第四届量子场论及其应用研讨会

Bootstrap SU(3) YM theory on Lattice

杨刚

中国科学院理论物理研究所

W.I.P with Yuanhong Guo, Zeyu Li, Guorui Zhu

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What is bootstrap

Difficulties:

- Do not know fundamental Lagrangian
- Do not know how to do computation (e.g., non-perturbative)
- Known methods too complicated

Universal physical properties → Bootstrap → **Constrain physical results directly**

What is bootstrap

字面意思：
拔靴带，
靴袢

What is bootstrap

“Pull yourself up by your bootstraps”

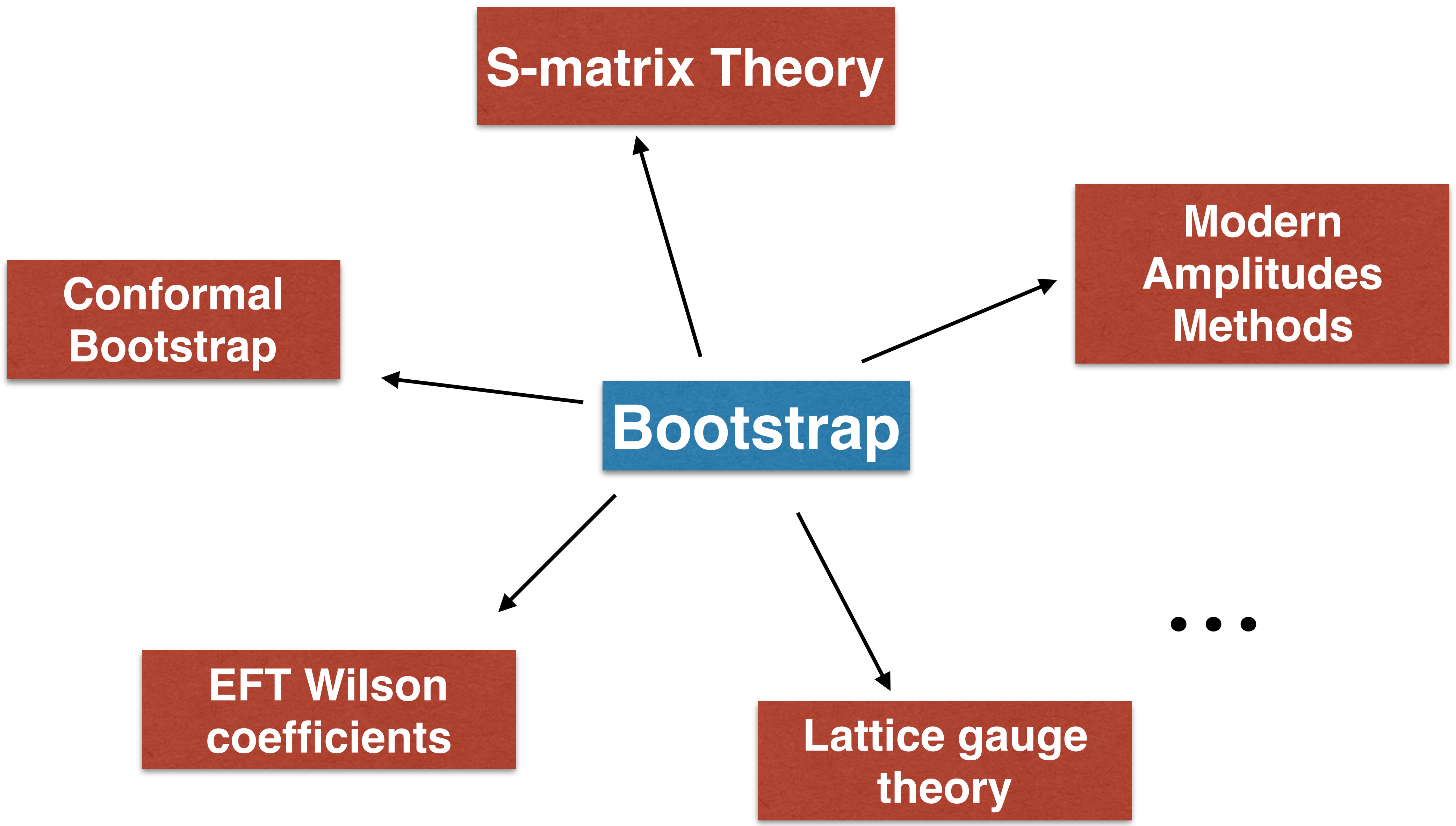
The Surprising Adventures of
Baron Munchausen, (1781)

What is bootstrap

中国武侠轻功 “梯云纵”：

陆小凤更吃惊，脚尖点地，身子立刻蹿起。大殿上的横梁离地十丈。没有人能一掠十丈。他身子蹿起，**左足足尖在右足足背上一点**，竟施展出武林中久已绝传的“梯云纵”绝顶轻功，居然掠上了横梁。

Bootstrap


S-matrix Theory
Modern Amplitudes Methods
Conformal Bootstrap
Bootstrap
EFT Wilson coefficients
Lattice gauge theory
...

Bootstrap lattice gauge theory

First proposed by Anderson and Kruczenski in 2016, and later improved significantly by Kazakov and Zheng.

Anderson, Kruczenski, arXiv:1612.08140

Kazakov, Zheng, arXiv:2203.11360, 2404.16925

Bootstrap in the framework of lattice YM theory

- First principle computation: from fundamental action
- Universal: no special symmetry, and for general dimensions
- A method with "analytic" control

Basic strategy

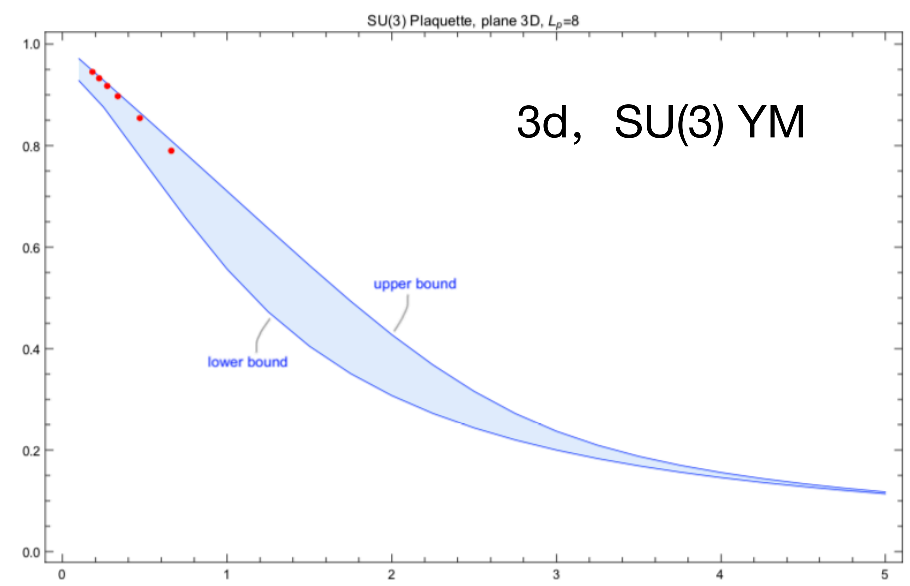
Positivity property

+

Schwinger-Dyson equations

→

Constrain the value of physical quantities



Main results

Previous studies focus on large N YM, or SU(2) YM, where only single-trace Wilson loops operators are necessary.

Anderson, Kruczenski, arXiv:1612.08140

Kazakov, Zheng, arXiv:2203.11360, 2404.16925

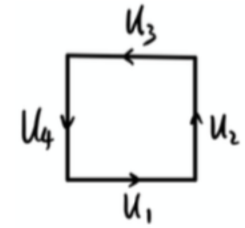
(Li, Zhou, arXiv:2024.17071 for U(1))

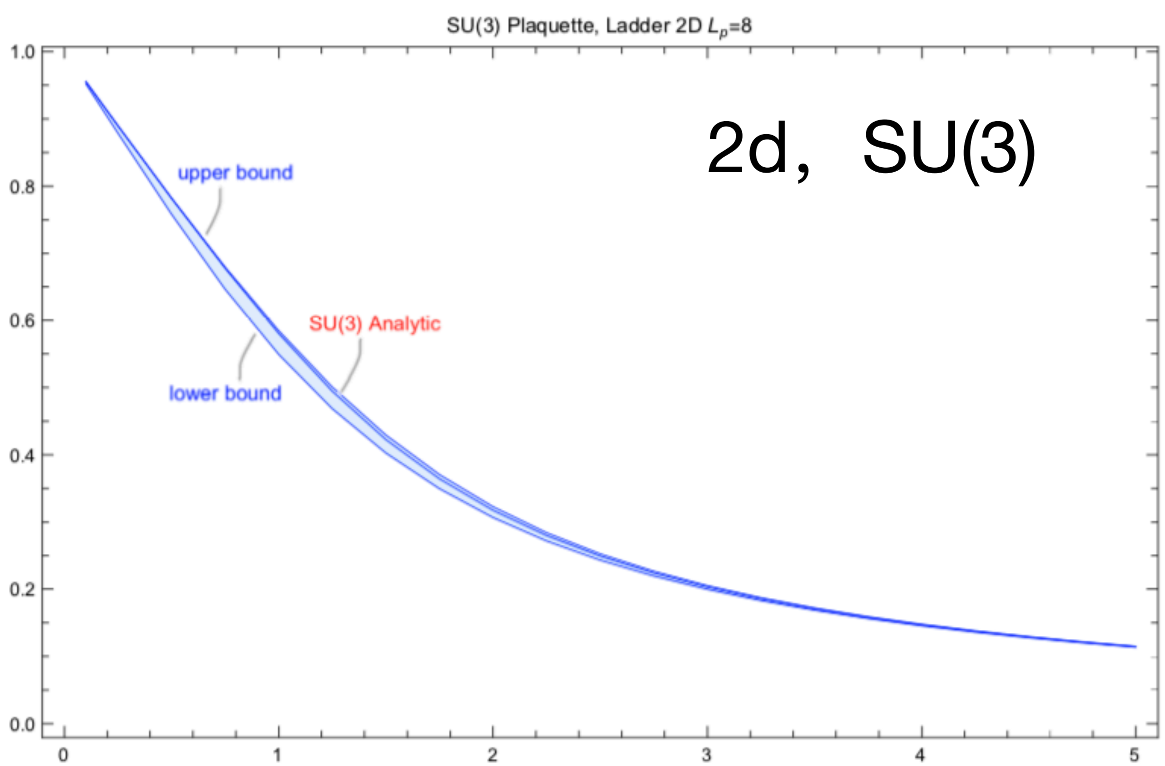
We consider for the first time the SU(3) case which involves single- and double-trace operators. We show that the method works as well as in the SU(2) case.

$$\mathrm{tr}U\mathrm{tr}V = \mathrm{tr}UV + \mathrm{tr}UV^\dagger, \quad \forall U, V \in SU(2)$$

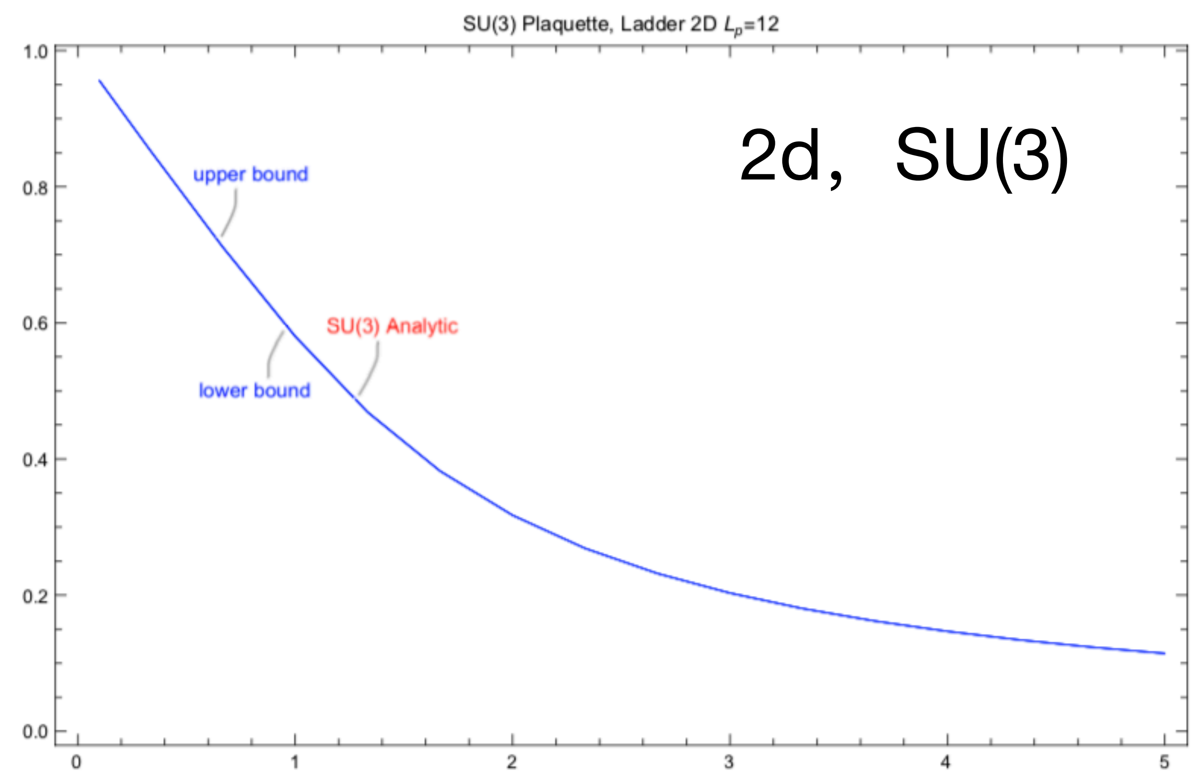
$$\mathrm{tr}U\mathrm{tr}V\mathrm{tr}W = -\mathrm{tr}WVU - \mathrm{tr}UVW + \mathrm{tr}U\mathrm{tr}VW + \mathrm{tr}UV\mathrm{tr}W + \mathrm{tr}UW\mathrm{tr}V + \mathrm{tr}U^\dagger V\mathrm{tr}U^\dagger W - \mathrm{tr}U^\dagger VU^\dagger W, \qquad \forall U, V, W \in SU(3)$$

Results： 2d

2d plaquette： $U_P = U_1U_2U_3U_4 =$ 



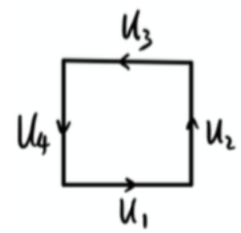
(a) SU(3), $L_{\text{path}} \leq 8$

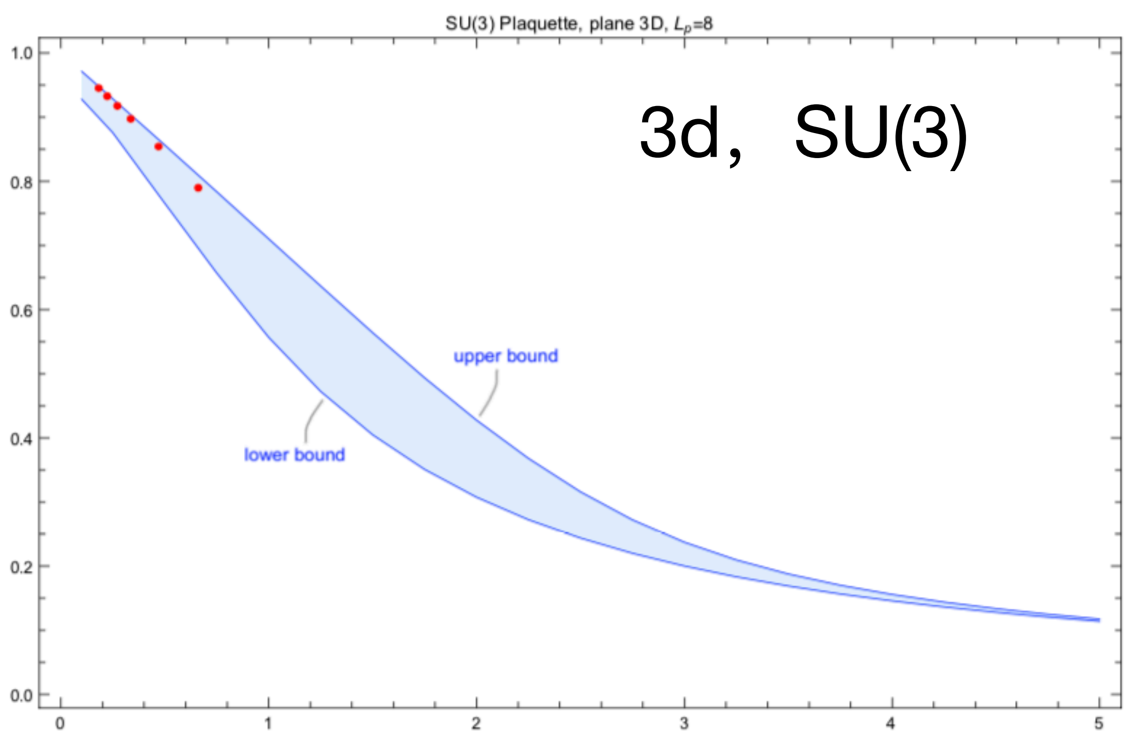


(b) SU(3), $L_{\text{path}} \leq 12$

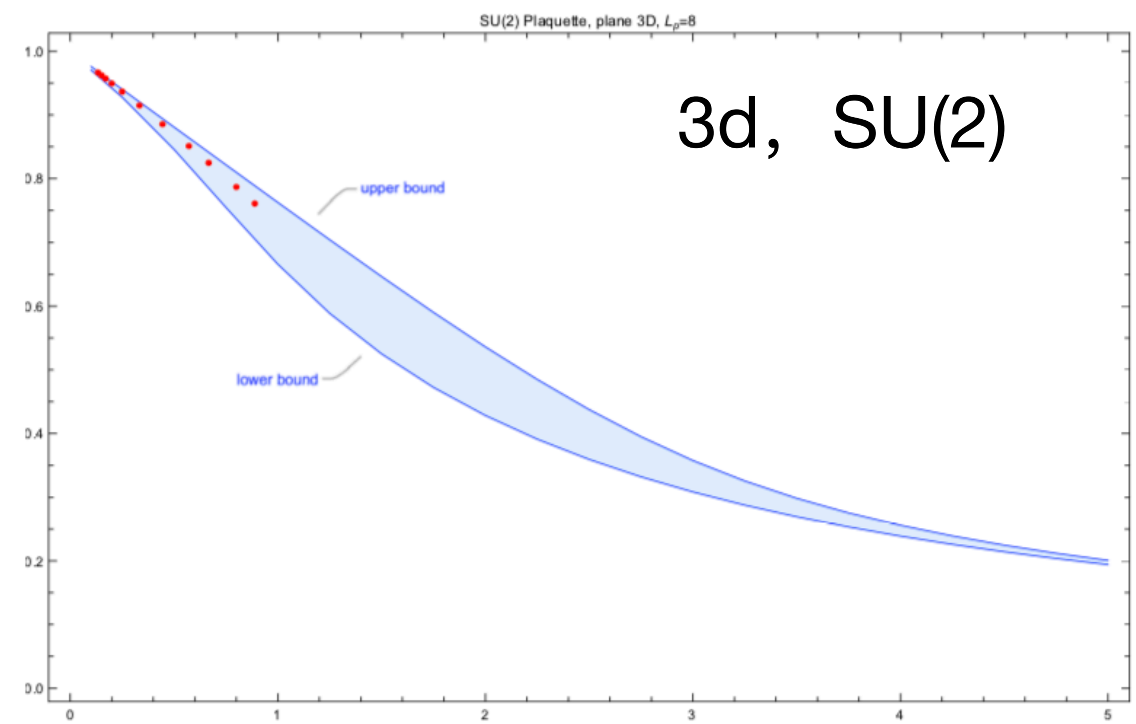
x-axis is coupling constant, y-axis is the expectation value of plaquette.

Results： 3d

3d plaquette： $U_P = U_1 U_2 U_3 U_4 =$ 



(a) SU(3), $L_{\text{path}} \leq 8$



(b) SU(2), $L_{\text{path}} \leq 8$

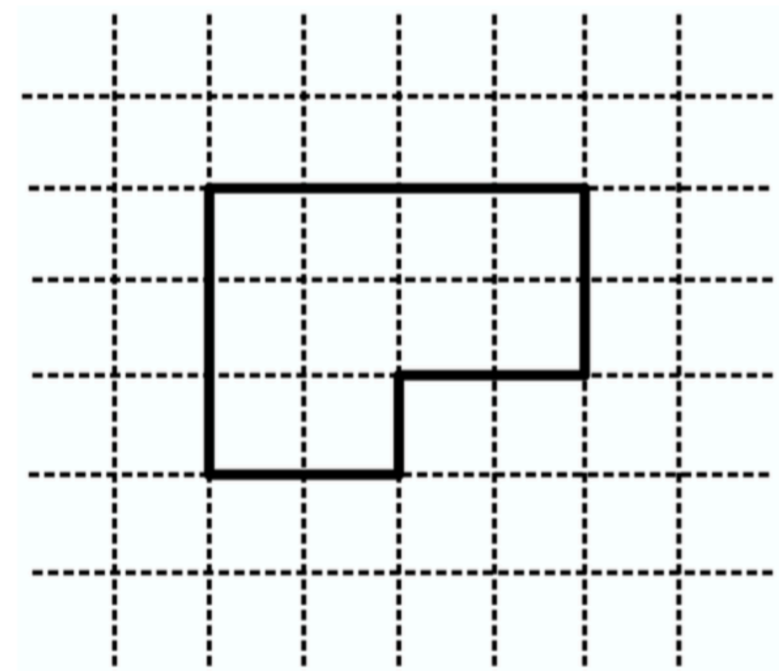
Outline

- Background
- Method
- Results
- Outlook

Outline

- Background
- **Method**
- Results
- Outlook

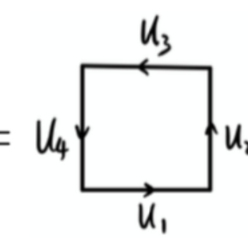
Setup of the theory



Action:

$$S = -\frac{N}{2\lambda}\sum_P \mathrm{tr}(U_P + U_P^\dagger)$$

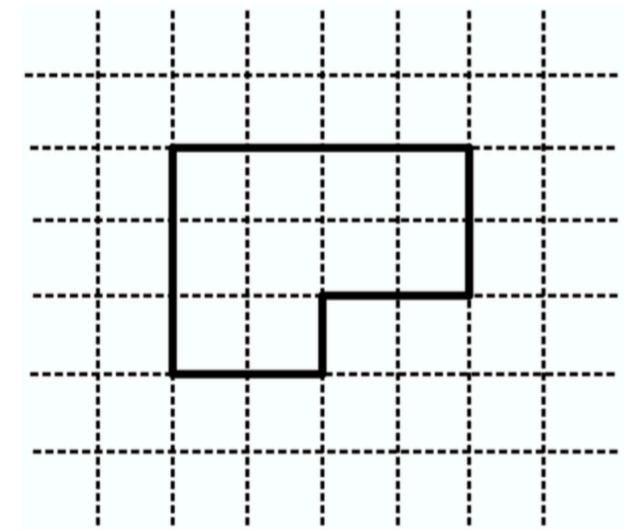
$$U_P = U_1 U_2 U_3 U_4 =$$



Partition function:

$$Z = \int \mathscr{D}U e^{-S} = \int \prod_{x,\hat{\mu}} dU_{\hat{\mu}}(x) e^{+\frac{N}{2\lambda}\sum_P \mathrm{tr}(U_P + U_P^\dagger)}$$

Physical quantity

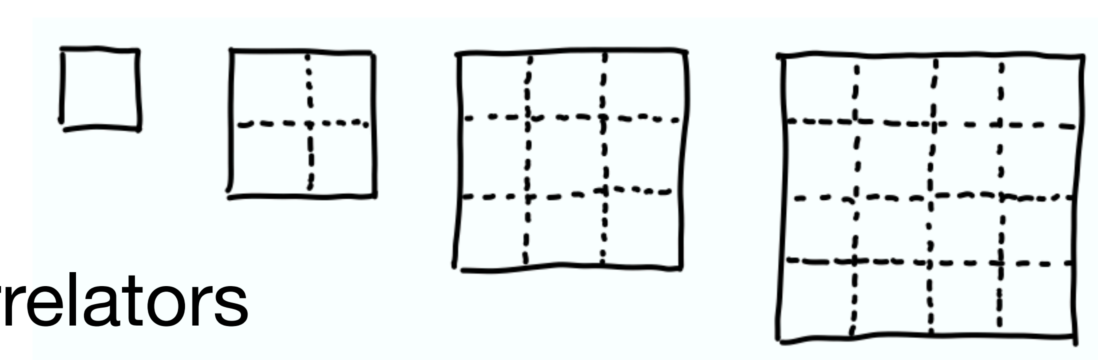


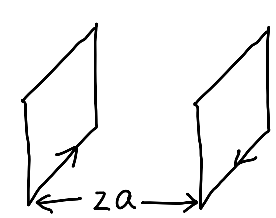
Expectation value:

$$\langle \mathcal{O} \rangle = Z^{-1} \int \mathcal{D}U e^{-S} \mathcal{O} = Z^{-1} \int \prod_{x,\hat{\mu}} dU_{\hat{\mu}}(x) e^{+\frac{N}{2\lambda} \sum_P \mathrm{tr}(U_P + U_P^\dagger)} \mathcal{O}$$

The fundamental quantities is the Wilson loops.

- Area law for confinement
- Glueball spectrum from loop correlators





Main strategy

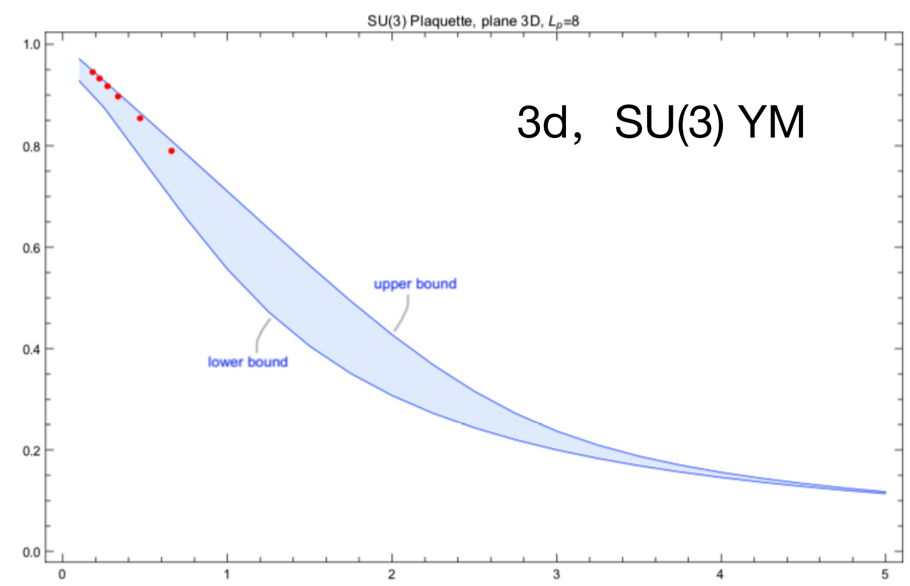
Fundamental positivity property

+

Schwinger-Dyson equations

→

Constrain the value of Wilson loop operators



Positivity

Consider operator: $\mathcal{O} = \sum_{i=1}^{N} \alpha_i o_i$

Positivity: $\langle \mathcal{O}\mathcal{O} \rangle = \sum_{i,j} \alpha_i \alpha_j \langle o_i o_j \rangle \geq 0$ For any α_i, α_j !

$w_{i,j} = \langle o_i o_j \rangle$ → $\left(w_{i,j} \right)_{N \times N} \succeq 0$

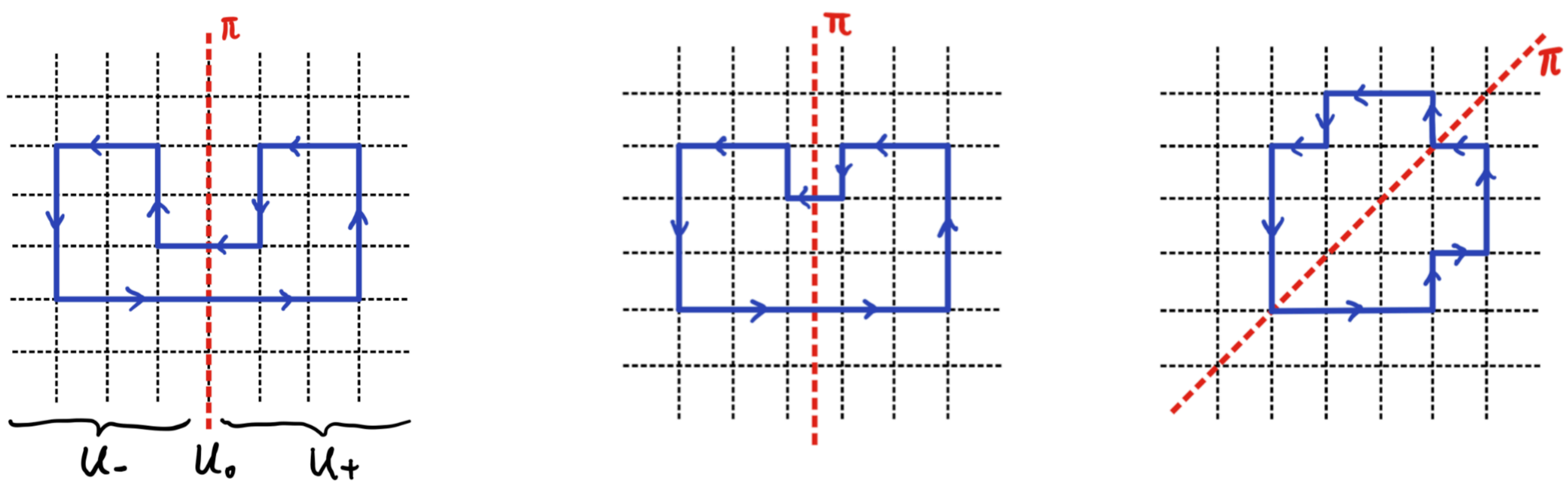
$$\begin{pmatrix} w_0 & w_1 & w_2 & w_3 & \cdots \\ w_1 & w_2 & w_3 & w_4 & \cdots \\ w_2 & w_3 & w_4 & w_5 & \cdots \\ w_3 & w_4 & w_5 & w_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \succeq 0 \quad \longleftrightarrow \quad \begin{vmatrix} w_0 & w_1 & w_2 & \cdots & w_n \\ w_1 & w_2 & w_3 & \cdots & w_{n+1} \\ w_2 & w_3 & w_4 & \cdots & w_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \\ w_n & w_{n+1} & w_{n+2} & \cdots & w_{2n} \end{vmatrix} \geq 0 \quad \forall n \geq 0$$

Positivity in QFT

Hermitian positivity: $\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0$ $\quad \int \mathcal{D}U \, e^{-S} \, |f(U)|^2 \geq 0$

Reflection positivity: **K. Osterwalder and R. Schrader**, "Axioms for Euclidean Green's functions", 1973

$\langle \mathcal{O}_+^R \mathcal{O}_+ \rangle \geq 0$ $\quad \left| \int \mathcal{D}U_+ \, e^{-S_+} f(U_+) \right|^2 \geq 0$

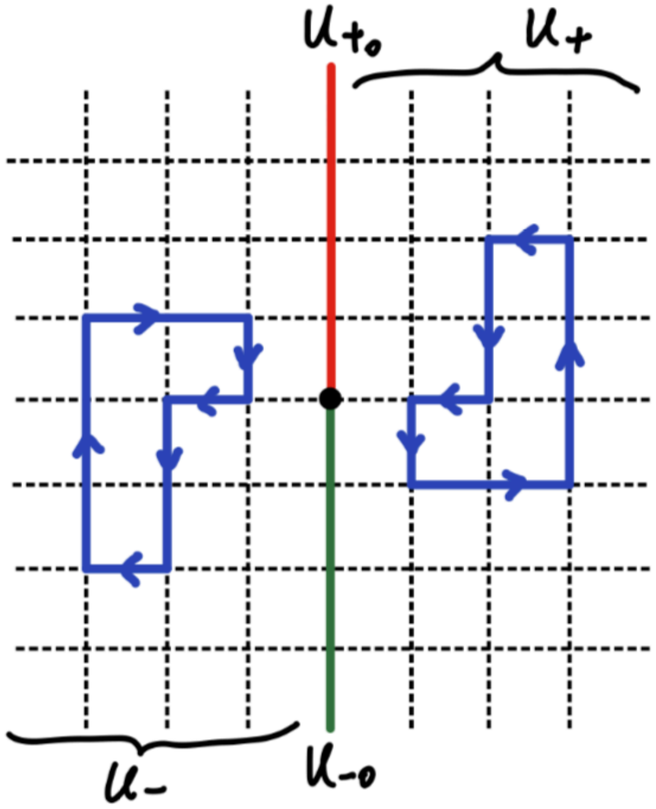


Positivity in QFT

Reflection positivity:

$$\langle \mathcal{O}_+^R \mathcal{O}_+ \rangle \geq 0$$

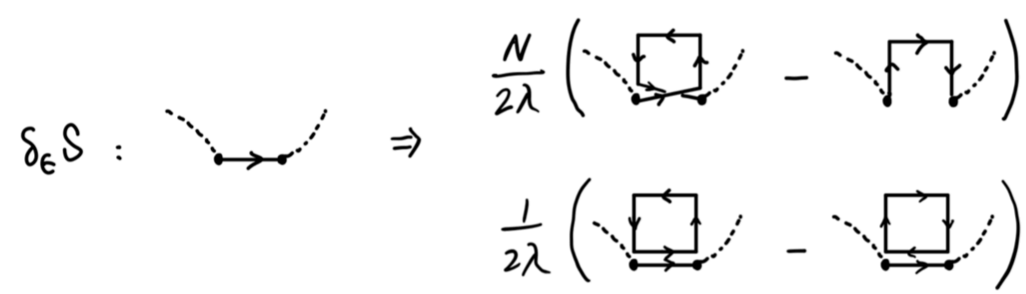
A twist version of Reflection positivity (twist-RP): exact in 2d lattice, or in continuum theory.



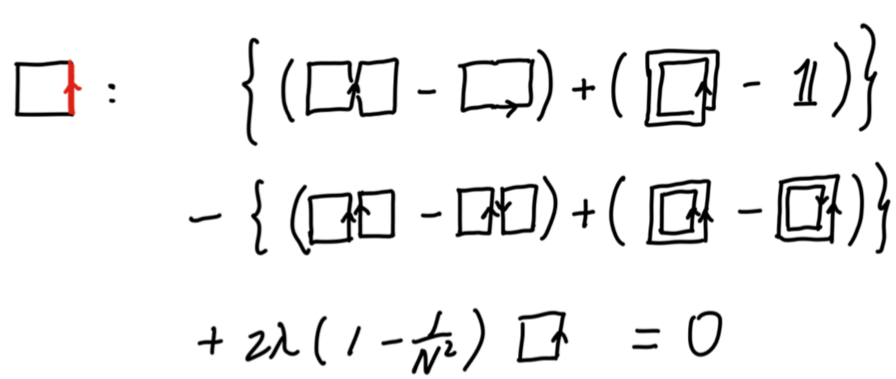
Loop equations

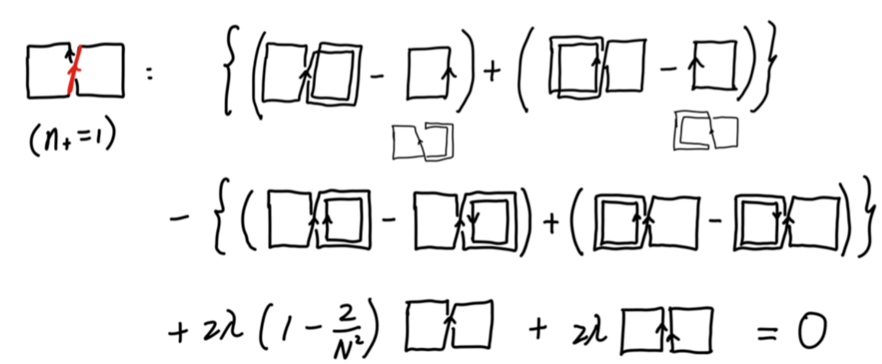
Schwinger-Dyson equation for Wilson loops:

$$\int \mathcal{D}U \, \delta_{\epsilon(\mu)} \left[e^{-S} \, W^{ab}_{\vec{x}}(\mu, C_\mu) \right] = 0 \longrightarrow - \langle W^{ab}_{\vec{x}} \, \delta_\epsilon S \rangle + \langle \delta_\epsilon W^{ab}_{\vec{x}} \rangle = 0$$



Examples:

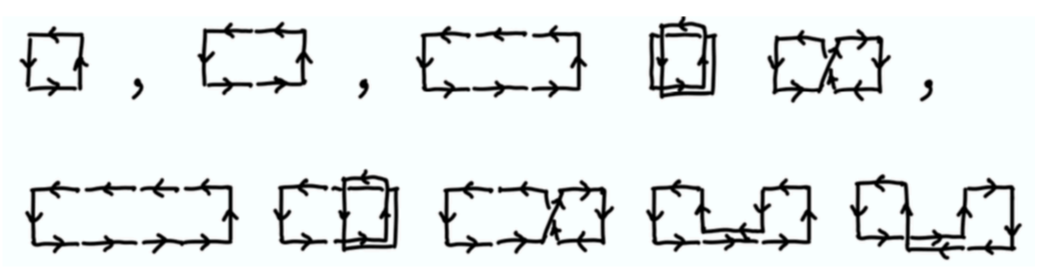




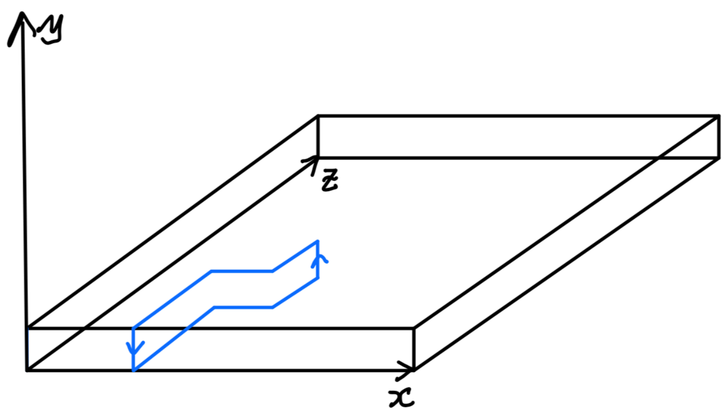
Dimension reduction

Consider operators that are in (d-1) sub-plane.

2d loops, extended in only one direction:



3d loops in a 2d sub-plane



Compare number of loops:

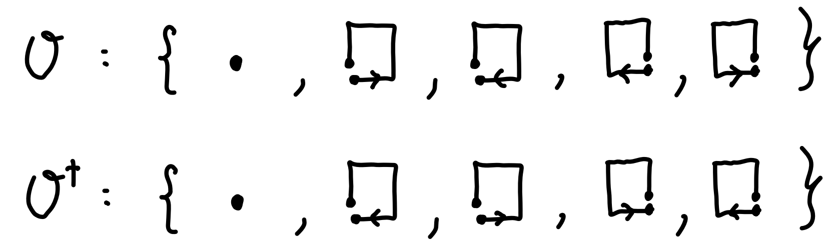
Length of loops	4	6	8	10	12	14	16	18	20
# of general loops	1	1	7	15	95	465	3,217	21,762	159,974
# of ladder-type loops	1	1	3	5	13	32	90	268	867

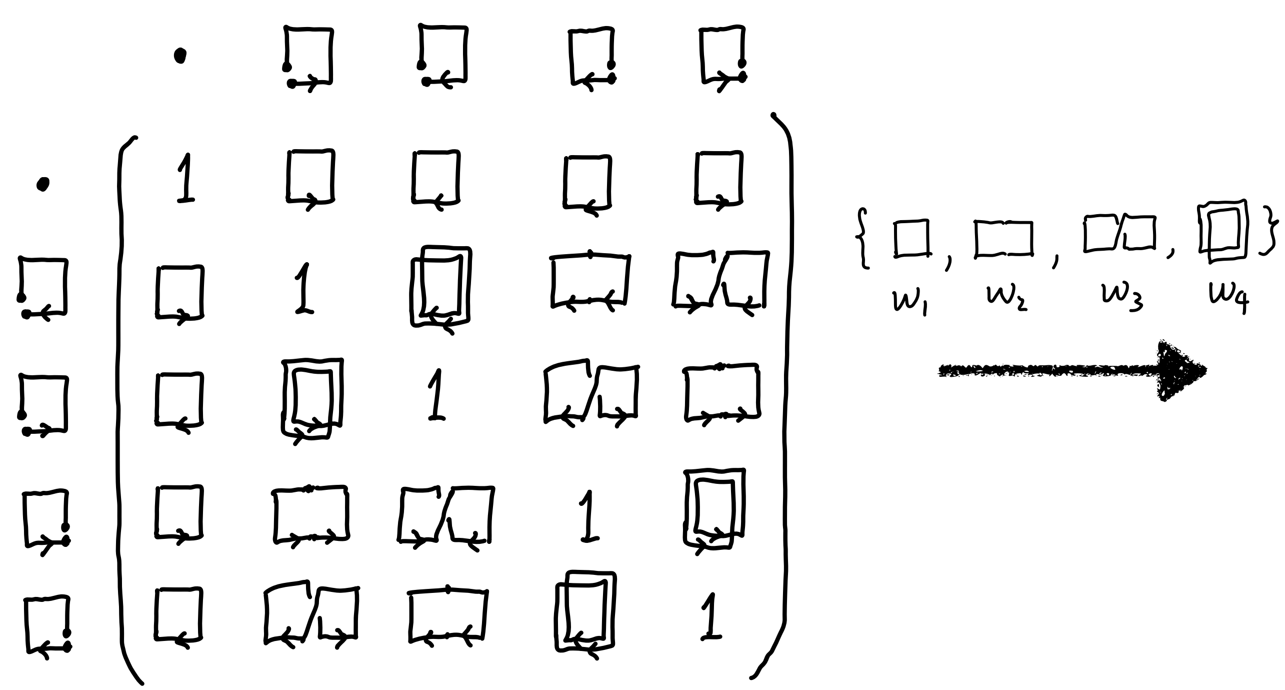
Question： does this reduction have good convergence? （Yes！）

An explicit example

Hermitian positivity:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0$$



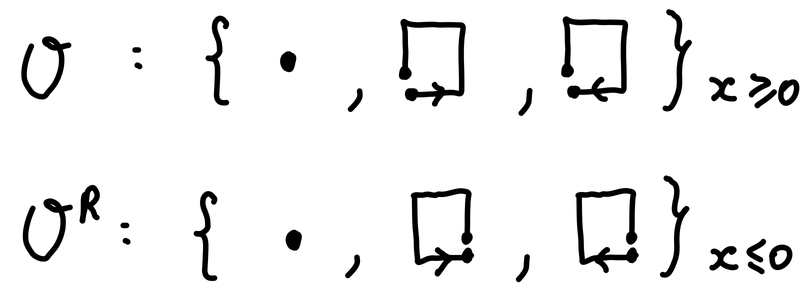


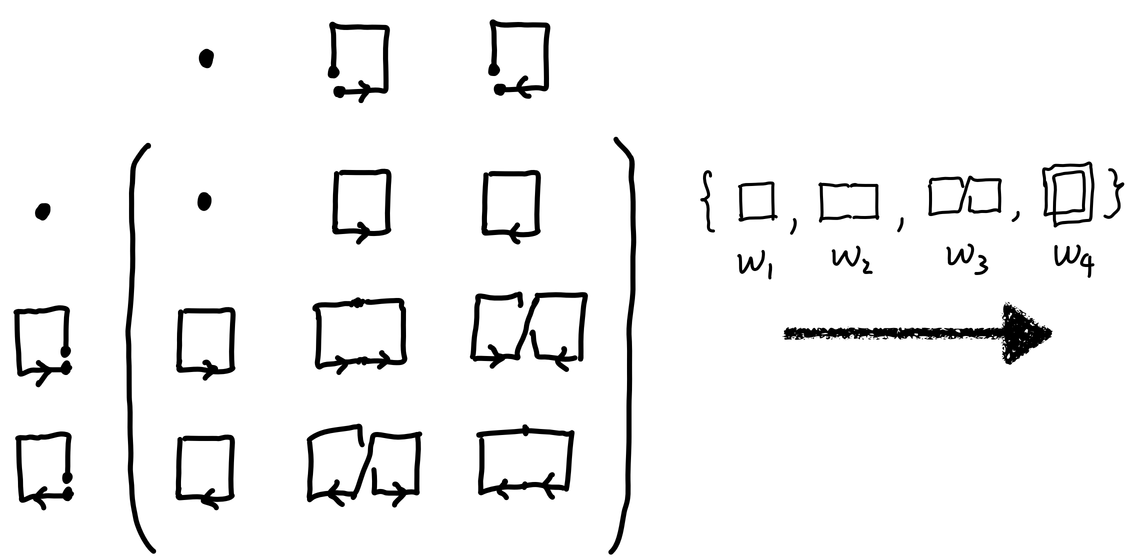
$$\begin{pmatrix} 1 & w_1 & w_1 & w_1 & w_1 \\ w_1 & 1 & w_4 & w_2 & w_3 \\ w_1 & w_4 & 1 & w_3 & w_2 \\ w_1 & w_2 & w_3 & 1 & w_4 \\ w_1 & w_3 & w_2 & w_4 & 1 \end{pmatrix} \succeq 0$$

An explicit example

Reflection positivity:

$$\langle \mathcal{O}_+^R \mathcal{O}_+ \rangle \geq 0$$

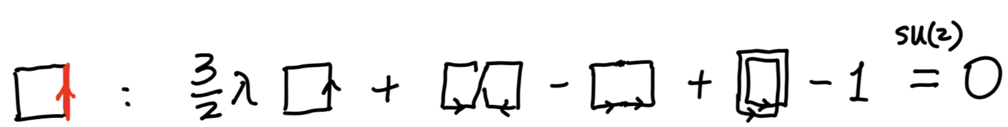




$$\begin{pmatrix} 1 & w_1 & w_1 \\ w_1 & w_2 & w_3 \\ w_1 & w_3 & w_2 \end{pmatrix} \succeq 0$$

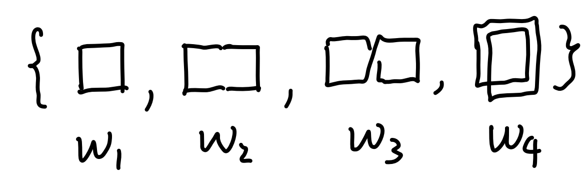
An explicit example

$$\begin{pmatrix} 1 & w_1 & w_1 & w_1 & w_1 \\ w_1 & 1 & w_4 & w_2 & w_3 \\ w_1 & w_4 & 1 & w_3 & w_2 \\ w_1 & w_2 & w_3 & 1 & w_4 \\ w_1 & w_3 & w_2 & w_4 & 1 \end{pmatrix} \succeq 0 \qquad \begin{pmatrix} 1 & w_1 & w_1 \\ w_1 & w_2 & w_3 \\ w_1 & w_3 & w_2 \end{pmatrix} \succeq 0$$



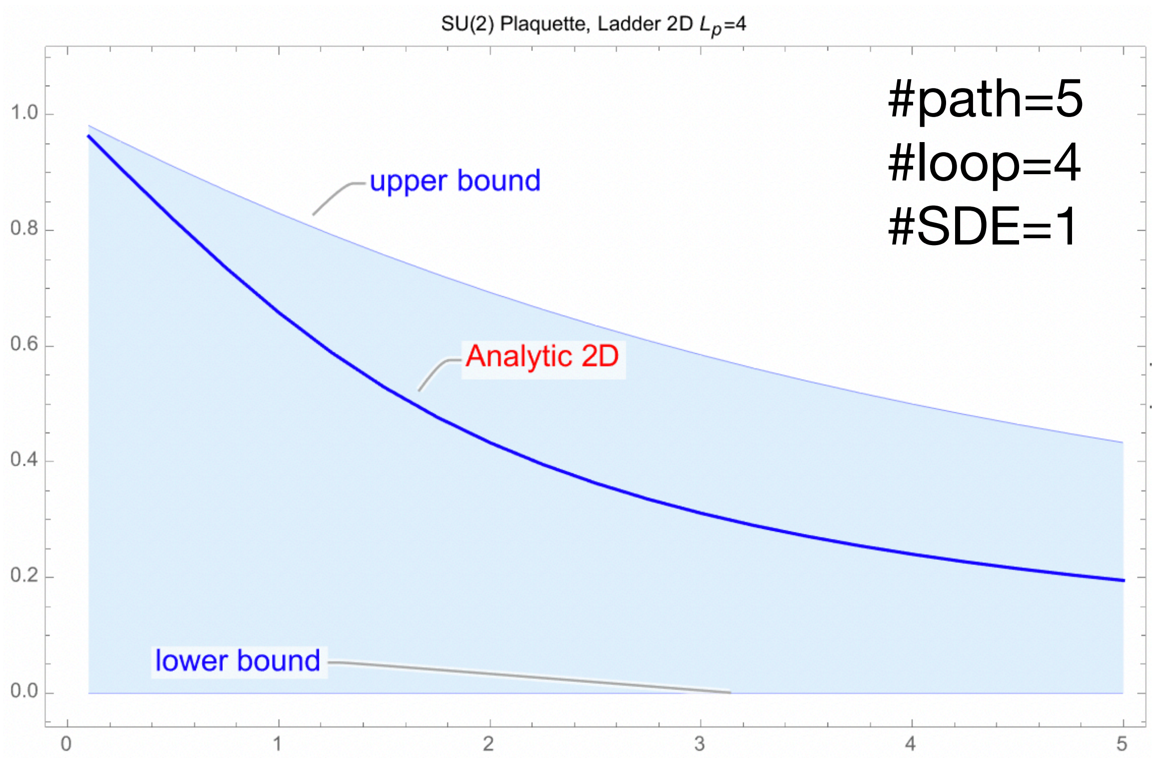
$$\frac{3}{2}\lambda w_1 + w_3 - w_2 + w_4 - 1 = 0$$

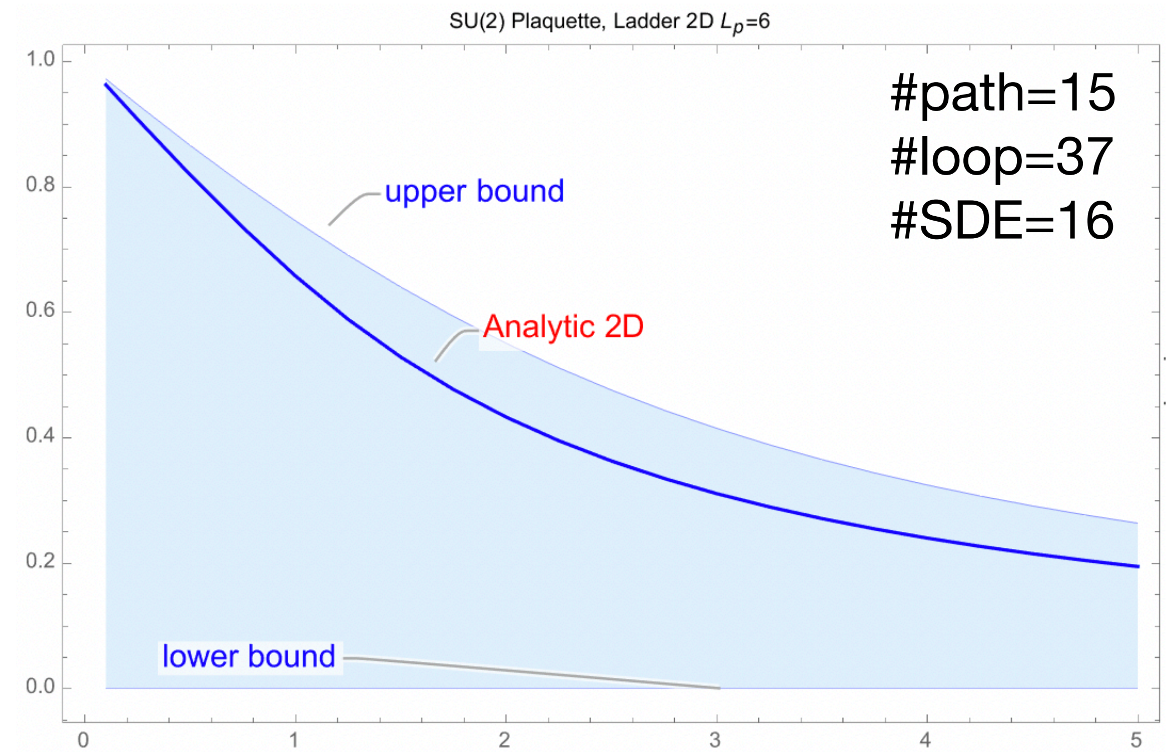
Constrain the value of Wilson loops



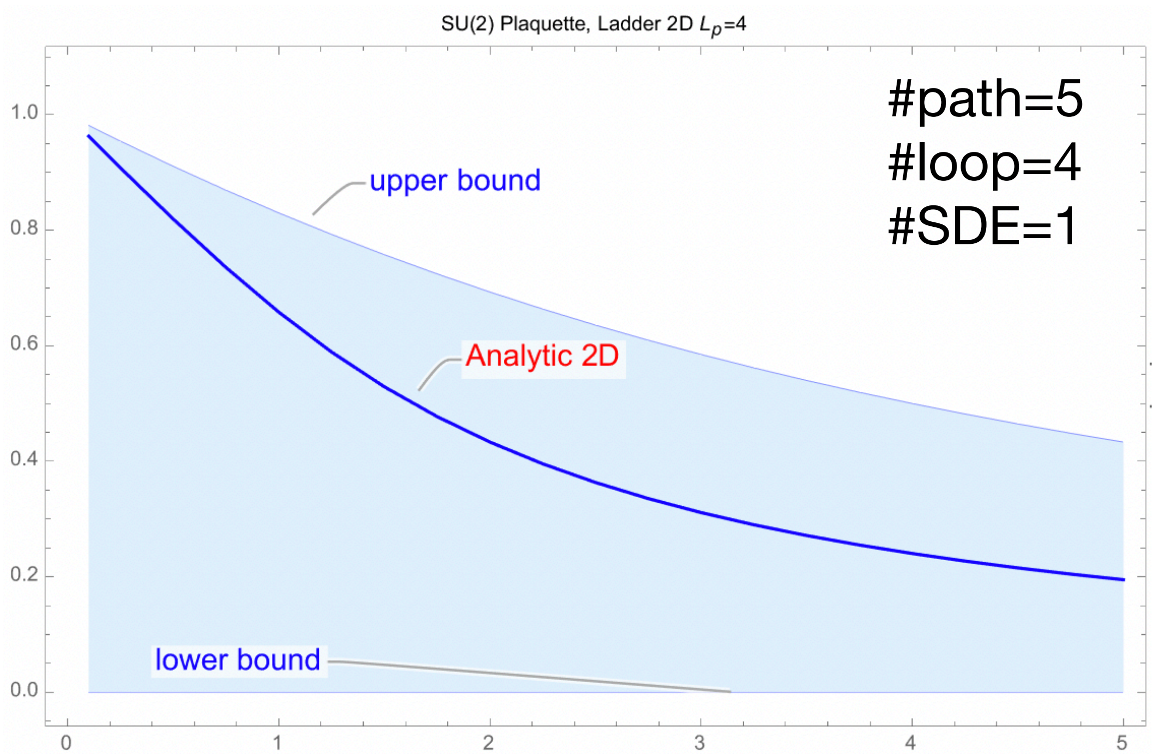
Here only 1 loop equation is related.

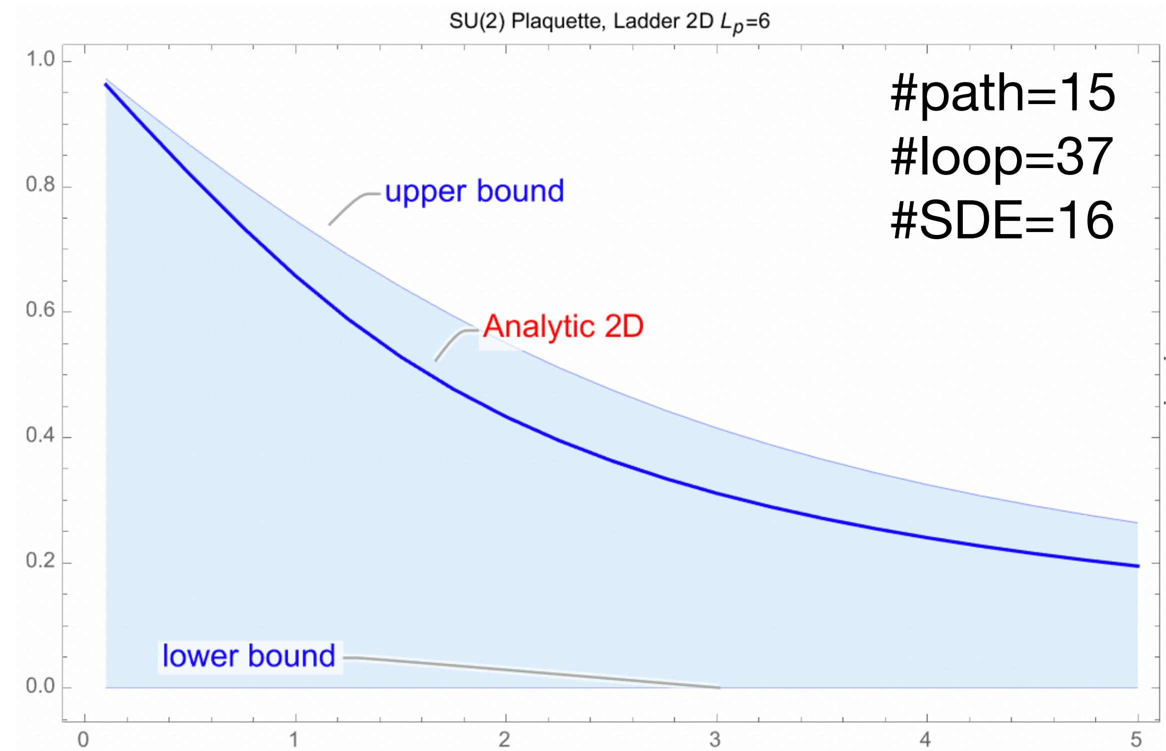
An explicit example

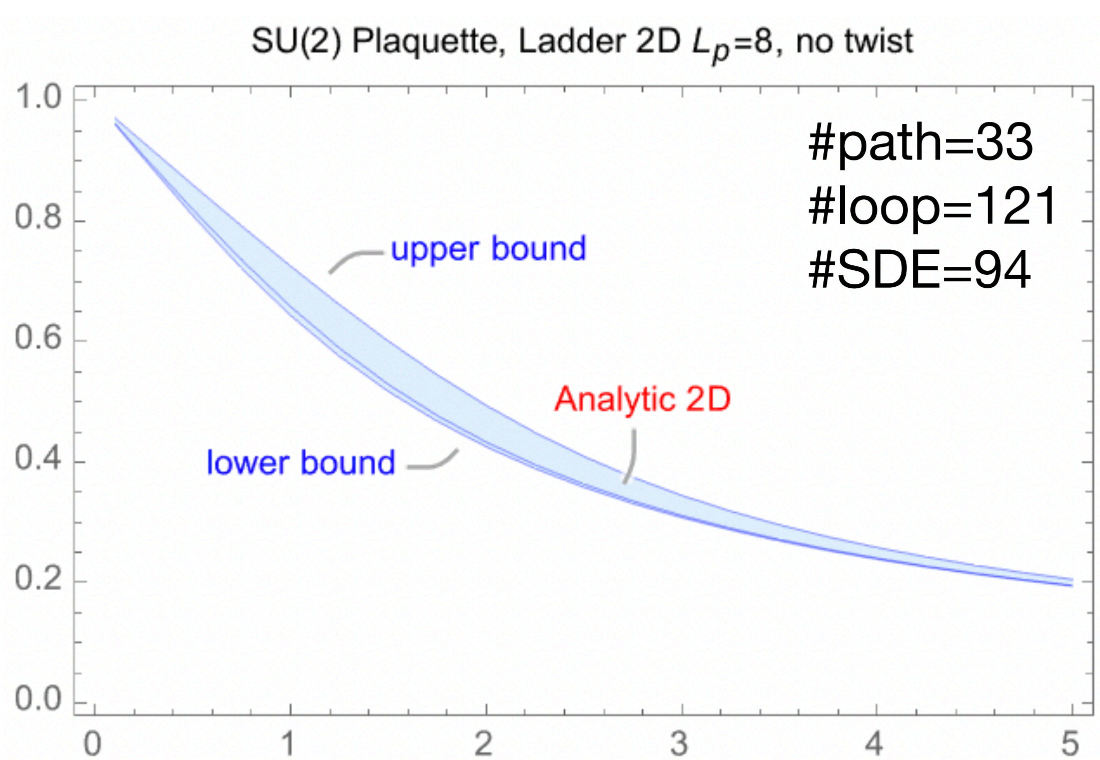


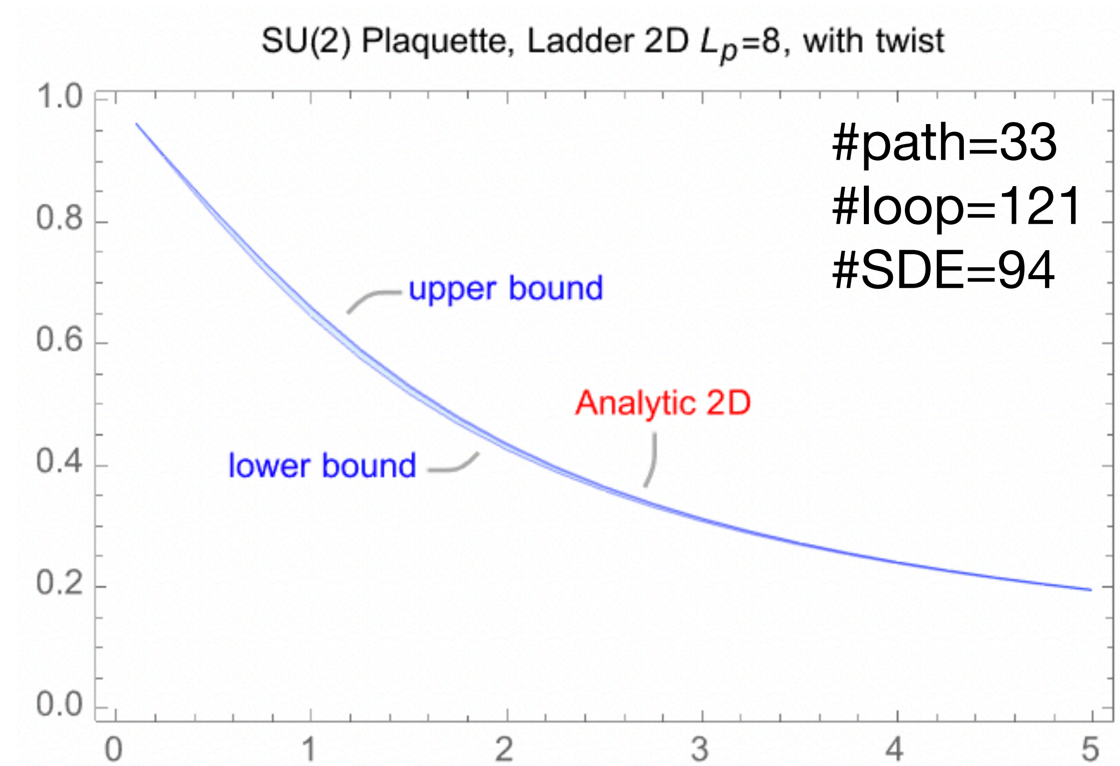


An explicit example









Semi-definite programming

A mathematical problem: "Semi-definite Programming"

- Simple problem: using Mathematica directly

```
SemidefiniteOptimization[w[1], {{(positiveMatrix) ≽_{S_+^{positiveMatrix//Length}} 0}, loopEquations}, loopVariables]
```

- Complicated problem: need specific software

MOSEK

Our solver is the answer for all your LPs, QPs, SOCPs, SDPs, and MIPs.
Includes interfaces to C, C++, Java, MATLAB, .NET, Python, Julia, Rust and R.

Outline

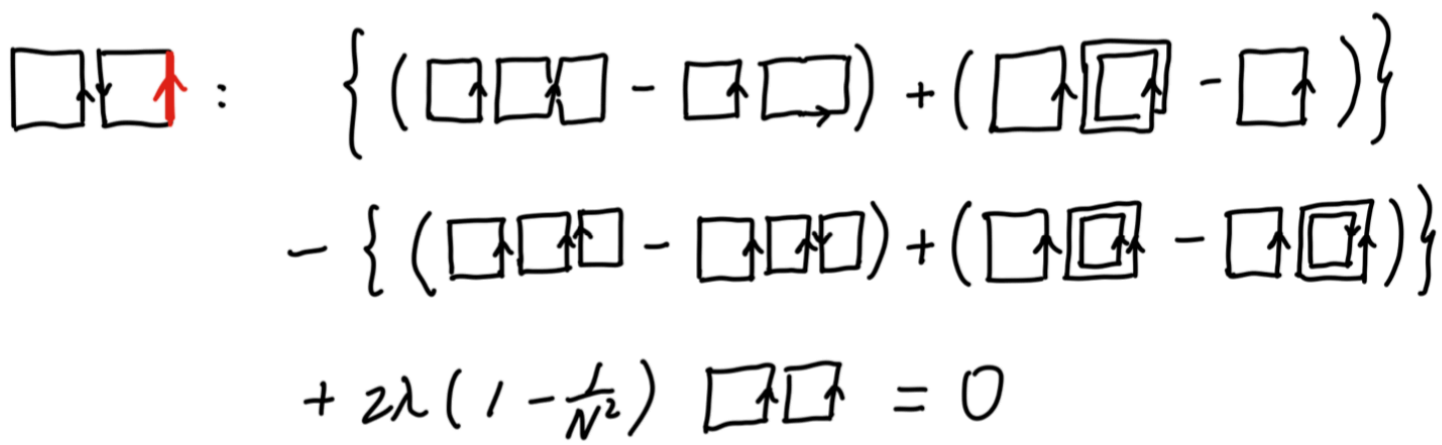
- Background
- Method
- Results: SU(3)
- Outlook

SU(3) YM theory

Need to consider multiple-trace operators.

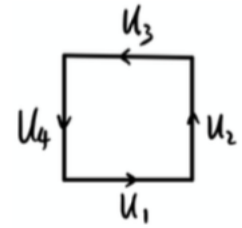
Number of loop variables increase significantly.

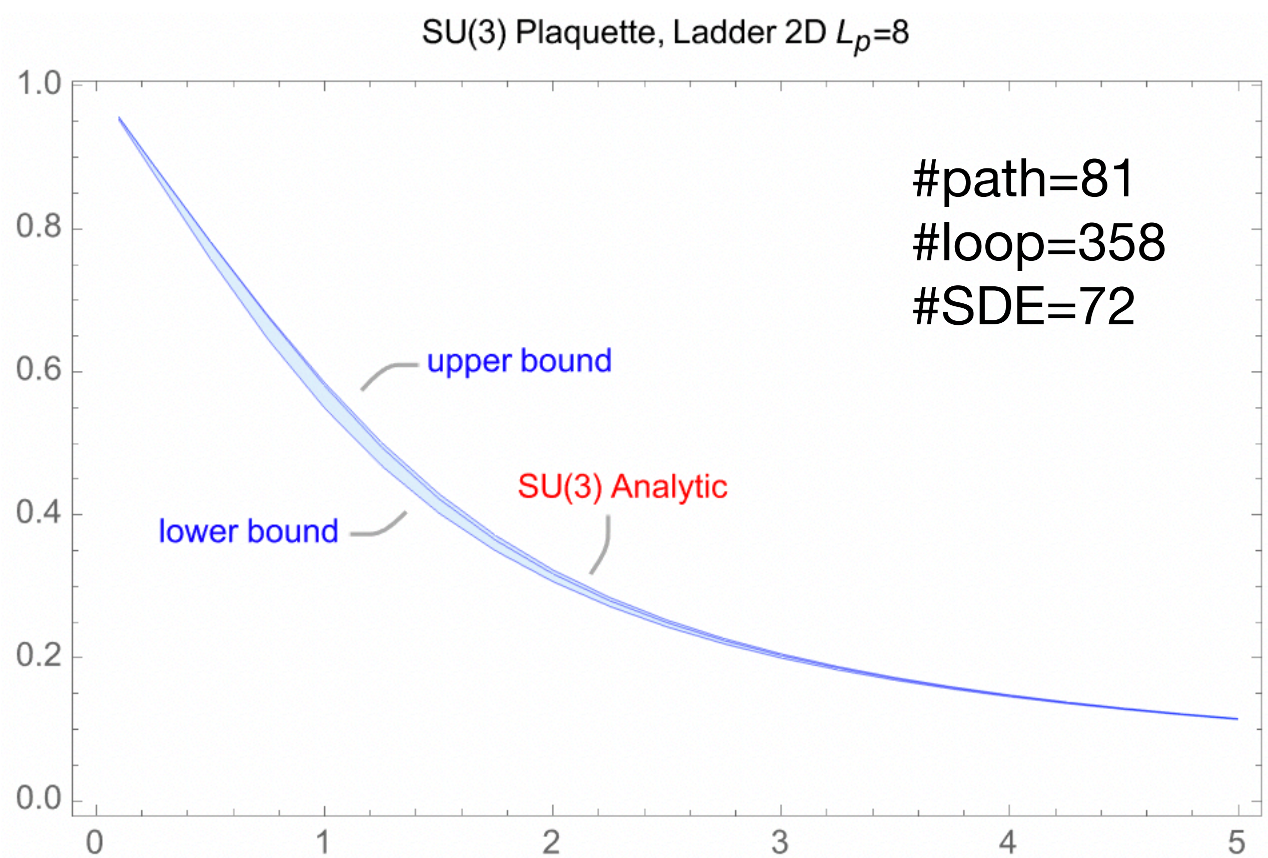
Example of SDE:

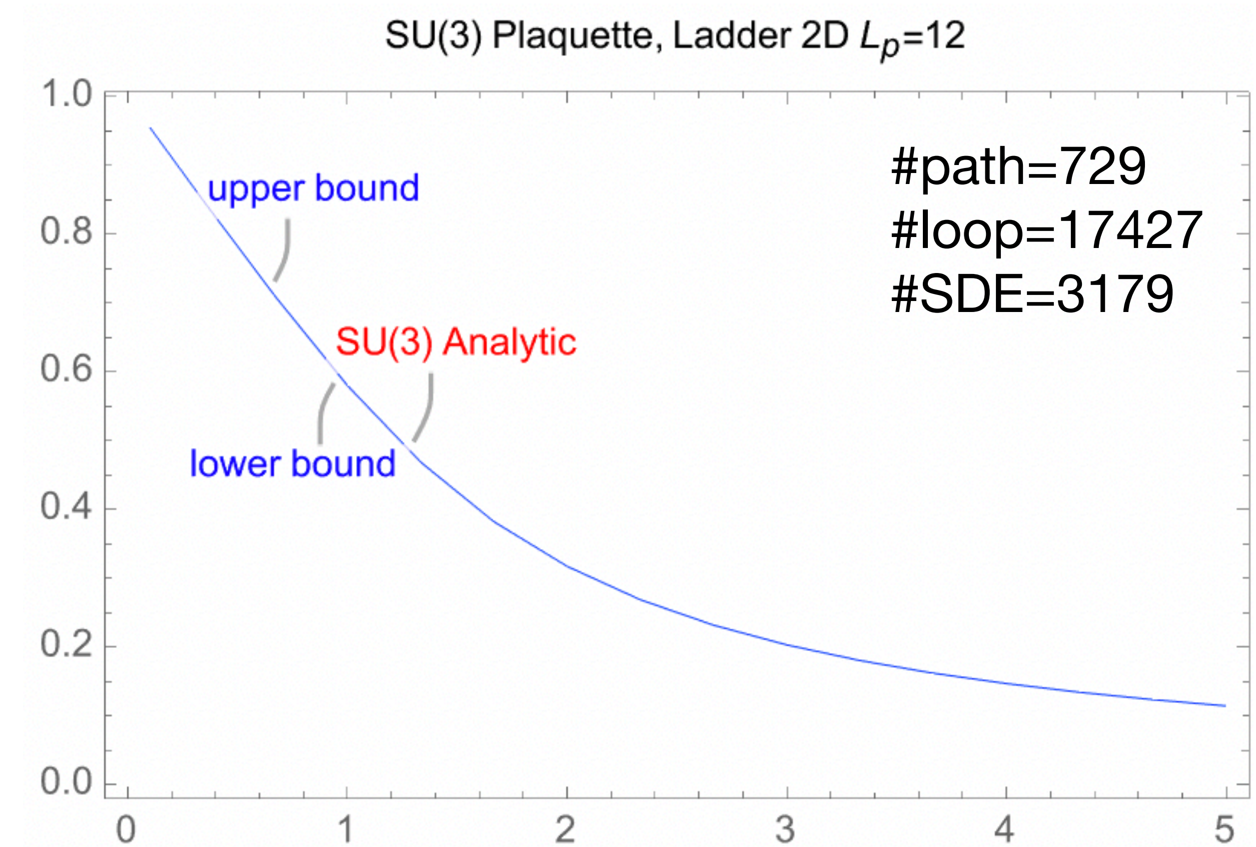


Good convergence?

Results： 2d SU(3)

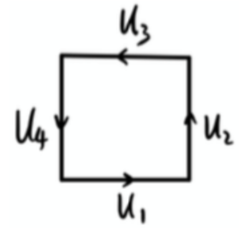
2d plaquette： $U_P = U_1 U_2 U_3 U_4 =$ 

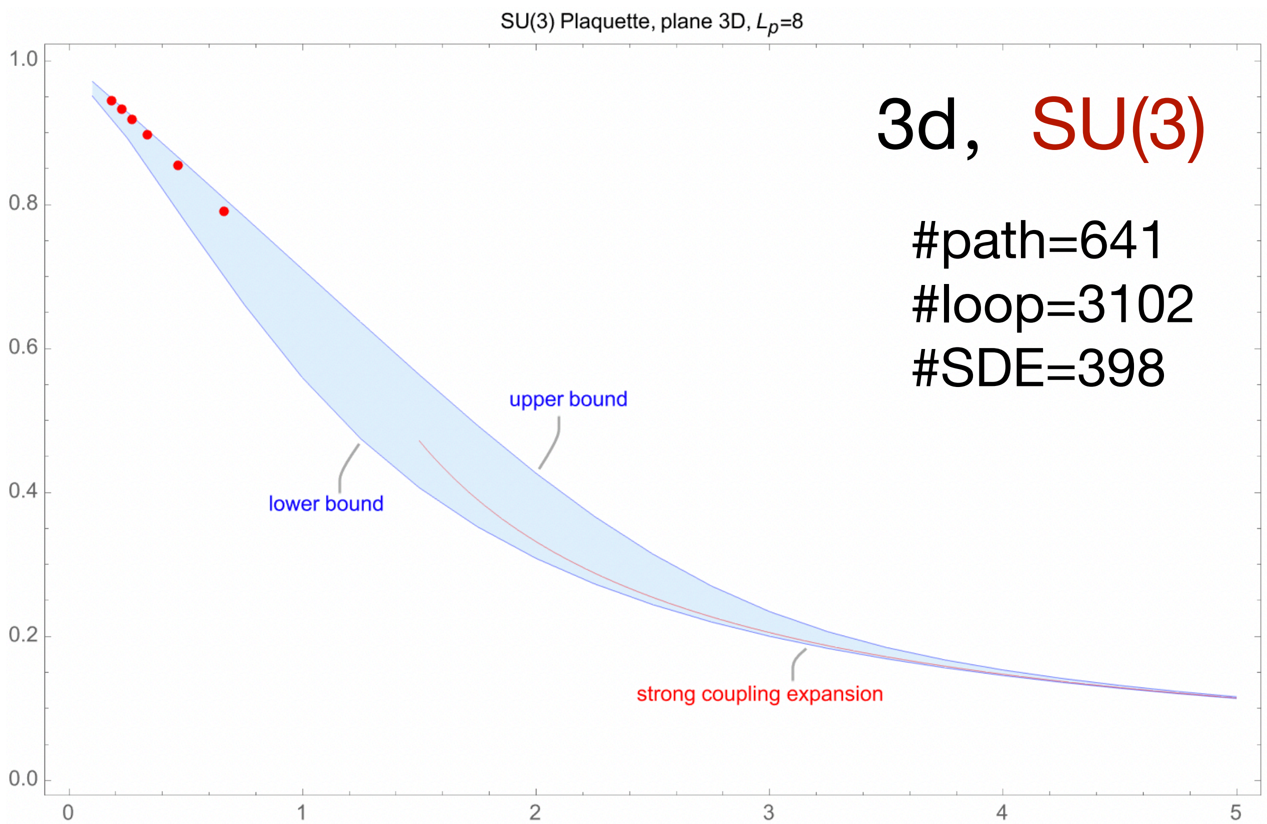


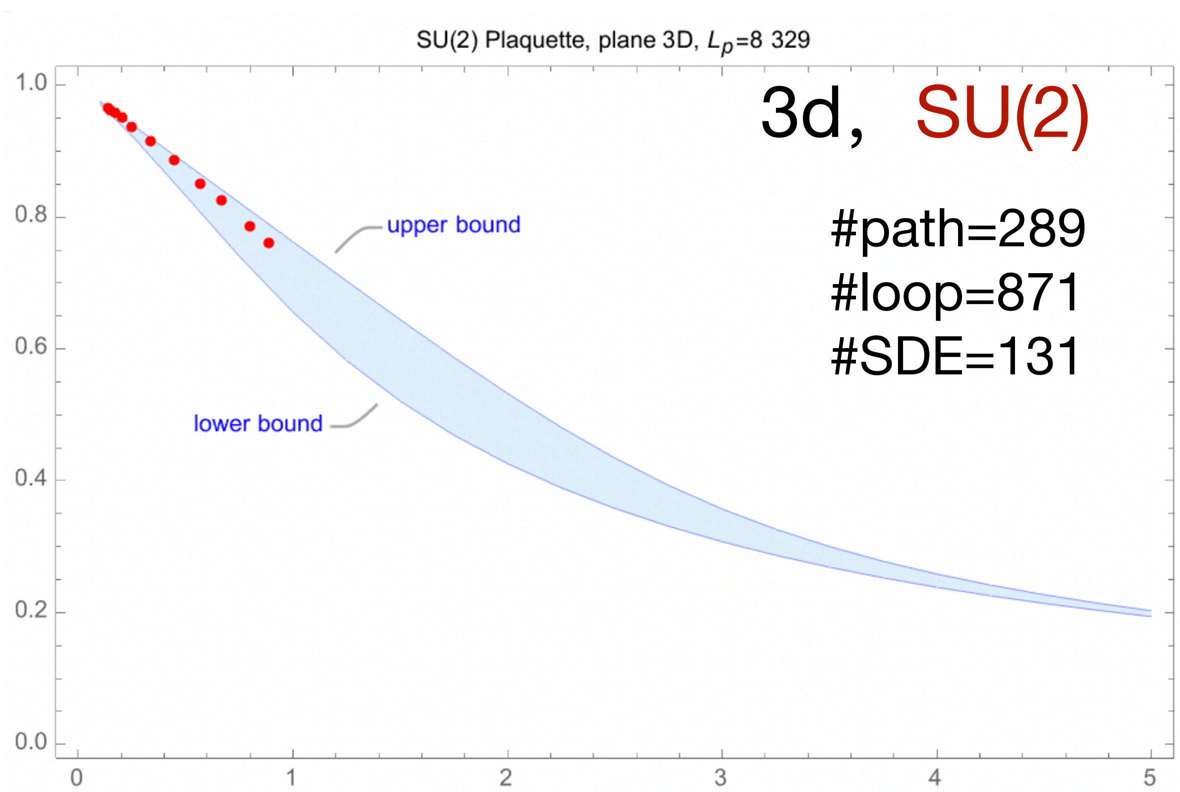


~ 3 digits precision

Results： 3d SU(3)

3d plaquette： $U_P = U_1 U_2 U_3 U_4 =$ 

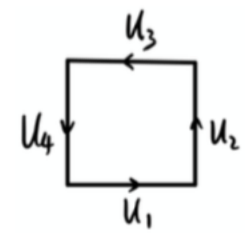


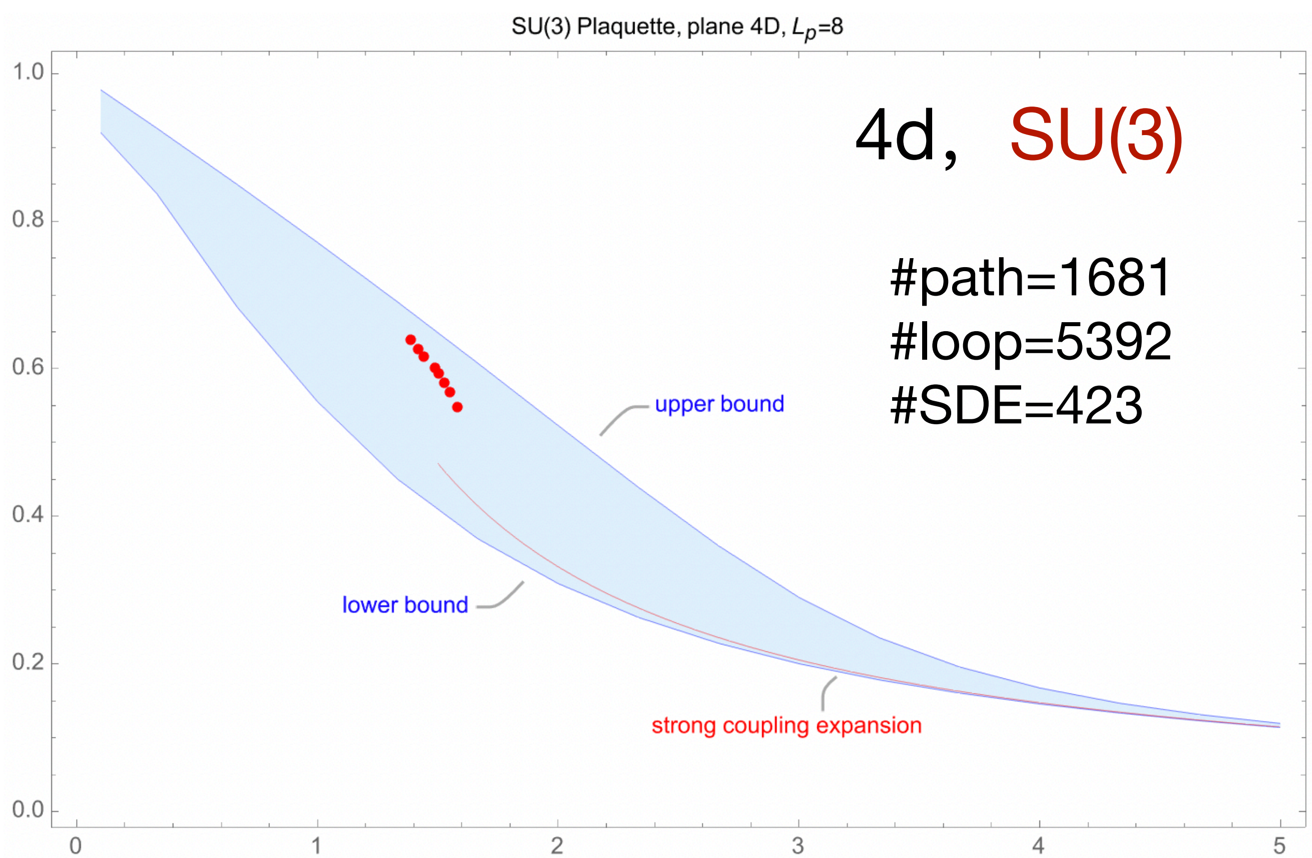


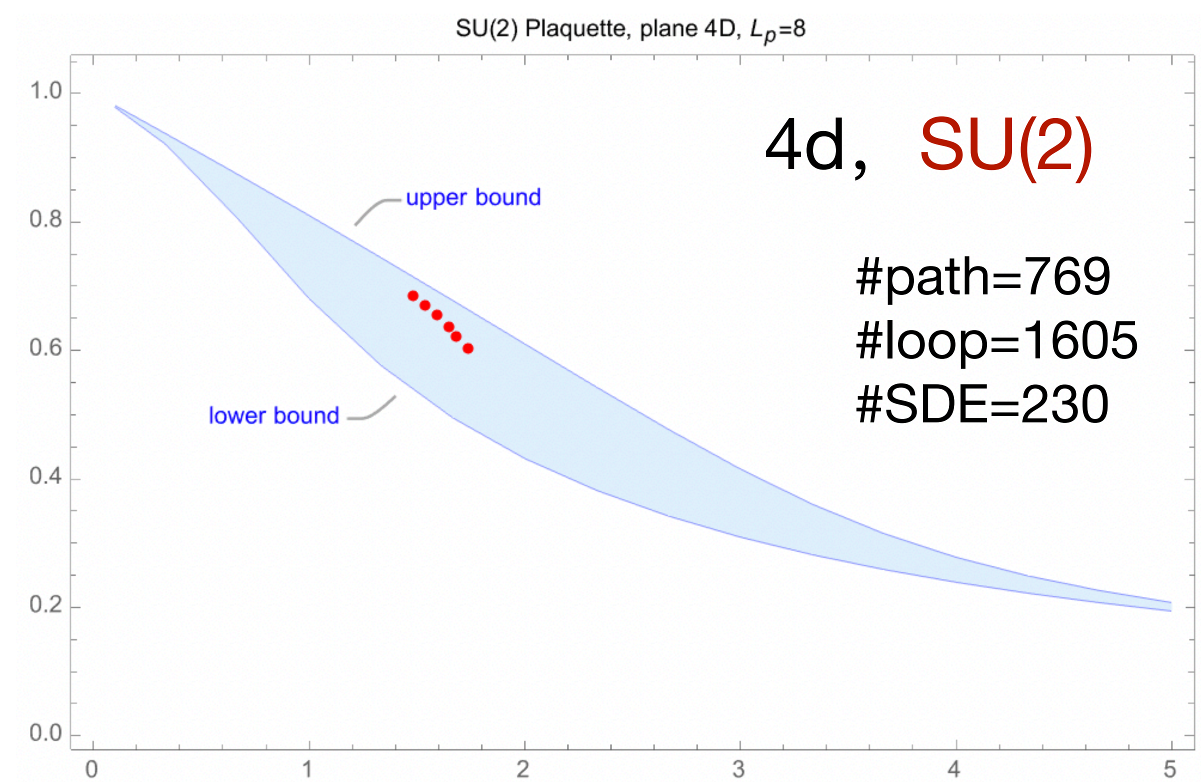
SU(3) and SU(2) is similar

Red dots from MC data: Athenodoroua, Teper, arXiv:1609.03873 (3d), 2106.00364 (4d)

Results： 4d SU(3)

4d plaquette： $U_P = U_1 U_2 U_3 U_4 =$ 





SU(3) and SU(2) is similar

Red dots from MC data: Athenodoroua, Teper, arXiv:1609.03873 (3d), 2106.00364 (4d)

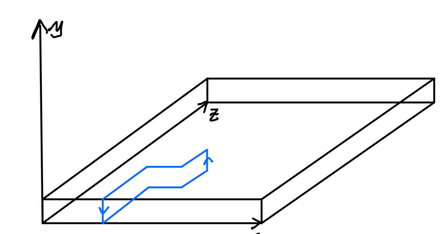
Summary

Bootstrap methods play increasingly important roles in various studies.

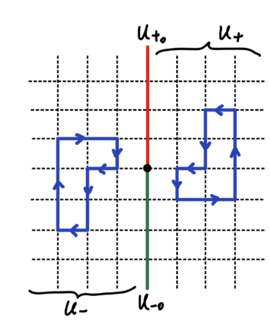
Our new progress for bootstrapping lattice gauge theory:

- Consider SU(3) that involves multi-trace operators for the first time. Show that the method works well.

- Consider (d-1) sub-plane operators. Show that there is good convergence property.



- Introduce a new RP: twist-RP, which is exact in 2d lattice and in continuum theory.



Outlook

Challenges:

- How to make “good” truncation for the loops?
- How to derive the SDE efficiently?
- How to improve the SDP computation?
- New constraints?

Generalization: higher N, QCD with dynamic quarks, …

Postdoc position @ ITP-CAS

1 or 2 postdoctoral positions @ ITP-CAS, Beijing

Research area:

"non-perturbative QCD" and "quantum gravity"

Candidates with background of high-loop computations or bootstrap methods are particularly encouraged to apply.

Thank you for your attention!

Reduction of high-trace operators

SU(2): only single-trace operators

$$\mathrm{tr}U\mathrm{tr}V = \mathrm{tr}UV + \mathrm{tr}UV^\dagger, \quad \forall U, V \in SU(2)$$

SU(3): need single- and double-trace operators

$$\mathrm{tr}U\mathrm{tr}V\mathrm{tr}W = -\mathrm{tr}WVU - \mathrm{tr}UVW + \mathrm{tr}U\mathrm{tr}VW + \mathrm{tr}UV\mathrm{tr}W + \mathrm{tr}UW\mathrm{tr}V$$
$$+ \mathrm{tr}U^\dagger V\mathrm{tr}U^\dagger W - \mathrm{tr}U^\dagger VU^\dagger W, \qquad \forall U, V, W \in SU(3)$$

The SU(3) matrix satisfies

$$1 = \det(X) = \frac{1}{3!}\epsilon^{a_1a_2a_3}\epsilon_{b_1b_2b_3}X_{a_1}{}^{b_1}X_{a_2}{}^{b_2}X_{a_3}{}^{b_3}, \qquad \forall X \in SU(3). \tag{A.1}$$

Expanding the RHS we have

$$\mathrm{tr}(X)\mathrm{tr}(X)\mathrm{tr}(X) - 3\mathrm{tr}(X)\mathrm{tr}(XX) + 2\mathrm{tr}(XXX) = 6, \qquad \forall X \in SU(3). \tag{A.2}$$

For general triple-trace operator, using

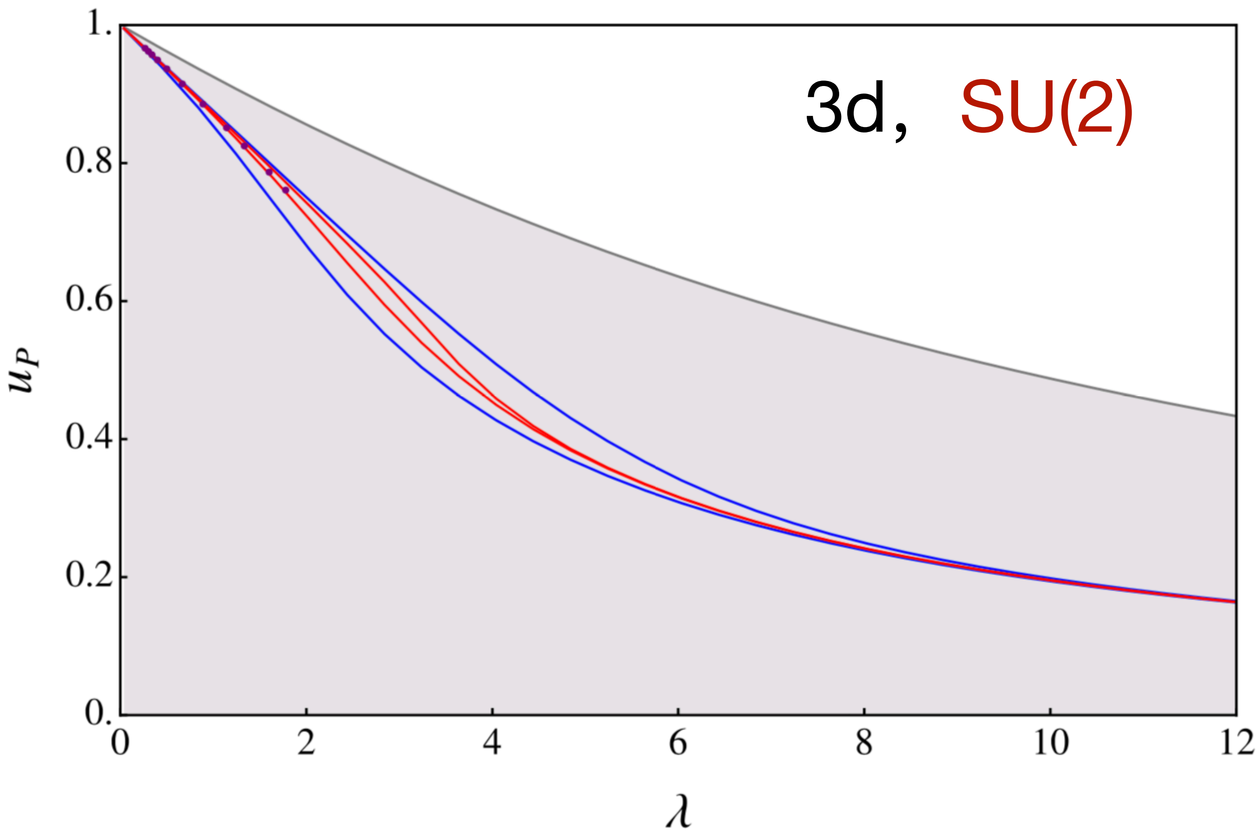
$$\delta^{a_1a_2a_3}_{b_1b_2b_3}X_{a_1}{}^{b_1}Y_{a_2}{}^{b_2}Z_{a_3}{}^{b_3} = \delta^{a_1a_2a_3}_{b_1b_2b_3}X_{a_1}{}^{b_1}Y_{a_2}{}^{b_2}Z_{a_3}{}^{b_3}\det(X^\dagger), \qquad \forall X, Y, Z \in SU(2), \tag{A.3}$$

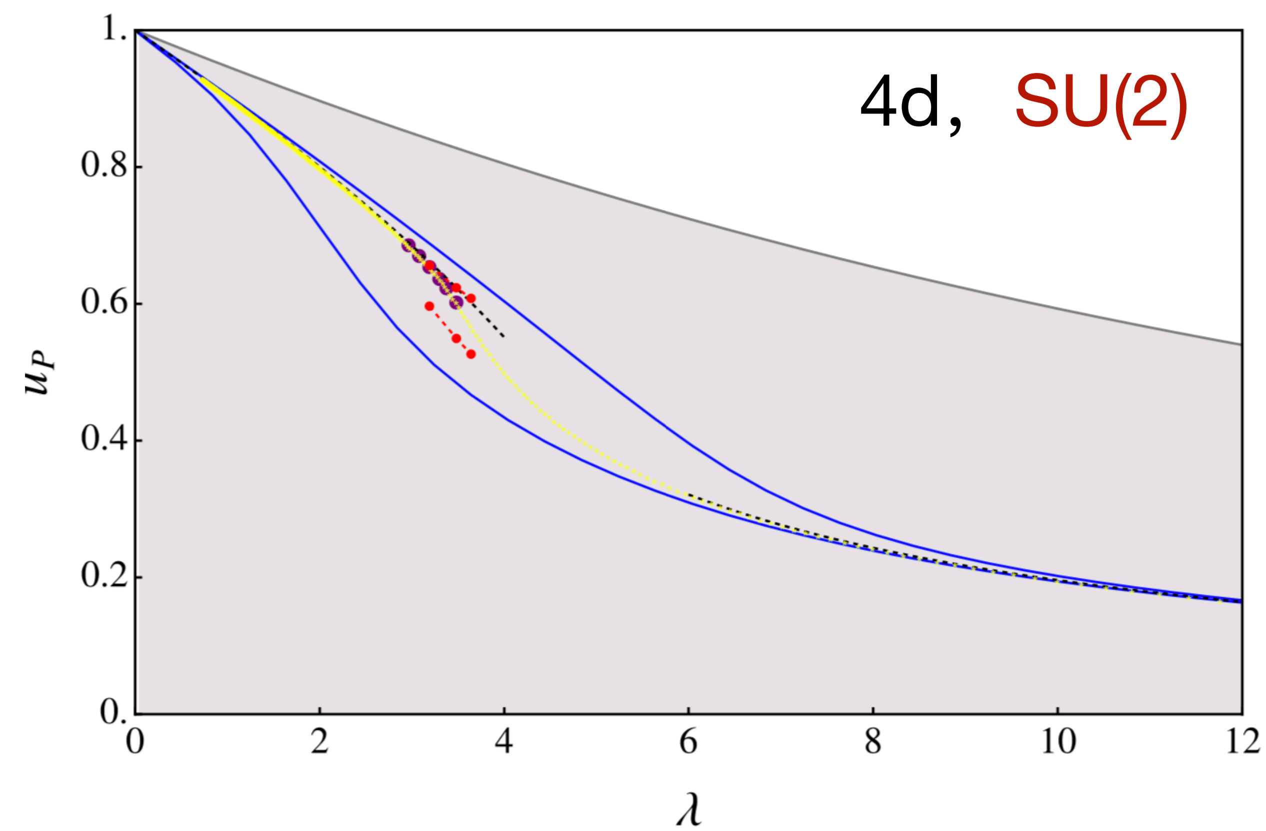
we can prove the following identity

$$\begin{aligned}\mathrm{tr}(X)\mathrm{tr}(Y)\mathrm{tr}(Z) = &-\mathrm{tr}(XYZ) - \mathrm{tr}(ZYX) + \mathrm{tr}(X)\mathrm{tr}(YZ) + \mathrm{tr}(Y)\mathrm{tr}(ZX) + \mathrm{tr}(Z)\mathrm{tr}(XY)\\ &+ \mathrm{tr}(X^\dagger Y)\mathrm{tr}(X^\dagger Z) - \mathrm{tr}(X^\dagger YX^\dagger Z), \qquad \forall X, Y, Z \in SU(3),\end{aligned} \tag{A.4}$$

SU(2) results

Kazakov, Zheng, arXiv:2404.16925





Dim	#var	#LE	#SDPvar	#Block	Blocksize	Memory	Time
2	8335	14591	1044	18	~ 50	$\sim$ 100 MB	$\sim$ 1 s
3	174387	98032	93561	38	~ 200	220 GB	$\sim$ 1 days
4	343851	152149	211912	40	~ 300	1 TB	$\sim$ 20 days

Well-known methods

Exactly solvable model:

- low dimensional models
- specific symmetries: supersymmetry, conformal symmetry

Effective field theories:

- Chiral perturbation,
- heavy quark effective theory,
- Nambu-Jona-Lasinio (NJL) model, etc

Lattice simulation: most successful quantitative method

“positivity bootstrap” strategy

Result: plaquette-correlator

3d double-trace operators: