Tame multi-leg Feynman integrals beyond one loop

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第四届量子场论及其应用研讨会

2024/11/17, 华南师范大学

Outline

I. Introduction

II. Explore asymptotic expansion

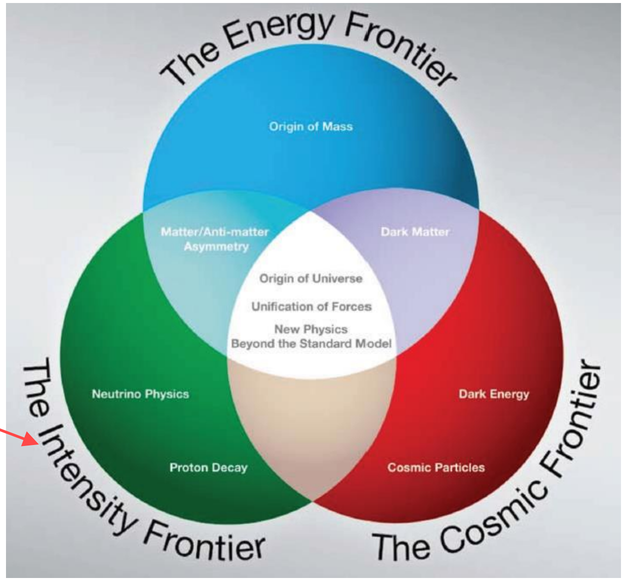
III. Explore iterative reduction

IV. Summary and outlook

Precision: gateway to discovery

- **New particles/physics have not been discovered yet at LHC**

  - **Currently main strategy: search anomalous deviations from theory**
  - **Interplay between exp. and th.**



To make full use of data: theoretical errors should be much smaller than experimental errors, ideally:

$$\boxed{Error_{th} < \frac{1}{3} Error_{exp}}$$

High-precision test of Higgs boson

➢ High-precision data expected

$$\mu \equiv \frac{\sigma_{\mathrm{Exp}}}{\sigma_{\mathrm{SM}}}$$

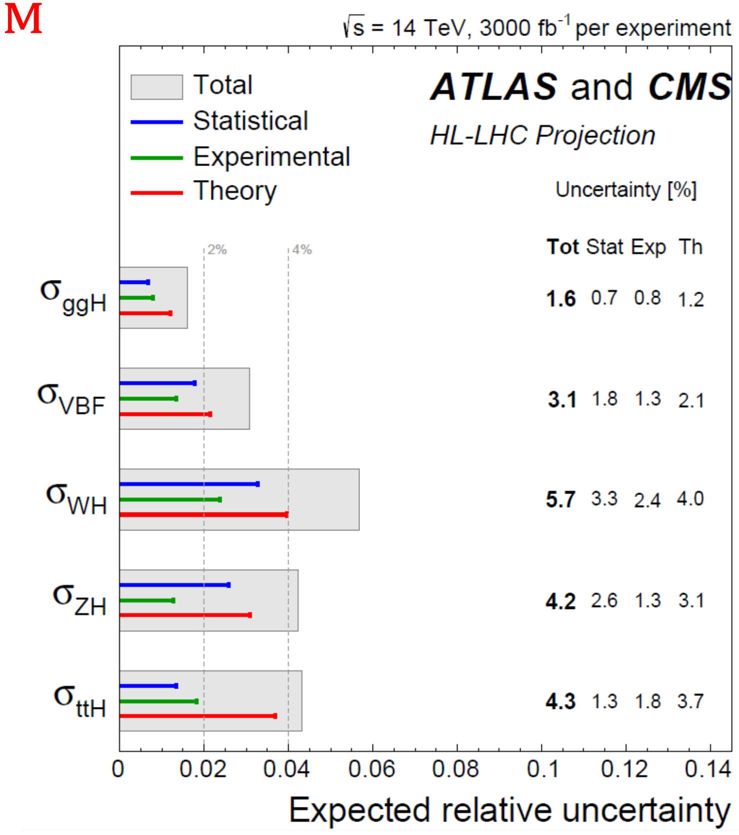
- **2012:** $\mu = 1.4 \pm 0.3(\mathrm{exp}) \pm (\mathrm{negligible\ th})$ PLB2012
- **2022:** $\mu = 1.05 \pm 0.04(\mathrm{exp}) \pm 0.04(\mathrm{th})$ Nature2022
- **Theorists also working hard, but experimentalists are excellent…**

➢ Looking ahead

- **Run III of LHC (22-25)**
- **HL-LHC (29-40): expect $O(1\%)$ uncertainty**
- **Requirement: reducing theoretical uncertainties by at least a factor of 5-10 (1-2 higher orders in α_s !)**

Eg., DGLAP : 3-loop@2004 → 4-loop@202x See Tongzhi Yang's talk

Can theorists keep up (α_s^2 in 16 years)?



CERN2019
Assuming theoretical errors halved

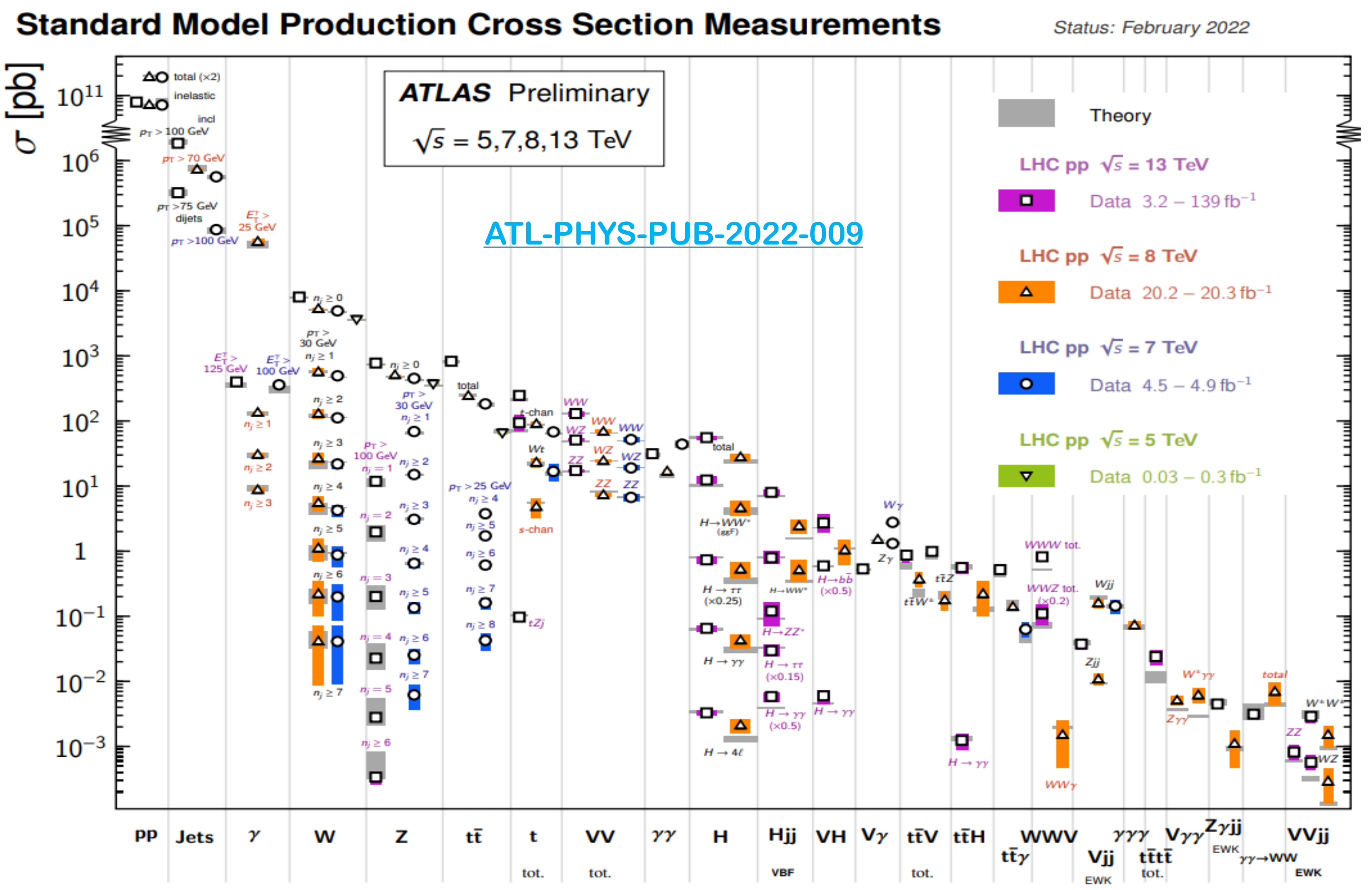
Era of precision physics

➢ High-precision data

- Many observables probed at precent level precision

➢ QCD cor. requirement

- Most processes: N2LO
- Many processes: N3LO
- Some processes: N4LO
- A few processes: N5LO



Current status of perturbative calcualtion

➢ Accomplished processes

- NLO solved, automatic codes exist: MadGraph, Helac, etc

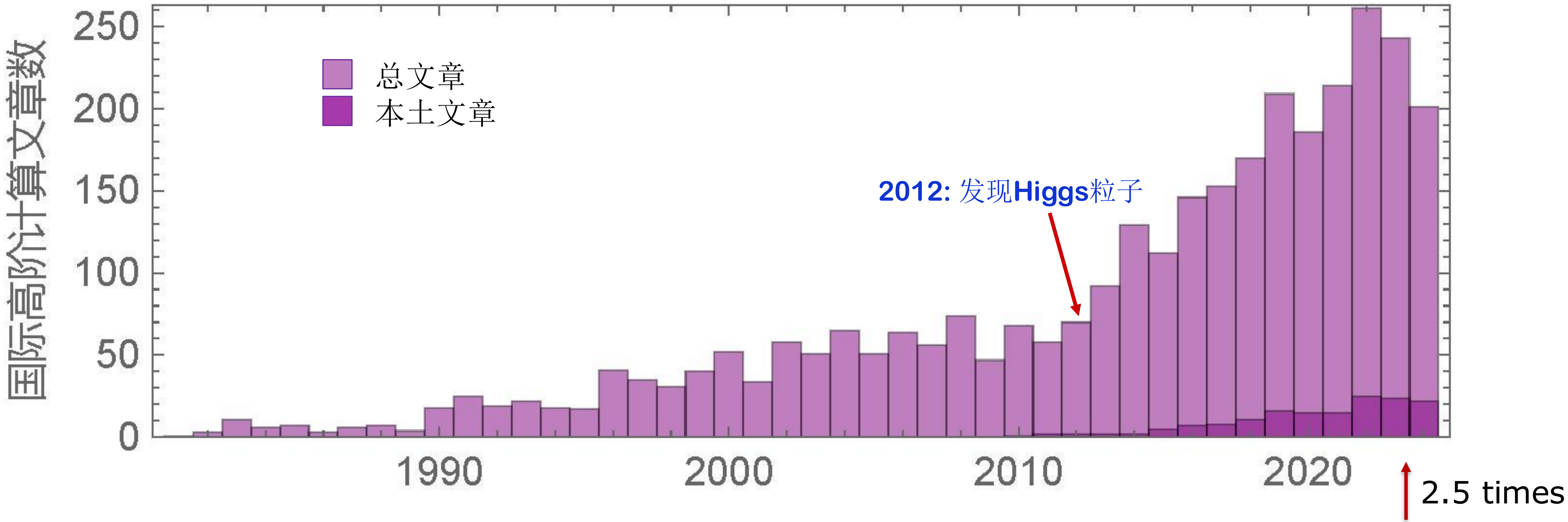
Legs / Order	2→1	2→2	2→3	2→4	2→5	2→6
NLO	★★★	★★★	★★★	★★★	★★★	★★★
N2LO	★★★	★★	★	?	?	
N3LO	★★	★	?			
N4LO	★	?				
N5LO	?					

Automatic high computation is highly demanded!!!

➢ A “billion-dollar project”

- LHC cost about 10 billion dollars
- It is waste of money unless having high-precision computation

High-order community



- In 10 years: high-order papers increased by 2.5 times
- The number of papers is almost unchanged for the whole hep-ph field

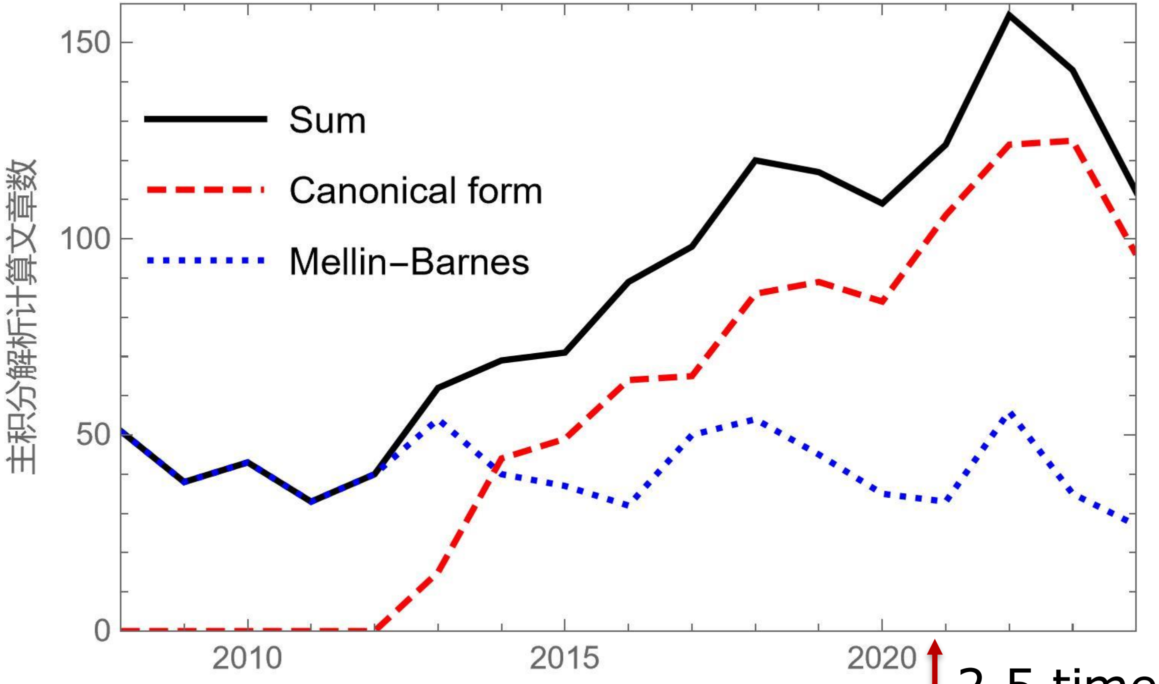
Main bottleneck: Feynman integrals computation

$$\sum_{\vec{\nu}'} Q^{\vec{\nu}jk}_{\vec{\nu}'}(D,\vec{s})I_{\vec{\nu}'}(D,\vec{s}) = 0$$

1) Reduce loop integrals to basis (Master Integrals)

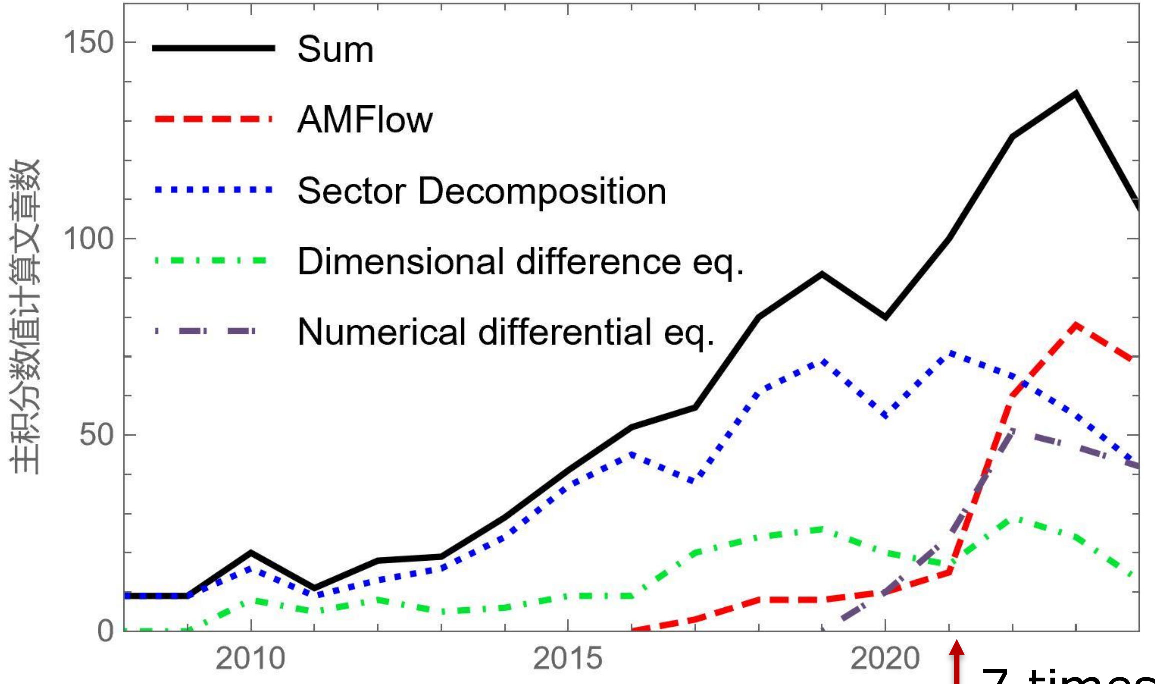
2) Calculate MIs

Computation of MIs



2.5 times

- ✓ Efficient for numerical evaluation
- ✓ High precision
- × High manpower cost
- × Only applicable for some processes



7 times

- ✓ Applicable for any process
- ✓ Low manpower cost for AMF/SD
- ✓ High precision except for SD
- ✓ Efficient by combining AMF+differential eq.

But, all depend on IBP!!!

Reduction: the bottleneck!

➢ The state-of-the-art IBP method: very challenging

- Four-loop $g + g \to H$ (NNLP in HTL): 860 days (wall time!) Davies, Herren, Steinhauser, PRL2020

➢ Improvements for IBPs

- Syzygy equations: trimming IBP system
  - Gluza, Kajda, Kosower, PRD2011
  - Böhm, Georgoudis, Larsen, Schulze, Zhang, PRD2018
  - NeatIBP: Wu, et al. CPC2024
- Block-triangular form: search simple IBP system
  - Liu, YQM, PRD2019
  - Guan, Liu, YQM, CPC2020
  - Blade: Guan, Liu YQM, Wu, 2405.14621

Improve efficiency by a hundredfold $\approx$ half order in α_s

We need to calculate two more orders in α_s! How?

Way to bypass IBP

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Large auxiliary mass expansion

➢ **Introduction of auxiliary mass**

刘霄： 2017.11 – 2019.12

$$\mathcal{M}(D,\vec{s},\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha + \mathrm{i}\eta)^{\nu_\alpha}}$$

$$\mathcal{D}_\alpha \equiv q_\alpha^2 - m_\alpha^2$$

➢ **Asymptotic expansion**

$$\mathcal{M}(D,\vec{s},\eta) = \eta^{LD/2-\sum_\alpha \nu_\alpha} \sum_{\mu_0=0}^{\infty} \eta^{-\mu_0} \mathcal{M}_{\mu_0}^{\mathrm{bub}}(D,\vec{s})$$

➢ **Ansatz and matching**

$$\sum_{i=1}^{n} Q_i(D,\vec{s},\eta)\, \mathcal{M}_i(D,\vec{s},\eta) = 0$$

$$Q_i(D,\vec{s},\eta) = \sum_{(\lambda_0,\vec{\lambda})\in\Omega_{d_i}^{r+1}} Q_i^{\lambda_0\ldots\lambda_r}(D)\, \eta^{\lambda_0} s_1^{\lambda_1} \cdots s_r^{\lambda_r}$$

- Expensive for expansion for many variables and complex expression
- Byproduct: block-triangular form

Liu, YQM, PRD2019
Guan, Liu, YQM, CPC2020
Blade: Guan, Liu YQM, Wu, 2405.14621

Large spacetime expansion

➢ Quantities present in all Feynman integrals

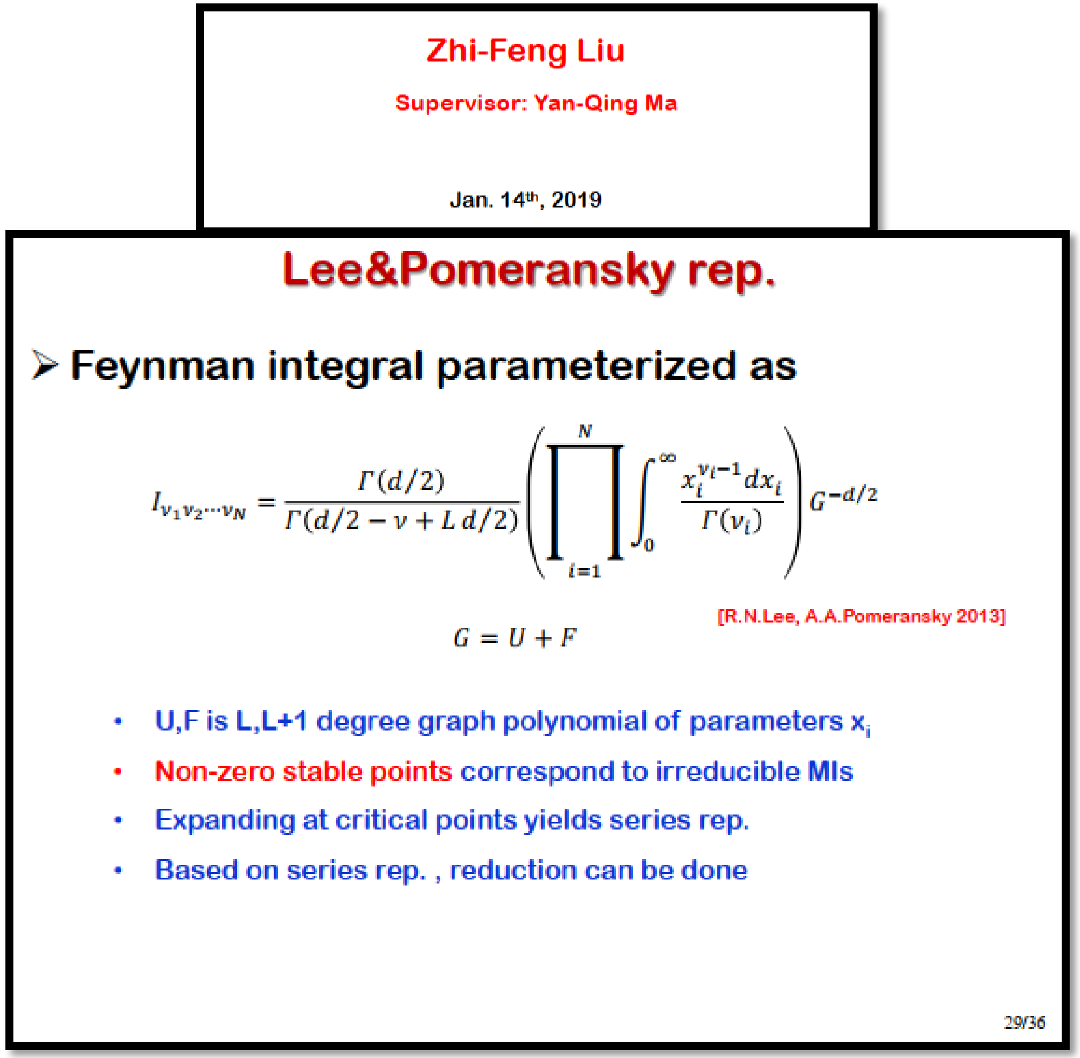
- Space-time dimension $D \to 4$ and
  Feynman prescription $\eta \to 0^+$
- Asymptotic expansion of FIs at $D \to \infty$
  (No η, simpler final expression)

刘霄、刘志峰： 2018.07 – 2019.01

➢ Later we realized: it has been studied by Baikov long time ago

Baikov, PLB2006

Again, expensive for expansion for many variables and complex expression



Zhi-Feng Liu

Supervisor: Yan-Qing Ma

Jan. 14th, 2019

Lee&Pomeransky rep.

➢ Feynman integral parameterized as

$$I_{\nu_1\nu_2\cdots\nu_N} = \frac{\Gamma(d/2)}{\Gamma(d/2-\nu+L\,d/2)}\left(\prod_{i=1}^{N}\int_0^\infty \frac{x_i^{\nu_i-1}dx_i}{\Gamma(\nu_i)}\right)G^{-d/2}$$

[R.N.Lee, A.A.Pomeransky 2013]

$$G = U + F$$

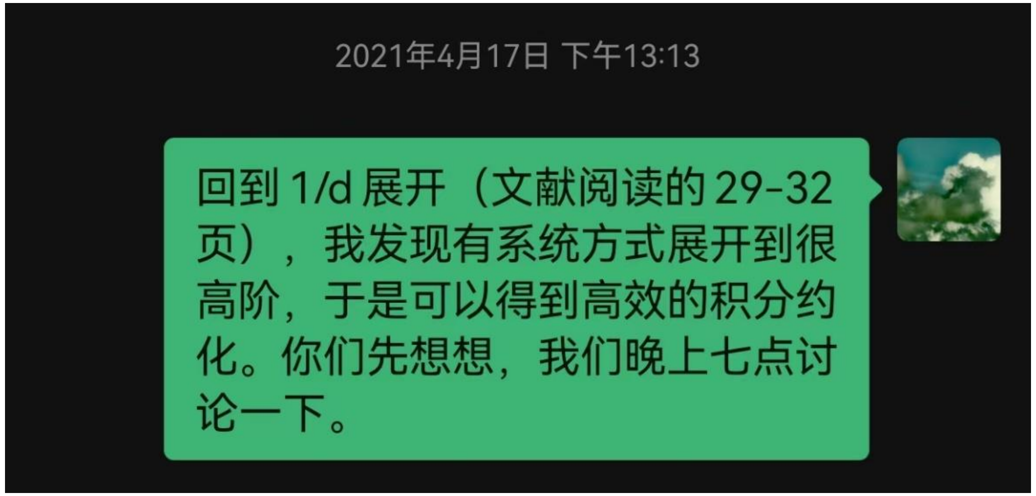
- U,F is L,L+1 degree graph polynomial of parameters x_i
- Non-zero stable points correspond to irreducible MIs
- Expanding at critical points yields series rep.
- Based on series rep. , reduction can be done

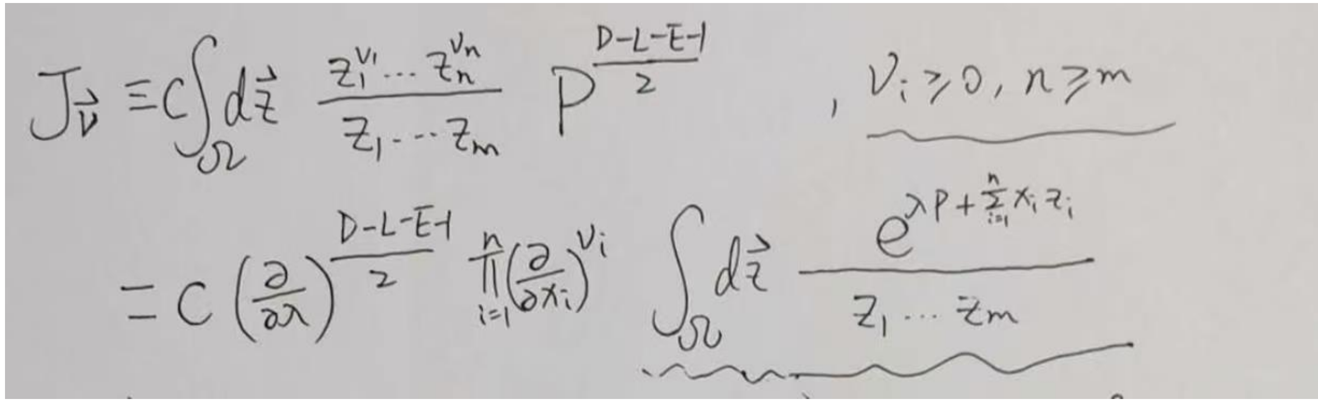
29/36

λ-space expansion

刘志峰、武文浩：**2021.04-2021.10**

➢ Squeeze singularities: generating function





D **to** λ**: Laplace transformation**
η **to** λ**: Mellin transformation**

➢ $1/\lambda$ expansion to construct λ-DEs

- **Very simple singularities:** 0 **and** ∞
- **Much less unknown parameters**
- **But, still expensive for expansion for many variables**

- **Byproduct: generating function in** D**-space**   Guan, Li, YQM, PRD2023

λ-space: to compute MIs (like AMFlow)

见东山： **2021.08-2022.10**

武文浩、黄瑞钧、母道明： **2022.08-2024.03**

本科生毕业论文（设计）

题目 费曼积分计算的新方法

姓名与学号 见东山 3180105527

指导教师 朱华星、马滟青

年级与专业 2018级物理学

所在学院 物理学院

递交日期 2022 年 5 月

本科生毕业论文

题目： 微扰量子色动力学方法研究及应用

Study and Application of Perturbative

Quantum Chromodynamics

姓　　名： 母道明

学　　号： 1900011408

院　　系： 物理学院

专　　业： 物理学

导师姓名： 马滟青

二〇二三 年 五 月

Finding:

Perfect for one-loop, but very hard for multiloop

Byproduct of λ-space: one-loop(-like) MIs

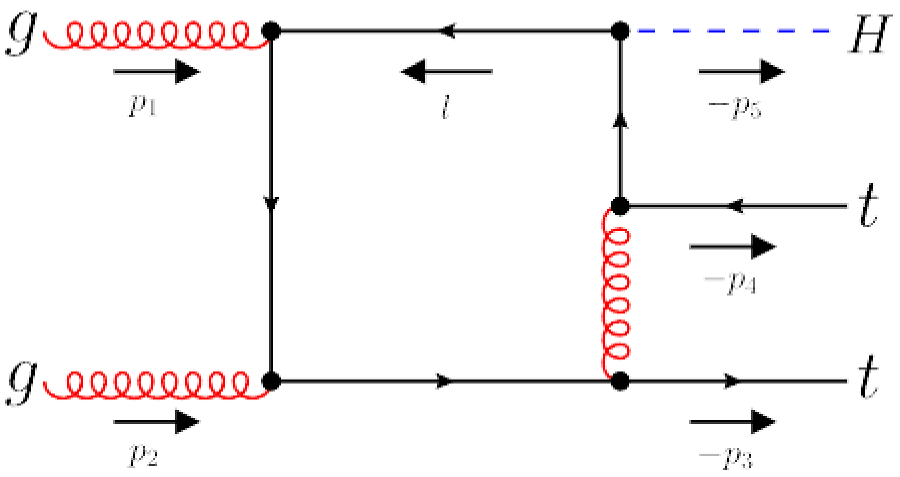
- **Dimension-changing transformation (DCT)** Huang, Jian, YQM, Mu, Wu (In preparation)

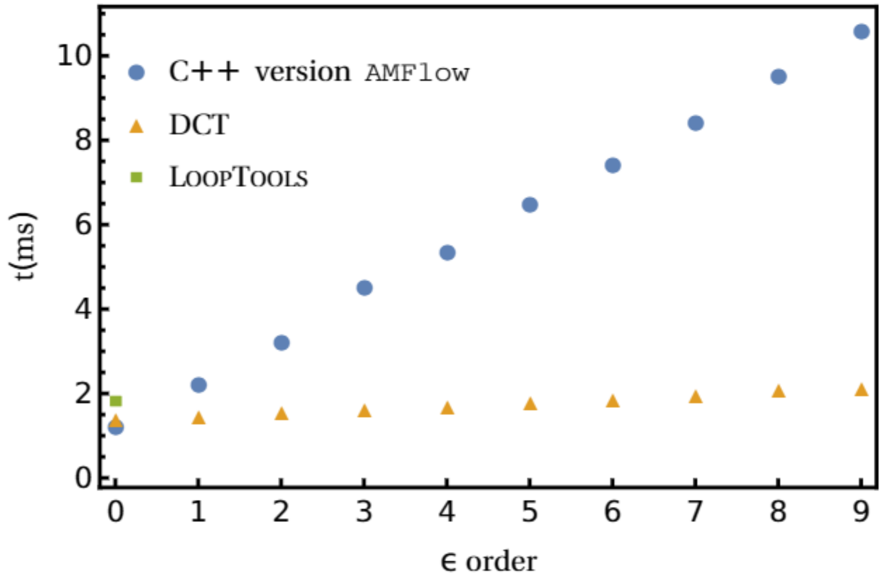
$$I^D_{\vec{\nu}}(\eta) \equiv \int \frac{\mathrm{d}^D l}{\mathrm{i}\pi^{D/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}_\alpha - \eta)^{\nu_\alpha}}$$

$$I^D_{\vec{\nu}} = \int \frac{\mathrm{d}^{D-D_0} l_\perp}{\pi^{(D-D_0)/2}} \int \frac{\mathrm{d}^{D_0} l_\parallel}{\mathrm{i}\pi^{D_0/2}} \prod_{\alpha=1}^{N} \frac{1}{(\mathcal{D}^\parallel_\alpha - l_\perp^2)^{\nu_\alpha}}$$

$$\boxed{I^D_{\vec{\nu}} = \frac{1}{\Gamma(\delta/2)} \int_0^\infty \mathrm{d}\eta\, \eta^{\delta/2-1} I^{D_0}_{\vec{\nu}}(\eta)}$$

- **Very efficient: any number of legs, any spacetime dimension**





Lessons: 2017-now

- **Reduction using asymptotic is very hard**
  - May need to expand to about 100 orders
  - Suppose we have n integration variables, the number of terms is about 10^{2n}
  - For cutting-edge problems, $n > 10$
- **One-loop and one-loop-like FIs can be perfectly computed using DCT**
  - So what?

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Iterative computation of MIs

Feynman parameter integration through differential equations

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[2] *Theoretical Physics Department, CERN, 1211 Geneva, Switzerland*

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ABSTRACT: We present a new method for numerically computing generic multi-loop Feynman integrals. The method relies on an iterative application of Feynman's trick for combining two propagators. Each application of Feynman's trick introduces a simplified Feynman integral topology which depends on a Feynman parameter that should be integrated over. For each integral family, we set up a system of differential equations which we solve in terms of a piecewise collection of generalized series expansions in the Feynman parameter. These generalized series expansions can be efficiently integrated term by term, and segment by segment. This approach leads to a fully algorithmic method for computing Feynman integrals from differential equations, which does not require the manual determination of boundary conditions. Furthermore, the most complicated topology that appears in the method often has less master integrals than the original one. We illustrate the strength of our method with a five-point two-loop integral family.

$$\frac{1}{D_i^{\nu_i} D_j^{\nu_j}} = \frac{\Gamma(\nu_i+\nu_j)}{\Gamma(\nu_i)\Gamma(\nu_j)} \int_0^1 dx \, \frac{x^{\nu_i-1}(1-x)^{\nu_j-1}}{(D_i x + D_j(1-x))^{\nu_i+\nu_j}}$$

The integrand of x is easier to compute because there is one less denominator

Iterative reduction

关鑫、黄礼鸿： **2022.07-2023.08**

- **Integrate out parameters one by one**
- **Reduce integrand to corresponding MIs at each step**

$$J(\vec{\nu}; D) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\Gamma(\nu_1)\cdots\Gamma(\nu_K)} \int \prod_{i=1}^{K} (x_i^{\nu_i - 1} \mathrm{d}x_i) \delta(1 - X) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$

Finding:

Too many variables: hard to reconstruct complex intermediate expression

Lessons after 7-year study

- Reduction is very hard, no matter using any method, the reason: too many integration variables

  - IBP
  - Asymptotic expansion
  - Iterative reduction
  - Intersection number

困难守恒原理

No precision, no New Physics!!!

没有未来了。

A possible simplification?

End of 2023?

- **Feynman parametrization**

$$J(\vec{\nu}; D) = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\Gamma(\nu_1)\cdots\Gamma(\nu_K)} \int \prod_{i=1}^{K} (x_i^{\nu_i - 1} \mathrm{d}x_i) \delta(1 - X) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$

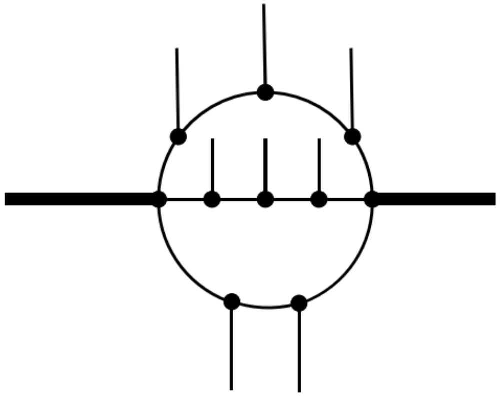
- U: degree L in the Feynman parameters x_i
- F: degree $L+1$

$$\mathcal{U} = \sum_{T \in T(G)} \prod_{e_i \notin T} x_i$$

- **Will things be simpler if we fix U unintegrated?**

$$J(\vec{\nu}; D) = \int [\mathrm{d}\mathbf{X}] \prod_{a=1}^{B} X_a^{\nu_a - 1} \mathcal{U}^{\nu - \frac{(L+1)D}{2}} I_{\vec{\nu}}^{\frac{LD}{2}}(\vec{X})$$

X_a: the summation of Feynman parameter for the a-th branch



Surprise!

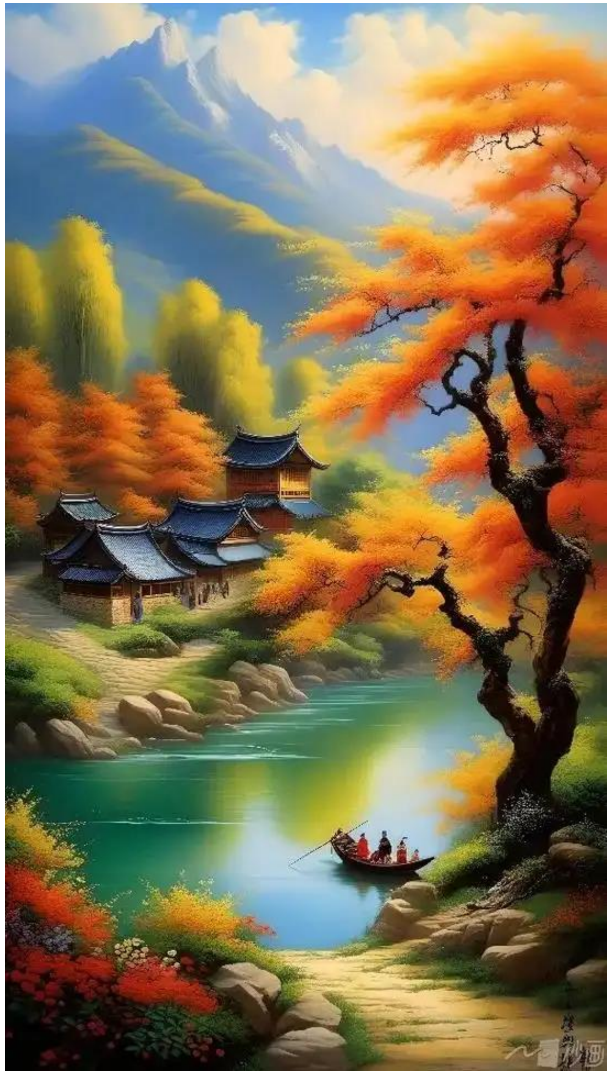
黄礼鸿、黄瑞钧： **2024.04-now**

➢ **The integrand is as simple as one-loop FIs!**

$$J(\vec{\nu};D)=\int[\mathrm{d}\mathbf{X}]\prod_{a=1}^{B}X_a^{\nu_a-1}\mathcal{U}^{\nu-\frac{(L+1)D}{2}}I_{\vec{\nu}}^{\frac{LD}{2}}(\vec{X})$$

A new representation

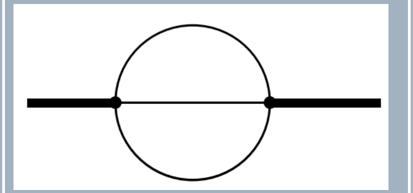
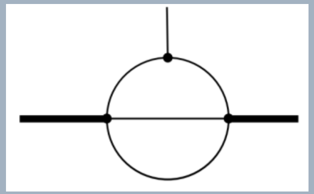
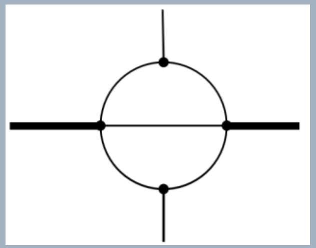
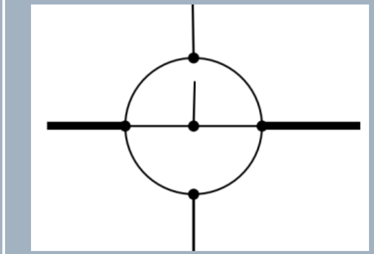
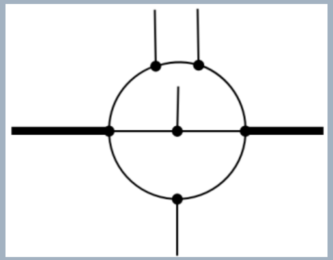
- Because F is then degree 2
- Integrand can be computed using DCT



➢ **Much less unintegrated parameters!**

- 2 loops: $B=2$
- 3 loops: $B=5$

Example: compute MIs

					
DCT/pt (ms)	0.12	0.19	0.47	2.73	7.89
Total time (CPU)	<1min	<1min	<3min	3min	12h

- **A very very preliminary implementation**
- **Much more to improve**

Summary and outlook

- Find novel structure of FIs: one-loop-like integrals followed by integration over a few variables

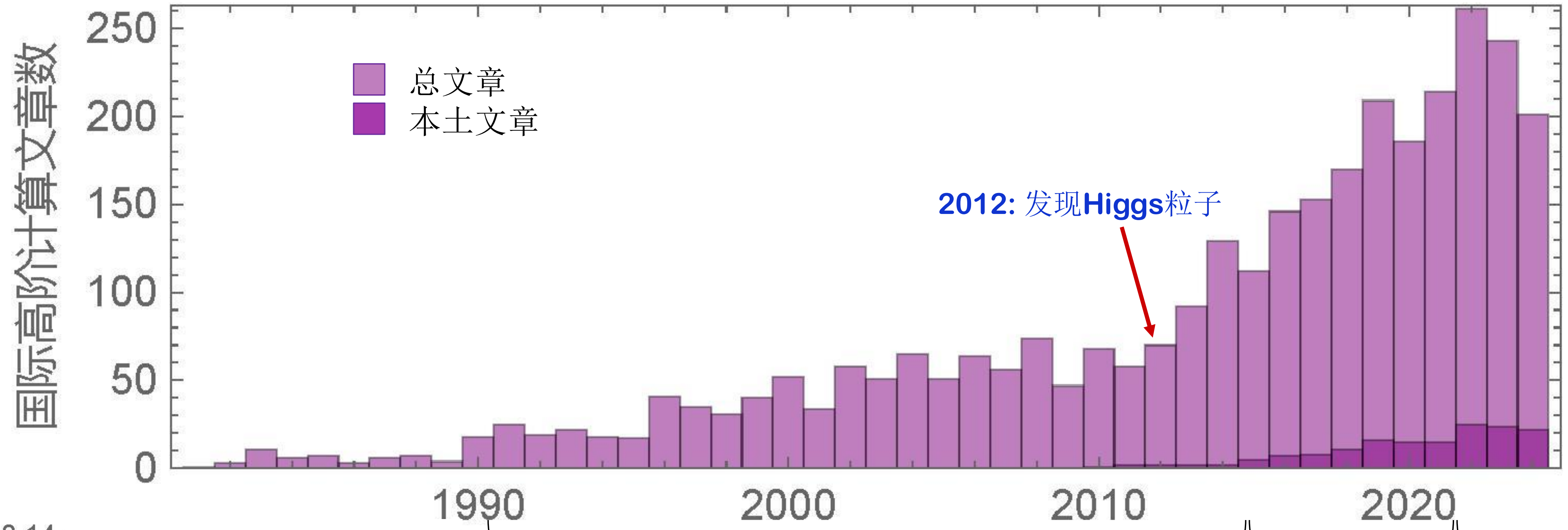
- All previous FIs techniques can be applied to the novel representation

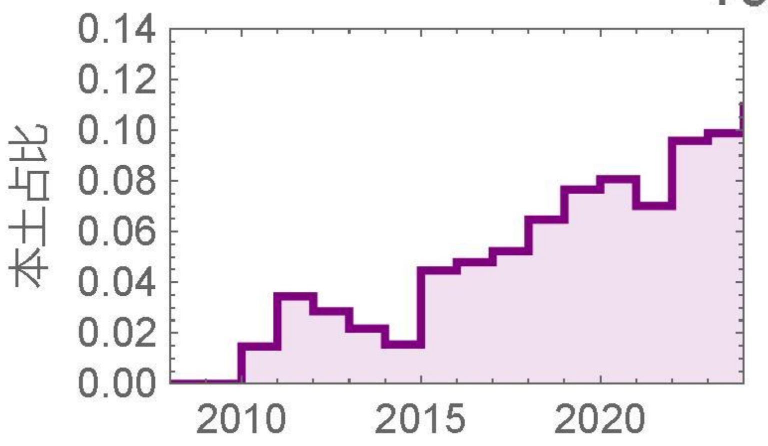
- Optimistic to overcome multiloop multileg FIs computation, and to meet the requirement of high-precision LHC data

- Stay tuned

Thank you !

High-order community





1990-2014 萌芽期：吴丹迪、郑汉青、廖益、李新强、冯波、李重生团队、乔从丰团队、贾宇团队等

2015-2021 追赶期：大量回国年轻人及本土培养年轻人加入

2022-202x并跑期：产生多个国际影响力工作