Combinatorial Geometry and Feynman Integrals

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第四届量子场论及其应用研讨会

广州

2024.11.17

Literatures

I. Tai-Fu Feng, Chao-Hsi Chang, Jian-Bin Chen, Hai-Bin Zhang

GKZ-hypergeometric systems for Feynman integrals

NPB 953 (2020) 114952 [arXiv: 1912.01726]

II. Tai-Fu Feng, Hai-Bin Zhang, Yan-Qing Dong, Yang Zhou

GKZ-system of the 2-loop self energy with 4 propagators

EPJC 83 (2023)4, 314 [arXiv: 2209.15194]

III. Hai-Bin Zhang, Tai-Fu Feng

1: GKZ hypergeometric systems of the three vacuum Feynman integrals

JHEP 2305 (2023) 075 [arXiv: 2303.02795]

2: GKZ hypergeometric systems of the four vacuum Feynman integrals [2403.13025]

IV. Tai-Fu Feng, Hai-Bin Zhang, Chao-Hsi Chang

Feynman integrals of Grassmannians

PRD 106 (2022) 116025 [arXiv: 2206.04224]

V. Tai-Fu Feng, Yang Zhou, Hai-Bin Zhang

Gauss relations in Feynman integrals [arXiv: 2407.10287]

Contents

I. Embedding of Feynman Integrals in Grassmannians

- Feynman integrals involving several energy scales can be given by some finite linear combinations of generalized hypergeometric functions.
- Any commonly used functions of one indeterminate of analysis can be expressed as the Gauss function

$$
{}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} \middle| \, x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! \, (c)_n} x^n \,, \tag{1.1}
$$

  where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer notation.
- For the given parameters $a, \ b, \ c$, there are 24 hypergeometric series solutions totally of the partial differential equation (PDE) which can be written as the GKZ-system on the Grassmannians $G_{2,4}$.

I. Embedding of Feynman Integrals in Grassmannians

- In α-parameterization, the Feynman integral of one-loop self-energy is

$$
\begin{aligned}
& iA_{1SE}(p^2, m_1^2, m_2^2) \\
= & -\left(\Lambda_{\rm RE}^2\right)^{2-D/2}\int_0^\infty d\alpha_1 d\alpha_2 \int \frac{d^D q}{(2\pi)^D} \exp\Big\{i\Big[\alpha_1(q^2-m_1^2) \\
& +\alpha_2((q+p)^2-m_2^2)\Big]\Big\} \\
= & \frac{i^{2-D/2}\exp\left\{\frac{i\pi(2-D)}{4}\right\}\Gamma(2-D/2)\left(\Lambda_{\rm RE}^2\right)^{2-D/2}}{(4\pi)^{D/2}} \\
& \times\int_S \omega_3(t)\delta(t_1t_2+t_1t_3+t_2t_3)(t_1t_2)^{1-D/2}t_3^{D/2-1} \\
& \times\left[t_1m_1^2+t_2m_2^2+t_3p^2\right]^{D/2-2}\,, \qquad (1.2)
\end{aligned}
$$

- The hyperplane S is given by the equation $t_3+1=0$, and $\omega_3(t)=t_1dt_2dt_3-t_2dt_1dt_3+t_3dt_1dt_2$ is the volume element in the projective plane P^2, respectively.

I. Embedding of Feynman Integrals in Grassmannians

- The integral can be embedded in the subvariety of the Grassmannian $G_{3,6}$

$$\xi' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & r_1 \\ 0 & 1 & 0 & 1 & 1 & r_2 \\ 0 & 0 & 1 & 1 & 1 & r_3 \end{pmatrix}, \tag{1.3}$$

with $r_1 = m_1^2,\ r_2 = m_2^2,\ r_3 = p^2$.

- Row: 1: integration variable t_1, 2: t_2, 3: t_3, respectively.
- Column: 1: the power function $t_1^{1-D/2}$, 2: $t_2^{1-D/2}$, 3: $t_3^{D/2-1}$, 6: the power of the linear polynomial $t_1 m_1^2 + t_2 m_2^2 + t_3 p^2$.
- The polynomial under δ function is taken as the fourth and fifth columns of the subvariety of the Grassmannian $G_{3,6}$.

I. Embedding of Feynman Integrals in Grassmannians

- Because the fourth and fifth columns in the matroid Eq.(1.3) coalesce into a same point in projective space P^2, ξ'^{1S} is reduced to the subvariety of the Grassmannian $G_{3,5}$ represented by the matroid ξ of size 3×5

$$\xi = \begin{pmatrix} 1 & 0 & 0 & 1 & r_1 \\ 0 & 1 & 0 & 1 & r_2 \\ 0 & 0 & 1 & 1 & r_3^2 \end{pmatrix}. \tag{1.4}$$

  with the exponent vector $\beta_{(1S)} = (2 - \frac{D}{2},\ 2 - \frac{D}{2},\ \frac{D}{2},\ -1,\ \frac{D}{2} - 1) \in C^5$.

- Similarly the Feynman integral of 1-loop massless triangle diagram is embedded in the subvariety of the Grassmannian $G_{3,5}$ represented by the matroid in Eq.(1.4) with $r_{1,2} = p_{1,2}^2,\ r_3 = p_3^2 = (p_1 + p_2)^2$ and the exponent vector $\beta_{(1T)} = (1,\ 1,\ 1,\ \frac{D}{2} - 2,\ 1 - \frac{D}{2}) \in C^5$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

The hypergeometric function on the general stratum of the Grassmannian $G_{3,5}$ with the splitting coordinates in Eq.(1.4) satisfies the GKZ-system as

$$\begin{aligned}
&\Big\{\vartheta_{1,4}+\vartheta_{1,5}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_1\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\Big\{\vartheta_{2,4}+\vartheta_{2,5}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_2\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\Big\{\vartheta_{3,4}+\vartheta_{3,5}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=-\beta_3\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\Big\{\vartheta_{1,4}+\vartheta_{2,4}+\vartheta_{3,4}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=(\beta_4-1)\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,\\
&\Big\{\vartheta_{1,5}+\vartheta_{2,5}+\vartheta_{3,5}\Big\}\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})=(\beta_5-1)\Phi(\boldsymbol{\beta},\ \boldsymbol{\xi})\ ,
\end{aligned}\tag{2.1}$$

where the Euler operators $\vartheta_{i,j}=\xi_{i,j}\partial/\partial\xi_{i,j}$, and the exponent vector $\boldsymbol{\beta}=(\beta_1,\cdots,\beta_5)\in C^5$ satisfying $\sum\beta_i=2$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

Corresponding to the Grassmannian $G_{3,5}$ represented by the matroid in Eq.(1.4), the exponent matrix is generally written as

$$\begin{pmatrix} \beta_1 - 1 & 0 & 0 & \alpha_{1,4} & \alpha_{1,5} \\ 0 & \beta_2 - 1 & 0 & \alpha_{2,4} & \alpha_{2,5} \\ 0 & 0 & \beta_3 - 1 & \alpha_{3,4} & \alpha_{3,5} \end{pmatrix} . \tag{2.2}$$

where

$$\begin{aligned} & \sum_{i=1}^{5} \beta_i = 2, \quad \sum_{j=1}^{3} \alpha_{j,4} = \beta_4 - 1, \quad \sum_{j=1}^{3} \alpha_{j,5} = \beta_5 - 1 \\ & \alpha_{j,4} + \alpha_{j,5} = -\beta_j, \quad j = 1, 2, 3. \end{aligned} \tag{2.3}$$

Six indeterminate exponents satisfy four independent linear constraints.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

Let $\mathcal{N} = \{1, \cdots, 5\}$ denoting the set of indices of the columns in Eq.(1.4). Choosing the spanning subset $\mathcal{B}$ of the vector subspace C^3 in the vector space C^5 and the integer lattice on the complement $\mathcal{N} \backslash \mathcal{B}$, one gets the hypergeometric function accordingly.
For example as $\mathcal{B} = \{1, 2, 3\}$, there are 12 choices on the matrix of integer lattice whose submatrix composed of the fourth- and fifth columns is formulated as $\pm n_1 E_3^{(i)} \pm n_2 E_3^{(j)}$, where $n_{1,2} \geq 0$, $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$, and other elements are all zero.

$$\text{Integer lattice } (0_{3\times 3} \,|\, \pm n_1 E_3^{(i)} \pm n_2 E_3^{(j)}): \quad E_3^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix},$$

$$E_3^{(2)} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{pmatrix}, E_3^{(3)} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}.$$

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Corresponding to the integer lattice

$$\left(0_{3\times 3}\middle| n_1 E_3^{(1)} + n_2 E_3^{(2)}\right) = \begin{pmatrix} 0 & 0 & 0 & \boxed{n_2} & -n_2 \\ 0 & 0 & 0 & \boxed{n_1} & -n_1 \\ 0 & 0 & 0 & -n_1 - n_2 & n_1 + n_2 \end{pmatrix}, \qquad (2.4)$$

- the exponents are given by the matrix

$$||\boldsymbol{\alpha}|| = \begin{pmatrix} \beta_1 - 1 & 0 & 0 & \boxed{0} & -\beta_1 \\ 0 & \beta_2 - 1 & 0 & \boxed{0} & -\beta_2 \\ 0 & 0 & \beta_3 - 1 & \beta_4 - 1 & 1 - \beta_3 - \beta_4 \end{pmatrix}, \qquad (2.5)$$

where $\alpha_{1,4} = \alpha_{2,4} = 0$ because $n_{1,2}$ are nonnegative.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The generalized hypergeometric function is

$$\Phi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) = A^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(r_1)^{-\beta_1}(r_2)^{-\beta_2}(r_3)^{1-\beta_3-\beta_4} \times \varphi^{(1)}_{\{1,2,3\}}\left(\boldsymbol{\beta},\ \frac{r_3}{r_2},\ \frac{r_3}{r_1}\right),$$

$$\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \sum_{n_1,n_2} c^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2)x_1^{n_1}x_2^{n_2}\ , \tag{2.6}$$

where

$$A^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(1-\beta_2)\Gamma(2-\beta_3-\beta_4)}\ ,$$

$$c^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2) = \frac{(\beta_2)_{n_1}(\beta_1)_{n_2}(1-\beta_4)_{n_1+n_2}}{n_1!n_2!(2-\beta_3-\beta_4)_{n_1+n_2}}\ . \tag{2.7}$$

with the Pochhammer notation $(a)_n = \Gamma(a+n)/\Gamma(a)$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

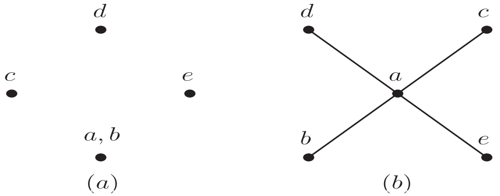


Figure: 1 The geometric configurations of the hypergeometric functions on the projective plane P^2, where the points $a, \cdots, e$ denote the indices of columns of the 3×5 exponent matrix.

The geometric representation of the function $\Phi^{(1)}_{\{1,2,3\}}$ is drown in Fig.1(a) where $\{a, b\} = \{3, 4\}$ and $\{c, d, e\} = \{1, 2, 5\}$, which the determinant of any 2×2 minor of the submatrix consisted of the third and fourth columns is zero.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Corresponding to the integer lattice

$$(0_{3\times 3} \mid n_1 E_3^{(1)} + n_2 E_3^{(3)}) = \begin{pmatrix} 0 & 0 & 0 & \boxed{n_2} & -n_2 \\ 0 & 0 & 0 & n_1 - n_2 & -n_1 + n_2 \\ 0 & 0 & 0 & -n_1 & \boxed{n_1} \end{pmatrix}, \tag{2.8}$$

- the exponents are given by the matrix

$$||\boldsymbol{\alpha}|| = \begin{pmatrix} \beta_1 - 1 & 0 & 0 & \boxed{0} & -\beta_1 \\ 0 & \beta_2 - 1 & 0 & \beta_3 + \beta_4 - 1 & \beta_1 + \beta_5 - 1 \\ 0 & 0 & \beta_3 - 1 & -\beta_3 & \boxed{0} \end{pmatrix} \tag{2.9}$$

where $\alpha_{1,4} = \alpha_{3,5} = 0$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The generalized hypergeometric function is formulated as

$$\Phi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) = A^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta})(r_1)^{-\beta_1}(r_2)^{\beta_1+\beta_5-1} \times \varphi^{(2)}_{\{1,2,3\}}\left(\boldsymbol{\beta},\ \frac{r_3}{r_2},\ \frac{r_2}{r_1}\right),$$

$$\varphi^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \sum_{n_1,n_2} c^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2) x_1^{n_1} x_2^{n_2}, \tag{2.10}$$

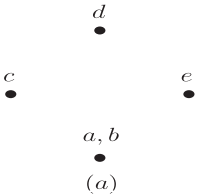
- Where

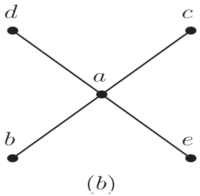
$$A^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{\Gamma(\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(1-\beta_3)\Gamma(\beta_1+\beta_5)\Gamma(\beta_3+\beta_4)},$$

$$c^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta},\ n_1,n_2) = \frac{(-)^{n_1+n_2}(\beta_3)_{n_1}(\beta_1)_{n_2}}{n_1!n_2!(\beta_1+\beta_5)_{-n_1+n_2}(\beta_3+\beta_4)_{n_1-n_2}}. \tag{2.11}$$

Note that $1/(a)_{-n} = (-1)^n(1-a)_n$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$





- The geometric representation of the hypergeometric function is determined by the exponent matrix presented in Eq.(2.9), the determinants of the submatrices $\det(||\alpha||_{\{1,2,5\}}) = \det(||\alpha||_{\{2,3,4\}}) = 0$.
- The geometric representation of the function $\Phi^{(2)}_{\{1,2,3\}}$ is drown in Fig.1(b) where $a = 2,\ \{b, c\} = \{1, 5\}$ and $\{d, e\} = \{3, 4\}$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Taking the affine spanning $\mathcal{B} = \{2,4,5\}$, one finds $\det(\xi_{\{2,4,5\}}) = r_1 - r_3$ which differs from $\det(\xi_{\{1,2,3\}}) = 1$, where $\det(\xi_{\{2,4,5\}})$ denotes the determinant of the 3×3 minor of the matrix in Eq.(1.4) composed of the 2nd, 4th and 5th columns.
- In addition,

$$\xi^{-1}_{\{2,4,5\}} \cdot \xi = \begin{pmatrix} -\frac{r_3-r_2}{r_3-r_1} & 1 & -\frac{r_2-r_1}{r_3-r_1} & 0 & 0 \\ \frac{r_3}{r_3-r_1} & 0 & -\frac{r_1}{r_3-r_1} & 1 & 0 \\ -\frac{1}{r_3-r_1} & 0 & \frac{1}{r_3-r_1} & 0 & 1 \end{pmatrix}. \tag{2.12}$$

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- Corresponding to the integer lattice

$$\left(0_{3\times\mathcal{B}}\middle|\left(n_1E_3^{(1)}+n_2E_3^{(3)}\right)_{\mathcal{N}\setminus\mathcal{B}}\right)$$
$$=\begin{pmatrix} \boxed{n_2} & 0 & -n_2 & 0 & 0 \\ n_1-n_2 & 0 & -n_1+n_2 & 0 & 0 \\ -n_1 & 0 & \boxed{n_1} & 0 & 0 \end{pmatrix}, \qquad (2.13)$$

- the exponents are given by the matrix

$$||\boldsymbol{\alpha}||$$
$$=\begin{pmatrix} \boxed{0} & \beta_2-1 & -\beta_2 & 0 & 0 \\ \beta_1+\beta_5-1 & 0 & \beta_2+\beta_3-1 & \beta_4-1 & 0 \\ -\beta_5 & 0 & \boxed{0} & 0 & \beta_5-1 \end{pmatrix}, \qquad (2.14)$$

where $\alpha_{1,1}=\alpha_{3,3}=0$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- The generalized hypergeometric function is

$$\begin{aligned}
\Phi^{(2)}_{\{2,4,5\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) &= A^{(2)}_{\{2,4,5\}}(\boldsymbol{\beta})\frac{(r_3-r_1)^{2-\beta_1-\beta_3}(r_3)^{\beta_1+\beta_5-1}(r_1-r_2)^{-\beta_2}}{\det(\boldsymbol{\xi}_{\{2,4,5\}})} \\
&\quad\times(-r_1)^{\beta_2+\beta_3-1}\varphi^{(2)}_{\{2,4,5\}}\Big(\boldsymbol{\beta},\ \frac{r_3}{r_1},\ \frac{r_1(r_3-r_2)}{r_3(r_1-r_2)}\Big) \\
&= A^{(2)}_{\{2,4,5\}}(\boldsymbol{\beta})(r_3-r_1)^{1-\beta_1-\beta_3}(r_3)^{\beta_1+\beta_5-1}(r_1-r_2)^{-\beta_2} \\
&\quad\times(-r_1)^{\beta_2+\beta_3-1}\varphi^{(2)}_{\{2,4,5\}}\Big(\boldsymbol{\beta},\ \frac{r_3}{r_1},\ \frac{(1-r_2/r_3)}{\boxed{(1-r_2/r_1)}}\Big)\ ,
\end{aligned}$$

$$\varphi^{(2)}_{\{2,4,5\}}(\boldsymbol{\beta},\ x_1,\ x_2) = \varphi^{(2)}_{\{1,2,3\}}(\widehat{(124)}\widehat{(35)}\boldsymbol{\beta},\ x_1,\ x_2)\ . \tag{2.15}$$

- In order to obtain the analytical expressions in the whole domain of definition, we present the fundamental solution systems under all possible affine spanning $\mathcal{B}$.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

- In these hypergeometric functions, $\varphi_{\mathcal{B}}^{(i)}$, $i = 1, 3, 5, 8, 10, 12$ are the first Appell functions, while $\varphi_{\mathcal{B}}^{(j)}$, $j = 2, 4, 6, 7, 9, 11$ are the Horn functions.
- It is easy to find that the convergent regions of $\varphi_{\{1,2,3\}}^{(1)}$, $\varphi_{\{1,2,3\}}^{(2)}$, and $\varphi_{\{1,2,3\}}^{(3)}$ have nonempty intersections in a connected component of definition domain, thus they constitute a fundamental solution system in the proper nonempty subset of the parameter space.
- The linear combinations of hypergeometric functions on the different nonempty proper subsets of the parameter space are regarded as analytic continuations of each other.

II. Affine coordinates, Integer Lattice and Geometric representations of $G_{3,5}$

$$
\begin{aligned}
\Psi(\boldsymbol{\beta},\ \boldsymbol{\xi}) &= \sum_{i=\{1,2,3\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{1,5,6\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{3,7,8\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{4,5,12\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{8,9,10\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) \\
&= \sum_{i=\{10,11,12\}} C^{(i)}(\boldsymbol{\beta})\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi})\ . \qquad (2.16)
\end{aligned}
$$

Using the Gauss inverse relations below, we can derive the combinatorial coefficients uniquely, then continue the analytic expressions to the whole domain of definition of the Feynman integral by the Gauss-Kummer relations.

III. Generalized Gauss inverse relations

- The Gauss inverse relations include the following analytic continuation together with its various variants

$$
{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, x\right) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}\,{}_2F_1\left(\begin{matrix} a, 1+a-c \\ 1+a-b \end{matrix} \,\middle|\, \frac{1}{x}\right) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}\,{}_2F_1\left(\begin{matrix} b, 1+b-c \\ 1-a+b \end{matrix} \,\middle|\, \frac{1}{x}\right). \tag{3.1}
$$

- Note that this transformation satisfies the idempotent property. Performing the inverse transformation on the terms of the right side, one finds that the sum of the results after transformation is exactly the term on the left side.

III. Generalized Gauss inverse relations

The Gauss inverse relations, i.e. the analytic continuation formulas from one connected component to another in the domain of definition, are obtained through the Mellin-Barnes's contour on the corresponding complex plane.

The Mellin-Barnes representation of the hypergeometric function $\varphi^{(1)}_{\{1,2,3\}}$ is

$$\begin{aligned}&\frac{\Gamma(\beta_2)\Gamma(\beta_1)\Gamma(1-\beta_4)}{\Gamma(2-\beta_3-\beta_4)}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x_1,\ x_2)\\&=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\frac{\Gamma(\beta_2+s_1)\Gamma(\beta_1+s_2)\Gamma(1-\beta_4+s_1+s_2)}{\Gamma(2-\beta_3-\beta_4+s_1+s_2)}\\&\quad\times\Gamma(-s_1)\Gamma(-s_2)(-x_1)^{s_1}(-x_2)^{s_2}ds_1\bigwedge ds_2\ .\end{aligned} \tag{3.2}$$

III. Generalized Gauss inverse relations

Performing the transformation $\beta_1 + s_2 = -s_2'$ on the complex plane s_2, we rewrite the Barnes's contour integral in the right-handed of above equation as

$$
\begin{aligned}
& \frac{(-x_2)^{-\beta_1}}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \frac{\Gamma(\beta_2 + s_1)\Gamma(\beta_1 + s_2')\Gamma(1 - \beta_1 - \beta_4 + s_1 - s_2')}{\Gamma(\beta_2 + \beta_5 + s_1 - s_2')} \\
& \times \Gamma(-s_1)\Gamma(-s_2')(-x_1)^{s_1}(-x_2)^{-s_2'} ds_1 \bigwedge ds_2' \\
= & \frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(1 - \beta_1 - \beta_4)}{\Gamma(\beta_2 + \beta_5)} (-x_2)^{-\beta_1} \varphi_{\{1,2,3\}}^{(4)}(\boldsymbol{\beta}, x_1, \frac{1}{x_2}) \,. \qquad (3.3)
\end{aligned}
$$

III. Generalized Gauss inverse relations

Under the affine transformation $1-\beta_4+s_1+s_2=-s'_2$ on the complex plane s_2, the Barnes's contour integral in the right-handed of Eq.(3.2) is formulated as

$$\begin{aligned}&\frac{(-x_2)^{\beta_4-1}}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\frac{\Gamma(\beta_2+s_1)\Gamma(\beta_1+\beta_4-1-s_1-s'_2)\Gamma(1-\beta_4+s_1+s'_2)}{\Gamma(1-\beta_3-s'_2)}\\&\times\Gamma(-s_1)\Gamma(-s'_2)(-x_1)^{s_1}(-x_2)^{-s_1-s'_2}ds_1\bigwedge ds'_2\\=&\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(\beta_2)\Gamma(1-\beta_4)}{\Gamma(1-\beta_3)}(-x_2)^{\beta_4-1}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{1}{x_2},\ \frac{x_1}{x_2})\ .\end{aligned}\tag{3.4}$$

III. Generalized Gauss inverse relations

Then the residue theorem implies the following equation: Gauss inverse relations

$$\begin{aligned}
&\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}, x_1, x_2) \\
&= \frac{\Gamma(1-\beta_1-\beta_4)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_4)}(-x_2)^{-\beta_1}\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta}, x_1, \frac{1}{x_2}) \\
&\quad + \frac{\Gamma(\beta_1+\beta_4-1)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_1)\Gamma(1-\beta_3)}(-x_2)^{\beta_4-1}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{1}{x_2}, \frac{x_1}{x_2}) \,. \qquad (3.5)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}, x_1, x_2) \\
&= \frac{\Gamma(1-\beta_2-\beta_4)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_1+\beta_5)\Gamma(1-\beta_4)}(-x_1)^{-\beta_2}\varphi^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{1}{x_1}, x_2) \\
&\quad + \frac{\Gamma(\beta_2+\beta_4-1)\Gamma(2-\beta_3-\beta_4)}{\Gamma(\beta_2)\Gamma(1-\beta_3)}(-x_1)^{\beta_4-1}\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{1}{x_1}, \frac{x_2}{x_1}) \,. \qquad (3.6)
\end{aligned}$$

III. Generalized Gauss inverse relations

- The images of the generalized hypergeometric functions under the map of inverse transformation of certain variable are the linear combinations of the generalized hypergeometric solutions of the GKZ-system in the same affine spanning.
- The method presented here generalizes the approach adopted in the work *A new development of the theory of hypergeometric functions* by E. W. Barnes, published in Proc. London. Math. Soc. **6**(1907)141-177, and can be used to derive the analytic continuations of any generalized hypergeometric functions. For example, we can derive the analytic continuations of the Pochhammer functions ${}_{p+1}F_p$, and verify those continuations satisfying the idempotent property accordingly.

IV. Generalized Gauss adjacent relations

- The independent Gauss adjacent relations are the following two equations

$$
\begin{aligned}
&c\ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\ \middle|\ x\right) = a\ x\ {}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix}\ \middle|\ x\right) + c\ {}_2F_1\left(\begin{matrix} a, b-1 \\ c \end{matrix}\ \middle|\ x\right) , \\
&(a-c+1)\ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}\ \middle|\ x\right) \\
&= a\ {}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}\ \middle|\ x\right) - (c-1)\ {}_2F_1\left(\begin{matrix} a, b \\ c-1 \end{matrix}\ \middle|\ x\right) ,
\end{aligned}
\tag{4.1}
$$

  together with two equations obtained by the interchanging $a \leftrightarrow b$ in the above equations.
- For the GKZ-system on the Grassmannian, the adjacent relations of the hypergeometric functions are determined by $G_{k,n}$ and its dual $G_{k,n}^{\perp}$.

IV. Generalized Gauss adjacent relations

- For $G_{3,5}$, the dual variety of the Grassmannian ξ in Eq.(1.4) is given by the matroid

$$\boldsymbol{\xi}_{\perp} = \begin{pmatrix} -1 & -1 & -1 & 1 & 0 \\ -r_1 & -r_2 & -r_3 & 0 & 1 \end{pmatrix} . \tag{4.2}$$

- Corresponding to $\boldsymbol{\beta} + e_1 = (1+\beta_1,\ \beta_2,\ \cdots,\beta_5)$, we obtain three independent adjacent relations among $\Phi^{(i)}_{\{1,2,3\}},\ i \in \{1,\cdots,12\}$.

$$\begin{aligned}
&\beta_1 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) + (\beta_4 - 1)\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_1 - e_4,\ \boldsymbol{\xi}) \\
&+(\beta_5 - 1) r_1 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_1 - e_5,\ \boldsymbol{\xi}) \equiv 0 , \\
&\beta_2 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) + (\beta_4 - 1)\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_2 - e_4,\ \boldsymbol{\xi}) \\
&+(\beta_5 - 1) r_2 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_2 - e_5,\ \boldsymbol{\xi}) \equiv 0 , \\
&\beta_3 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \boldsymbol{\xi}) + (\beta_4 - 1)\Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_3 - e_4,\ \boldsymbol{\xi}) \\
&+(\beta_5 - 1) r_3 \Phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta} + e_3 - e_5,\ \boldsymbol{\xi}) \equiv 0 .
\end{aligned} \tag{4.3}$$

V. Generalized Gauss-Kummer relations

- The third type Gauss relations are derived through Kummer's classification, which can be written as

$$\begin{aligned} {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, x\right) &= (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix}\,\middle|\, x\right) \\ &= (1-x)^{-a}\,{}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix}\,\middle|\, \frac{x}{x-1}\right) \\ &= (1-x)^{-b}\,{}_2F_1\left(\begin{matrix} c-a,\ b \\ c \end{matrix}\,\middle|\, \frac{x}{x-1}\right) \end{aligned} \tag{5.1}$$

and its various variants.

V. Generalized Gauss-Kummer relations

For the GKZ-system on the Grassmannian, the generalized hypergeometric solutions corresponding to the same geometric representation are proportional to each other in the intersection of their convergent regions.

Corresponding to the geometric representation shown in Fig.1(a) with $\{a,b\}=\{3,5\}$, $\{c,d,e\}=\{1,2,4\}$, we derive the following six solutions of the GKZ-system presented in Eq.(2.1) which are proportional to each other in the intersection of their convergent regions,

$$\Phi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}) \sim \Phi^{(1)}_{\{1,2,5\}}(\boldsymbol{\beta}) \sim \Phi^{(5)}_{\{1,3,4\}}(\boldsymbol{\beta}) \sim \Phi^{(10)}_{\{1,4,5\}}(\boldsymbol{\beta}) \sim \Phi^{(10)}_{\{2,4,5\}}(\boldsymbol{\beta}) \sim \Phi^{(5)}_{\{2,3,4\}}(\boldsymbol{\beta}) \,. \tag{5.2}$$

V. Generalized Gauss-Kummer relations

Dividing each function by a common power factor and requiring equality to each other on the concrete principal value plane, we obtain the generalized Gauss-Kummer relations as

$$\begin{aligned}
&\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta},\ x,\ y) \\
&= (1-y)^{-\beta_1}(1-x)^{-\beta_2}\varphi^{(1)}_{\{1,2,5\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ \frac{y}{y-1}) \\
&= (1-x)^{\beta_5-1}\varphi^{(5)}_{\{1,3,4\}}(\boldsymbol{\beta},\ \frac{x}{x-1},\ \frac{x-y}{x-1}) \\
&= (1-y)^{\beta_5-1}\varphi^{(5)}_{\{2,3,4\}}(\boldsymbol{\beta},\ \frac{y}{y-1},\ \frac{y-x}{y-1}) \\
&= (1-x)^{1-\beta_2-\beta_3}(1-y)^{-\beta_1}\varphi^{(10)}_{\{1,4,5\}}(\boldsymbol{\beta},\ x,\ \frac{x-y}{1-y}) \\
&= (1-y)^{1-\beta_1-\beta_3}(1-x)^{-\beta_2}\varphi^{(10)}_{\{2,4,5\}}(\boldsymbol{\beta},\ y,\ \frac{y-x}{1-x})\ ,
\end{aligned} \tag{5.3}$$

with $x=r_2/r_3,\ y=r_1/r_3$.

VI. The analytic expressions for 1-loop self energy

In this scheme, we obtain the analytic expressions of a Feynman integral in its whole domain of definition through the following steps.

- After embedding the Feynman integral on a variety (a special stratum) of the Grassmannian $G_{k,n}$ $(k < n)$, we construct all hypergeometric solutions for the general stratum of the Grassmannian $G_{k,n}$ under all possible affine spanning.
- We derive the inverse and adjacent relations among hypergeometric solutions under the same affine spanning, and the Gauss-Kummer relations among hypergeometric solutions from different affine spanning.

VI. The analytic expressions for 1-loop self energy

- In the neighborhood of the regular singularities, we write the Feynman integral as a finite linear combinations of the canonical series solutions for our special stratum under same affine spanning.

- The combination coefficients are obtained by the reduced Gauss inverse relations among the canonical series solutions, then the analytic expressions of the Feynman integral are continued to its whole domain of definition.

VI. The analytic expressions for 1-loop self energy

In the example of 1-loop self energy, its Feynman integral is embedded in the general stratum of $G_{3,5}$.

- The exponent vector

$$\boldsymbol{\beta} = \boldsymbol{\beta}_{(1S)} = (2 - \frac{D}{2},\ 2 - \frac{D}{2},\ \frac{D}{2},\ -1,\ \frac{D}{2} - 1) \in C^5\ . \tag{6.1}$$

  Where D is the time-space dimension in dimensional regularization.

- The boundary conditions:

$$\begin{aligned} iA_{1SE}(p^2, 0, 0) &= \frac{i\Gamma(2-\frac{D}{2})\Gamma^2(\frac{D}{2}-1)}{(4\pi)^{D/2}\Gamma(D-2)}\Big(\frac{-p^2}{\Lambda_{\rm RE}^2}\Big)^{\frac{D}{2}-1}\ , \\ iA_{1SE}(0, m^2, 0) = iA_{1SE}(0, 0, m^2) &= \frac{i\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2}-1)}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}\Big(\frac{m^2}{\Lambda_{\rm RE}^2}\Big)^{\frac{D}{2}-1}\ , \end{aligned} \tag{6.2}$$

  which are used to obtain the combinatorial coefficients. Here $\Lambda_{\rm RE}$ is the renormalization scale.

VI. The analytic expressions for 1-loop self energy

- $|p^2| < m_1^2 < m_2^2$

$$\begin{aligned} & A_{1SE}(p^2, m_1^2, m_2^2) \\ = & C^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{-\beta_2}(p^2)^{1-\beta_3-\beta_4}\varphi^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{p^2}{m_1^2}) \\ & + C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_5-1}\varphi^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}) \\ & + C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_2+\beta_5-1}(m_2^2)^{-\beta_2}\varphi^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_1^2}, \frac{m_1^2}{m_2^2}) \end{aligned} \tag{6.3}$$

- $m_2^2 < |p^2| < m_1^2$

$$\begin{aligned} & A_{1SE}(p^2, m_1^2, m_2^2) \\ = & C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_5-1}\varphi^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_1^2}, \frac{m_2^2}{m_1^2}) \\ & + C^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(p^2)^{\beta_1+\beta_5-1}\varphi^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{p^2}{m_1^2}) \\ & + C^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{1-\beta_2-\beta_4}(p^2)^{-\beta_3}\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_2^2}{m_1^2}) \end{aligned} \tag{6.4}$$

VI. The analytic expressions for 1-loop self energy

- $m_2^2 < m_1^2 < |p^2|$

$$
\begin{aligned}
&A_{1SE}(p^2, m_1^2, m_2^2) \\
&= C^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{-\beta_1}(m_2^2)^{1-\beta_2-\beta_4}(p^2)^{-\beta_3}\varphi^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_2^2}{m_1^2}) \\
&+C^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{\beta_3+\beta_5-1}(p^2)^{-\beta_3}\varphi^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_1^2}{p^2}, \frac{m_2^2}{m_1^2}) \\
&+C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})(p^2)^{\beta_5-1}\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_1^2}{p^2})
\end{aligned}
\tag{6.5}
$$

- $m_1^2 < m_2^2 < |p^2|$

$$
\begin{aligned}
&A_{1SE}(p^2, m_1^2, m_2^2) \\
&= C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})(p^2)^{\beta_5-1}\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_1^2}{p^2}) \\
&+C^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_3+\beta_5-1}(p^2)^{-\beta_3}\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_2^2}{p^2}, \frac{m_1^2}{m_2^2}) \\
&+C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{1-\beta_1-\beta_4}(m_2^2)^{-\beta_2}(p^2)^{-\beta_3}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_1^2}{p^2}, \frac{m_1^2}{m_2^2})
\end{aligned}
\tag{6.6}
$$

VI. The analytic expressions for 1-loop self energy

- $m_1^2 < |p^2| < m_2^2$

$$
\begin{aligned}
& A_{1SE}(p^2, m_1^2, m_2^2) \\
= & C^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{-\beta_2}(p^2)^{\beta_2+\beta_5-1}\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{p^2}) \\
& + C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_2^2)^{\beta_5-1}\varphi^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}) \\
& + C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta})(m_1^2)^{1-\beta_1-\beta_4}(m_2^2)^{-\beta_2}(p^2)^{-\beta_3}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}, \frac{m_1^2}{p^2}, \frac{m_1^2}{m_2^2})
\end{aligned}
\tag{6.7}
$$

- Using the boundary conditions in Eq.(6.2), we have

$$
\begin{aligned}
& C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) = C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{\Gamma(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}\Gamma(\frac{D}{2})}, \\
& C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}) = \frac{(-1)^{D/2-2}\Gamma^2(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})}{(4\pi)^{D/2}\Gamma(D-2)}.
\end{aligned}
\tag{6.8}
$$

VI. The analytic expressions for 1-loop self energy

- Other coefficients are linear combinations of the above coefficients through the Gauss inverse relations.
- In order to derive other combinatorial coefficients, we apply the Gauss-inverse relations.
- Performing the inverse transformation of suitable variables in Eq.(6.3) and Eq.(6.4), for example, one gets

$$
\begin{aligned}
C_{\{1,2,3\}}^{(3)}(\boldsymbol{\beta}) =& (-1)^{\beta_5-1}\frac{\Gamma(\beta_1+\beta_5-1)\Gamma(2-\beta_2-\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)}C_{\{1,2,3\}}^{(5)}(\boldsymbol{\beta}) \\
&+(-1)^{-\beta_2}\frac{\Gamma(\beta_1+\beta_5-1)\Gamma(\beta_2+\beta_5)}{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}C_{\{1,2,3\}}^{(6)}(\boldsymbol{\beta})\;, \\
C_{\{1,2,3\}}^{(2)}(\boldsymbol{\beta}) =& (-1)^{-\beta_1}\frac{\Gamma(1-\beta_1-\beta_5)\Gamma(2-\beta_2-\beta_5)}{\Gamma(\beta_3+\beta_4)\Gamma(1-\beta_5)}C_{\{1,2,3\}}^{(5)}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_3+\beta_4-1}\frac{\Gamma(\beta_2+\beta_5)\Gamma(1-\beta_1-\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_1)}C_{\{1,2,3\}}^{(6)}(\boldsymbol{\beta})\;,
\end{aligned}
\tag{6.9}
$$

VI. The analytic expressions for 1-loop self energy

- $$C_{\{1,2,3\}}^{(5)}(\boldsymbol{\beta}) = (-1)^{-\beta_1}\frac{\Gamma(\beta_1+\beta_5)\Gamma(\beta_2+\beta_5-1)}{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}C_{\{1,2,3\}}^{(2)}(\boldsymbol{\beta})$$
$$+(-1)^{\beta_5-1}\frac{\Gamma(\beta_2+\beta_5-1)\Gamma(2-\beta_1-\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_1)}C_{\{1,2,3\}}^{(3)}(\boldsymbol{\beta}),$$
$$C_{\{1,2,3\}}^{(6)}(\boldsymbol{\beta}) = (-1)^{\beta_3+\beta_4-1}\frac{\Gamma(\beta_1+\beta_5)\Gamma(1-\beta_2-\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)}C_{\{1,2,3\}}^{(2)}(\boldsymbol{\beta})$$
$$+(-1)^{-\beta_2}\frac{\Gamma(2-\beta_1-\beta_5)\Gamma(1-\beta_2-\beta_5)}{\Gamma(\beta_3+\beta_4)\Gamma(1-\beta_5)}C_{\{1,2,3\}}^{(3)}(\boldsymbol{\beta}), \quad (6.10)$$

- thus

$$C_{\{1,2,3\}}^{(6)}(\boldsymbol{\beta}) = (-1)^{\beta_2}\frac{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5-1)\Gamma(\beta_2+\beta_5)}C_{\{1,2,3\}}^{(3)}(\boldsymbol{\beta})$$
$$+(-1)^{\beta_2+\beta_5}\frac{\Gamma(2-\beta_2-\beta_5)\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)\Gamma(\beta_2+\beta_5)}C_{\{1,2,3\}}^{(5)}(\boldsymbol{\beta}),$$
$$C_{\{1,2,3\}}^{(2)}(\boldsymbol{\beta}) = (-1)^{\beta_1+\beta_5}\frac{\Gamma(2-\beta_1-\beta_5)\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(\beta_2)\Gamma(\beta_1+\beta_5)}C_{\{1,2,3\}}^{(3)}(\boldsymbol{\beta})$$
$$+(-1)^{\beta_1}\frac{\Gamma(1-\beta_3-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5)\Gamma(\beta_2+\beta_5-1)}C_{\{1,2,3\}}^{(5)}(\boldsymbol{\beta}). \quad (6.11)$$

VI. The analytic expressions for 1-loop self energy

- In a similar way, the combinatorial coefficients of $\phi^{(i)}_{\{1,2,3\}}(\beta)$, $i = 4, 7, 9, 11$ are respectively written as

$$
\begin{aligned}
C^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_2+\beta_5}\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_5)\Gamma(2-\beta_2-\beta_5)}{\Gamma(\beta_2+\beta_5)\Gamma(\beta_3)\Gamma(1-\beta_2)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_2}\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_2+\beta_5)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\;, \\
C^{(7)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_1+\beta_5}\frac{\Gamma(2-\beta_1-\beta_5)\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_3)\Gamma(1-\beta_1)\Gamma(\beta_1+\beta_5)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_1}\frac{\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\;, \\
C^{(9)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_3}\frac{\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5-1)\Gamma(\beta_3+\beta_5)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_3+\beta_5}\frac{\Gamma(1-\beta_2-\beta_4)\Gamma(\beta_5)\Gamma(2-\beta_3-\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_3)\Gamma(\beta_3+\beta_5)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\;, \\
C^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_3}\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_2+\beta_5-1)\Gamma(\beta_3+\beta_5)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_3+\beta_5}\frac{\Gamma(1-\beta_1-\beta_4)\Gamma(2-\beta_3-\beta_5)\Gamma(\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_3)\Gamma(\beta_3+\beta_5)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta}) \quad (6.12)
\end{aligned}
$$

VI. The analytic expressions for 1-loop self energy

- The combinatorial coefficients of $\phi^{(i)}_{\{1,2,3\}}(\boldsymbol{\beta}),\ i=1,8,12$ are respectively written as

$$\begin{aligned}
C^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_1+\beta_5-1}\frac{\Gamma(\beta_3+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(\beta_3)\Gamma(\beta_1+\beta_5-1)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{1-\beta_2-\beta_5}\frac{\Gamma(\beta_3+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_2)\Gamma(\beta_3)\Gamma(\beta_2+\beta_5-1)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_1+\beta_2}\frac{\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(2-\beta_3-\beta_4)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\ , \\
C^{(8)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_1+\beta_5-1}\frac{\Gamma(\beta_2+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(1-\beta_1)\Gamma(\beta_2)\Gamma(\beta_1+\beta_5-1)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_1+\beta_3}\frac{\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(2-\beta_2-\beta_4)\Gamma(\beta_2+\beta_5-1)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{1-\beta_3-\beta_5}\frac{\Gamma(\beta_2+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_2)\Gamma(1-\beta_3)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\ , \\
C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta}) =& (-1)^{\beta_2+\beta_3}\frac{\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1+\beta_5-1)\Gamma(2-\beta_1-\beta_4)}C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_2+\beta_5-1}\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_2)\Gamma(\beta_2+\beta_5-1)}C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta}) \\
&+(-1)^{\beta_3+\beta_5-1}\frac{\Gamma(\beta_1+\beta_4-1)\Gamma(1-\beta_4)\Gamma(\beta_5)}{\Gamma(\beta_1)\Gamma(1-\beta_3)\Gamma(\beta_3+\beta_5-1)}C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})\ . \qquad (6.13)
\end{aligned}$$

VII. The expressions for a 2-loop massless triangle

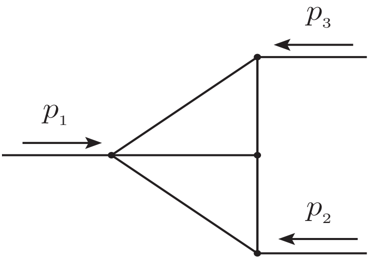


Figure: A 2-loop massless triangle diagram whose Feynman integral can be embedded in $G_{3,5}$.

VII. The expressions for a 2-loop massless triangle

In α-parametrization, the Feynman integral of the 2-loop massless triangle diagram drawn in Fig.2 is written as

$$
\begin{aligned}
iA_{WE}(p_1^2, p_2^2, p_3^2) = & \frac{\left(\Lambda_{\mathrm{RE}}^2\right)^{5-D}\Gamma(5-D)\exp\{i\pi(1-\frac{D}{2})\}}{(4\pi)^D(-i)^{4-D}} \int_S \omega_5(t) \\
& \times[(t_1+t_3)(t_2+t_4)+(t_1+t_2+t_3+t_4)t_5]^{5-3D/2} \\
& \times\Big[t_1t_2t_5p_1^2 + t_1(t_2t_3+t_3t_4+t_3t_5+t_4t_5)p_2^2 \\
& +t_2(t_1t_4+t_3t_4+t_3t_5+t_4t_5)p_3^2\Big]^{D-5}, \qquad (7.1)
\end{aligned}
$$

The integral can be embedded in the matroid $\boldsymbol{\xi}^{WE}$ of size 3×5

$$
\boldsymbol{\xi}^{WE} = \begin{pmatrix} 1 & 0 & 0 & 1 & p_1^2 \\ 0 & 1 & 0 & 1 & p_2^2 \\ 0 & 0 & 1 & 1 & p_3^2 \end{pmatrix}, \qquad (7.2)
$$

where the splitting coordinates $r_1 = p_1^2$, $r_2 = p_2^2$, and $r_3 = p_3^2$, respectively.

VII. The expressions for a 2-loop massless triangle

For convenience we take the exponent vector $\boldsymbol{\beta} = \boldsymbol{\beta}_{(WE)} + 2\Delta\ \mathbf{e}_1 - \Delta\ \mathbf{e}_2 - \Delta\ \mathbf{e}_3$ in Eq.(2.1), where $\boldsymbol{\beta}_{(WE)} = (1,\ 1,\ 1,\ 3-D,\ D-4) \in \mathbf{C}^5$, and Δ is a nonzero c-number. Certainly at the end of calculation, we take the limit $\Delta \to 0$. The boundary values of the Feynman integral of the 2-loop massless triangle are

$$A_{WE}(p^2, p^2, 0) = A_{WE}(p^2, 0, p^2) = \Big(\frac{\Lambda^2_{\rm RE}}{-p^2}\Big)^{5-D}\mathcal{B}_1\ ,$$
$$A_{WE}(0, p^2, p^2) = \Big(\frac{\Lambda^2_{\rm RE}}{-p^2}\Big)^{5-D}\mathcal{B}_2\ , \tag{7.3}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are derived through the Gegenbauer polynomial technique.

$$\begin{aligned}\mathcal{B}_1 =& \frac{\Gamma^2(\frac{D}{2}-1)\Gamma(4-D)}{(4\pi)^D\Gamma(\frac{D}{2})}\Big\{(3-D)\Gamma(2-\frac{D}{2})\\ &+\frac{\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}-2)}{\Gamma(\frac{3D}{2}-5)}\Big[\frac{2(4-D)}{6-D}-1\Big]\Big\}\ ,\\ \mathcal{B}_2 =& \frac{2\Gamma^3(\frac{D}{2}-1)\Gamma(5-D)}{(4\pi)^D\Gamma(D-2)\Gamma(\frac{3D}{2}-5)}\sum_{n=0}^{\infty}\frac{\Gamma(D-2+n)}{n!(\frac{D}{2}-1+n)^2}\\ &\times\Big\{\frac{1}{(1+n)(3-\frac{D}{2}+n)}+\frac{1}{(1+n)(D-3+n)}\\ &+\frac{1}{(D-3+n)(\frac{3D}{2}-5+n)}\Big\}\ .\end{aligned} \tag{7.4}$$

VII. The expressions for a 2-loop massless triangle

- $|p_3^2| < |p_2^2| < |p_1^2|$

$$
\begin{aligned}
&\left(\Lambda^2_{\rm RE}\right)^{5-D} A_{WE}(p_1^2, p_2^2, p_3^2) \\
&= C^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_1^2)^{-\beta_1}(p_2^2)^{-\beta_2}(p_3^2)^{1-\beta_3-\beta_4}\varphi^{(1)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{p_3^2}{p_2^2},\ \frac{p_3^2}{p_1^2}\Big) \\
&+ C^{(2)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_1^2)^{-\beta_1}(p_2^2)^{\beta_1+\beta_5-1}\varphi^{(2)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{p_3^2}{p_2^2},\ \frac{p_2^2}{p_1^2}\Big) \\
&+ C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_1^2)^{\beta_5-1}\varphi^{(3)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{p_3^2}{p_1^2},\ \frac{p_2^2}{p_1^2}\Big)\ .
\end{aligned}
\tag{7.5}
$$

- $|p_3^2| < |p_1^2| < |p_2^2|$

$$
\begin{aligned}
&\left(\Lambda^2_{\rm RE}\right)^{5-D} A_{WE}(p_1^2, p_2^2, p_3^2) \\
&= C^{(1)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_1^2)^{-\beta_1}(p_2^2)^{-\beta_2}(p_3^2)^{1-\beta_3-\beta_4}\varphi^{(1)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{p_3^2}{p_2^2},\ \frac{p_3^2}{p_1^2}\Big) \\
&+ C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_2^2)^{\beta_5-1}\varphi^{(5)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{p_3^2}{p_2^2},\ \frac{p_1^2}{p_2^2}\Big) \\
&+ C^{(6)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_1^2)^{\beta_2+\beta_5-1}(p_2^2)^{-\beta_2}\varphi^{(6)}_{\{1,2,3\}}\Big(\boldsymbol{\beta},\ \frac{p_3^2}{p_1^2},\ \frac{p_1^2}{p_2^2}\Big)\ .
\end{aligned}
\tag{7.6}
$$

VII. The expressions for a 2-loop massless triangle

- $|p_2^2| < |p_3^2| < |p_1^2|$

$$
\begin{aligned}
&\left(\Lambda_{\rm RE}^2\right)^{5-D} A_{WE}(p_1^2, p_2^2, p_3^2) \\
&= C_{\{1,2,3\}}^{(3)}(\boldsymbol{\beta})(p_1^2)^{\beta_5 - 1} \varphi_{\{1,2,3\}}^{(3)}\Big(\boldsymbol{\beta}, \frac{p_3^2}{p_1^2}, \frac{p_2^2}{p_1^2}\Big) \\
&\quad + C_{\{1,2,3\}}^{(7)}(\boldsymbol{\beta})(p_1^2)^{-\beta_1}(p_3^2)^{\beta_1+\beta_5-1} \varphi_{\{1,2,3\}}^{(7)}\Big(\boldsymbol{\beta}, \frac{p_2^2}{p_3^2}, \frac{p_3^2}{p_1^2}\Big) \\
&\quad + C_{\{1,2,3\}}^{(8)}(\boldsymbol{\beta})(p_1^2)^{-\beta_1}(p_2^2)^{1-\beta_2-\beta_4}(p_3^2)^{-\beta_3} \varphi_{\{1,2,3\}}^{(8)}\Big(\boldsymbol{\beta}, \frac{p_2^2}{p_3^2}, \frac{p_2^2}{p_1^2}\Big) \; .
\end{aligned} \tag{7.7}
$$

- $|p_2^2| < |p_1^2| < |p_3^2|$

$$
\begin{aligned}
&\left(\Lambda_{\rm RE}^2\right)^{5-D} A_{WE}(p_1^2, p_2^2, p_3^2) \\
&= C_{\{1,2,3\}}^{(8)}(\boldsymbol{\beta})(p_1^2)^{-\beta_1}(p_2^2)^{1-\beta_2-\beta_4}(p_3^2)^{-\beta_3} \varphi_{\{1,2,3\}}^{(8)}\Big(\boldsymbol{\beta}, \frac{p_2^2}{p_3^2}, \frac{p_2^2}{p_1^2}\Big) \\
&\quad + C_{\{1,2,3\}}^{(9)}(\boldsymbol{\beta})(p_1^2)^{\beta_3+\beta_5-1}(p_3^2)^{-\beta_3} \varphi_{\{1,2,3\}}^{(9)}\Big(\boldsymbol{\beta}, \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_1^2}\Big) \\
&\quad + C_{\{1,2,3\}}^{(10)}(\boldsymbol{\beta})(p_3^2)^{\beta_5-1} \varphi_{\{1,2,3\}}^{(10)}\Big(\boldsymbol{\beta}, \frac{p_2^2}{p_3^2}, \frac{p_1^2}{p_3^2}\Big) \; .
\end{aligned} \tag{7.8}
$$

VII. The expressions for a 2-loop massless triangle

- $|p_1^2| < |p_2^2| < |p_3^2|$

$$
\begin{aligned}
&\left(\Lambda^2_{\rm RE}\right)^{5-D} A_{WE}(p_1^2, p_2^2, p_3^2) \\
&= C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_3^2)^{\beta_5-1}\varphi^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{p_2^2}{p_3^2},\ \frac{p_1^2}{p_3^2}) \\
&+C^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_2^2)^{\beta_3+\beta_5-1}(p_3^2)^{-\beta_3}\varphi^{(11)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{p_2^2}{p_3^2},\ \frac{p_1^2}{p_2^2}) \\
&+C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_1^2)^{1-\beta_1-\beta_4}(p_2^2)^{-\beta_2}(p_3^2)^{-\beta_3}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{p_1^2}{p_3^2},\ \frac{p_1^2}{p_2^2})\ .
\end{aligned}
\tag{7.9}
$$

- $|p_1^2| < |p_3^2| < |p_2^2|$

$$
\begin{aligned}
&\left(\Lambda^2_{\rm RE}\right)^{5-D} A_{WE}(p_1^2, p_2^2, p_3^2) \\
&= C^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_2^2)^{-\beta_2}(p_3^2)^{\beta_2+\beta_5-1}\varphi^{(4)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{p_3^2}{p_2^2},\ \frac{p_1^2}{p_3^2}) \\
&+C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_2^2)^{\beta_5-1}\varphi^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{p_3^2}{p_2^2},\ \frac{p_1^2}{p_2^2}) \\
&+C^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta})(p_1^2)^{1-\beta_1-\beta_4}(p_2^2)^{-\beta_2}(p_3^2)^{-\beta_3}\varphi^{(12)}_{\{1,2,3\}}(\boldsymbol{\beta},\ \frac{p_1^2}{p_3^2},\ \frac{p_1^2}{p_2^2})\ .
\end{aligned}
\tag{7.10}
$$

VII. The expressions for a 2-loop massless triangle

The boundary conditions give

$$\begin{aligned}
&-2\Gamma(5-D)\Gamma(D-3)C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})+\Gamma(5-D)\Gamma(D-3)C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})=\mathcal{B}_1\;,\\
&-2\Gamma(5-D)\Gamma(D-3)C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})+\Gamma(5-D)\Gamma(D-3)C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})=\mathcal{B}_1\;,\\
&-\frac{1}{2}\Gamma(5-D)\Gamma(D-3)C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})-\frac{1}{2}\Gamma(5-D)\Gamma(D-3)C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})=\mathcal{B}_2\;.
\end{aligned}\tag{7.11}$$

Then we have

$$\begin{aligned}
&C^{(3)}_{\{1,2,3\}}(\boldsymbol{\beta})=-\frac{\mathcal{B}_1+\mathcal{B}_2}{2\Gamma(5-D)\Gamma(D-3)}\;,\\
&C^{(5)}_{\{1,2,3\}}(\boldsymbol{\beta})=C^{(10)}_{\{1,2,3\}}(\boldsymbol{\beta})=-\frac{\mathcal{B}_2}{\Gamma(5-D)\Gamma(D-3)}\;.
\end{aligned}\tag{7.12}$$

VIII. Summary

- After embedding Feynman integral on a variety of Grassmannian $G_{k,n}$, we construct all hypergeometric solutions under all possible affine spanning.
- We derive the inverse and adjacent relations among hypergeometric solutions under the same affine spanning, and the Gauss-Kummer relations among hypergeometric solutions from different affine spanning.
- In the neighborhood of the regular singularities, we write Feynman integral as a finite linear combinations of the canonical series solutions under same affine spanning.
- The combination coefficients are obtained by the reduced Gauss inverse relations, then the analytic expressions of the Feynman integral are continued to its whole domain of definition.

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THANKS!