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Matrix moment approach to positivity bounds and UV reconstruction from IR

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based on [2411.xxxxx] with 周双勇

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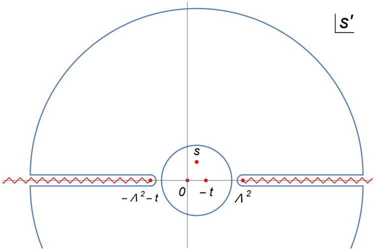
Positivity Bounds Setup

- Locality: Froissart bound $|A(s \to \infty, t)| \sim s^{2-\epsilon}$.
- Unitarity: $S^\dagger S = 1, 0 \leq \mathrm{Im}\, A^\ell(s) \leq |A^\ell(s)|^2 \leq 1$.
- Analyticity: $A(s,t)$ have clear analytic structure in the complex s plane for fixed t.

Snowmass White Paper: UV Constraints on IR Physics, de Rham, Kundu, Reece, Tolley & SYZ, 2203.06805

Twice-subtracted dispersion relation

$$A_{\dots}(s,t) = \int_{\Lambda^2}^{\infty} \frac{\mathrm{d}s'}{s'^2} \left[\frac{s^2\, \mathrm{Im}\, A_{\dots}}{s'-s} + \frac{s^2\, \mathrm{Im}\, A_{\dots}}{s'-u} \right]$$



Sum Rules: $g_{\dots}^{i,j} \equiv\sim \partial_s^i \partial_t^j A_{\dots}(s,t)$

$$g_{\dots}^{i,j} \sim \sum_\ell \int_{\Lambda^2}^{\infty} (\cdots\cdots)\, \mathrm{Im}\, A_{\dots}^{\ell}(s')\mathrm{d}s',$$

Example: $g^{2,1} = \sum_\ell \int_{\Lambda^2}^{\infty} \frac{\ell(\ell+1)}{s'^4} \mathrm{Im}\, A^\ell(s')\mathrm{d}s'$

EFT coefficients:

$$A_{\mathbf{1234}}(s,t) \sim \sum_{i,j} g^{i,j}_{\mathbf{1234}} (s + \frac{t}{2})^i t^j$$

Wilson coefficients:

$$\mathcal{L}_{\text{Vec}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + a_1 (F_{\mu\nu} F^{\mu\nu})^2 + a_2 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 + \cdots \quad (1)$$

$$\mathcal{L}_{\text{Grav}} = R + \frac{\lambda_3}{3!} R^{(3)} + \frac{\lambda_4}{2^3} (R^{(2)})^2 + \frac{\tilde{\lambda}_4}{2^3} (R\tilde{R})^2 + \cdots \quad (2)$$

We constrain $g^{i,j}_{\mathbf{1234}}$ not $a_1, a_2, \lambda_3, \lambda_4, \tilde{\lambda}_4, \cdots$.

Arkani-Hamed, Huang & Huang, *talks in 2017*, 2012.15849

Bellazzini, Miro, Rattazzi, Riembau & Riva, 2011.00037

$$g^{2i,0} = \int_{\Lambda^2}^{\infty} \frac{2\mathrm{d}s'}{\pi (s')^{2i+1}} \,\mathrm{Im}\, A(s',0) \xrightarrow{x_1 \equiv \Lambda^2/s'} g^{2i,0} = \Lambda^{-4i} \int_0^1 x_1^{2i} \mathrm{d}\mu(x_1)$$

A standard Hausdorff moment problem!

Solvability condition:

Positive-semidefiness of Hankel matrices generated by $1, x_1, 1-x_1$:

$$\mathcal{H}(g^{2i,0}) \succeq 0 \quad \& \quad \mathcal{H}^{\text{shift}}(g^{2i,0}) \succeq 0 \quad \& \quad \mathcal{H}(g^{2i,0}) - \mathcal{H}^{\text{shift}}(g^{2i,0}) \succeq 0$$

$$\mathcal{H}(g^{2i,0}) \equiv \begin{pmatrix} g^{2,0} & g^{4,0} & g^{6,0} & \cdots \\ g^{4,0} & g^{6,0} & g^{8,0} & \cdots \\ g^{6,0} & g^{8,0} & g^{10,0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathcal{H}^{\text{shift}}(g^{2i,0}) \equiv \begin{pmatrix} g^{4,0} & g^{6,0} & g^{8,0} & \cdots \\ g^{6,0} & g^{8,0} & g^{10,0} & \cdots \\ g^{8,0} & g^{10,0} & g^{12,0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Main observation:

Chiang, Huang, Li, Rodina & Weng, 2105.02862

$$g^{i,j} = \sum_{j'} V^{i,j}_{\gamma_1,\gamma_2} \mathfrak{m}^{\gamma_1,\gamma_2}, \mathfrak{m}^{\gamma_1,\gamma_2} = \int_{\mathcal{K}} x_1^{\gamma_1} x_2^{\gamma_2} \mathrm{d}\mu(x_1, x_2), x_1 \sim \frac{1}{s'}, x_2 \sim \frac{J^2}{s'}, J^2 = \ell(\ell+1)$$

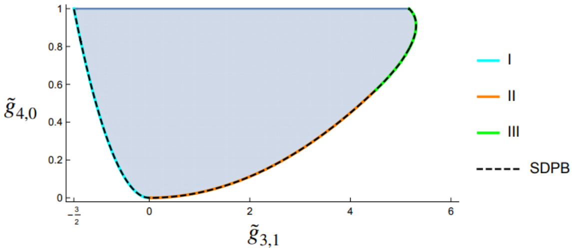
Necessary condition: Hankel matrices generated by any positive polynomial on $\mathcal{K}$.

$q(x_1) = c_{00} + c_{10} x_1 + c_{01} x_2 + \cdots,$
$v^T = (c_{00}, c_{10}, c_{01}, \cdots),$

$$0 \leq \int_{\mathcal{K}} p(x_1, x_2) q(x_1, x_2)^2 \mathrm{d}\mu \sim v^T \mathcal{H}_p v$$

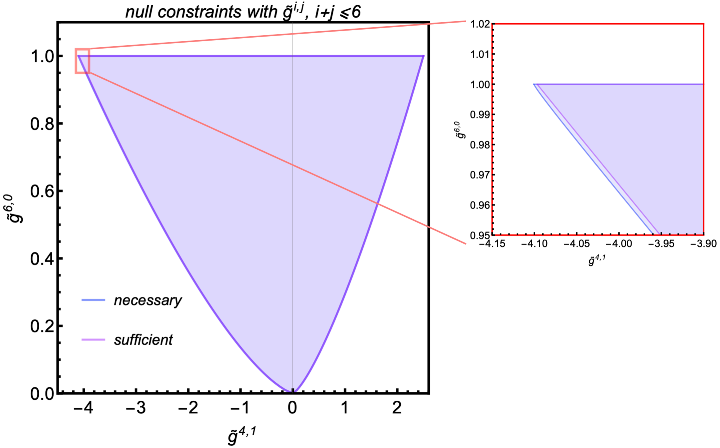
Agrees with numerical method in simple cases

Chiang, Huang, Li, Rodina & Weng, 2105.02862



Solvability conditions
(Necessary and sufficient)

- A set of positive polynomials (only depend on $\mathcal{K}$) to generate Hanekl matrices;
- The flat extension to restore full moment sequence from the truncated moment sequence.



(1) Minor changes: real-valued $\Longrightarrow$ matrix-valued (vast math literature)

(2) Major difficulty: Legendre polynomial $\Longrightarrow$ Wigner d-matrix: $(\cos\theta = 1 + 2t/s)$

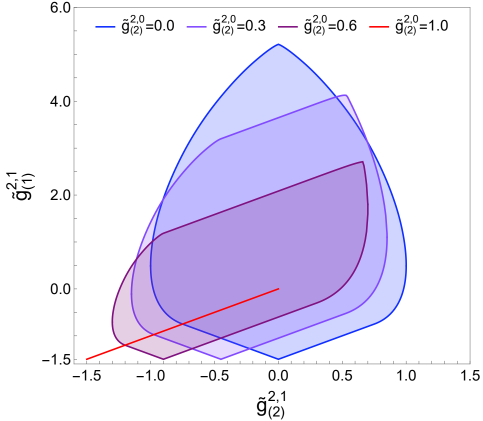
$$d^{\ell}_{h_{12}h_{43}}(\theta) \equiv \sum_{k=0}^{\infty} \mathcal{D}^{k}_{\ell,h_{12},h_{43}}(t/s)^k, \quad \mathcal{D}^{k}_{\ell,h_{12},h_{43}} \sim \sqrt{\prod_{i=1}^{|h|} \left(J^2 - i(i-1)\right)^{-\operatorname{sign} h}}$$

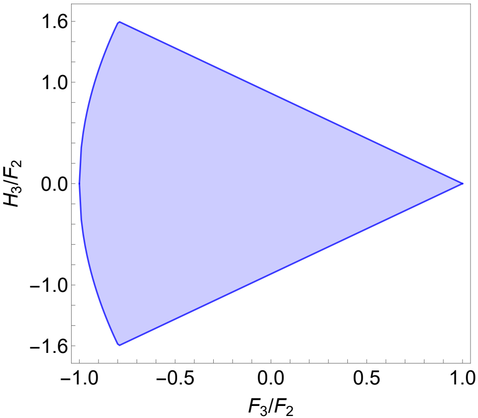
For spin-1, **generalized** moments: $\tilde{\mathfrak{m}}^{\gamma_1\gamma_2}_{\mathbf{1234}} = \int_{\mathcal{K}} \sqrt{x_2(x_2 - 2x_1)}\, x_1^{\gamma_1} x_2^{\gamma_2} \mathrm{d}\mu(x_1, x_2)$

Solved by a lift: $\mathcal{K} \to \mathcal{K}' = \{(x_1, x_2, x_3) | (x_1, x_2) \in \mathcal{K}, x_3^2 = x_2^2 - 2x_1x_2\}$
$(x_1, x_2) \mapsto (x_1, x_2, \sqrt{x_2(x_2 - 2x_1)})$

$$\tilde{\mathfrak{m}}^{\gamma_1\gamma_2\gamma_3}_{\mathbf{1234}} = \int_{\mathcal{K}'} x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3} \mathrm{d}\mu'(x_1, x_2, x_3)$$

Our formulation results agree with previous numerical results



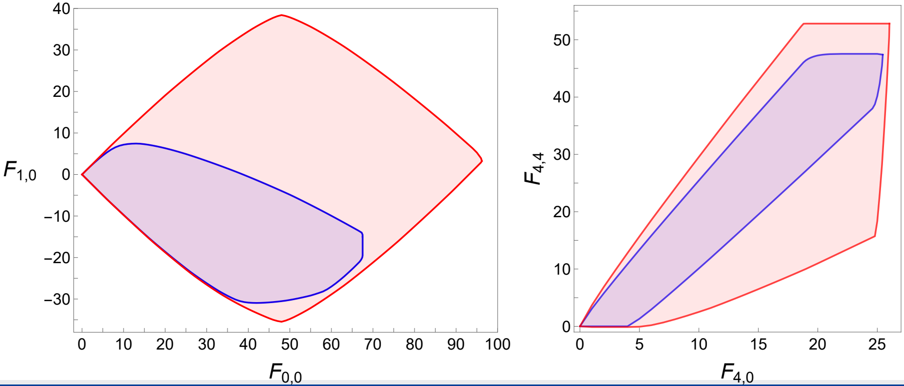


Du, Zhang & SYZ, 2111.01169

Henriksson, McPeak, Russo & Vichi, 2203.08164

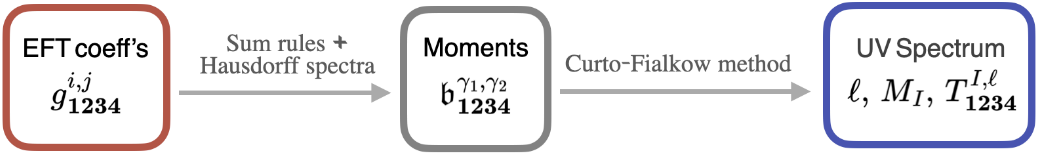
Spin 2 (Improved)

$$0 \leq \mathrm{Im}\, A^{\ell}(s) \leq 1, \qquad A_{++--}(s,t) = \sum \frac{8\pi G_N}{stu} + \alpha_1 \frac{1}{s} + \alpha_2 \frac{t^2}{s} + \sum_{k,q} F_{k,q} s^{k-q} t^q$$



IR to UV Spectrum

From IR EFT coefficients to UV spectrum,



Forward limit: Determinacy of Hausdorff moment problem

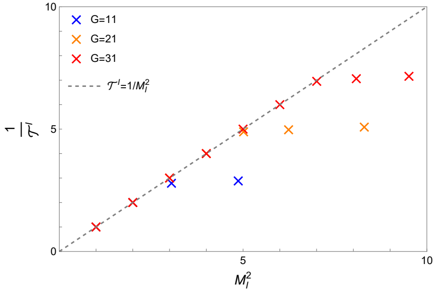
$\implies$ unique mass spectrum

$\implies$ unique moment sequence

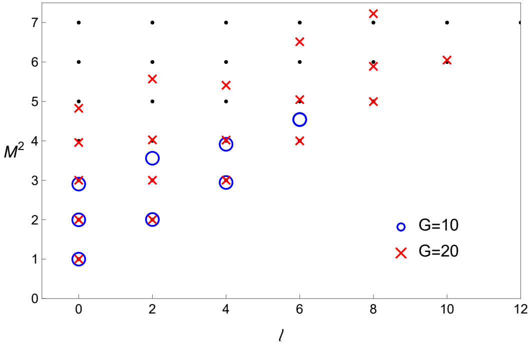
Non-forward limit: Tchakaloff's theorem $\implies$ a special UV completion

$$\mathrm{Im}\, A^{\ell}_{1234}(s') \sim \sum_{I,\ell} T^{I,\ell}_{1234}\,\delta(s' - M_I^2)$$ for any EFT satisfying positivity bounds

Better with more precise IR



Mass spectrum of Veneziono amplitude



Spectrum of Visrosoro amplitude

Determinacy: Uniqueness of representing measure of moment problem

$$\Downarrow$$

Uniqueness of UV completion

Assign $\mathfrak{c}^{I,\gamma}_{1234} \sim \sum_{\ell} T^{I,\ell}_{1234} \left(\ell(\ell+1)\right)^{\gamma}$ to every mass

every $\mathfrak{c}^{\gamma}$, $\mathfrak{c}^{\gamma+1}/\mathfrak{c}^{\gamma} \sim \gamma^2 \Longrightarrow$ boundary of unique UV completion.

UV completion of string amplitude is unique

because of $\ell \leq M_I^2 - 1$ or $2M_I^2 - 2$.

- Matrix moment problem as alternative approach to get positivity bounds - vast math literature
- Generalized matrix moment problem to include multiple fields and spins
- Matrix L moment problem to give improved bounds
- Inverse problem: exact UV spectrum from EFT data

Thank you for your listening!