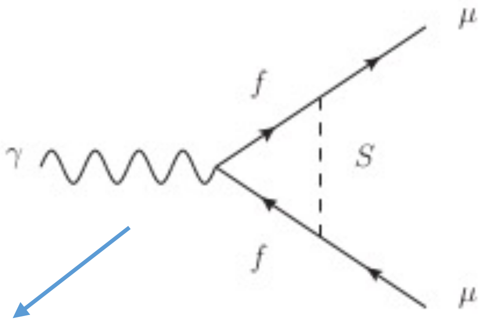
Contributions to $(g-2)_\mu$

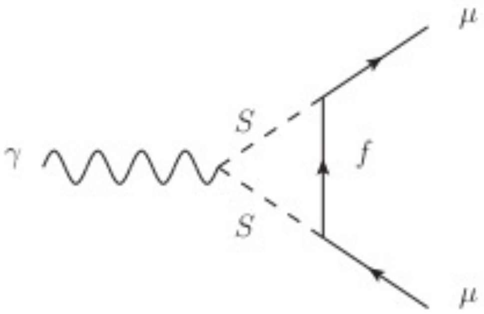
canonical interactions at one-loop level

$Q_f + Q_{S(V)} = -1$

$\mu f S(V)$ interactions $\quad \bar{\mu}(y_L\omega_- + y_R\omega_+)fS + \bar{\mu}\gamma^\mu(g_L\omega_- + g_R\omega_+)fV_\mu + \text{h.c.}$

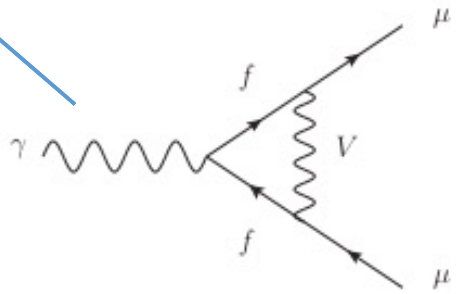
standard QED interactions $\quad eQ_f\bar{f}\gamma^\mu f A_\mu + ieQ_S[S^\dagger(\partial^\mu S) - (\partial^\mu S)^\dagger S]A_\mu + ieQ_V\{[V_{\mu\nu}(V^\mu)^\dagger - (V_{\mu\nu})^\dagger V^\mu]A^\nu + V_\mu(V_\nu)^\dagger A^{\mu\nu}\}$

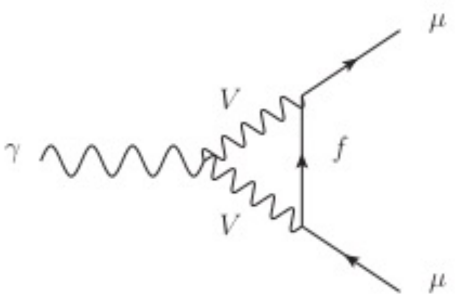




$ff\gamma$ vertex mediated $\propto Q_f$

$SS\gamma(VV\gamma)$ vertex mediated $\propto Q_{S(V)}$





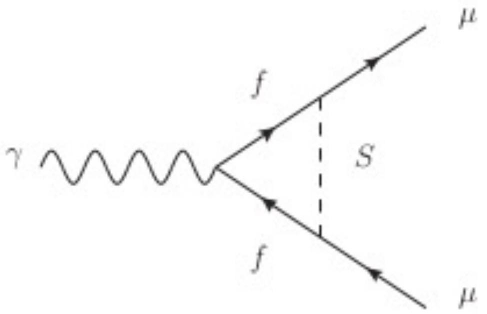
general structure of the one-loop contributions

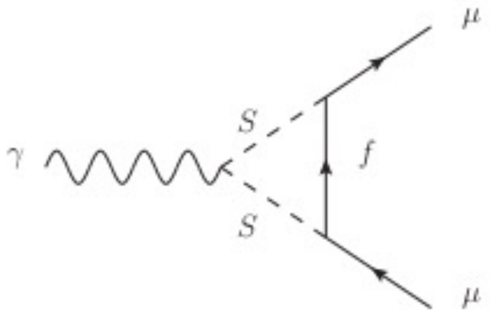
$$\Delta a_\mu = \frac{N_C m_\mu^2}{8\pi^2}\left\{\sum_S \frac{(|y_L|^2+|y_R|^2)(-Q_f I_{LL}^f - Q_S I_{LL}^S) + [y_L(y_R)^* + y_R(y_L)^*](-Q_f I_{LR}^f - Q_S I_{LR}^S)}{m_S^2} + \sum_V \frac{(|g_L|^2+|g_R|^2)(-Q_f L_{LL}^f - Q_V L_{LL}^V) + [g_L(g_R)^* + g_R(g_L)^*](-Q_f L_{LR}^f - Q_V L_{LR}^V)}{m_V^2}\right\}$$

contributions in chiral basis

chiral symmetry breaking
$\overline{\mu_L}\sigma^{\mu\nu}\mu_R + \overline{\mu_R}\sigma^{\mu\nu}\mu_L$

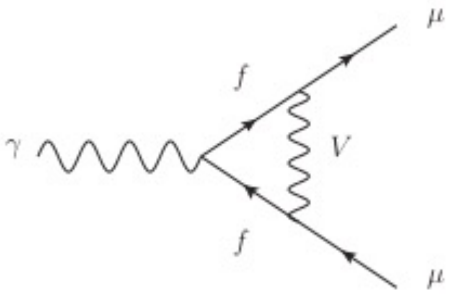
I_{LL}^f, I_{LR}^f

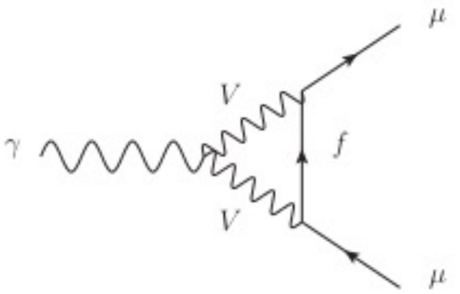




I_{LL}^S, I_{LR}^S

L_{LL}^f, L_{LR}^f





L_{LL}^V, I_{LR}^V

three different representations

- PaVe representation

  tensor integral and scalar integral

- Integral representation

  not unique $\int_0^1 dx(1-2x) = \int_0^1 dx(x^2 + ax - \frac{1}{3} - \frac{a}{2}) = 0$

- Special function representation

  pre-defined function $f(k^2, m_0^2, m_1^2) \equiv \lim_{\epsilon \to 0^+} \int_0^1 dx \frac{1}{x m_0^2 + (1-x) m_1^2 - x(1-x)k^2 - i\epsilon}$

analytic form of the loop functions: scalar mediator case

$\Longrightarrow$ **PaVe Representation**

$$I^f_{LL} = \lim_{k^2\to 0} m_S^2[C_{11}(m_\mu^2,k^2,m_\mu^2,m_S^2,m_f^2,m_f^2)+C_{12}(m_\mu^2,k^2,m_\mu^2,m_S^2,m_f^2,m_f^2)+C_1(m_\mu^2,k^2,m_\mu^2,m_S^2,m_f^2,m_f^2)]$$

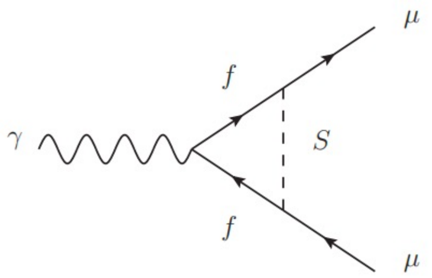
$$= \frac{m_S^2}{16m_\mu^4}[(3m_S^2+m_\mu^2-5m_f^2)B_0(0,m_f^2,m_f^2)+2m_S^2B_0(0,m_S^2,m_S^2)+(5m_f^2-m_\mu^2-5m_S^2)B_0(m_\mu^2,m_f^2,m_S^2)$$
$$+(3m_S^4-2m_S^2m_\mu^2-6m_S^2m_f^2+3m_f^4-2m_f^2m_\mu^2-m_\mu^4)C_0(m_\mu^2,0,m_\mu^2,m_S^2,m_f^2,m_f^2)+2(m_S^2-m_\mu^2-m_f^2)]$$

$$= \frac{1}{2}\int_0^1 dx\frac{x^2(1-x)m_S^2}{(1-x)m_S^2-x(1-x)m_\mu^2+xm_f^2}$$

$\Longrightarrow$ **Integral Representation**

$$= \frac{m_S^2}{4m_\mu^6}\Big\{[m_f^6-(3m_S^2+2m_\mu^2)m_f^4+(3m_S^4+m_S^2m_\mu^2+m_\mu^4)m_f^2+m_S^4m_\mu^2-m_S^6]f(m_\mu^2,m_f^2,m_S^2)$$
$$+[-m_f^4+(2m_S^2+m_\mu^2)m_f^2-m_S^4]\log\frac{m_f^2}{m_S^2}+m_\mu^2(2m_f^2-2m_S^2-m_\mu^2)\Big\}.$$

$\Longrightarrow$ **Special Function Representation**



$$I^f_{LR} = \boxed{\frac{m_f m_S^2}{m_\mu}}\lim_{k^2\to 0} C_1(m_\mu^2,k^2,m_\mu^2,m_S^2,m_f^2,m_f^2)$$

$$= \frac{m_f m_S^2}{4m_\mu^3}[B_0(m_\mu^2,m_f^2,m_S^2)-B_0(0,m_f^2,m_f^2)+(m_f^2-m_S^2-m_\mu^2)C_0(m_\mu^2,0,m_\mu^2,m_S^2,m_f^2,m_f^2)]$$

$$= \frac{m_f}{2m_\mu}\int_0^1 dx\frac{x^2m_S^2}{(1-x)m_S^2-x(1-x)m_\mu^2+xm_f^2}$$

$$= \frac{m_f m_S^2}{4m_\mu^5}\Big\{[m_f^4-2m_f^2(m_S^2+m_\mu^2)+m_S^4+m_\mu^4]f(m_\mu^2,m_f^2,m_S^2)+(m_S^2+m_\mu^2-m_f^2)\log\frac{m_f^2}{m_S^2}+2m_\mu^2\Big\}.$$

$$I_{LL}^S = -m_S^2 \lim_{k^2\to 0}[C_1(m_\mu^2,k^2,m_\mu^2,m_f^2,m_S^2,m_S^2) + C_{11}(m_\mu^2,k^2,m_\mu^2,m_f^2,m_S^2,m_S^2) + C_{12}(m_\mu^2,k^2,m_\mu^2,m_f^2,m_S^2,m_S^2)]$$

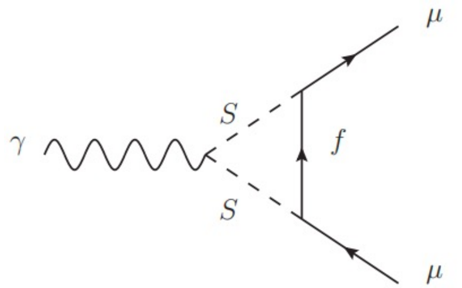
$$= \frac{m_S^2}{16m_\mu^4}[(5m_S^2 - m_\mu^2 - 3m_f^2)B_0(0,m_S^2,m_S^2) - 2m_f^2 B_0(0,m_f^2,m_f^2) + (5m_f^2 + m_\mu^2 - 5m_S^2)B_0(m_\mu^2,m_f^2,m_S^2)$$

$$+ (-3m_S^4 + 2m_S^2m_\mu^2 + 6m_S^2m_f^2 - 3m_f^4 + 2m_f^2m_\mu^2 + m_\mu^4)C_0(m_\mu^2,0,m_\mu^2,m_f^2,m_S^2,m_S^2) + 2(m_S^2 + m_\mu^2 - m_f^2)]$$

$$= -\frac{1}{2}\int_0^1 dx \frac{x(1-x)^2 m_S^2}{(1-x)m_S^2 - x(1-x)m_\mu^2 + xm_f^2}$$

$$= \frac{m_S^2}{4m_\mu^6}\Big\{[m_f^6 - (3m_S^2 + m_\mu^2)m_f^4 + (3m_S^2 - m_\mu^2)m_S^2m_f^2 - m_S^2(m_\mu^2 - m_S^2)^2]f(m_\mu^2,m_f^2,m_S^2)$$

$$+ [-m_f^4 + 2m_S^2m_f^2 + m_S^2m_\mu^2 - m_S^4]\log\frac{m_f^2}{m_S^2} + m_\mu^2(2m_f^2 - 2m_S^2 + m_\mu^2)\Big\}.$$



$$I_{LR}^S = \frac{m_f m_S^2}{2m_\mu}\lim_{k^2\to 0}[C_0(m_\mu^2,k^2,m_\mu^2,m_f^2,m_S^2,m_S^2) + 2C_1(m_\mu^2,k^2,m_\mu^2,m_f^2,m_S^2,m_S^2)]$$

$$= \frac{m_f m_S^2}{4m_\mu^3}[B_0(m_\mu^2,m_f^2,m_S^2) - B_0(0,m_S^2,m_S^2) + (m_S^2 + m_\mu^2 - m_f^2)C_0(m_\mu^2,0,m_\mu^2,m_f^2,m_S^2,m_S^2)]$$

$$= -\frac{m_f}{2m_\mu}\int_0^1 dx \frac{x(1-x)m_S^2}{(1-x)m_S^2 - x(1-x)m_\mu^2 + xm_f^2}$$

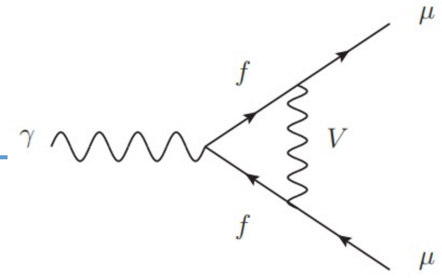
$$= \frac{m_f m_S^2}{4m_\mu^5}\Big\{[m_f^4 - m_f^2(2m_S^2 + m_\mu^2) + m_S^4 - m_S^2m_\mu^2]f(m_\mu^2,m_f^2,m_S^2) + (m_S^2 - m_f^2)\log\frac{m_f^2}{m_S^2} + 2m_\mu^2\Big\}.$$

analytic form of the loop functions: vector mediator case

$$
\begin{aligned}
L^f_{LL} &= \lim_{k^2\to 0}\Big\{m_V^2[(D-2)(C_{11}+C_{12})+(D+2)C_1+2C_0] \\
&- [(D+2)C_{001}+2C_{00}+m_\mu^2(C_{111}+3C_{112}+C_{11}+C_{12})-k^2(C_{112}+C_{12})-m_f^2(C_{11}+C_{12})]\Big\} \\
&= -\frac{1}{16m_\mu^4}\Big\{[5m_f^4+m_f^2(7m_V^2-4m_\mu^2)-m_\mu^4-6m_V^4+11m_\mu^2m_V^2]B_0(0,m_f^2,m_f^2)-2m_V^2(m_f^2+m_\mu^2+2m_V^2)B_0(0,m_V^2,m_V^2) \\
&+ [m_\mu^4+m_\mu^2(4m_f^2-9m_V^2)-5(m_f^4+m_f^2m_V^2-2m_V^4)]B_0(m_\mu^2,m_f^2,m_V^2)+2(m_f^2+m_\mu^2-m_V^2)(m_f^2+m_\mu^2+2m_V^2) \\
&[-3m_f^6+7m_f^4m_\mu^2+m_f^2(3m_V^2-5m_\mu^2)(3m_V^2+m_\mu^2)+m_\mu^6-6m_V^6+17m_\mu^2m_V^4-12m_\mu^4m_V^2]C_0(m_\mu^2,0,m_\mu^2,m_V^2,m_f^2,m_f^2)\Big\}
\end{aligned}
$$

$$
= -\frac{1}{2}\int_0^1 dx\,\frac{x[x(1+x)m_f^2-x(1-x)m_\mu^2+2(1-x)(2-x)m_V^2]}{(1-x)m_V^2-x(1-x)m_\mu^2+xm_f^2}
$$

$$
\begin{aligned}
&= \frac{1}{4m_\mu^6}\Big\{[m_f^8-m_f^6(m_V^2+3m_\mu^2)+m_f^4(3m_\mu^4+2m_\mu^2m_V^2-3m_V^4)+m_f^2(-m_\mu^6-m_\mu^4m_V^2-4m_\mu^2m_V^4+5m_V^6) \\
&- 3m_\mu^4m_V^4+5m_\mu^2m_V^6-2m_V^8]f(m_\mu^2,m_f^2,m_V^2)+m_\mu^2[2m_f^4+m_f^2(2m_V^2-3m_\mu^2)-(m_\mu^2-2m_V^2)^2] \\
&+ [-m_f^6+2m_f^4m_\mu^2+m_f^2(-m_\mu^4-2m_\mu^2m_V^2+3m_V^4)+3m_\mu^2m_V^4-2m_V^6]\log\frac{m_f^2}{m_V^2}\Big\}.
\end{aligned}
$$



$$
\begin{aligned}
L^f_{LR} &= \frac{m_f}{m_\mu}\cdot\lim_{k^2\to 0}[-m_V^2(D\cdot C_1+2C_0)+(D+2)C_{001}+2C_{00}+m_\mu^2(C_{111}+3C_{112})-k^2(C_{112}+C_{12})] \\
&= \frac{m_f}{m_\mu}\cdot\frac{1}{8m_\mu^2}\Big\{(3m_f^2-3m_\mu^2+5m_V^2)B_0(0,m_f^2,m_f^2)-2m_V^2B_0(0,m_V^2,m_V^2)-3(m_f^2-m_\mu^2+m_V^2)B_0(m_\mu^2,m_f^2,m_V^2) \\
&+ 2(m_f^2+m_\mu^2-m_V^2)-[m_f^4-2m_\mu^2(m_f^2-4m_V^2)+4m_f^2m_V^2+m_\mu^4-5m_V^4]C_0(m_\mu^2,0,m_\mu^2,m_V^2,m_f^2,m_f^2)\Big\}
\end{aligned}
$$

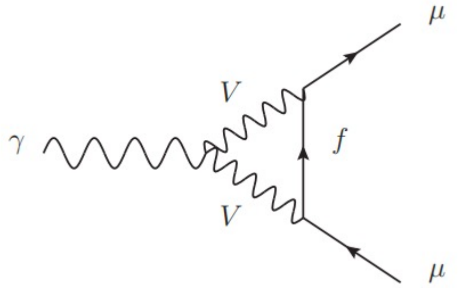
$$
= \frac{m_f}{2m_\mu}\int_0^1 dx\,\frac{x[xm_f^2-x(1-2x)m_\mu^2+4(1-x)m_V^2]}{(1-x)m_V^2-x(1-x)m_\mu^2+xm_f^2}
$$

$$
\begin{aligned}
&= \frac{m_f}{4m_\mu^5}\Big\{[-m_f^6+3m_f^4m_\mu^2+3m_f^2(-m_\mu^4+m_V^4)+m_\mu^6+3m_\mu^2m_V^4-2m_V^6]f(m_\mu^2,m_f^2,m_V^2) \\
&+ 2m_\mu^2(-m_f^2+2m_\mu^2-2m_V^2)+[m_f^4+m_f^2(-2m_\mu^2+m_V^2)+m_\mu^4+m_\mu^2m_V^2-2m_V^4]\log\frac{m_f^2}{m_V^2}\Big\}.
\end{aligned}
$$

$$L^V_{LL} = -\frac{1}{2m_V^2}\lim_{k^2\to 0}\Big\{m_f^2(2m_V^2-k^2)C_0 + [2m_V^2((D-6)m_V^2+m_\mu^2+3m_f^2) - k^2(m_\mu^2+3m_f^2-2m_V^2)]C_1$$
$$+ [2m_V^2((D-2)m_V^2+m_\mu^2+m_f^2) - k^2(m_\mu^2+m_f^2)](C_{11}+C_{12})\Big\}$$
$$= \frac{1}{16m_\mu^4}\Big\{-2m_f^2(m_\mu^2+m_f^2+2m_V^2)B_0(0,m_f^2,m_f^2) - [m_\mu^4+m_\mu^2(13m_V^2-4m_f^2)+3m_f^4+m_f^2m_V^2-10m_V^4]B_0(0,m_V^2,m_V^2)$$
$$+ [m_\mu^4+m_\mu^2(13m_V^2-2m_f^2)+5(m_f^4+m_f^2m_V^2-2m_V^4)]B_0(m_\mu^2,m_f^2,m_V^2) + 2(m_\mu^4+3m_V^2m_\mu^2-m_f^4-m_f^2m_V^2+2m_V^4)$$
$$+ [m_\mu^6-m_\mu^4(5m_f^2+12m_V^2)+m_\mu^2(7m_f^4-12m_f^2m_V^2+17m_V^4)-3(m_f^6-3m_f^2m_V^4+2m_V^6)]C_0(m_\mu^2,0,m_\mu^2,m_f^2,m_V^2,m_V^2)\Big\}$$

$$= \frac{1}{2}\int_0^1 dx\,\frac{(1-x)[x(1+x)m_f^2 - x(1-x)m_\mu^2 + 2(1-x)(2-x)m_V^2]}{(1-x)m_V^2 - x(1-x)m_\mu^2 + xm_f^2}$$

$$= \frac{1}{4m_\mu^6}\Big\{[m_f^8 - m_f^6(m_V^2+2m_\mu^2) + m_f^4(m_\mu^4+2m_\mu^2m_V^2-3m_V^4) + m_f^2m_V^4(5m_V^2-7m_\mu^2)$$
$$+ m_V^2(3m_\mu^2-2m_V^2)(m_\mu^2-m_V^2)^2]f(m_\mu^2,m_f^2,m_V^2) + m_\mu^2[2m_f^4+m_f^2(2m_V^2-m_\mu^2)+m_\mu^4+8m_\mu^2m_V^2-4m_V^4]$$
$$+ [-m_f^6+m_f^4m_\mu^2+3m_f^2m_V^2(-m_\mu^2+m_V^2)-3m_\mu^4m_V^2+5m_\mu^2m_V^4-2m_V^6]\log\frac{m_f^2}{m_V^2}\Big\}.$$



$$L^V_{LR} = \frac{m_f}{4m_\mu m_V^2}\cdot\lim_{k^2\to 0}\Big\{[2m_V^2((4-D)m_V^2+m_\mu^2+m_f^2) - k^2(m_\mu^2+m_f^2)]C_0 + 4m_\mu^2(2m_V^2-k^2)(C_{11}+C_{12})$$
$$+ [4m_V^2(-Dm_V^2+3m_\mu^2+m_f^2) - 2k^2(3m_\mu^2+m_f^2-2m_V^2)]C_1\Big\}$$
$$= \frac{m_f}{m_\mu}\cdot\frac{1}{8m_\mu^2}\Big\{2m_f^2B_0(0,m_f^2,m_f^2) + (3m_V^2+m_f^2-m_\mu^2)B_0(0,m_V^2,m_V^2) + (m_\mu^2-3m_f^2-3m_V^2)B_0(m_\mu^2,m_f^2,m_V^2)$$
$$+ [m_\mu^4+2m_\mu^2(4m_V^2-m_f^2)+m_f^4+4m_f^2m_V^2-5m_V^4]C_0(m_\mu^2,0,m_\mu^2,m_f^2,m_V^2,m_V^2) + 2m_f^2-2m_\mu^2-2m_V^2\Big\}$$

agree with J. P. Leveille (1978)

disagree with F. Jegerlehner (2009)

$$= -\frac{m_f}{2m_\mu}\int_0^1 dx\,\frac{(1-x)[xm_f^2 + x(2x-1)m_\mu^2 + 4(1-x)m_V^2]}{(1-x)m_V^2 - x(1-x)m_\mu^2 + xm_f^2}$$
$$= -\frac{m_f}{2m_\mu}\int_0^1 dx\,\frac{x^2m_f^2 + (3-2x)(1-x)m_V^2}{(1-x)m_V^2 - x(1-x)m_\mu^2 + xm_f^2}$$

$$= \frac{m_f}{4m_\mu^5}\Big\{[-m_f^6+2m_f^4m_\mu^2+m_f^2(-m_\mu^4-m_\mu^2m_V^2+3m_V^4)-3m_\mu^4m_V^2+5m_\mu^2m_V^4-2m_V^6]f(m_\mu^2,m_f^2,m_V^2)$$
$$-2m_\mu^2(m_f^2+2m_V^2) + [m_f^4+m_f^2(-m_\mu^2+m_V^2)+3m_\mu^2m_V^2-2m_V^4]\log\frac{m_f^2}{m_V^2}\Big\}.$$

Vector : $Q_V = -2x^2(1+x-2\epsilon) + \lambda^2(1-\epsilon)^2 Q_S$

Axialvector : $Q_A = -2x^2(1+x+2\epsilon) + \lambda^2(1+\epsilon)^2 Q_P$

arXiv version

Vector: $Q_V = 2x^2(1+x-2\epsilon) - \lambda^2(1-\epsilon)^2 Q_S$

Axialvector: $Q_A = 2x^2(1+x+2\epsilon) - \lambda^2(1+\epsilon)^2 Q_P$

journal version

book

difference may appear for light charged gauge boson

The singularity of loop integrals

Landau singularity (LS)
leading order (LO)
next (N)

- PaVe representation

$B_0(m_\mu^2, m_0^2, m_1^2)$

- LO-LS
  - $m_\mu^2 = (m_0 + m_1)^2$ (normal threshold)
  - $m_\mu^2 = (m_0 - m_1)^2$ (pseudo threshold, unphysical region)
- NLO-LS: $m_0 = 0$, $m_1 = 0$

$C_0(m_\mu^2, 0, m_\mu^2, m_0^2, m_1^2, m_1^2)$

- LO-LS: $m_\mu^2 = (m_0 + m_1)^2$, $m_\mu^2 = (m_0 - m_1)^2$ → pseudo threshold
- NLO-LS: $m_1 = 0$, $m_\mu^2 = (m_0 + m_1)^2$, $m_\mu^2 = (m_0 - m_1)^2$ → pseudo threshold
- NNLO-LS: $m_0 = 0$, $m_1 = 0$

- Integral representation

zero points of $xm_0^2 - x(1-x)m_\mu^2 + (1-x)m_1^2$ between 0 and 1

- Special function representation

$$f(k^2, m_0^2, m_1^2)$$

singularities in all

$$m_0 = 0$$
$$m_1 = 0$$
$$m_\mu^2 = (m_0 + m_1)^2$$

Expansion of the loop functions

three mass scales: $m_\mu,\ m_f,\ m_{S(V)}$

- Hierarchical mass expansion:

SUSY vector-like fermion ← $m_\mu \ll m_f, m_{S(V)}$ $\quad m_{S(V)} \ll m_f, m_\mu \quad$ $m_f \ll m_\mu, m_{S(V)}$ → W in the SM

$m_f, m_{S(V)} \ll m_\mu \quad m_f, m_\mu \ll m_{S(V)} \quad m_\mu, m_{S(V)} \ll m_f$

- Degenerate mass case:

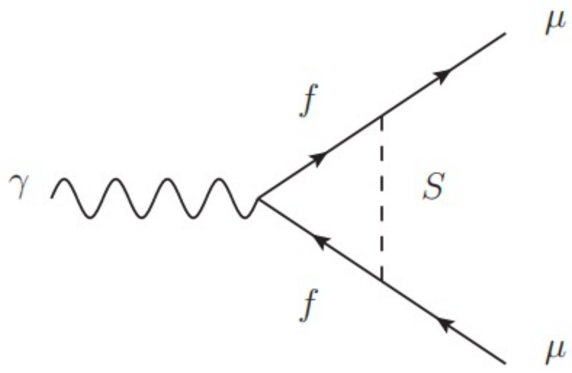
$m_f = m_\mu \quad m_\mu = m_{S(V)} \quad m_f = m_{S(V)}$

$m_f = m_\mu = m_{S(V)}$

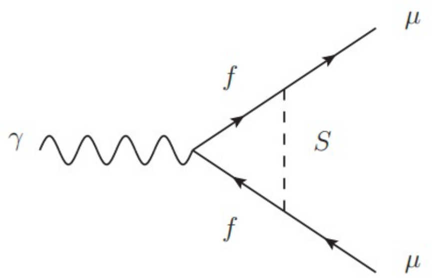
Z and Higgs in the SM ← $m_f = m_\mu$

example of hierarchical mass expansion: $m_\mu \ll m_f, m_S$

$$I^f_{LL} = \frac{m_S^2}{12(m_S^2 - m_f^2)^4}(2m_S^6 + 3m_S^4 m_f^2 - 6m_S^2 m_f^4 + m_f^6 + 6m_S^4 m_f^2 \log\frac{m_f^2}{m_S^2})$$
$$+ \frac{m_\mu^2 m_S^2}{24(m_S^2 - m_f^2)^6}\left[3m_S^8 + 44m_S^6 m_f^2 - 36m_S^4 m_f^4 - 12m_S^2 m_f^6 + m_f^8 + 12m_S^4 m_f^2(3m_f^2 + 2m_S^2)\log\frac{m_f^2}{m_S^2}\right]$$
$$+ \frac{m_\mu^4 m_S^2}{40(m_S^2 - m_f^2)^8}\left[4m_S^{10} + 155m_S^8 m_f^2 + 80m_S^6 m_f^4 - 220m_S^4 m_f^6 - 20m_S^2 m_f^8 + m_f^{10}\right.$$
$$\left. + 60m_S^4 m_f^2(m_S^4 + 4m_S^2 m_f^2 + 2m_f^4)\log\frac{m_f^2}{m_S^2}\right] + \mathcal{O}(m_\mu^6).$$

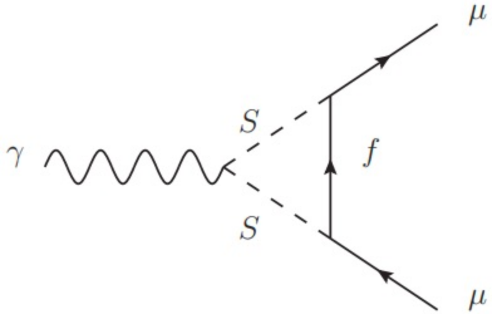


$$I^f_{LL} \approx \begin{cases} \frac{1}{6}(1 + \frac{11m_f^2}{2m_S^2} - \frac{6m_f^2}{m_S^2}\log\frac{m_S}{m_f}), & \text{for } m_\mu \ll m_f \ll m_S \\ \frac{m_S^2}{12m_f^2}(1 - \frac{2m_S^2}{m_f^2}), & \text{for } m_\mu \ll m_S \ll m_f \end{cases}$$



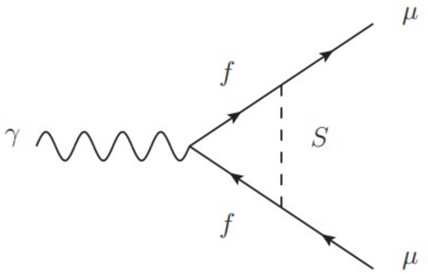
behaviours of $\frac{m_\mu^2}{m_S^2} I^f_{LL(LR)}$ under different mass limits

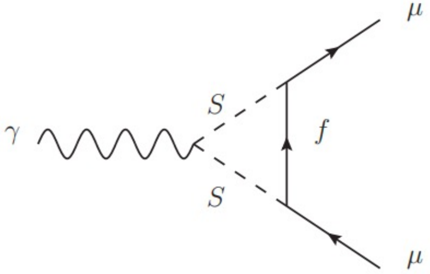
		$\frac{m_\mu^2}{m_S^2} I^f_{LL}$	$\frac{m_\mu^2}{m_S^2} I^f_{LR}$
Chiral limit $m_\mu \to 0$		0	0
Chiral limit $m_f \to 0$	$m_\mu \neq m_S$	$-\frac{1}{4} - \frac{m_S^2}{2m_\mu^2} - \frac{m_S^4}{2m_\mu^4}\log(1-\frac{m_\mu^2}{m_S^2})$	0
	$m_\mu = m_S$	$-\frac{3}{4} + \frac{1}{2}\log\frac{m_\mu}{m_f}$	$\frac{\pi}{4}$
Limit of $m_S \to 0$	$m_\mu \neq m_f$	$-\frac{1}{4} + \frac{m_f^2}{2m_\mu^2} + \frac{m_f^2(m_f^2 - m_\mu^2)}{2m_\mu^4}\log(1-\frac{m_\mu^2}{m_f^2})$	$\frac{m_f}{2m_\mu}[1+(\frac{m_f^2}{m_\mu^2}-1)\log(1-\frac{m_\mu^2}{m_f^2})]$
	$m_\mu = m_f$	$\frac{1}{4}$	$\frac{1}{2}$
Decoupling limit $m_f \to \infty$		0	0
Decoupling limit $m_S \to \infty$		0	0



behaviours of $\frac{m_\mu^2}{m_S^2} I^S_{LL(LR)}$ under different mass limits

		$\frac{m_\mu^2}{m_S^2} I^S_{LL}$	$\frac{m_\mu^2}{m_S^2} I^S_{LR}$
Chiral limit $m_\mu \to 0$		0	0
Chiral limit $m_f \to 0$	$m_\mu \neq m_S$	$\frac{1}{4} - \frac{m_S^2}{2m_\mu^2} + \frac{m_S^2(m_\mu^2 - m_S^2)}{2m_\mu^4}\log(1-\frac{m_\mu^2}{m_S^2})$	0
	$m_\mu = m_S$	$-\frac{1}{4}$	0
Limit of $m_S \to 0$	$m_\mu \neq m_f$	$\frac{1}{4} + \frac{m_f^2}{2m_\mu^2} + \frac{m_f^4}{2m_\mu^4}\log(1-\frac{m_\mu^2}{m_f^2})$	$\frac{m_f}{2m_\mu}[1+\frac{m_f^2}{m_\mu^2}\log(1-\frac{m_\mu^2}{m_f^2})]$
	$m_\mu = m_f$	$\frac{1}{4}(3-2\log\frac{m_\mu}{m_S})$	$\frac{1}{2}(1-\log\frac{m_\mu}{m_S})$
Decoupling limit $m_f \to \infty$		0	0
Decoupling limit $m_S \to \infty$		0	0





leading order formulae of $\Delta a_\mu/(\frac{N_C m_\mu^2}{8\pi^2 m_S^2})$ in different scenarios

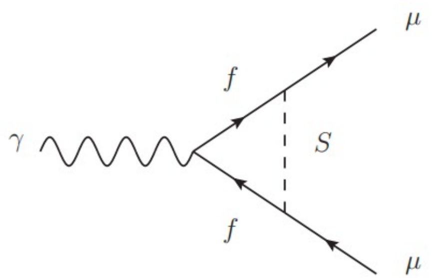
hierarchical mass

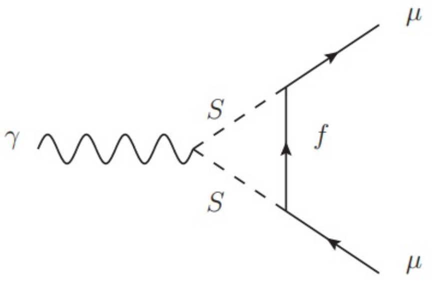
$m_\mu \ll m_f \ll m_S$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)(-\frac{1}{6}Q_f+\frac{1}{12}Q_S)+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_f}{m_\mu}[-(\log\frac{m_S}{m_f}-\frac{3}{4})Q_f+\frac{1}{4}Q_S]$
$m_\mu \ll m_S \ll m_f$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{m_S^2}{m_f^2}(-\frac{1}{12}Q_f+\frac{1}{6}Q_S)+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_S^2}{4m_f m_\mu}(-Q_f+Q_S)$
$m_S \ll m_f \ll m_\mu$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{m_S^2}{4m_\mu^2}(Q_f-Q_S)$ $+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_S^2 m_f}{2m_\mu^3}[(2\log\frac{m_\mu}{m_f}-i\pi-1)Q_f-Q_S]$
$m_S \ll m_\mu \ll m_f$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{m_S^2}{m_f^2}(-\frac{1}{12}Q_f+\frac{1}{6}Q_S)+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_S^2}{4m_\mu m_f}(-Q_f+Q_S)$
$m_f \ll m_\mu \ll m_S$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)(-\frac{1}{6}Q_f+\frac{1}{12}Q_S)+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_f}{m_\mu}[-(\log\frac{m_S}{m_f}-\frac{3}{4})Q_f+\frac{1}{4}Q_S]$
$m_f \ll m_S \ll m_\mu$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{m_S^2}{4m_\mu^2}(Q_f-Q_S)$ $+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_S^2 m_f}{2m_\mu^3}[(2\log\frac{m_\mu}{m_f}-i\pi-1)Q_f-Q_S]$

degenerate mass

$m_S \ll m_f = m_\mu$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{m_S^2}{4m_\mu^2}[-Q_f+(2\log\frac{m_\mu}{m_S}-3)Q_S]$ $+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_S^2}{2m_\mu^2}[-Q_f+(\log\frac{m_\mu}{m_S}-1)Q_S]$
$m_f = m_\mu \ll m_S$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)(-\frac{1}{6}Q_f+\frac{1}{12}Q_S)+[y_L(y_R)^*+y_R(y_L)^*]\cdot[-(\log\frac{m_S}{m_\mu}-\frac{3}{4})Q_f+\frac{1}{4}Q_S]$
$m_f \ll m_\mu = m_S$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot[(\frac{3}{4}-\frac{1}{2}\log\frac{m_\mu}{m_f})Q_f+\frac{1}{4}Q_S]$ $+[y_L(y_R)^*+y_R(y_L)^*]\cdot[-\frac{\pi}{4}Q_f+\frac{m_f}{2m_\mu}(\log\frac{m_\mu}{m_f}-1)Q_S]$
$m_\mu = m_S \ll m_f$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{m_\mu^2}{m_f^2}(-\frac{1}{12}Q_f+\frac{1}{6}Q_S)+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_\mu}{4m_f}(-Q_f+Q_S)$
$m_\mu \ll m_f = m_S$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{1}{24}(-Q_f+Q_S)+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_S}{m_\mu}(-\frac{1}{6}Q_f+\frac{1}{12}Q_S)$
$m_f = m_S \ll m_\mu$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{m_S^2}{4m_\mu^2}(Q_f-Q_S)$ $+[y_L(y_R)^*+y_R(y_L)^*]\cdot\frac{m_S^3}{2m_\mu^3}[(2\log\frac{m_\mu}{m_S}-1-i\pi)Q_f-Q_S]$

$m_f = m_\mu = m_S$	$(\lvert y_L\rvert^2+\lvert y_R\rvert^2)\cdot\frac{2\sqrt{3}\pi-9}{36}(-Q_f+Q_S)$ $+[y_L(y_R)^*+y_R(y_L)^*]\cdot(-\frac{9-\sqrt{3}\pi}{18}Q_f+\frac{2\sqrt{3}\pi-9}{18}Q_S)$





$m_\mu \ll m_f \ll m_S$	$\lvert y_{L(R)}\rvert^2 \cdot (-\frac{1}{6}Q_f + \frac{1}{12}Q_S)$	$Q_f < -\frac{1}{3}$ or $Q_S > -\frac{2}{3}$
$m_\mu \ll m_S \ll m_f$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{m_S^2}{m_f^2}(-\frac{1}{12}Q_f + \frac{1}{6}Q_S)$	$Q_f < -\frac{2}{3}$ or $Q_S > -\frac{1}{3}$
$m_S \ll m_f \ll m_\mu$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{m_S^2}{4m_\mu^2}(Q_f - Q_S)$	$Q_f > -\frac{1}{2}$ or $Q_S < -\frac{1}{2}$
$m_S \ll m_\mu \ll m_f$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{m_S^2}{m_f^2}(-\frac{1}{12}Q_f + \frac{1}{6}Q_S)$	$Q_f < -\frac{2}{3}$ or $Q_S > -\frac{1}{3}$
$m_f \ll m_\mu \ll m_S$	$\lvert y_{L(R)}\rvert^2 \cdot (-\frac{1}{6}Q_f + \frac{1}{12}Q_S)$	$Q_f < -\frac{1}{3}$ or $Q_S > -\frac{2}{3}$
$m_f \ll m_S \ll m_\mu$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{m_S^2}{4m_\mu^2}(Q_f - Q_S)$	$Q_f > -\frac{1}{2}$ or $Q_S < -\frac{1}{2}$

hierarchical mass

$m_S \ll m_f = m_\mu$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{m_S^2}{4m_\mu^2}[-Q_f + (2\log\frac{m_\mu}{m_S} - 3)Q_S]$	$Q_f \lesssim -1$ or $Q_S \gtrsim 0$
$m_f = m_\mu \ll m_S$	$\lvert y_{L(R)}\rvert^2 \cdot (-\frac{1}{6}Q_f + \frac{1}{12}Q_S)$	$Q_f < -\frac{1}{3}$ or $Q_S > -\frac{2}{3}$
$m_f \ll m_\mu = m_S$	$\lvert y_{L(R)}\rvert^2 \cdot [(\frac{3}{4} - \frac{1}{2}\log\frac{m_\mu}{m_f})Q_f + \frac{1}{4}Q_S]$	$Q_f \lesssim 0$ or $Q_S \gtrsim -1$
$m_\mu = m_S \ll m_f$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{m_\mu^2}{m_f^2}(-\frac{1}{12}Q_f + \frac{1}{6}Q_S)$	$Q_f < -\frac{2}{3}$ or $Q_S > -\frac{1}{3}$
$m_\mu \ll m_f = m_S$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{1}{24}(-Q_f + Q_S)$	$Q_f < -\frac{1}{2}$ or $Q_S > -\frac{1}{2}$
$m_f = m_S \ll m_\mu$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{m_S^2}{4m_\mu^2}(Q_f - Q_S)$	$Q_f > -\frac{1}{2}$ or $Q_S < -\frac{1}{2}$
$m_f = m_\mu = m_S$	$\lvert y_{L(R)}\rvert^2 \cdot \frac{2\sqrt{3}\pi - 9}{36}(-Q_f + Q_S)$	$Q_f < -\frac{1}{2}$ or $Q_S > -\frac{1}{2}$

doubly charged scalar $\bar{\mu}\mu^c S$

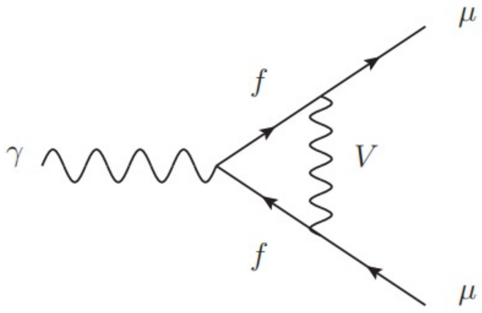
degenerate mass

electric charge bounded by $\Delta a_\mu > 0$ with pure left-handed (right-handed) couplings in different scenarios

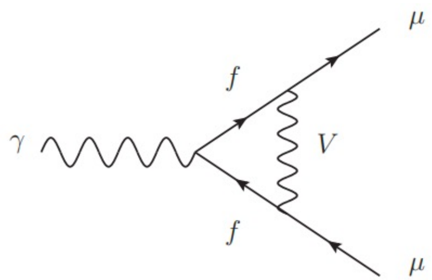
Expansion of the loop functions: vector mediator case

example of degenerate mass case: $m_\mu = m_f$

$$L_{LL}^f = \begin{cases} -\frac{1}{2m_\mu^6}\left[m_\mu^6 - 3m_\mu^4 m_V^2 + 2m_\mu^2 m_V^4 + m_V^2(m_\mu^4 - 3m_\mu^2 m_V^2 + m_V^4)\log\frac{m_\mu^2}{m_V^2}\right] \\ -\frac{m_V^3}{m_\mu^6\sqrt{m_V^2-4m_\mu^2}}(5m_\mu^4 - 5m_V^2 m_\mu^2 + m_V^4)\log\frac{m_V+\sqrt{m_V^2-4m_\mu^2}}{2m_\mu}, & \text{for } m_\mu < \frac{m_V}{2} \\ -\frac{1}{2m_\mu^6}\left[m_\mu^6 - 3m_\mu^4 m_V^2 + 2m_\mu^2 m_V^4 + m_V^2(m_\mu^4 - 3m_\mu^2 m_V^2 + m_V^4)\log\frac{m_\mu^2}{m_V^2}\right] \\ -\frac{m_V^3}{m_\mu^6\sqrt{4m_\mu^2-m_V^2}}(5m_\mu^4 - 5m_\mu^2 m_V^2 + m_V^4)\arctan\sqrt{\frac{4m_\mu^2}{m_V^2}-1}, & \text{for } m_\mu > \frac{m_V}{2} \\ 20\log 2 - \frac{29}{2}, & \text{for } m_\mu = \frac{m_V}{2} \end{cases}$$

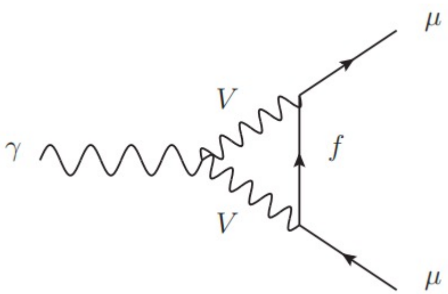


$$L_{LL}^f \approx \begin{cases} -\frac{2}{3} + \frac{m_\mu^2}{4m_V^2} + \frac{m_\mu^4}{60m_V^4}(137 - 120\log\frac{m_V}{m_\mu}), & \text{for } m_\mu \ll m_V \\ -\frac{1}{2} + \frac{m_V^2}{2m_\mu^2}(3 - 2\log\frac{m_\mu}{m_V}) - \frac{5\pi m_V^3}{4m_\mu^3} + \frac{m_V^4}{4m_\mu^4}(1 + 12\log\frac{m_\mu}{m_V}), & \text{for } m_V \ll m_\mu \end{cases}$$



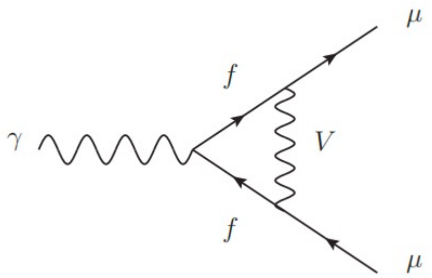
behaviours of $\frac{m_\mu^2}{m_V^2} L^f_{LL(LR)}$ under different mass limits

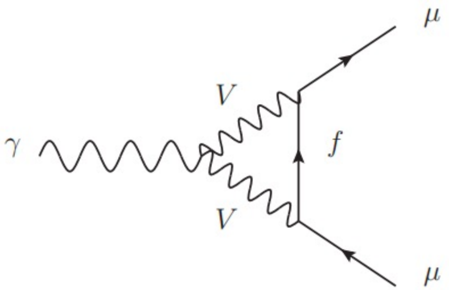
		$\frac{m_\mu^2}{m_V^2} L^f_{LL}$	$\frac{m_\mu^2}{m_V^2} L^f_{LR}$
Chiral limit $m_\mu \to 0$		0	0
Chiral limit $m_f \to 0$	$m_\mu \neq m_V$	$-\frac{(m_\mu^2 - 2m_V^2)^2}{4m_\mu^2 m_V^2} + \frac{m_V^2(3m_\mu^2 - 2m_V^2)}{2m_\mu^4} \log(1 - \frac{m_\mu^2}{m_V^2})$	0
	$m_\mu = m_V$	$-\frac{1}{4} - \frac{1}{2} \log \frac{m_\mu}{m_f}$	$\frac{\pi}{4}$
Limit of $m_V \to 0$	$m_\mu \neq m_f$	$\frac{1}{4m_\mu^4 m_V^2} \Big[m_\mu^2(2m_f^4 - 3m_f^2 m_\mu^2 - m_\mu^4) + 2m_f^2(m_f^2 - m_\mu^2)^2 \log(1 - \frac{m_\mu^2}{m_f^2})\Big] + 1 + \frac{m_f^2}{m_\mu^2} \log(1 - \frac{m_\mu^2}{m_f^2})$	$\frac{m_f}{m_\mu}\Big\{\frac{1}{2m_\mu^2 m_V^2}\Big[-m_f^2 m_\mu^2 + 2m_\mu^4 - (m_f^2 - m_\mu^2)^2 \cdot \log(1 - \frac{m_\mu^2}{m_f^2})\Big] - \frac{1}{2} - \frac{m_f^2 + m_\mu^2}{2m_\mu^2} \log(1 - \frac{m_\mu^2}{m_f^2})\Big\}$
	$m_\mu = m_f$	$-\frac{m_\mu^2}{2m_V^2} + \frac{3}{2} - \log \frac{m_\mu}{m_V}$	$\frac{m_\mu^2}{2m_V^2} + \log \frac{m_\mu}{m_V} - 1$
Decoupling limit $m_f \to \infty$		$-\frac{5m_\mu^2}{12m_V^2}$	$\frac{m_f m_\mu}{4m_V^2}$
Decoupling limit $m_V \to \infty$		0	0



behaviours of $\frac{m_\mu^2}{m_V^2} L^V_{LL(LR)}$ under different mass limits

		$\frac{m_\mu^2}{m_V^2} L^V_{LL}$	$\frac{m_\mu^2}{m_V^2} L^V_{LR}$
Chiral limit $m_\mu \to 0$		0	0
Chiral limit $m_f \to 0$	$m_\mu \neq m_V$	$\frac{1}{4m_\mu^4 m_V^2}\Big[m_\mu^6 + 8m_\mu^4 m_V^2 - 4m_\mu^2 m_V^4 + 2m_V^2(m_\mu^2 - m_V^2)(2m_V^2 - 3m_\mu^2) \log \frac{m_V^2 - m_\mu^2}{m_V^2}\Big]$	0
	$m_\mu = m_V$	$\frac{5}{4}$	0
Limit of $m_V \to 0$	$m_\mu \neq m_f$	$\frac{1}{4m_\mu^4 m_V^2}\Big[m_\mu^2(2m_f^4 - m_f^2 m_\mu^2 + m_\mu^4) + 2m_f^4(m_f^2 - m_\mu^2) \cdot \log \frac{m_f^2 - m_\mu^2}{m_f^2}\Big] + \frac{1}{2m_\mu^2(m_f^2 - m_\mu^2)}\Big[3m_\mu^4 \log \frac{m_f^2}{m_V^2} + (3m_f^4 + 3m_\mu^4) \log \frac{m_f^2 - m_\mu^2}{m_f^2} + m_\mu^2(3m_f^2 - 4m_\mu^2)\Big]$	$\frac{m_f}{m_\mu}\Big\{\frac{m_f^2}{2m_\mu^2 m_V^2}\Big[-m_\mu^2 + (m_\mu^2 - m_f^2) \log \frac{m_f^2 - m_\mu^2}{m_f^2}\Big] + \frac{1}{2m_\mu^2(m_f^2 - m_\mu^2)}\Big[m_\mu^2(2m_\mu^2 - m_f^2) - 3m_\mu^4 \log \frac{m_f^2}{m_V^2} - (3m_\mu^4 + 2m_\mu^2 m_f^2 + m_f^4) \log \frac{m_f^2 - m_\mu^2}{m_f^2}\Big]\Big\}$
	$m_\mu = m_f$	$\frac{m_\mu^2}{2m_V^2} + \frac{\pi m_\mu}{2m_V} + 2 - 3 \log \frac{m_\mu}{m_V}$	$-\frac{m_\mu^2}{2m_V^2} - \frac{\pi m_\mu}{2m_V} + 2 \log \frac{m_\mu}{m_V} - \frac{1}{2}$
Decoupling limit $m_f \to \infty$		$\frac{m_\mu^2}{3m_V^2}$	$-\frac{m_f m_\mu}{4m_V^2}$
Decoupling limit $m_V \to \infty$		0	0





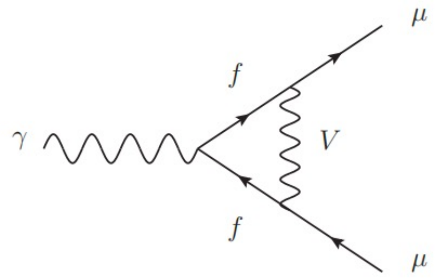
leading order formulae of $\Delta a_\mu / (\frac{N_C m_\mu^2}{8\pi^2 m_V^2})$ in different scenarios

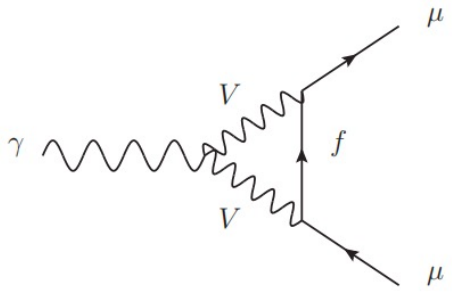
hierarchical mass

$m_\mu \ll m_f \ll m_V$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)\cdot(\frac{2}{3}Q_f - \frac{5}{6}Q_V) + [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_f}{m_\mu}(-Q_f + Q_V)$
$m_\mu \ll m_V \ll m_f$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{5}{12}Q_f - \frac{1}{3}Q_V) + [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_f}{4m_\mu}(-Q_f + Q_V)$
$m_V \ll m_f \ll m_\mu$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{1}{4}Q_f - \frac{1}{4}Q_V)$ $+ [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_f}{m_\mu}[(\log\frac{m_\mu}{m_f} - \frac{1}{2}i\pi - 1)Q_f + \frac{m_f^2}{m_\mu^2}(-\log\frac{m_\mu}{m_f} + \frac{1}{2}i\pi + \frac{1}{2})Q_V]$
$m_V \ll m_\mu \ll m_f$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{5}{12}Q_f - \frac{1}{3}Q_V) + [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_f}{4m_\mu}(-Q_f + Q_V)$
$m_f \ll m_\mu \ll m_V$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{2}{3}Q_f - \frac{5}{6}Q_V) + [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_f}{m_\mu}(-Q_f + Q_V)$
$m_f \ll m_V \ll m_\mu$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{1}{4}Q_f - \frac{1}{4}Q_V)$ $+ [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_f}{m_\mu}[(\log\frac{m_\mu}{m_f} - \frac{1}{2}i\pi - 1)Q_f + \frac{m_V^2}{m_\mu^2}(-3\log\frac{m_\mu}{m_V} + \frac{3}{2}i\pi + 1)Q_V]$

degenerate mass

$m_V \ll m_f = m_\mu$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{1}{2}Q_f - \frac{1}{2}Q_V) + [g_L(g_R)^* + g_R(g_L)^*](-\frac{1}{2}Q_f + \frac{1}{2}Q_V)$
$m_f = m_\mu \ll m_V$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{2}{3}Q_f - \frac{5}{6}Q_V) + [g_L(g_R)^* + g_R(g_L)^*](-Q_f + Q_V)$
$m_f \ll m_\mu = m_V$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)\cdot[Q_f(\frac{1}{4} + \frac{1}{2}\log\frac{m_\mu}{m_f}) - \frac{5}{4}Q_V]$ $+ [g_L(g_R)^* + g_R(g_L)^*]\cdot[-\frac{\pi}{4}Q_f + \frac{m_f}{2m_\mu}(\log\frac{m_\mu}{m_f} + 2)Q_V]$
$m_\mu = m_V \ll m_f$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{5}{12}Q_f - \frac{1}{3}Q_V) + [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_f}{4m_\mu}(-Q_f + Q_V)$
$m_\mu \ll m_f = m_V$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{13}{24}Q_f - \frac{17}{24}Q_V) + [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_V}{m_\mu}(-\frac{1}{2}Q_f + \frac{3}{4}Q_V)$
$m_f = m_V \ll m_\mu$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{1}{4}Q_f - \frac{1}{4}Q_V)$ $+ [g_L(g_R)^* + g_R(g_L)^*]\cdot\frac{m_V}{m_\mu}[(\log\frac{m_\mu}{m_V} - \frac{1}{2}i\pi - 1)Q_f + \frac{m_V^2}{m_\mu^2}(-4\log\frac{m_\mu}{m_V} + 2i\pi + \frac{3}{2})Q_V]$
$m_f = m_\mu = m_V$	$(\lvert g_L\rvert^2 + \lvert g_R\rvert^2)(\frac{\sqrt{3}\pi}{9}Q_f + \frac{2\sqrt{3}\pi - 18}{9}Q_V) + [g_L(g_R)^* + g_R(g_L)^*](\frac{9 - 4\sqrt{3}\pi}{18}Q_f + \frac{27 - 2\sqrt{3}\pi}{18}Q_V)$





$m_\mu \ll m_f \ll m_V$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{2}{3}Q_f - \frac{5}{6}Q_V)$	$Q_f > -\frac{5}{9}$ or $Q_V < -\frac{4}{9}$
$m_\mu \ll m_V \ll m_f$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{5}{12}Q_f - \frac{1}{3}Q_V)$	$Q_f > -\frac{4}{9}$ or $Q_V < -\frac{5}{9}$
$m_V \ll m_f \ll m_\mu$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{1}{4}Q_f - \frac{1}{4}Q_V)$	$Q_f > -\frac{1}{2}$ or $Q_V < -\frac{1}{2}$
$m_V \ll m_\mu \ll m_f$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{5}{12}Q_f - \frac{1}{3}Q_V)$	$Q_f > -\frac{4}{9}$ or $Q_V < -\frac{5}{9}$
$m_f \ll m_\mu \ll m_V$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{2}{3}Q_f - \frac{5}{6}Q_V)$	$Q_f > -\frac{5}{9}$ or $Q_V < -\frac{4}{9}$
$m_f \ll m_V \ll m_\mu$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{1}{4}Q_f - \frac{1}{4}Q_V)$	$Q_f > -\frac{1}{2}$ or $Q_V < -\frac{1}{2}$

hierarchical mass

W boson $\overline{\mu_L}\gamma^\mu \nu_L W_\mu^-$

$m_V \ll m_f = m_\mu$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{1}{2}Q_f - \frac{1}{2}Q_V)$	$Q_f > -\frac{1}{2}$ or $Q_V < -\frac{1}{2}$
$m_f = m_\mu \ll m_V$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{2}{3}Q_f - \frac{5}{6}Q_V)$	$Q_f > -\frac{5}{9}$ or $Q_V < -\frac{4}{9}$
$m_f \ll m_\mu = m_V$	$\lvert g_{L(R)}\rvert^2 \cdot [Q_f(\frac{1}{4} + \frac{1}{2}\log\frac{m_\mu}{m_f}) - \frac{5}{4}Q_V]$	$Q_f \gtrsim 0$ or $Q_V \lesssim -1$
$m_\mu = m_V \ll m_f$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{5}{12}Q_f - \frac{1}{3}Q_V)$	$Q_f > -\frac{4}{9}$ or $Q_V < -\frac{5}{9}$
$m_\mu \ll m_f = m_V$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{13}{24}Q_f - \frac{17}{24}Q_V)$	$Q_f > -\frac{17}{30}$ or $Q_V < -\frac{13}{30}$
$m_f = m_V \ll m_\mu$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{1}{4}Q_f - \frac{1}{4}Q_V)$	$Q_f > -\frac{1}{2}$ or $Q_V < -\frac{1}{2}$
$m_f = m_\mu = m_V$	$\lvert g_{L(R)}\rvert^2 \cdot (\frac{\sqrt{3}\pi}{9}Q_f + \frac{2\sqrt{3}\pi - 18}{9}Q_V)$	$Q_f > -\frac{6\sqrt{3} - 2\pi}{6\sqrt{3} - \pi} \approx -0.57$ or $Q_V < -\frac{\pi}{6\sqrt{3} - \pi} \approx -0.43$

degenerate mass

electric charge bounded by $\Delta a_\mu > 0$ with pure left-handed (right-handed) couplings in different scenarios

Applications and examples

example 1: one-loop contributions in the SM

interaction: $-e\bar{\mu}\gamma^{\mu}\mu\, A_{\mu}$

$g_L = g_R = -e$

$$\Delta a_{\mu}^{\gamma} = \lim_{m_\gamma \to 0} \frac{m_\mu^2}{8\pi^2 m_\gamma^2}\{(|g_L|^2 + |g_R|^2)L_{LL}^f(m_\mu, m_\mu, m_\gamma) + [g_L(g_R)^* + g_R(g_L)^*]L_{LR}^f(m_\mu, m_\mu, m_\gamma)\}$$

$$\Delta a_{\mu}^{\gamma} = \lim_{m_\gamma \to 0} \frac{e^2 m_\mu^2}{4\pi^2 m_\gamma^2}[-\frac{1}{2m_\mu^6}(m_\mu^6 - 3m_\mu^4 m_\gamma^2 + m_\gamma^2 m_\mu^4 \log\frac{m_\mu^2}{m_\gamma^2}) + \frac{1}{2m_\mu^4}(m_\mu^4 - 2m_\mu^2 m_\gamma^2 + m_\gamma^2 m_\mu^2 \log\frac{m_\mu^2}{m_\gamma^2})] = \frac{e^2}{8\pi^2}.$$

photon contribution

$m_V \ll m_f = m_\mu$

interaction: $\frac{g}{\sqrt{2}}\overline{\mu_L}\gamma^{\mu}\nu_L W_{\mu}^- + h.c.$

$g_L = \frac{g}{\sqrt{2}}, \qquad g_R = 0$

$$\Delta a_{\mu}^{W} = \frac{m_\mu^2}{8\pi^2 m_W^2}|g_L|^2 L_{LL}^V(m_\mu, m_\nu, m_W).$$

$$\Delta a_{\mu}^{W} \approx \frac{g^2 m_\mu^2}{16\pi^2 m_W^2}[\frac{1}{4} + \frac{2m_W^2}{m_\mu^2} - \frac{m_W^4}{m_\mu^4} + \frac{m_W^2}{2m_\mu^2}(1 - \frac{m_W^2}{m_\mu^2})(\frac{2m_W^2}{m_\mu^2} - 3)\log(1 - \frac{m_\mu^2}{m_W^2})]$$

$$\approx \frac{5g^2 m_\mu^2}{96\pi^2 m_W^2}(1 + \frac{m_\mu^2}{5m_W^2} + \frac{9m_\mu^4}{100m_W^4}).$$

W contribution

$m_f \ll m_\mu \ll m_V$

interaction: $\frac{g}{c_W}[(-\frac{1}{2} + s_W^2)\overline{\mu_L}\gamma^{\mu}\mu_L + s_W^2\overline{\mu_R}\gamma^{\mu}\mu_R]Z_{\mu} + h.c.$

$g_L = \frac{g}{c_W}(-\frac{1}{2} + s_W^2), \qquad g_R = \frac{g}{c_W}s_W^2$

$$\Delta a_{\mu}^{Z} = \frac{m_\mu^2}{8\pi^2 m_Z^2}\{(|g_L|^2 + |g_R|^2)L_{LL}^f(m_\mu, m_\mu, m_Z) + [g_L(g_R)^* + g_R(g_L)^*]L_{LR}^f(m_\mu, m_\mu, m_Z)\}.$$

$$\Delta a_{\mu}^{Z} \approx \frac{g^2 m_\mu^2}{8\pi^2 m_Z^2 c_W^2}\{[(s_W^2 - \frac{1}{2})^2 + s_W^4][-\frac{2}{3} + \frac{m_\mu^2}{4m_Z^2} + \frac{m_\mu^4}{60m_Z^4}(137 - 120\log\frac{m_Z}{m_\mu})] + s_W^2(2s_W^2 - 1)[1 + \frac{m_\mu^2}{6m_Z^2}(11 - 12\log\frac{m_Z}{m_\mu}) + \frac{m_\mu^4}{12m_Z^4}(89 - 120\log\frac{m_Z}{m_\mu})]\}$$

$$\approx \frac{g^2 m_\mu^2}{48\pi^2 m_Z^2 c_W^2}(4s_W^4 - 2s_W^2 - 1).$$

Z contribution

$m_f = m_\mu \ll m_V$

interaction: $(-m_\mu/v)\bar{\mu}\mu h$

$y_L = y_R = -m_\mu/v$

$$\Delta a_{\mu}^{h} = \frac{m_\mu^2}{8\pi^2 m_h^2}\{(|y_L|^2 + |y_R|^2)I_{LL}^f(m_\mu, m_\mu, m_h) + [y_L(y_R)^* + y_R(y_L)^*]I_{LR}^f(m_\mu, m_\mu, m_h)\}$$

$$\Delta a_{\mu}^{h} \approx \frac{m_\mu^4}{4\pi^2 m_h^2 v^2}[\log\frac{m_h}{m_\mu} - \frac{7}{12} + \frac{m_\mu^2}{m_h^2}(3\log\frac{m_h}{m_\mu} - \frac{13}{8}) + \frac{m_\mu^4}{m_h^4}(9\log\frac{m_h}{m_\mu} - \frac{201}{40})]$$

Higgs contribution

$m_f = m_\mu \ll m_S$

example 2: compute the muon MDM induced by doubly charged mediators

interaction: $\bar{\mu}(y_L\omega_- + y_R\omega_+)\mu^C S + \bar{\mu}\gamma^\mu(g_L\omega_- + g_R\omega_+)\mu^C V_\mu + \text{h.c.}$

$Q_f = Q_{\mu^C} = -Q_\mu = 1$

$Q_S = Q_V = -2$

$f = \mu^C$

symmetry factor 2 from each vertex

$$\Delta a_\mu = \frac{m_\mu^2}{2\pi^2}\Big\{\frac{(|y_L|^2+|y_R|^2)(-I^f_{LL}+2I^S_{LL}) + [y_L(y_R)^* + y_R(y_L)^*]\,(-I^f_{LR}+2I^S_{LR})}{m_S^2}$$
$$+ \frac{(|g_L|^2+|g_R|^2)(-L^f_{LL}+2L^V_{LL}) + [g_L(g_R)^* + g_R(g_L)^*]\,(-L^f_{LR}+2L^V_{LR})}{m_V^2}\Big\}.$$

$m_f = m_\mu \ll m_{S(V)}$

$$\Delta a_\mu \approx \frac{m_\mu^2}{2\pi^2 m_S^2}\Big\{-\frac{1}{3}(|y_L|^2+|y_R|^2) + [y_L(y_R)^* + y_R(y_L)^*]\cdot(-\log\frac{m_S}{m_\mu}+\frac{1}{4})\Big\}$$
$$+ \frac{m_\mu^2}{2\pi^2 m_V^2}\Big\{\frac{7}{3}(|g_L|^2+|g_R|^2) - 3\,[g_L(g_R)^* + g_R(g_L)^*]\Big\}.$$

agree with F. S. Queiroz and W. Shepherd, New Physics Contributions to the Muon Anomalous Magnetic Moment: A Numerical Code, Phys. Rev. D 89, 095024 (2014), arXiv:1403.2309 [hep-ph].

example 3: compute the dark matter MDM

interaction: $\bar{\chi}(y_L\omega_- + y_R\omega_+) f S^* + \text{h.c.}$

$m_\mu \to m_\chi$ $\qquad$ $Q_f + Q_{S^*} = Q_f - Q_S = Q_\chi = 0$

$$\Delta a_\mu = \frac{m_\chi^2 Q_f}{8\pi^2 m_S^2}\{(|y_L|^2 + |y_R|^2)(-I_{LL}^f + I_{LL}^S) + [y_L(y_R)^* + y_R(y_L)^*]\,(-I_{LR}^f + I_{LR}^S)\}$$

$$\Delta a_\mu = -\frac{Q_f}{16\pi^2}\Big\{(|y_L|^2 + |y_R|^2)\cdot[\frac{-m_f^4 + (2m_S^2 + m_\chi^2)m_f^2 + m_S^2(m_\chi^2 - m_S^2)}{2m_\chi^2} f(m_\chi^2, m_f^2, m_S^2)$$
$$+ \frac{m_f^2 - m_S^2}{m_\chi^2}\log\frac{m_f}{m_S} - 1] + \frac{m_f}{m_\chi}[y_L(y_R)^* + y_R(y_L)^*]\cdot[\frac{-m_f^2 + m_S^2 + m_\chi^2}{2} f(m_\chi^2, m_f^2, m_S^2) + \log\frac{m_f}{m_S}]\Big\}$$

agree with A. Ibarra, M. Reichard and G. Tomar, Probing Dark Matter Electromagnetic Properties in Direct Detection Experiments, arXiv:2408.15760 [hep-ph].

example 4: compute the muon MDM from axion-like particle interactions

diagonal interaction: $y_P a\bar{\mu}(i\gamma^5)\mu$

$y_L = -iy_P, \qquad y_R = iy_P$

$m_f = m_\mu$ →

$$\Delta a_\mu = \frac{m_\mu^2 y_P^2}{4\pi^2 m_a^2}[I_{LL}^f(m_\mu, m_\mu, m_a) - I_{LR}^f(m_\mu, m_\mu, m_a)]$$

↓

$$\Delta a_\mu = \frac{m_\mu^2 y_P^2}{4\pi^2 m_a^2}[I_{LL}^f(m_\mu, m_\mu, m_a) - I_{LR}^f(m_\mu, m_\mu, m_a)]$$

$$= \frac{y_P^2}{16\pi^2} \cdot \begin{cases} \left[-\frac{m_\mu^2+2m_a^2}{m_\mu^2} + \frac{2m_a^2(m_a^2-m_\mu^2)}{m_\mu^4}\log\frac{m_a}{m_\mu} + \frac{2m_a^3(3m_\mu^2-m_a^2)}{m_\mu^4\sqrt{m_a^2-4m_\mu^2}}\log\frac{m_a+\sqrt{m_a^2-4m_\mu^2}}{2m_\mu}\right], & \text{for } 2m_\mu < m_a \\ \left[-\frac{m_\mu^2+2m_a^2}{m_\mu^2} + \frac{2m_a^2(m_a^2-m_\mu^2)}{m_\mu^4}\log\frac{m_a}{m_\mu} + \frac{2m_a^3(3m_\mu^2-m_a^2)}{m_\mu^4\sqrt{4m_\mu^2-m_a^2}}\arctan\sqrt{\frac{4m_\mu^2}{m_a^2}-1}\right], & \text{for } 2m_\mu > m_a \end{cases}$$

off-diagonal interaction: $a\bar{\mu}(y_L\omega_- + y_R\omega_+)f + \text{h.c.}$

$y_L = -iy_P, \qquad y_R = iy_P$

→

$$\Delta a_\mu = \frac{m_\mu^2}{8\pi^2 m_a^2}\{(|y_L|^2 + |y_R|^2)I_{LL}^f(m_\mu, m_f, m_a) + [y_L(y_R)^* + y_R(y_L)^*]I_{LR}^f(m_\mu, m_f, m_a)\}$$

f is τ lepton

$$\Delta a_\mu \approx \frac{m_\mu^2}{8\pi^2 m_a^2}[y_L(y_R)^* + y_R(y_L)^*]I_{LR}^f(m_\mu, m_\tau, m_a)$$

$$\approx \frac{m_\mu m_\tau}{32\pi^2}[y_L(y_R)^* + y_R(y_L)^*] \cdot \begin{cases} \frac{1}{(m_a^2-m_\tau^2)^3}(-3m_a^4 + 4m_a^2 m_\tau^2 - m_\tau^4 - 2m_a^4\log\frac{m_\tau^2}{m_a^2}), & \text{for } m_\mu \ll m_\tau, m_a \\ \frac{1}{m_\tau^2}[1 + \frac{1}{3m_\tau^2}(m_\mu^2 - 3m_a^2)], & \text{for } m_\mu, m_a \ll m_\tau \end{cases}$$

f is electron

$$\Delta a_\mu \approx \frac{m_\mu^2}{8\pi^2 m_a^2}(|y_L|^2 + |y_R|^2)I_{LL}^f(m_\mu, m_e, m_a)$$

$$\approx -\frac{1}{32\pi^2}(|y_L|^2 + |y_R|^2) \cdot \begin{cases} \frac{1}{m_\mu^4}[m_\mu^4 + 2m_a^2 m_\mu^2 + 2m_a^4\log\frac{m_a^2-m_\mu^2}{m_a^2}], & \text{for } m_e \ll m_\mu, m_a \text{ and } m_\mu < m_a \\ [1 + \frac{2}{m_\mu^2}(m_a^2 - m_e^2 - i\pi m_e^2 + m_e^2\log\frac{m_\mu^2}{m_e^2})], & \text{for } m_e, m_a \ll m_\mu \end{cases}$$

agree with A. M. Galda and M. Neubert, ALP-LEFT Interference and the Muon (g-2), JHEP 11, 015 (2023), arXiv:2308.01338 [hep-ph].

Summary and conclusions

- Complete analytic and expansion formulae for the muon magnetic dipole moment
  - a useful reference for cross check at one-loop level
- Universality of the formulae (spin-half fermion)
  - valid for canonical and quasi-canonical interactions: proper substitutions of masses, couplings, and electric charge conservation conditions
- Equivalent representations of the contributions
  - PaVe representation, integral representation, special function representation
  - singularities
- Approximations of the contributions
  - clear scope of applications
  - correctness: compared to the existing results and cross checked by the Package-X
- Physical implications of new physics
  - not just list of mathematical formulae
  - physics discussions: decoupling behaviour, chiral limit, electric charge bounds with pure left-handed (right-handed) couplings

Thanks!

Comments and criticisms are welcome.

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Backups

What is anomalous magnetic moment?

magnetic moment ↑

$$\vec{\mu_l} = g_l \frac{Q_l}{2m_l} \vec{S}$$

$$g_l = 2$$

Dirac equation (1928)

anomalous magnetic moment ↑

$$g_l = 2(1 + a_l)$$

$\gamma \ell^+ \ell^-$ vertex $\quad (-ieQ)\bar{u}(p_1)\left[\gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}}{2m}(p_1 - p_2)_\nu F_2(q^2)\right]u(p_2)$

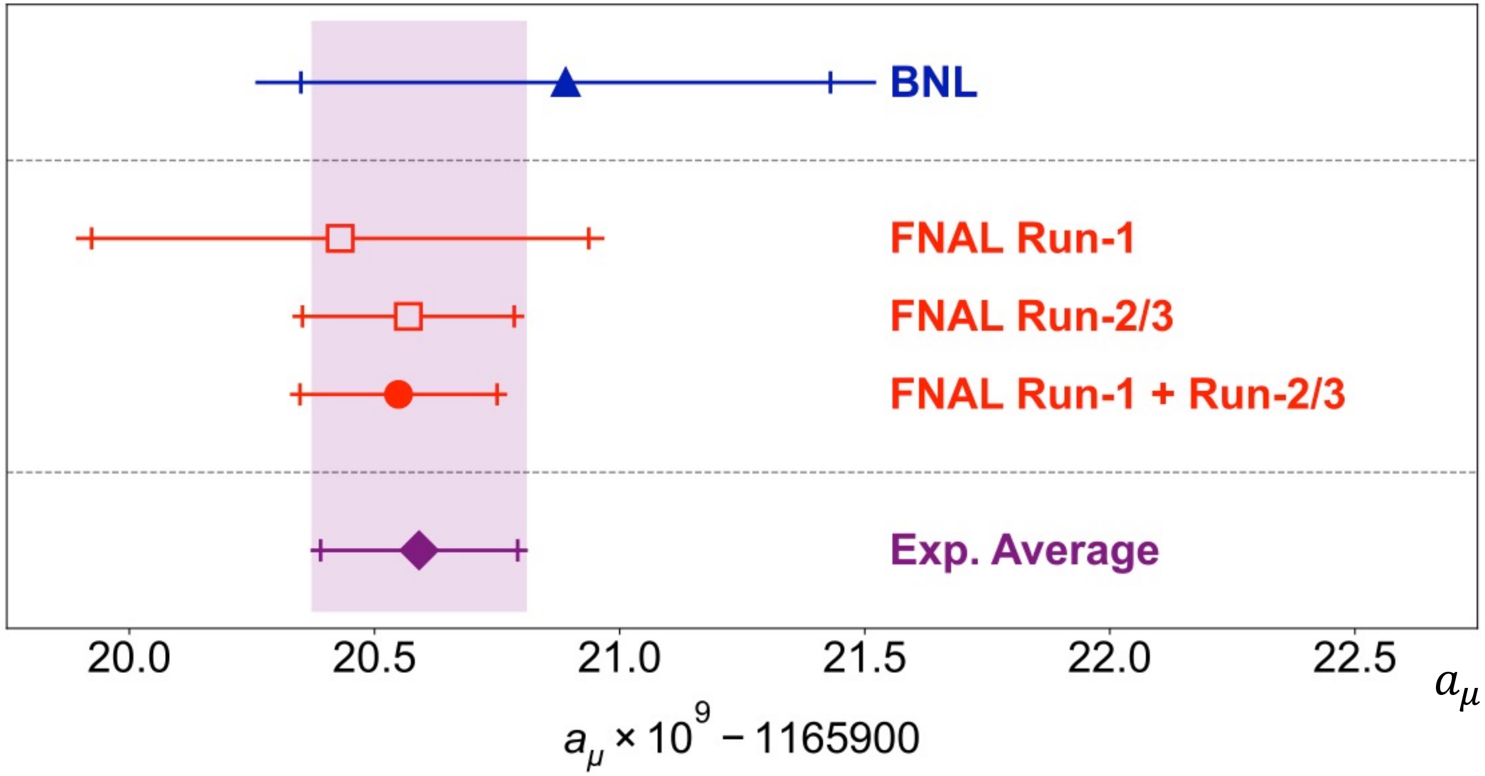
$(-ieQ)$ → tree level

$F_2(q^2)$ → form factor $a_l = F_2(0)$

- QED (**Schwinger,** 1948) $\quad a_l = \dfrac{\alpha}{2\pi}$
- SM and beyond the SM (BSM)

muon g-2 anomaly

G. W. Bennett et al. [Muon g-2], Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL, Phys. Rev. D 73, 072003 (2006), arXiv:hep-ex/0602035.
B. Abi et al. [Muon g-2], Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm, Phys. Rev. Lett. 126, 141801 (2021), arXiv:2104.03281.
D. P. Aguillard et al. [Muon g-2], Measurement of the Positive Muon Anomalous Magnetic Moment to 0.20 ppm, Phys. Rev. Lett. 131, 161802 (2023), arXiv:2308.06230.



FNAL muon g-2 experiment (2023)

$a_\mu(\mathrm{BNL}) = 116592080(63)\times 10^{-11}$

$a_\mu(\mathrm{FNAL\,1}) = 116592040(54)\times 10^{-11}$

$a_\mu(\mathrm{FNAL\ 2/3}) = 116592057(25)\times 10^{-11}$

$a_\mu(\mathrm{FNAL\ 1 + 2/3}) = 116592055(24)\times 10^{-11}$

$a_\mu(\mathrm{Exp}) = 116592059(22)\times 10^{-11}$

$a_\mu(\mathrm{SM}) = 116591810(43)\times 10^{-11}$ T. Aoyama *et al.* (2020)

$a_\mu(\mathrm{Exp}) - a_\mu(\mathrm{SM}) = (249 \pm 48)\times 10^{-11}$ 5σ

A. Boccaletti, *et al.* High precision calculation of the hadronic vacuum polarisation contribution to the muon anomaly, arXiv:2407.10913 [hep-lat].

- Without UV completion of the each scenario
  - hierarchical mass

$$m_\mu \ll m_f, m_{S(V)} \quad m_{S(V)} \ll m_f, m_\mu \quad m_f \ll m_\mu, m_{S(V)}$$
$$m_f, m_{S(V)} \ll m_\mu \quad m_f, m_\mu \ll m_{S(V)} \quad m_\mu, m_{S(V)} \ll m_f$$

  - degenerate mass

$$m_f = m_\mu \qquad m_\mu = m_{S(V)} \qquad m_f = m_{S(V)}$$
$$m_f = m_\mu = m_{S(V)}$$

- Renormalizable interactions at one-loop level
  - two-loop contributions
  - non-renormalizable interactions

- Understanding of the imaginary part
  - If EFT breakdown, is a_μ well defined for light loop particles?
  - Does the imaginary part lead to observable effects?

$$\pm \frac{e a_\mu}{4m_\mu} \bar{\mu} \sigma^{\mu\nu} \mu F_{\mu\nu}$$