

All Five-point Kaluza–Klein Correlators and Hidden 8d Symmetry in $\text{AdS}_5 \times S^3$

袁 野 (Ellis Ye Yuan)

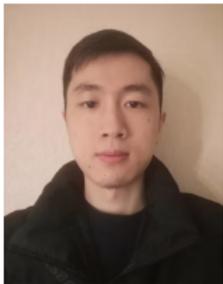


第四届量子场论及其应用研讨会

华南师范大学 2024.11.17

arxiv:2408.12260

collaboration with



黄中杰
Zhongjie Huang



王波
Bo Wang

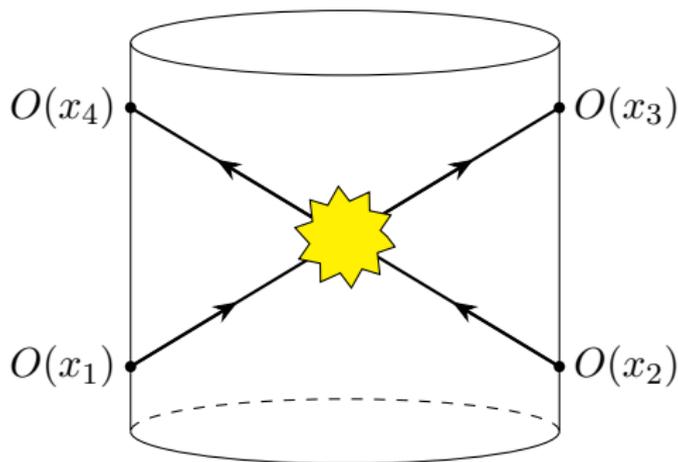


张家荣
Jiarong Zhang

Also collaboration with 曹趣 (Qu Cao) at the early stage.

Holographic correlators

An important class of observables in the context of AdS/CFT.



- ▶ Bulk: perturbative scattering of particles in AdS.
- ▶ Boundary: correlation of local operators in strongly-coupled CFT.

Perturbative expansion in the bulk

$$\mathcal{G} = \begin{array}{cccc} \text{disconnected} & \text{tree} & \text{1-loop} & \text{2-loop} \\ \mathcal{G}^{(0)} & \mathcal{G}^{(1)} & \mathcal{G}^{(2)} & \mathcal{G}^{(3)} \\ + (\text{stringy corrections}) \end{array} + a + a^2 + a^3 + \dots$$

Traditional approach: **Witten diagrams**.

- ▶ Bulk-to-bulk propagators are non-trivial functions of AdS invariant distance.
- ▶ Integrals inside AdS is more complicated.
- ▶ **Tremendous amount of interaction vertices.**

About Kaluza–Klein modes

- ▶ Typical AdS/CFT models: $\text{AdS}_n \times (\text{internal space})$.
- ▶ Particles can move not only purely in AdS, but also in the internal space.
- ▶ Boundary:
A sequence of 1/2-BPS operators O_p , indexed by a Kaluza–Klein charge p .

The resulting bulk interaction is **RICH** and **DIFFICULT**.

organizing principles

Modern approach

- ▶ Modern approach to the computation of holographic correlators is realized by **BOOTSTRAP**.
- ▶ This is pioneered by the work of Rastelli and Zhou, which computes all $\langle O_p O_q O_r O_s \rangle$ in $\text{AdS}_5 \times S^5$ [‘16, ‘17].
 - ▶ Graviton + KK compactifications
- ▶ **Reduced correlator**

$$\langle O_p O_q O_r O_s \rangle \xrightarrow[\text{symmetry}]{\text{superconformal}} \mathcal{H}_{pqrs}.$$

- ▶ Bootstrap leads to **a single unified formula for all KK correlators!**

Modern approach: Mellin space

For conformal correlators it is useful to introduce

Mellin amplitudes [Mack, '09][Penedones, '11]

$$\langle O_{p_1}(x_1) \cdots O_{p_n}(x_n) \rangle_{\text{connected}} = \int [d\gamma] \mathcal{A}(\gamma) \prod_{i < j} \frac{\Gamma(\gamma_{ij})}{((x_i - x_j)^2)^{\gamma_{ij}}}.$$

γ_{ij} : Mellin variables, constrained by

$$\gamma_{ij} = \gamma_{ji}, \quad \sum_{j=1}^n \gamma_{ij} = 0, \quad \gamma_{ii} \equiv -p_i.$$

- ▶ $\Delta_i + \Delta_j - 2\gamma_{ij}$ resembles Mandelstam variables.
- ▶ Locations of poles \Leftrightarrow twists of operators in the OPE.
- ▶ Factorize on poles like scattering amplitudes.
- ▶ Much simpler structure: e.g., for contact diagrams $\mathcal{M} = \text{polynomial}$.

Modern approach: bootstrap

- ▶ At 4 points, study Mellin amplitude for reduced correlator

$$\mathcal{H}_{pqrs} = \int [d\delta] \mathcal{M}_{pqrs}(\rho) \prod_{i < j} \frac{\Gamma(\delta_{ij})}{((x_i - x_j)^2)^{\delta_{ij}}}$$

($\delta_{ii} \equiv 2 - p_i$ due to shifts caused by \mathcal{R})

- ▶ Set up ansatz

$$\mathcal{M}_{pqrs} = \sum_{i,j,k} \frac{a_{ijk} \sigma^i \tau^i}{(s - s_m + 2i)(t - t_m + 2j)(\tilde{u} - \tilde{u}_m + 2k)}$$

R-symmetry factor

Range determined by selection rules

- ▶ a_{ijk} solved by symmetries, flat space limit, asymptotics, polynomial behavior or residues, **FACTORIZATION**, etc.

Use of KK correlators: hidden structures

- ▶ 4-point KK correlators possess a **hidden 10d conformal symmetry** [Caron-Huot, Trinh, '18].
- ▶ Relation between reduced correlators

$$\mathcal{H}_{pqrs} = \mathcal{D}_{pqrs} \mathcal{H}_{2222}.$$

- ▶ Alternatively, expressed in terms of a generating function [Alday, Zhou, '19]

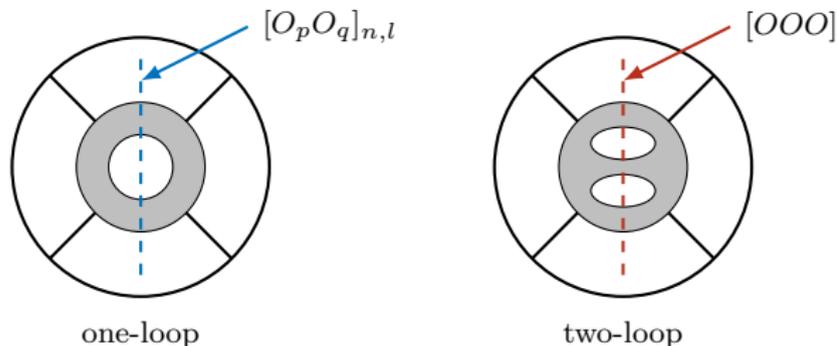
$$\mathbf{H}((x_i - x_j)^2, y_i \cdot y_j) = \mathcal{H}_{2222}((x_i - x_j)^2 + y_i \cdot y_j).$$

Expand and extract all R-symmetry structures for \mathcal{H}_{pqrs} .

Q1: Does the hidden structure extend to higher points?

Use of KK correlators: loop corrections in AdS

- ▶ Computations of loop-level scattering are realized by **UNITARITY** + **BOOTSTRAP** [Aharony, et al, '16].
- ▶ Lower-loop CFT data \Rightarrow higher-loop singularities.



- ▶ For $\mathcal{H}_{2222}^{2\text{-loop}}$, a structural observation helps avoid the input of $[OOO]$ [Huang, EYY, '21][Drummond, Paul, '22].

Q2: CFT data for triple-particle operators?

In this work, we bootstrap
all **five-point** Kaluza–Klein correlators

Model to be studied

- ▶ Instead of gravitons, we choose to study gluons.
- ▶ Can be realized on an $\text{AdS}_5 \times \text{S}^3$ background [Alday, et al, '21]
 - ▶ by inserting probe D7 branes in $\text{AdS}_5 \times \text{S}^5$.
 - ▶ by D3 branes probing F-theory singularity.
- ▶ $\mathcal{N} = 2$ SCFT on the boundary.
- ▶ (Scalar) gluons + Kaluza–Klein tower

$$\mathcal{O}_p^I(x; v, \bar{v}) \equiv \mathcal{O}_p^{I; \alpha_1 \dots \alpha_p; \beta_1 \dots \beta_{p-1}} v_{\alpha_1} \cdots v_{\alpha_p} \bar{v}_{\beta_1} \cdots \bar{v}_{\beta_{p-2}}.$$

Twist p . I - gauge (bulk). v - $\text{SU}_R(2)$. \bar{v} - $\text{SU}_L(2)$.

- ▶ Later we abbreviate

$$x_{ij}^2 \equiv (x_i - x_j)^2, \quad v_{ij} \equiv \epsilon_{\alpha\beta} v_i^\alpha v_j^\beta, \quad \bar{v}_{ij} \equiv \epsilon_{\alpha\beta} \bar{v}_i^\alpha \bar{v}_j^\beta.$$

A first look at the structure

- ▶ Decomposition according to color traces

$$\langle O_{p_1}(x_1) \cdots O_{p_n}(x_n) \rangle = \sum_{\sigma \in S_n / Z_2} \text{tr}[T^{I_{\sigma(1)}} \cdots T^{I_{\sigma(n)}}] G[\sigma] + \cdots .$$

We focus on $G_n \equiv G[12 \cdots n]$.

- ▶ **Extremality**: complexity of R-symmetry structures (assuming p_n is the largest twist)

$$2\mathcal{E} = p_1 + p_2 + \cdots + p_{n-1} - p_n.$$

- ▶ Correlators are non-vanishing only for $\mathcal{E} \geq n - 2$.
The largest twist cannot be **TOO** large!
 \implies **selection rules**.
- ▶ It is a good strategy to focus on fixed \mathcal{E} at a time.
So we work with $\mathcal{E} = 3, 4, 5, \dots$ for five-point correlators.

Four-points vs higher-points

- ▶ At four points, many of the results in graviton scattering have their analogues in gluon scattering.
- ▶ A **hidden 8d conformal symmetry** exists. [Alday, et al, '21]
- ▶ Superconformal symmetry constrains that

$$G_4 = G_4^{\text{free}} + \underbrace{\left(V_{1234} x_{13}^2 x_{24}^2 + V_{1342} x_{14}^2 x_{23}^2 + V_{1423} x_{12}^2 x_{34}^2 \right)}_{R_{1234}} H_4,$$

with $V_{1234} \equiv v_{12}v_{23}v_{34}v_{41}$. R_{1234} is fully permutation invariant.

- ▶ **No higher-point analogue is known.**

Q3: Write $G_{n>4}$ in a way trivializing superconformal?

We obtain a **unified** formula at five points.

Q1: Does the hidden structure extend to higher points?

Yes at five points.

Q2: CFT data for triple-particle operators?

Leave for future work.

Q3: Write $G_{n>4}$ in a way trivializing superconformal?

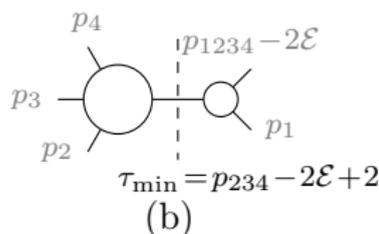
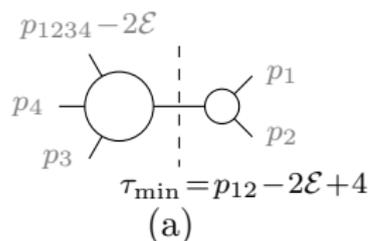
Interesting observations.

OPE selection rules

- ▶ Mellin amplitude has poles at

$$\gamma_{12} = \frac{p_1 + p_2 - \tau}{2} - k.$$

- ▶ Bounded from below by $\Gamma(\gamma_{12})$ in the Mellin transform.
- ▶ Upper bound (or min of τ) set by the min extremality of sub-amplitudes (consider exchange of O_τ)



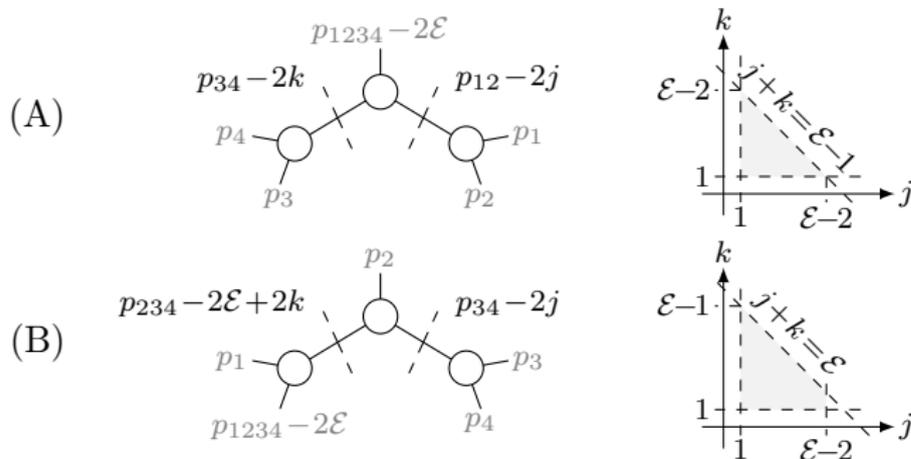
- ▶ Allowed poles in a single channel

$$\text{channel (12)} : \gamma_{12} - j, \quad j = 1, 2, \dots, \mathcal{E} - 2,$$

$$\text{channel (15)} : \gamma_{15} - p_1 + k, \quad k = 1, 2, \dots, \mathcal{E} - 1.$$

OPE selection rules

- For consecutive OPEs, a further constraint from the min extremality of the middle sub-amplitude



- There are also other component fields in the vector multiplet of O (discussed later). Their twists are greater than O , hence do not affect the above counting.

Ansatz

- ▶ At 5 points there are five independent Mellin variables. We choose them to be $\{\gamma_{12}, \gamma_{23}, \gamma_{34}, \gamma_{45}, \gamma_{15}\}$.
- ▶ Kinematic bases
 - ▶ Simultaneous poles

$$(A) : \frac{\{1, \gamma_{23}, \gamma_{45}, \gamma_{15}\}}{(\gamma_{12} - j)(\gamma_{34} - k)}, \quad j, k \geq 1, j+k \leq \mathcal{E} - 1,$$

$$(B) : \frac{\{1, \gamma_{12}, \gamma_{23}, \gamma_{45}\}}{(\gamma_{34} - 1)(\gamma_{15} - p_1 + k)}, \quad j, k \geq 1, j+k \leq \mathcal{E},$$

- ▶ Single poles

$$(a) : \frac{\{1\}}{\gamma_{12} - j}, \quad j = 1, 2, \dots, \mathcal{E} - 2,$$

$$(b) : \frac{\{1\}}{\gamma_{15} - p_1 + k}, \quad k = 1, 2, \dots, \mathcal{E} - 1.$$

Name the whole list \mathcal{K} (collecting all channels).

Ansatz

- ▶ R/L-structure bases: work in a specific frame

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ z_1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ z_2 \end{pmatrix}, v_5 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Similarly for \bar{v} 's (with z replaced by w).

- ▶ In principle, can work out compatible R/L-structures for each kinematic basis element, but this is not economic.
- ▶ In practice, simply list all R/L-structures for a given \mathcal{E}

$$z_1^m z_2^n w_1^i w_2^j, \quad m + n \leq \mathcal{E} - 1, \quad i + j \leq \mathcal{E} - 3$$

Name the whole list \mathcal{I} .

- ▶ **Full ansatz:** take the direct product

$$\mathcal{K} \otimes \mathcal{I}$$

and make linear combinations with unknown coefficients.

Operators in the exchange and factorizations

- ▶ We solve the ansatz solely by studying **FACTORIZATIONS**.
- ▶ Operators that can be exchanged

operator	\mathcal{O}_p	\mathcal{J}_p^μ	\mathcal{F}_p
twist	p	p	$p + 2$
Lorentz spin	0	1	0
$SU(2)_R$ spin	$\frac{p}{2}$	$\frac{p}{2} - 1$	$\frac{p}{2} - 2$
$SU(2)_L$ spin	$\frac{p}{2} - 1$	$\frac{p}{2} - 1$	$\frac{p}{2} - 1$

- ▶ Need correlators at lower points

$$\langle JOO \rangle, \quad \langle FOO \rangle, \quad \langle JOOO \rangle, \quad \langle FOOO \rangle.$$

- ▶ We work out these correlators from $\langle OOO \rangle$ and $\langle OOOO \rangle$ using analytic superspace techniques.

Operators in the exchange and factorizations

- ▶ Factorization on scalar (O or F) poles (assuming point $1, 2, \dots, k$ are on the left) [Fitzpatrick, et al, '11]

$$M \sim \frac{1}{\gamma_{i,i+1} - \frac{p_i + p_{i+1} - p}{2} + m} \frac{\Gamma(p) m!}{(p-1)_m} M_{L,m} M_{R,m},$$

$$M_{L,m} = \sum_{\substack{i_{ab} \geq 0, \\ \sum i_{ab} = m}} M_L(\gamma_{ab} + i_{ab}) \prod_{1 \leq a < b \leq k} \frac{(\gamma_{ab})_{i_{ab}}}{i_{ab}!}.$$

- ▶ Similar factorization for spin 1 (J) has also been studied before [Goncalves, et al, '14].
- ▶ A pole can receive contributions from **primaries** ($m = 0$) and conformal **descendants** of other primaries ($m > 0$). Need to sum them up when comparing the ansatz.
- ▶ Gluing $SU(2)_R$ and $SU(2)_L$ structures.

With this bootstrap procedure, we obtain formulas for

$$\langle O_p O_q O_r O_s O_{p+q+r+s-2\mathcal{E}} \rangle$$

with $\mathcal{E} = 3, 4, 5, 6$, respectively.

The direct computational result looks **cumbersome...**

Extremality 3 (3.5 pages)

The image shows a screenshot of the LibreOffice Writer application window. The title bar reads "mE3 — LibreOffice Writer". The menu bar includes "File", "Edit", "View", "Insert", "Format", "Styles", "Table", "Form", "Tools", "Window", and "Help". The toolbar contains various icons for text formatting, alignment, and editing. Below the toolbar, the text style is set to "Preformatted Text" and the font is "Liberation Mono" at "10 pt". The main editing area displays four pages of text, which is extremely dense and illegible due to the small font size. The status bar at the bottom indicates "Pages 1 - 4 of 4", "2,566 words, 13,376 characters", "Default Page Style", and "English (USA)". The system tray at the very bottom shows the taskbar with various application icons and the system clock showing "11:22".

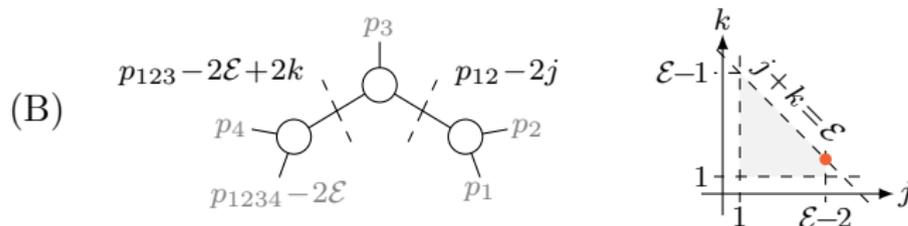
Extremality 4 (44.3 pages)

The screenshot displays the LibreOffice Writer interface. The title bar reads "mE4 - LibreOffice Writer". The menu bar includes "File", "Edit", "View", "Insert", "Format", "Styles", "Table", "Form", "Tools", "Window", and "Help". The toolbar contains various icons for text formatting, alignment, and editing. The document content is a dense grid of text, appearing to be a highly repetitive or algorithmically generated document. The status bar at the bottom indicates "Pages 1 - 45 of 45", "32,418 words, 173,473 characters", "Default Page Style", and "English (USA)". The system tray at the bottom right shows the date and time as "11:22".

... yet there are **STRUCTURES** buried deep inside!

Hints for a unified formula

- Choose some simultaneous poles sitting at the corner of the poles grid, e.g., at $\gamma_{12} = \mathcal{E} - 2$ and $\gamma_{45} = p_4 - 2$



- Keeping track of the change as \mathcal{E} varies, this term reads

$$A_{\{p_i\}} \supset \frac{t_{12}^{\mathcal{E}-3}}{(\mathcal{E}-3)!} \frac{t_{15}^{p_1-\mathcal{E}+1}}{(p_1-\mathcal{E}+1)!} \frac{t_{25}^{p_2-\mathcal{E}+1}}{(p_2-\mathcal{E}+1)!} \frac{t_{35}^{p_3-2}}{(p_3-2)!} \frac{t_{45}^{p_4-2}}{(p_4-2)!}$$

$$\times \frac{(p_4-2) v_{12}^2 v_{34} v_{45} v_{53} \gamma_{23}}{(\gamma_{12} - \mathcal{E} + 2)(\gamma_{45} - p_4 + 2)},$$

Here $t_{ij} \equiv v_{ij} \bar{v}_{ij}$.

- Factors in blue suggest a “Mellin” transformation on S^3 .

[Vieira, Aprile, ‘20]

Hints for a unified formula

- ▶ Generalized Mellin amplitude (Mellin transform also on S^3)

$$\mathcal{F}_{\{p\}} = \sum_{n_{ij}} \int [d\gamma_{ij}] \widetilde{\mathcal{M}}_{\{p\}}(\gamma, n) \prod_{i < j} \frac{t_{ij}^{n_{ij}}}{\Gamma(1 + n_{ij})} \frac{\Gamma(\gamma_{ij})}{((x_i - x_j)^2)^{\gamma_{ij}}},$$

Summation is truncated by $1/\Gamma(1 + n_{ij})$.

The previous expression as a term in this summation.

- ▶ If $\widetilde{\mathcal{M}}_{\{p\}}$ is independent of $\{p\}$, then we can construct a generating function ($\rho_{ij} \equiv \gamma_{ij} - n_{ij}$)

$$\mathbf{F} = \sum_{p_i=0}^{\infty} \mathcal{F}_{\{p\}} = \int [d\rho] \widetilde{\mathcal{M}}(\rho_{ij}) \prod_{i < j} \frac{\Gamma(\rho_{ij})}{(x_{ij}^2 - t_{ij})^{\rho_{ij}}}$$

- ▶ A manifestation of [hidden 8d structures](#).

Hints for a unified formula

- ▶ At 4 points

$$G_4 = G_4^{\text{free}} + \underbrace{(V_{1234} x_{13}^2 x_{24}^2 + V_{1342} x_{14}^2 x_{23}^2 + V_{1423} x_{12}^2 x_{34}^2)}_{R_{1234}} H_4,$$

$$\mathbf{A}_4 = \sum_{n_{ij}} \left(\hat{R}_{1234} \circ \tilde{M}_4 \right) \prod_{i < j} \frac{t_{ij}^{n_{ij}}}{\Gamma(1 + n_{ij})}, \quad \tilde{M}_4 = \frac{1}{(\rho_{12} - 1)(\rho_{14} - 1)}.$$

- ▶ The Mellin space operator

$$\hat{R}_{1234} = V_{1234} \hat{\gamma}_{13} \hat{\gamma}_{24} + V_{1342} \hat{\gamma}_{14} \hat{\gamma}_{23} + V_{1423} \hat{\gamma}_{12} \hat{\gamma}_{34},$$

acts as multiplication and shift

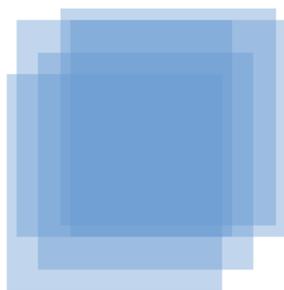
$$\hat{\gamma}_{ij} \circ F(\gamma_{ij}, n_{ij}) = \gamma_{ij} F(\gamma_{ij} + 1, n_{ij}),$$

and is again permutation invariant.

A dozen of days later ...



A MASS.



A pile of simpler stuff?



The ATOM.

Formula (for planar correlators)

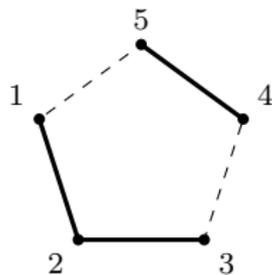
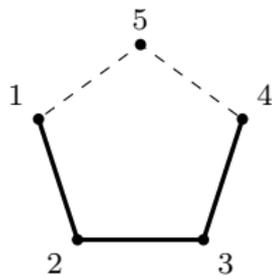
Five-point generating function

$$\mathbf{A}_5 = \widehat{R}^{(1)} \circ \widetilde{M}_5^{(1)} + \widehat{R}^{(2)} \circ \widetilde{M}_5^{(2)} + (\text{cyclic}),$$

where

$$\widetilde{M}_5^{(1)} = -\frac{1}{5(\rho_{12} - 1)(\rho_{23} - 1)(\rho_{34} - 1)},$$

$$\widetilde{M}_5^{(2)} = -\frac{2}{5(\rho_{12} - 1)(\rho_{23} - 1)(\rho_{45} - 1)},$$



Formula (for planar correlators)

$$\begin{aligned}\widehat{R}^{(1)} = & \widehat{R}_{1234,5} + \widehat{R}_{2345,1} + \widehat{R}_{3451,2} + \widehat{R}_{4512,3} + \widehat{R}_{5123,4} \\ & + (v_{13,5} \hat{n}_{13} + v_{23,5} \hat{n}_{23} + v_{14,5} \hat{n}_{14} + v_{24,5} \hat{n}_{24}) \widehat{R}_{1234} \\ & + (v_{13,4} \hat{n}_{13} + v_{23,4} \hat{n}_{23} + v_{53,4} \hat{n}_{53}) \widehat{R}_{1235} \\ & + (v_{23,1} \hat{n}_{23} + v_{24,1} \hat{n}_{24} + v_{25,1} \hat{n}_{25}) \widehat{R}_{2345},\end{aligned}$$

$$\begin{aligned}\widehat{R}^{(2)} = & \widehat{R}_{4512,3} + \widehat{R}_{4513,2} + \widehat{R}_{4521,3} + \widehat{R}_{4523,1} + \widehat{R}_{4531,2} \\ & + \widehat{R}_{4532,1} + (v_{14,5} \hat{n}_{14} + v_{24,5} \hat{n}_{24} + v_{34,5} \hat{n}_{34}) \widehat{R}_{1234} \\ & + (v_{51,4} \hat{n}_{51} + v_{52,4} \hat{n}_{52} + v_{53,4} \hat{n}_{53}) \widehat{R}_{1235}.\end{aligned}$$

Here $v_{ij,k} \equiv v_{ik}v_{jk}/v_{ij}$, and $\hat{n}_{ij} \circ F(\gamma_{ij}, n_{ij}) = n_{ij} F(\gamma_{ij}, n_{ij} - 1)$.

An additional elementary **five-label** \widehat{R} operator

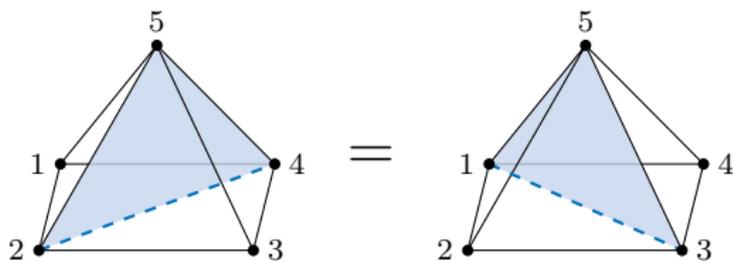
$$\begin{aligned}\widehat{R}_{1234,5} = & V_{12345} \hat{\gamma}_{14} \hat{\gamma}_{25} \hat{\gamma}_{35} + V_{12354} \hat{\gamma}_{34} \hat{\gamma}_{15} \hat{\gamma}_{25} \\ & + V_{12534} \hat{\gamma}_{23} \hat{\gamma}_{15} \hat{\gamma}_{45} + V_{15234} \hat{\gamma}_{12} \hat{\gamma}_{35} \hat{\gamma}_{45}\end{aligned}$$

Comments on the five-label \widehat{R}

- ▶ $\widehat{R}_{1234,5}$ enjoys **cyclic permutation invariance** in (1234), and **switches sign under reflection**.
- ▶ It can be (non-uniquely) decomposed onto four-label \widehat{R} 's

$$\begin{aligned}\widehat{R}_{1234,5} &= v_{24,1}\widehat{\gamma}_{15}\widehat{R}_{2345} + v_{42,3}\widehat{\gamma}_{35}\widehat{R}_{1245}, \\ &= v_{13,4}\widehat{\gamma}_{45}\widehat{R}_{1345} + v_{31,2}\widehat{\gamma}_{25}\widehat{R}_{1235}.\end{aligned}$$

- ▶ Geometric intuition: cutting a **pyramid**



Outlook

- ▶ CFT data
- ▶ Higher points
- ▶ Graviton scattering
- ▶ Weak coupling

Thank you very much!

Questions & comments are welcome.