Two Loop Spacelike Collinear Limit of $\mathcal{N} = 4$ Super Yang-Mills Amplitudes

(Towards Three-Loop-Five-Point Feynman Integrals analytic computation)

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Base on Two-Loop Spacelike Splitting Amplitude for $\mathcal{N} = 4$ Super-Yang-Mills Theory [2406.14604]

Johannes Henn, Rourou Ma, **YQX**, Kai Yan, Yang Zhang, Hua Xing Zhu

and An Analytic Computation of Three-Loop Five-Point Feynman Integrals [2411.xxxxx]

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Collinear Limit Factorization : Motivation

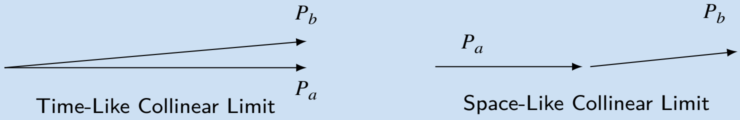
Factorization is a key property of gauge theory

$$(n\text{-point Amplitude}) \overset{\text{Colliner Limit}}{\Longrightarrow} \underbrace{(\text{Spliting-Amplitude})}_{\text{two-particle pole}} ((n-1)\text{-point Amplitude})$$

A slogan is,

"Scattering amplitude factorizes as the splitting amplitude and an amplitude with collinear particles replaced by a particle with the momenta P."

Possible violation? ... Timelike vs Spacelike



Time-like Collinear Factorization

Timelike: Collinear particles are both outgoing $p_a \cdot p_b > 0$. $(+,-,-,-)$

$$\text{Tree Level:} \quad |\mathcal{M}^{(0)}(p_a,p_b,...,p_n)\rangle \xrightarrow{a\|b} \mathbf{Sp}^{(0)}(p_a,p_b;P)|\mathcal{M}^{(0)}(P,...,\hat{a},...,\hat{b},p_n)\rangle$$

$$\text{One Loop:} \quad |\mathcal{M}^{(1)}(p_a,p_b,...,p_n)\rangle \xrightarrow{a\|b} \mathbf{Sp}^{(1)}(p_a,p_b;P)|\mathcal{M}^{(0)}(P,...,\hat{a},...,\hat{b},p_n)\rangle + \mathbf{Sp}^{(0)}(p_a,p_b;P)|\mathcal{M}^{(1)}(P,...,\hat{a},...,\hat{b},p_n)\rangle$$

$$\text{Two Loop:} \quad |\mathcal{M}^{(2)}(p_a,p_b,...,p_n)\rangle \xrightarrow{a\|b} \mathbf{Sp}^{(2)}(p_a,p_b;P)|\mathcal{M}^{(0)}(P,...,\hat{a},...,\hat{b},p_n)\rangle + \mathbf{Sp}^{(1)}(p_a,p_b;P)|\mathcal{M}^{(1)}(P,...,\hat{a},...,\hat{b},p_n)\rangle + \mathbf{Sp}^{(0)}(p_a,p_b;P)|\mathcal{M}^{(2)}(P,...,\hat{a},...,\hat{b},p_n)\rangle$$

Splitting amplitude is independent of the quantum number of non-collinear particles.

Catani, de Florian, Rodrigo, JHEP 07 (2012) 026

Space-like Collinear Factorization

One particle is incoming, one particle is outgoing $\Rightarrow \quad p_a \cdot p_b < 0.$

Splitting amplitude depends on the quantum number of non-collinear particles!

One-Loop example:

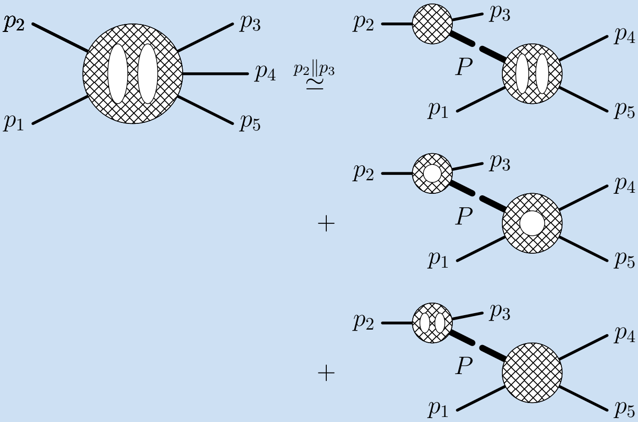
$$\begin{aligned}|\mathcal{M}^{(1)}(p_a,p_b,...,p_n)\rangle &\xrightarrow{a\|b} \mathbf{Sp}^{(1)}(p_a,p_b;P;\overbrace{p_1,\cdots,\hat{p}_a,\cdots,\hat{p}_b,\cdots p_n}^{\text{non-collinear particles}})|\mathcal{M}^{(0)}(P,...,\hat{a},...,\hat{b},p_n)\rangle \\ &+\mathbf{Sp}^{(0)}(p_a,p_b;P)|\mathcal{M}^{(1)}(P,...,\hat{a},...,\hat{b},p_n)\rangle\end{aligned}$$

Two-Loop example:

$$\begin{aligned}\left|\mathcal{M}^{(2)}(p_a,p_b,...,p_n)\right\rangle &\xrightarrow{a\|b} \mathbf{Sp}^{(2)}\left(p_a,p_b;P;p_1,\cdots,\hat{p}_a,\cdots,\hat{p}_b,\cdots p_n\right)\left|\mathcal{M}^{(0)}(P,...,\hat{a},...,\hat{b},p_n)\right\rangle \\ &+\mathbf{Sp}^{(1)}\left(p_a,p_b;P;p_1,\cdots,\hat{p}_a,\cdots,\hat{p}_b,\cdots p_n\right)\left|\mathcal{M}^{(1)}(P,...,\hat{a},...,\hat{b},p_n)\right\rangle \\ &+\mathbf{Sp}^{(0)}\left(p_a,p_b;P\right)\left|\mathcal{M}^{(2)}(P,...,\hat{a},...,\hat{b},p_n)\right\rangle.\end{aligned}$$

Catani, de Florian, Rodrigo, JHEP 07 (2012) 026

$$\left|\mathcal{M}^{(2)}(p_a, p_b, ..., p_n)\right\rangle \xrightarrow{a\parallel b} \mathbf{Sp}^{(2)}\left(p_a, p_b; P; p_1, \cdots, \hat{p}_a, \cdots, \hat{p}_b, \cdots p_n\right)\left|\mathcal{M}^{(0)}(P, ..., \hat{a}, ..., \hat{b}, p_n)\right\rangle$$
$$+\, \mathbf{Sp}^{(1)}\left(p_a, p_b; P; p_1, \cdots, \hat{p}_a, \cdots, \hat{p}_b, \cdots p_n\right)\left|\mathcal{M}^{(1)}(P, ..., \hat{a}, ..., \hat{b}, p_n)\right\rangle$$
$$+\, \mathbf{Sp}^{(0)}\left(p_a, p_b; P\right)\left|\mathcal{M}^{(2)}(P, ..., \hat{a}, ..., \hat{b}, p_n)\right\rangle .$$



Physical implications

Space-like collinear factorization violation

Process-dependent cross section collinear singularity

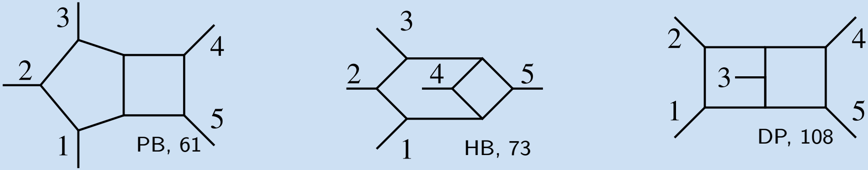
singularity not canceled by the (universal) parton distribution function

Our goals

- To explicitly compute two-loop collinear factorization violation terms.
- To find if there is collinear factorization violation for the cross section.

Analytic computation of five point Feynman integrals in the spacelike collinear limit

Two-Loop-Five-Points Massless Feynman Integrals



$(61+73+108)\times 5! = 29040$ integrals

$$\frac{\mathbb{Q}\text{-Number}}{\varepsilon^4} + \frac{\text{Log}}{\varepsilon^3} + \frac{\text{Li}_2}{\varepsilon^2} + \frac{\text{One-Fold-integration}}{\varepsilon^1} + \text{One-Fold-integration}$$

gitlab.com/pentagon-functions/PentagonMI

Challenge: Take the collinear limit for these results remains highly non-trivial.
Our strategy: Solve the Differential Equation Once again in Collinear Region.

Determine the boundary value of Feynman integrals for a point X_1 in the space-like collinear region
Solve the canonical differential equation analytically in the space-like collinear region

Abreu, Dixon, Herrmann, Page, Zeng, 10.1103/PhysRevLett.122.121603
Chicherin, Gehrmann, Henn, Wasser, Zhang and Zoia 10.1103/PhysRevLett.123.041603

Kinematics: $12 \to 345$

Scattering Region: $s_{12} > 0, s_{23} < 0, s_{34} > 0, s_{45} > 0, s_{15} < 0$

Spacelike collinear region for physics scattering $2 \parallel 3$

$$\{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\} \to \{sz, -4\delta^2, (1-z)xs, s, xs + c\delta\},$$

Namely: $p_2 \sim zP, p_3 \sim (1-z)P,\ z > 1,\ \delta \to 0$

The magic $c(y)$

$$c(y) := \frac{-8s^2 x(1+x)yz\sqrt{-s^3 x(1+x)(-1+z)z}}{1+y^2}$$

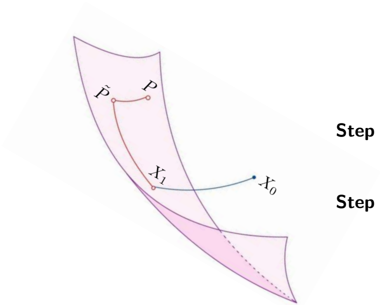
- Rationalized the square root.
- irrelevant for planar integrals, only appear in some of the nonplanar one.
- responsible for the violation.

$$\left(\frac{i+y}{i-y}\right)^4 = \frac{z_4 z_5}{\bar{z}_4 \bar{z}_5}$$

$$z_4 = \frac{\langle 23\rangle\langle 14\rangle}{\langle 12\rangle\langle 34\rangle}, \quad \bar{z}_4 = \frac{[23][14]}{[12][34]}$$

$$z_5 = \frac{\langle 23\rangle\langle 15\rangle}{\langle 12\rangle\langle 35\rangle}, \quad \bar{z}_4 = \frac{[23][15]}{[12][35]}$$

Solve Differential Equation in spacelike collinear region



Step 1: From the symmetric boundary point to a collinear boundary point:

$$\mathbf{f}(\lambda) = \left\{ \frac{4}{\lambda^2+1}, \frac{-4\lambda^2}{\lambda^2+1}, 1, \frac{2-2\lambda^2}{\lambda^2+1}, -1 \right\}$$

$$\underbrace{\{3,-1,1,1,-1\}}_{X_0} \to \underbrace{\{4,-4\delta^2,1,2,-1\}}_{X_1}$$

Step 2: From X_1 to a point $\tilde{P}$

$$\tilde{P} : \{sz, -4\delta^2, (1-z)sx, s, sx\}$$

Step 3: From $\tilde{P}$ to a generic point in spacelike collinear limit P

$$\tilde{P} : \{sz, -4\delta^2, (1-z)sx, s, sx + c\delta\}$$

$$c := \frac{-8s^2x(1+x)yz\sqrt{-s^3x(1+x)(-1+z)z}}{1+y^2}$$

- The boundary values at X_1 are polynomials of $\log\delta$, can be expressed in terms of the same set of transcendental constant for X_0, via 500 digits PSLQ.
- All master integrals solved in terms of s, z, x, δ and y, in terms of GPL functions up to weight 4.

Chicherin, Sotnikov 10.1007/JHEP12(2020)167

2loop 5point $\mathcal{N}=4$ Amplitudes Amplitudes structure

$$\mathcal{N}=4\ \text{Amplitude} = \sum_{k=-4}^{0}\sum_{\lambda=1}^{22}\sum_{i=1}^{6} \epsilon^k \times \mathbb{Q} \times (\text{Color Factor})_\lambda \times (\text{Parke-Taylor Factor})_i \times (\text{Pure Function})$$

$$\begin{aligned}
T_1 &= [\mathrm{Tr}(12345) - \mathrm{Tr}(15432)], T_7 = [\mathrm{Tr}(12543) - \mathrm{Tr}(13452)]\\
T_2 &= [\mathrm{Tr}(14325) - \mathrm{Tr}(15234)], T_8 = [\mathrm{Tr}(14523) - \mathrm{Tr}(13254)]\\
T_3 &= [\mathrm{Tr}(13425) - \mathrm{Tr}(15243)], T_9 = [\mathrm{Tr}(13524) - \mathrm{Tr}(14253)]\\
T_4 &= [\mathrm{Tr}(12435) - \mathrm{Tr}(15342)], T_{10} = [\mathrm{Tr}(12534) - \mathrm{Tr}(14352)]\\
T_5 &= [\mathrm{Tr}(14235) - \mathrm{Tr}(15324)], T_{11} = [\mathrm{Tr}(14532) - \mathrm{Tr}(12354)]\\
T_6 &= [\mathrm{Tr}(13245) - \mathrm{Tr}(15423)], T_{12} = [\mathrm{Tr}(13542) - \mathrm{Tr}(12453)]\\
T_{13} &= \mathrm{Tr}(12)[\mathrm{Tr}(345) - \mathrm{Tr}(543)], T_{18} = \mathrm{Tr}(13)[\mathrm{Tr}(245) - \mathrm{Tr}(542)],\\
T_{14} &= \mathrm{Tr}(23)[\mathrm{Tr}(451) - \mathrm{Tr}(154)], T_{19} = \mathrm{Tr}(24)[\mathrm{Tr}(351) - \mathrm{Tr}(153)]\\
T_{15} &= \mathrm{Tr}(34)[\mathrm{Tr}(512) - \mathrm{Tr}(215)], T_{20} = \mathrm{Tr}(35)[\mathrm{Tr}(412) - \mathrm{Tr}(214)],\\
T_{16} &= \mathrm{Tr}(45)[\mathrm{Tr}(123) - \mathrm{Tr}(321)], T_{21} = \mathrm{Tr}(41)[\mathrm{Tr}(523) - \mathrm{Tr}(325)],\\
T_{17} &= \mathrm{Tr}(51)[\mathrm{Tr}(234) - \mathrm{Tr}(432)], T_{22} = \mathrm{Tr}(52)[\mathrm{Tr}(134) - \mathrm{Tr}(431)],
\end{aligned}$$

Full Color!

$$\begin{aligned}
\mathrm{PT}_1 &= \mathrm{PT}(12345), \mathrm{PT}_2 = \mathrm{PT}(12354),\\
\mathrm{PT}_3 &= \mathrm{PT}(12453), \mathrm{PT}_4 = \mathrm{PT}(12534),\\
\mathrm{PT}_5 &= \mathrm{PT}(13425), \mathrm{PT}_6 = \mathrm{PT}(15423).
\end{aligned}$$

Collinear Limit for Parke-Taylor Factor: $\left\{\frac{\mathrm{PT}_1}{\mathrm{PT}_1}, \ldots, \frac{\mathrm{PT}_6}{\mathrm{PT}_1}\right\} \underset{\delta\to 0}{\sim} \left\{1, \frac{-x}{x+1}, 0, 0, 0, 1\right\}$

Parke-Taylor factors become linearly dependent, and some drop out.

2loop 5point $\mathcal{N}=4$ Amplitudes Spacelike Collinear Limit

$$\mathcal{A}_5(p_1,p_2,p_3,p_4,p_5) \xrightarrow{p_2\parallel p_3} \mathbf{Sp}\times\mathcal{A}_4(p_1,\tilde{P},p_4,p_5)$$

Splitting Amplitude:

$$\mathbf{Sp}^{(1)} = \left[\frac{\mu^2 z}{s_{ab}(1-z)}\right]^{\epsilon}\left\{2N_c\,\overline{r}_S^{(1)}(z+i0) + \mathbf{T}_a\cdot\mathbf{T}_{\text{in}}\,(2\pi i)\,c_1(\epsilon)\frac{1}{\epsilon}\right\}\mathbf{Sp}^{(0)}$$

$$\begin{aligned}\mathbf{Sp}^{(2)} = &\left[\frac{\mu^2 z}{s_{ab}(1-z)}\right]^{2\epsilon}\left\{4N_c^2\,\overline{r}_S^{(2)}(z+i0)\right.\\ &+N_c\,\mathbf{T}_a\cdot\mathbf{T}_{\text{in}}\,(2\pi i)\left[c_2(\epsilon)\,\frac{1}{\epsilon^3}+c_1^2(\epsilon)\left(-\frac{2}{\epsilon^2}\ln z+\frac{2}{\epsilon}\ln z\ln\left(\frac{z}{z-1}\right)\ -2\,\mathrm{Li}_3\left(1-\frac{1}{z}\right)-\ln(z)\ln^2\left(\frac{z}{z-1}\right)\right)\right]\\ &+\sum_{I\in\text{outgoing}}[\mathbf{T}_a\cdot\mathbf{T}_{\text{in}},\mathbf{T}_a\cdot\mathbf{T}_I]\,(2\pi i)\left[\left(\frac{1}{2\epsilon^2}-\frac{1}{2}\zeta_2\right)(\ln|z_I|^2+i\pi)+\frac{1}{6}\left(\ln^2\frac{z_I}{\bar{z}_I}+4\pi^2\right)\ln\frac{z_I}{\bar{z}_I}+2\zeta_3\right]\\ &\left.+\sum_{I\in\text{outgoing}}\{\mathbf{T}_a\cdot\mathbf{T}_{\text{in}},\mathbf{T}_a\cdot\mathbf{T}_I\}\,(2\pi^2)\left[\frac{1}{2\epsilon^2}-\frac{1}{2}\zeta_2\right]\right\}\mathbf{Sp}^{(0)}\end{aligned}$$

Dipole term is a pure phase ⇒ vanish in *Square Amplitude*.
Tripole term is anti-symmetric ⇒ vanish in *Color-Summed Square Amplitude*.

Factorization is restored.

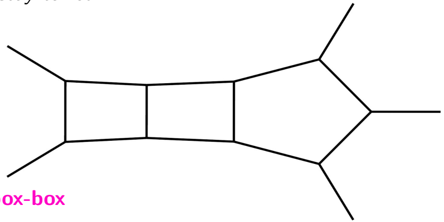
Dixon, Herrmann, Yan, Zhu,10.1007/JHEP05(2020)135

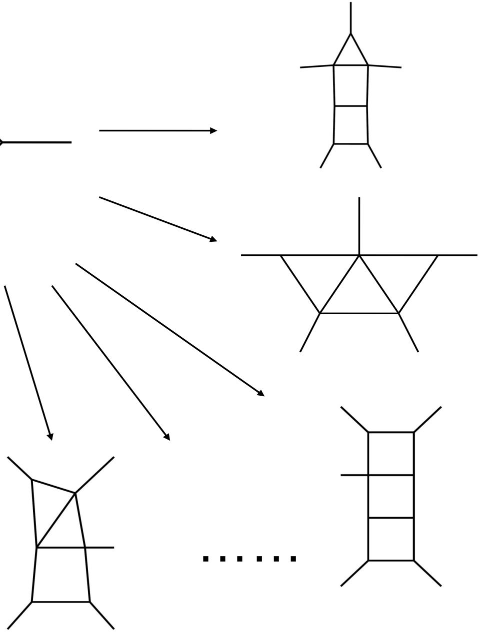
Towards Analytic Computation of

Three-Loop-Five-Point Feynman Integrals

The first step toward N3LO $2 \to 3$

[2411.XXXX] *stay tuned*





- **Pentagon-box-box**
- 316 Master Integrals.

√ Computational Algebraic Geometry implemented in NeatIBP.

√ All analytic till weight-6.

√ 5 scales and 1 $\sqrt{\text{square root}}$.

√ Use only $\log$, Li_2, Li_3 and their one-fold integration.

√ Full Analytic Boundary Values.

more advanced techniques...

Thank You!